Radiative Transfer of Polarized X-rays: Magnetized Thomson Scattering in Neutron Stars

by

Joseph A. Barchas

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree DOCTOR OF PHILOSOPHY

Approved, Thesis Committee:

Matthew G. Baring
Professor

Matthew Foster
Assistant Professor

Michael Wolf
Professor

Houston, Texas
May, 2017
ABSTRACT

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This thesis is a focused study of the polarization characteristics of radiative transfer in a strong magnetic field. The main process examined here is magnetized Compton scattering in a non-relativistic regime (i.e. magnetized Thomson scattering), and we focus on applying this study to predict polarization properties of the X-ray emission from magnetars. Magnetars are a highly magnetic sub-class of neutron stars, characterized by their extremely high surface magnetic fields, comparable to or exceeding the quantum critical field ($B_{cr} \approx 4.41 \times 10^{13}$ Gauss) at which an electron’s cyclotron energy and rest mass energy are equal. There are 29 known/candidate magnetars at this time, and they commonly exhibit persistent quasi-thermal surface emission in soft X-rays with flat tails extending into the hard X-rays up to around 150 keV, as well as transient bursting activity in hard X-rays attributed to magnetospheric flares.

Magnetized Thomson scattering refers to electron-photon scattering in a background magnetic field. The field introduces anisotropy to the problem, giving it a more complicated angular dependence. It also produces a strong frequency dependence to the cross section: it is resonant at the cyclotron frequency $\omega_B = eB/mc$. Additionally, electron motion perpendicular to the field becomes increasingly suppressed
at higher field strengths, leading to a reduction in the cross section for certain incoming photon angles when $\omega \ll \omega_B$. There are complicated polarization characteristics for the process as well, with the differential cross section depending on the initial and final polarization state of the photon. An important distinction occurs between photons that have a component of linear polarization parallel to the field and those that are fully orthogonal to it.

We explore this process in detail using a Monte Carlo simulation model, treating the transfer primarily in slab geometries, a common simplification. This allows a direct comparison with previous work and is an important step towards achieving more complicated geometries and scattering regimes. We fully map the frequency and angular dependence of this process in the optically thick regime, capturing both resonant and non-resonant properties of the scattering. We present results for a model of magnetar persistent surface emission, as well as a simple magnetar flare model. In both cases the transfer is purely due to magnetic Thomson scattering, and we superimpose the emission from regions of different temperature, density, and magnetic field. For magnetar surface emission, we see a phase-dependent linear polarization, forming either a single- or double-peaked pattern with a maximum level of roughly $\sim 25\%$, depending on the angle between the observer and the spin axis. This could have important implications for polarimetric determination of the effect of vacuum birefringence as polarized X-rays transfer through the magnetosphere to infinity. For the flare model we see much stronger polarization signals as the emission is coming from more localized regions, and it is highly dependent on viewing geometry and frequency.

A secondary process is also examined due to its importance in magnetized plasmas: the so-called generalized Faraday effect. This is analogous to the ordinary Faraday effect, where the phase lag caused by birefringence of circular eigenmodes of electromagnetic waves produces a constant rotation of the plane of linear polarization for a propagating wave. The generalized effect occurs when the eigenmodes are no longer
circular, and it produces a very similar rotation when viewed in terms of the Poincaré sphere. When this effect is assumed to be prolific, the transfer can be reformulated in terms of the normal modes in what is called the normal mode description. We explore the parameter space in which this description is valid, and develop a method to handle transfer in certain regimes where it is invalid. This prepares the way for incorporating such nuances in future developments of the magnetized radiative transfer problem.
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CHAPTER 1
Introduction

This thesis is a focused study of the polarization nature of X-rays scattered in highly magnetic environments, principally two main classes of neutron stars: pulsars and magnetars. Specifically, it is concerned with the polarization properties of magnetized Compton scattering and its consequences on the radiative transfer problem, which reveals an array of energy-dependent and angle-dependent characteristics. This is applicable to models that predict polarization observations coming from this kind of environment, and will be observable with future X-ray polarimetry missions.

In this thesis, we present results of a self-developed Monte Carlo radiative transfer model of the Compton scattering of polarized X-rays in an external magnetic field. Our model is built to handle arbitrary field orientations, and is in good agreement with the most detailed previous work (Whitney 1989), which we show explicitly for atmospheric slabs with vertical fields, a relatively simple case.

We show that localized patches of magnetized atmosphere put off strong polarization signals for a large range of frequencies and magnetic field orientations. The anisotropy induced by the magnetic field produces the strong angular dependence seen in intensity and polarization profiles. This dependence is further strengthened near the cyclotron resonance; more varied angular profiles are seen for this energy regime. There is also a general correlation between higher intensity and weaker polarization.

However, when integrating over an entire thermalized atmosphere, there is a strong depolarizing effect that occurs, mainly due to the different magnetic field directions
sampled. We show in Chapter 5 that even in the presence of this depolarizing effect, the field directions near the poles and equators are on the same axis, so their linear polarizations are able to survive the integration process and produce noticeable polarization signals, as high as 15%. However, it is the presence of localized hot spots, either on the surface or in the magnetosphere during flare activity, that induces the strongest polarization signals (> 35%) in bands where the hot emission dominates.

We also show that, at least in the soft X-ray band, models of polarized emission can produce significant polarization degrees *without* including vacuum birefringence effects in photon propagation through the magnetosphere: these help facilitate high degrees of polarization from entire stellar surfaces. Thus, the recent claims that high degrees of polarization seen in optical for the X-ray isolated neutron star RX J1856.5-3754 serve as evidence for the action of vacuum birefringence may be premature.

For magnetar flares, we use a two-zone thermal emission model inspired by the sum-of-two-blackbodies spectral fit common to these events. We see the polarization properties of each region dominating in the respective spectral regimes where their blackbody intensities dominate, hinting that spectrally resolved polarization observations can diagnose the field geometry of the emitting region in these flares.

Finally, we address the “generalized Faraday effect”: a propagation effect that occurs in magnetized plasmas due to birefringence of the normal modes. The usual Faraday effect seen in radio is produced by birefringence of circular normal modes. However, the normal modes are elliptical in general, making the description more complicated. We have explicitly addressed this general case with an aim of including its effect within the simulation in the near future.

The layout of this thesis is as follows. Chapter 1 gives an introductory review of the types of astrophysical sources that benefit from this type of analysis. Chapter
2 reviews the methods for describing polarization and derives the relevant scattering formulae. Chapter 3 describes the Monte Carlo method used to perform the numerical modeling, while Chapter 4 presents basic results and discussion providing insight into the nature of this process. Chapters 5 and 6 presents results relevant to magnetars, such as thermal surface emission and flare models. Chapter 7 reviews the related “generalized Faraday effect” and its importance to this problem. Conclusions are presented in Chapter 8.

1.1 Pulsars

Pulsars are understood to be neutron stars, the dense corpse of a massive star that has ended it’s fusion fuel-burning life in a core-collapse supernova event. Neutron stars, first theorized by Baade & Zwicky (1934), hold themselves up against gravitational collapse in a different way than other astrophysical objects: rather than a thermal pressure supplied by fusion processes in the core, it is the quantum degeneracy pressure of neutrons that counteracts gravity to result in a stable object. They are more massive than typical white dwarfs, which are another type of stellar remnant that uses a similar mechanism to counteract gravity: the quantum degeneracy pressure of electrons.

Neutron stars are incredibly dense, typically possessing radii of 10 – 20 km and masses about 1.5 – 2 times that of the sun. This corresponds to an average density of $10^{13} - 10^{15}$ g/cm$^3$, comparable to the density of an atomic nucleus, which averages about $10^{14}$ g/cm$^3$. As such, their interiors are home to exotic physics, and much study has gone into understanding the equation of state for such a dense system. This is important to determining the mass/radius relation as the equation of state determines the amount of pressure support. If angular momentum and magnetic flux of the
progenitor star are approximately conserved in the core-collapse supernova explosion, the neutron star is born with an incredibly high rotation rate \((P \sim 10 - 10^{-3}\text{ s})\) and strong magnetic field \((B \sim 10^9 - 10^{15}\text{ G})\).

There is an upper range to the mass of a neutron star, above which even neutron degeneracy pressure cannot hold the star back from collapse, and a black hole is formed. This is called the Tolman-Oppenheimer-Volkoff limit, and is about \(\sim 1.5 - 3M_\odot\), depending on assumptions about the equation of state above nuclear densities (Shapiro & Teukolsky 1983). Because of uncertainties in the equation of state, observations involving binary systems that allow a determination of the neutron star mass are quite useful to discriminating between theories of neutron star interiors. The modern observationally determined mass range is \(\sim 1 - 2M_\odot\) (Rawls et al. 2011).

Pulsars are a special observational class of neutron stars. First discovered in radio emission by Bell & Hewish (1967), they are characterized by their extremely regular pulsations. The rotating neutron star explanation became consensus because other potential explanations failed to accommodate the the range of periods observed: for example, neutron star pulsations not tied to rotation would be too fast, and rotating white dwarfs would be too slow. Though discovered in the radio waveband, pulsars have since been found to also emit in visible, X-ray, and/or gamma-ray frequencies. Over long periods of time, the pulsation rate of a given pulsar is observed to slowly decrease, aside from hiccups known as “glitches”, believed to be due to seismic events on the neutron star crust. The exact nature of the pulsed emission is still an active topic of study, but it is agreed that the pulsations are due to beams of radiation emanating above the magnetic poles, which are offset from the spin axis. The beamed emission pattern sweeps across interstellar space like a lighthouse with each rotation; when Earth is lucky enough to be in the path of this sweep, we see the neutron star
1.1: PULSARS

as a pulsar.

There are now over 2400 neutron stars known in the Milky Way, most of these radio pulsars*, with around 200 pulsars emitting at gamma-ray energies in excess of 1 GeV (see Abdo et al. 2013 for the second Fermi-LAT pulsar catalog). The typical magnetic fields of radio and gamma-ray pulsars range from $10^9$ Gauss to $10^{13}$ Gauss, as estimated by rotational “spin-down” attributed to electromagnetic torques enabled by the spinning stellar magnetic dipolar field.

1.1.1 Oblique Rotator Model

Early efforts to model pulsars were taken by Pacini (1967), who considered a neutron star with a dipole magnetic field and a rotation axis offset from the dipole axis. This oblique rotator model emits according to basic electromagnetism, with the emission powered solely by the rotational energy.

In vacuum, this produces monochromatic radiation at the pulsar frequency (i.e. Hz-kHz) which depletes the rotational energy of the neutron star. However, this monochromatic signal is likely to be reflected by circumstellar gas making it unable to reach us (Pacini 1968). Perhaps more importantly, such low frequency radiation is absorbed by the ionosphere, and cannot be observed on Earth.

A complete description of the vacuum field of a rotating star was first given by Deutsch (1955); the field is distorted by relativity, producing a “sweepback” effect where the co-rotation becomes relativistic (see the Appendix). The absence of plasma surrounding the pulsar is reasoned due to the strong gravity present. While this theory fails to predict observed emission properties, it does allow an estimate of the magnetic field strength and age.

*See the ATNF pulsar catalog at http://www.atnf.csiro.au/people/pulsar/psrcat/.
CHAPTER 1: INTRODUCTION

Ostriker & Gunn (1969) first derived the formula for pulsar characteristic age as a function of the period $P$ and period time derivative $\dot{P}$ based on this oblique rotator model (see Shapiro & Teukolsky 1983 for a detailed derivation). For a uniformly magnetized sphere of radius $R_{\text{NS}}$ and polar surface field $B_p$, we have a magnetic dipole moment $\mu_B = B_p R_{\text{NS}}^3/2$. For angular rotation rate $\Omega = 2\pi/P$, and with magnetic and rotational poles misaligned by angle $\alpha$, the Larmor formula gives

$$P_{\text{EM}} = \frac{2\mu_B^2}{3c^3} = \frac{B_p^2 R_{\text{NS}}^6 \Omega^4 \sin^2 \alpha}{6c^3}. \quad (1.1)$$

This corresponds to a drop in the rotational kinetic energy $E_{\text{rot}} = I\dot{\Omega}^2/2$, where $I$ is the moment of inertia:

$$-P_{\text{rot}} = -I\dot{\Omega} = 4\pi^2 P^{-3} \dot{P} \quad (1.2)$$

Equating the radiation energy loss with the spin energy loss, and solving for $B_p$, we have $B_p = \sqrt{6c^3 IP\dot{P}}/(2\pi R_{\text{NS}}^3 \sin \alpha)$ as the measure of the surface field at the pole, or

$$B_p = 6.4 \times 10^{11} G \left(\frac{I}{10^{45} \text{g cm}^2}\right)^{1/2} \left(\frac{P}{1\text{s}}\right)^{1/2} \left(\frac{\dot{P}}{10^{-12}}\right)^{1/2} \left(\frac{10^6 \text{cm}}{R_{\text{NS}}}\right)^3 \frac{1}{\sin \alpha}. \quad (1.3)$$

This estimate of $B_p$ guides the inference of neutron star field strengths from pulse timing observations. When assumptions are made about the values of $I$ and $\alpha$, this formula depends only on two observables, $P$ and $\dot{P}$. It is common to assume $I = 10^{45} \text{g cm}^2$, as well as $\sin \alpha \approx 1$ (orthogonal rotator) in this canonical field estimate, which generally underestimates the the polar field strength, serving as an estimate

\footnote{The equatorial magnetic field $B_{\text{eq}}$ has a simple relation to the polar field: $B_{\text{eq}} = B_p/2$.}
for the equatorial field strength instead.

The characteristic age $\tau$ is obtained by integrating the time evolution of the spin period. In general, the spin period evolves according to a power law: $\dot{\Omega} \propto \Omega^n$, where $n$ is known as the braking index. For the oblique rotator model we have

$$\dot{\Omega} = -\frac{B^2 R_6^6 \sin^2 \alpha}{6 I c^3} \Omega^3,$$

which gives $n = 3$. Assuming the initial period is much smaller than the current value, we have

$$\tau = \frac{\Omega}{(n-1)\dot{\Omega}} = \frac{P}{(n-1)\dot{P}}$$

For $n = 3$ we have $\tau = P/(2\dot{P})$; this is used as the canonical pulsar characteristic age.

To gauge the usefulness of this age estimate we can consider the Crab pulsar, which has a characteristic age of $\tau \approx 1250$ years. This is in reasonable agreement with the known supernova event that formed it in 1054 AD.

As opposed to assuming $n = 3$, the braking index can be obtained empirically by measuring the second period derivative for many pulsars, though the phenomenon of pulsar glitches prevent this measurement from being easy. Glitches refer to sudden increases in the spin rate followed by a relaxation toward a steadily, long-term, decreasing spin rate. They are likely to be caused by structural rearrangements in the moments of inertia between crust and interior, i.e. “starquakes”.

Solving for $n$ in the spin period evolution gives $n = \ddot{\Omega} \Omega / \dot{\Omega}^2$. However, empirical measurements of the second period derivative lead to braking indices that cover a vast range of $n$, of over $\pm 10000$, (for a review see Hobbs et al. 2010). To explain this, Blandford & Romani (1988) re-formulated the braking law with additional time dependence: $\dot{\Omega} = -K(t) \Omega^3$, where the time dependence can be due to magnetic field
evolution or a changing moment of inertia. Zhang & Xie (2012) showed how a simple analytical model of dipole decay modulated by an oscillation component agrees with the statistical properties of the observed braking indices in Hobbs et al. Also, for very young pulsars (< 1000 yr), gravitational wave emission is intense enough to provide significant energy losses, and leads to a higher braking index of 5 (Shapiro & Teukolsky 1983).

From the spin-down power $\dot{E}_{\text{rot}}$ we can also establish a relationship between the luminosity and the pulsar properties:

$$L \propto \frac{B_p^2}{P^4}. \quad (1.6)$$

From this we see that pulsars with large fields and shorter periods are more luminous, which biases detection towards these characteristics. To summarize, for observables $P$ and $\dot{P}$, we can form a dipole field strength $B_p \propto (P \dot{P})^{1/2}$, characteristic age $\tau \propto P/2\dot{P}$, and spin-down energy $\dot{E}_{\text{rot}} \propto \dot{P}P^{-3}$.

With the important relationships between the observables $P$ and $\dot{P}$, and these inferred quantities, it is informative to look at a so-called $P - \dot{P}$ diagram; a version dating from 2004 is shown in Fig. 1.1, with an updated diagram focusing on magnetars being displayed in Fig. 1.4. Contours on the plot show the characteristic field, age, and spin-down power from the oblique rotator model formulae. As pulsars age, they move to the lower right along field line contours in this diagram. The pulsar graveyard seen in the figure exists to the right of the “death line”, where essentially no pulsars are observed; the theoretical explanation for this empirically-observed feature is best understood in terms of more sophisticated emission models that we will review in Sec. 1.1.3.
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Figure 1.1: $P - \dot{P}$ diagram of pulsars (from Lorimer & Kramer 2004). Three sets of contours show the inferred magnetic fields, characteristic age, and luminosity, taken from the magnetic dipole model. The pulsar graveyard (to the right of the death line) is indicated in grey. Binary pulsars are marked with circled dots. Pulsars associated with supernova remnants are marked with stars. Pulsars with a lack of radio signal are indicated with solid triangles. SGR/AXP refers to the magnetar subpopulation, and are marked with open triangles.
The distribution can be divided into a main population in the center of the plot, and two sub-populations in the top right and bottom left. The main population consists mostly of normal radio pulsars. Those in the bottom left are known as millisecond pulsars; they are characterized by a short period combined with a very small spin-down rate. Current theory explains this population as being “recycled” older pulsars that regain angular momentum by accretion from a binary partner. They are fairly free of glitches and can be timed with very high precision; this has led to their use in gravitational wave astronomy as a network of high accuracy clocks known as a pulsar timing array (McWilliams et al. 2012). Pulsars in the top right of the figure are known as magnetars. They are characterized by an extremely high magnetic field: while normal pulsars have surface fields in the range $10^{11} - 10^{13}$ G, magnetar surface fields are typically $10^{13} - 10^{15}$ G, in excess of the quantum critical field at which the electron cyclotron energy equals its rest mass energy

$$B_{cr} = \frac{m^2 c^3}{e\hbar} = 4.4 \times 10^{13} \text{ G}. \quad (1.7)$$

This serves as a defining field scale that drives many of the physics considerations of this thesis.

### 1.1.2 Historical Co-rotating Plasma Models

The first observed pulsar radio spectra were understood as radiation from energetic particle beams near the pulsar emitting coherently. The pulse spin-down for the Crab pulsar had been measured, and the radio luminosity was far inferior to what would be implied by spin-down due to an oblique vacuum rotator. Thus, there must be some source of energetic particles in a pulsar which produces the coherent emission and
dominates the rotational energy loss.

Gunn & Ostriker (1969) proposed an acceleration mechanism involving efficient conversion of the near-wave radiation fields to particle kinetic energy; essentially a radiation pressure. This agreed reasonably with luminosity considerations, but did not address the pulsed emission at all. Meanwhile, Gold (1969) noted that in the vicinity of a rotating star possessing a magnetic field there would normally be a co-rotating magnetosphere. This extends out to some distance where external influence dominates and co-rotation ceases.

Because pulsars have fast rotation and strong magnetic fields, Gold realized that the distance out to which co-rotation is enforced may well be close to the “light cylinder”, \( R_{lc} = c/\Omega \), where co-rotation is at the speed of light.\(^1\) His proposed acceleration mechanism has plasma leaving from the neutron star surface, streaming through the co-rotating magnetosphere, and flung out near the light cylinder. The escaping particles stream along open field lines, thus they are focused along the magnetic poles; acceleration is produced by the enforcement of co-rotation.

Gold’s idea of emission near the light cylinder is ruled unlikely by Radhakrishnan & Cooke (1969), since the polarization sweep seen in radio and the small duty cycle of the pulses implies radio emission much closer to the magnetic pole. Radio pulse width constraints including relativistic aberration indicate that the radio emission altitude is around 10-100 stellar radii (Blaskiewicz et al. 1991). However, Gold inspired Goldreich & Julian (1969) to analyze the electrodynamics of an aligned rotator with a co-rotation condition.

\(^1\)The light cylinder also appears in the Deutsch solution (Deutsch 1955), despite the absence of a co-rotating plasma. It divides the solution into a near-field solution \( r \ll R_{lc} \) where the static dipole term dominates, a far-field solution \( r \gg R_{lc} \) where radiation dominates, and an intermediate-field solution \( r \approx R_{lc} \), where static, radiative, and inductive terms are important.
In the aligned vacuum rotator, the fields are unchanged by the rotation: the spin and dipole axis are aligned, and the solution is a static dipole with no radiation. When co-rotation is enforced inside the star by assuming the interior is perfectly conducting, a quadrupole electric field is produced in the outside vacuum. This implies an electric field strong enough to overcome gravity and strip charge from the surface. These features of the solution were omitted by Deutsch in his analysis.

Thus Goldreich & Julian proved that the region surrounding a neutron star cannot be empty; it must contain plasma. Furthermore, to a first approximation, the plasma will co-rotate with the star inside the light cylinder because of the strong magnetic field. Field lines which close within the light cylinder will contain a strictly co-rotating plasma (if gravity and particle inertia are ignored). Field lines which close beyond the light cylinder will be able to send particles to this region, where they are permanently lost to the star.

This latter region connects the magnetic poles to the light cylinder and is called a “polar cap” (see Sec. A.3 in the Appendix). In each polar cap there is a critical equipotential surface that is at the same voltage as the interstellar medium. Furthermore, the field lines are (approximately) equipotential. This divides each cap into two regions, each with positive/negative current along the field lines and corresponding to escaping protons/electrons, respectively. Self-consistent static solutions for the magnetosphere under these assumptions were later developed by Krause-Polstorff & Michel (1985). These arguments are not unique to the aligned case and will apply to an oblique rotator as well.
1.1.3 Modern Models

The current theoretical paradigm considers particle acceleration due to a magnetic-field-aligned electric field component $\vec{E}_\parallel$. This enhances the Goldreich & Julian model and the Gunn & Ostriker model by using an oblique co-rotating magnetosphere to explore radiation above radio frequencies. The particles are accelerated along magnetic field lines, producing a large amount of curvature radiation at $\gamma$-ray energies (see Sec. A.3 in the Appendix for more details). Pair production by curvature $\gamma$-rays is essential to generating the observed radio emission. This makes pulsars observed with both $\gamma$-ray and radio emission excellent tools for verifying emission models.

So-called polar cap models were developed by Sturrock (1971), Ruderman & Sutherland (1975), and Arons & Scharlemann (1979). These models have particles accelerated near the surface, producing curvature radiation at a few neutron star radii. They recognize that the magnetic fields present are strong enough to permit pair creation from a single photon $\gamma \rightarrow e^+e^-$. Momentum perpendicular to the magnetic field does not have to be conserved, as it can be absorbed by the global field structure. This process is single-vertex and thus dominates other photon conversion processes when in strong field environments. These pair-produced charges provide additional electric screening, which serves to limit the size of acceleration regions. This acceleration region is sometimes called an inner gap.

The attenuation coefficient for $\gamma \rightarrow e^+e^-$ in its simplest approximate form is an exponentially increasing function of photon energy (Erber 1966a), giving a super-exponential attenuation turnover. Because such sharp turnovers are not observed

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§The pulsar “death line” and graveyard shown in Fig. 1.1 can be understood in this context: this region of $P - \dot{P}$ phase space corresponds to pulsar properties such that pair production in the inner magnetosphere is no longer efficient enough to produce observable radio emission. Specifically, the field line curvature radius becomes too large.
in the majority of \( \gamma \)-ray pulsars observed by Fermi-LAT (Abdo et al. 2010), the emission can be assumed to come from higher altitudes where weaker magnetic fields inhibit single photon pair creation so that this \( \gamma \rightarrow e^+e^- \) process does not produce the turnover. This constraint has been widely used to indicate high altitudes of \( \gamma \)-ray emission for LAT pulsars (e.g., Story & Baring 2014). An example of observed exponential turnovers and and the range of cutoff energies is shown in Fig. 1.2.

Polar cap models are disfavored by modern Fermi-LAT observations; such a low-altitude emission predicts too small a beam size to produce the observed wide pulse profiles (Hirotani 2008). Population synthesis studies performed by Pierbattista et al. (2012) and Watters et al. (2009) show that the narrow polar-cap beams contribute at most only a handful of LAT pulsars. However, coherent radio emission is still expected to originate from low altitudes in the polar cap.

Modern observations favor high-altitude emission models such as the slot-gap model (e.g., Muslimov & Harding 2003) and outer gap model (Cheng et al. 1986, Romani 1996, Hirotani 2006). Both models have particles accelerated in electrostatic gaps that extend to the outer magnetosphere, near the light cylinder. An outer gap extends between the “null surface” where \( \vec{\Omega} \cdot \vec{B} = 0 \) and the light cylinder, while a slot gap extends between the polar cap surface and the light cylinder. Pair production can still occur, but is now due to a two-photon process \( \gamma\gamma \rightarrow e^+e^- \) seeded by surface X-rays, since the fields are too weak to support \( \gamma \rightarrow e^+e^- \). Pair production copiously takes place in outer gaps but is more limited in slot gaps.

Dyks et al. (2004) notes the magnetic field close to the light cylinder should be strongly influenced by magnetospheric currents and particle inertia: in this region the vacuum-retarded dipole is not a good approximation of the real magnetic field. However, it can approximate the actual magnetosphere in the limit of low particle
Figure 1.2: **Left panel:** Flux spectra $E^2 f(E)$, for photon energies $E$, for the Fermi-LAT pulsar J2021+3651, as presented in Abdo et al. (2009). The phase-averaged spectrum (open circle points) can be fit with a “Comptonized” spectral form $dN/dE \propto E^{-\alpha} \exp\{-E/E_c\}$, for $\alpha = 1.5 \pm 0.1$, and $E_c = 2.4 \pm 0.3 \pm 0.5$ GeV. The spectra from the first (open squares) and second (filled diamonds) pulse peaks also exhibit exponential turnovers. **Right panel:** The fitted exponential cutoff energies $E_{\text{cutoff}} \equiv E_c$ for the population of 46 pulsars in the first Fermi-LAT catalogue of gamma-ray pulsars (Abdo et al. 2010), as a function of pulsar light cylinder magnetic field $B_{\text{LC}}$. The histogram of $E_c$ values is projected along the right-hand axis. The narrow range of $E_c$ values is striking.

density, which luckily is expected within gaps. The average location of energy loss should take place near the light cylinder so that the rotating neutron star may lose enough angular momentum. Fig. 1.3 shows the geometry common to these models, as well as the specific location of the acceleration regions that differ between each model.

Once a model establishes the geometry and emissivity of an accelerating region, the $\gamma$-ray-producing curvature radiation can be analyzed. The usual strategy assumes the emission occurs tangent to the particle trajectory, which is following magnetic field lines in the co-rotating fame (Dyks et al. 2004, Romani & Watters 2010). If polarization is considered, the emission is assumed to be polarized such that its electric field vector is parallel to the particle’s instantaneous acceleration; propagation effects
1.2 Magnetars

A particularly exotic and topical subset of neutron stars are magnetars, which are cousins of gamma-ray pulsars that emit predominantly in X-rays between 0.1 keV and 1 MeV, mostly in steady quiescence. A central observational characteristic of the 29 known magnetars\(^4\) is that most of them undergo flaring outbursts of hard

1.2: MAGNETARS

X-rays with energies above 10 keV, sometimes enormously powerful, ranging from $10^{39} - 10^{46}$ ergs/sec. Magnetars are also distinguished by their comparatively long rotation periods, above 1 second, but most interestingly by the super-strong magnetic fields inferred to exist in their crusts, atmospheres and magnetospheres. Fields in the range of $10^{13} - 10^{15}$ are determined via their spin-down rate (e.g., Kouveliotou et al. 1998a), as the highest known magnetic fields in the Universe.

1.2.1 History

Magnetars were historically divided into two different classes of neutron stars: anomalous x-ray pulsars (AXPs) and soft gamma-ray repeaters (SGRs). AXPs were first discovered in soft x-rays and were considered to be a part of the galactic accreting binary pulsars. AXP luminosities range from $10^{34} - 10^{36}$ erg/s, falling within the range seen in accreting pulsars (Bildsten et al. 1997). In addition, their observed period derivatives imply a spin-down power significantly lower than their luminosity, ruling out conventional rotation-powered pulsars. As observations built, the absence of bright companions excluded standard binary systems powered by accretion as an explanation. With accretion and spin-down ruled out as the main power source, they were assigned their own peculiar class.

The first SGR (0525-66) was discovered when a hard x-ray / soft $\gamma$-ray burst was detected on March 5, 1979 by the Konus $\gamma$-ray burst detectors on the Soviet Venera 11 and Venera 12 spacecraft as they were in transit towards Venus (Mazets et al. 1979a). Another burst was seen from this source on March 6, and pulsations seen in the decaying tail of the bursts allowed it to be clearly identified as a flaring X-ray pulsar. In 1983, a similar source (SGR 1806-20) flared 12 times, and SGRs were soon established a sub-class of $\gamma$-ray bursts (Atteia et al. 1987). Duncan &
Thompson (1992) established the magnetar model to help explain the peculiarities of SGRs. Later, persistent x-ray pulsed emission was established for SGR 1806-20 by Kouveliotou et al. (1998b), and a super-strong magnetic field of $8 \times 10^{14}$ Gauss was inferred by the period derivative. Magnetars had been demonstrated to exist for the first time.

Many observational similarities soon began to emerge between AXPs and SGRs. Both lack evidence of binarity, occupy the same narrow region in $P - \dot{P}$ space (see Fig. 1.4), and have similar spectral properties in X-rays (Woods & Thompson 2006). In addition, AXP-like persistent emission was eventually detected from several SGRs, and SGR-like bursting was seen from several AXPs; this led to the current understanding that AXPs as well as SGRs are best explained as magnetars. As of 2014 there are 23 confirmed magnetars and 3 magnetar candidates (Olausen & Kaspi 2014b), with 3 additional candidates discovered in the past 3 years.

The emergent theoretical paradigm has at least 10% of neutron stars being born as magnetars, with activity powered by the decay of the super-strong ($> 10^{14}$ G) magnetic field for a period of $\sim 10^4$ years (Thompson et al. 2002). Spin-down is attributed to angular momentum carried away by magnetized outflows, as well as dipole radiation losses from a twisted magnetosphere. Such a twisted magnetosphere, shown in Fig. 1.5, has increased electrical current flowing across the light cylinder that produces a higher spin-down rate than pulsars with a dipolar field by up to 2 orders of magnitude. Because of the way spin-down rates are used to estimate field strength, this means a lower field estimate when this effect is included. Recent work spearheaded by Beloborodov (e.g., Parfrey et al. 2013) uses relativistic numerical MHD simulations to model these magnetospheres. The “fossil field” hypothesis is the most economical scenario for explaining magnetars’ super-strong fields using only flux.
Figure 1.4: $P - \dot{P}$ diagram showing the magnetar regime, taken from Viganò et al. (2013). Magnetars are shown in red, while rotation-powered pulsars are shown in blue. Two small classes related to magnetars are also shown: the high-B radio pulsars in cyan, and isolated neutron stars emitting in X-rays in purple. Solid lines show the evolutionary tracks of Viganò et al., while dashed lines are for the simpler dipole evolution.

conservation of the progenitor.

1.2.2 Persistent Emission

The steady emission properties characterizing magnetars occur mainly at X-ray energies below 10 keV. All confirmed magnetars exhibit pulsed, hot quasi-thermal emission with temperatures around 0.3-0.7 keV (e.g., see Perna et al. 2001); it is from this quiescent component that the spin-down field measurements are made. Because of its quasi-thermal spectrum, this emission is believed to be generated at the atmospheric
Figure 1.5: Comparison of a twisted, force-free magnetosphere (left panel) with a pure dipole (right panel), taken from Thompson et al. (2002). The twisted, force-free magnetosphere possesses a poloidal component, which causes given field line footpoints to twist an amount $\Delta \phi_{N-S}$ relative to the vacuum dipole. This gives a relative increase in spin-down torque of up to 2 orders of magnitude due to increased electrical current flowing across the light cylinder. (see also Parfrey et al. 2013)

surface of the neutron star.

Extending above the thermal signals are steep, non-thermal X-ray tails, usually resembling a power-law, that morph into flat X-ray tails above 10 keV (Kuiper et al. 2006a; den Hartog et al. 2008); these energetic tails are thought to be produced in their magnetospheres. The luminosity of both the thermal and non-thermal X-rays is commonly above $10^{35}\text{erg/sec}$, i.e. two decades higher than the solar luminosity. Specifically, roughly half are observed to have a near-constant x-ray luminosities in the $10^{34} - 10^{35}\text{erg/s}$ range. The overall X-ray pulsed fraction of a magnetar can be as high as 70% (Fernández & Thompson 2007).

Transient magnetars have much fainter quiescent emission in the $\lesssim 10^{32}\text{erg/s}$ range, though their luminosity can be comparable to the “constant” magnetars dur-
ing outburst activity lasting weeks to months (Rea & Esposito 2011). The intense stellar magnetic field profoundly impacts the formation of the spectra and polarization configuration of both these quiescent signals, since it controls both the plasma dispersion and the scattering opacity applicable to X-ray wavelengths. Unlike ordinary pulsars which emit radio pulses all the time, transient magnetars emit transient pulsed radio emission, usually following large flares (Camilo et al. 2006, Lyutikov 2002). This pulsed radio emission allows for independent distance estimates and precise position determinations, and pulse timing measurements in this band are of higher precision and can be done on shorter timescales than in X-rays.

The soft X-ray spectra of a typical magnetar is generally fit by a blackbody of temperature $\sim 0.3 - 0.6$ keV, connected to a soft power-law tail of index $\sim 2$-4 (Mereghetti et al. 2002, Perna et al. 2001). Magnetars have a hotter thermal component than rotation-powered X-ray pulsars; this emission comes from the neutron star surface, and this indicates that magnetars have hotter surfaces. In accreting X-ray pulsar systems it is the heated accretion columns that are detected rather than the atmosphere. Thus the magnetar spectrum in the $2 - 10$ keV range is softer than either accretion- or rotation-powered X-ray pulsars, as shown in Fig. 1.6. The non-thermal component below 10 keV is often physically interpreted as being produced by repeated cyclotron scattering of keV photons in the corona (e.g., Lyutikov & Gavriil 2006). Alternatively for some magnetars, a good fit can be obtained for a sum of two blackbodies of temperatures $\sim 0.3$ keV and $\sim 0.7$ keV, which has a simpler physical explanation as surface temperature gradients (Gotthelf & Halpern 2005) (see Fig 1.7 for an example). The magnetar best fit by two blackbodies is CXOU J0100-7211; it has very low interstellar absorption owing to its location in the Small Magellanic Cloud, allowing a deeper look into its soft X-ray emission. Tiengo et al. (2008) rejected the blackbody
Figure 1.6: Example X-ray spectra of different neutron star classes, taken from Mereghetti (2008a). The green curve shows the “standard-type” rotation-powered pulsar Geminga (adapted from Jackson & Halpern 2005). The red curve shows the accreting binary pulsar X Persei (adapted from Di Salvo et al. 1998). The blue curve shows the magnetar AXP 4U 0142+61 (adapted from Rea et al. 2007).

plus power-law fit for this object with high confidence.

Observations of the hard X-ray spectra of magnetars are unambiguously fit to a power-law component, with spectra extending further into hard X-rays than for accretion-powered pulsars, as shown in Fig.1.6. The power-law index is typically between $\sim 1 - 2$, making this emission a tangible fraction of the total energy output (Kuiper et al. 2004). Both pulsed and steady emission are seen in hard X-rays, with the pulsed spectra exceeding the steady spectra in hardness. No persistent $\gamma$-ray emission in the $0.1 - 10$ GeV range is seen from magnetars, though observations set upper bounds on any faint emission that may be present (Li et al. 2017). This implies
Figure 1.7: XMM-Newton spectra of XTE J1810-197, taken from Gotthelf & Halpern (2005). The earliest (Sep 2003) and latest (Mar 2004) epochs are shown, fitted either with the blackbody plus power-law model or a double blackbody model. The residuals from the best-fit models are also shown. (a) Blackbody plus power-law fit in Sep 2003. (b) Double blackbody fit in Sep 2003. (c) Blackbody plus power-law fit in Mar 2004. (d) Double blackbody fit in Mar 2004.

a turnover in hard X-rays (200-500 keV range).

Despite intrinsic faintness and locations in crowded and strongly absorbed regions of the galactic plane, optical and/or near-infrared counterparts have been found for roughly one third of magnetars. This emission is variable, but a lack of simultaneous observations in optical and X-ray during outburst activity makes the relation between flux changes unclear. Spatially extended emission has recently been seen in a single magnetar, J1834.9-0846, and is associated with a wind nebula (Younes et al. 2016). This provides a calorimeter for the net plasma wind output of the magnetar, which
is far above the standard dipole pulsar wind picture in a Crab-like pulsar.

Radio pulsations are seen from only 6 magnetars, despite magnetars lying above the pulsar death line (McLaughlin et al. 2004, Olausen & Kaspi 2014a). Compared to standard radio pulsars they show a flat spectrum, large flux variability on hour timescales, and transient behavior, making such emission likely to be connected with X-ray outbursts. For some objects elongated jet-like structures are seen, which are suggestive of outflows connected with bursting activity.

1.2.3 Flaring

Energetic outbursts of hard x-rays / soft γ-rays first seen in SGRs caused them to initially be classified as a sub-class of γ-ray bursts. Unlike γ-ray bursts, which are one-time events thought to be of extragalactic origin, flaring magnetars are characterized by multiple episodes of activity. Pulsations in the tail of the flare are detectable in some of the longer flares (> 10 s), though they not always clearly visible (see Fig. 1.9 for an example with detectable pulsations). Energetically, magnetar flares are divided into two classes: recurrent outbursts of peak luminosity \(10^{37} \text{ to } 10^{43} \text{ erg/s, and}\)

much rarer “giant” flares of peak luminosity \(10^{44} \text{ to } 10^{47} \text{ erg/s. Due to the historical connection, magnetars with high flaring activity are often renamed with an SGR designation.}\)

The recurrent outbursts have short durations, typically in the range \(\sim 0.01 - 1 \text{ s;}\)

the distribution of burst durations is fit by a log-normal distribution peaked at 0.1 s. A typical burst quickly reaches peak luminosity (within a handful of pulses), followed by a longer decay period. Good fits over the \(1 - 100 \text{ keV range can be obtained with two blackbodies with temperatures } 2 - 4 \text{ keV and } 8 - 12 \text{ keV (Feroci et al. 2004), as shown in Fig. 1.8 (see Lin et al. 2012). This burst was simultaneously detected with}\)
Figure 1.8: **Left panel:** Spectrum of the magnetar SGR J1550-5418 burst detected on 2009 January 22, as presented in Lin et al. (2012). The data is a joint fit spectrum using observations from *Swift/X-ray Telescope* (XRT) (1-10 keV, black markers) and *Fermi/Gamma-ray Burst Monitor* (GBM) (>10 keV, colored markers). Note the turnover in hard X-rays. A fit using the sum of two blackbodies is shown. **Right panel:** Background-subtracted, rapidly varying light curve of the burst, binned with a time resolution of 8 ms. Top & bottom show GBM & XRT data, respectively.

the Swift/XRT and Fermi/GBM telescopes; the high data quality is unprecedented in the modern era. A typical episode of magnetar flare activity lasts hours to days; over this time dozens to hundreds of individual short bursts can occur. Aside from bursting activity, this active phase can coincide with spectral hardening, pulse-profile changes, and glitching.

Giant flares are aptly named; their peak luminosities exceed even supernovae by orders of magnitude, making them potentially the most energetic events occurring within our galaxy. Such an energy release corresponds to 0.1 – 10% of the total energy stored within the magnetic fields. They are characterized by a short spike rising almost instantaneously ($\sim 10^{-3}$ s) and lasting for about 0.2-0.5 s. This is followed by an energetic, pulsating tail lasting several minutes after the initial burst.

Only three giant flares have so far been detected; the first was the March 1979 flare from SGR 0525-66 that led to the discovery of magnetars (see Mazets et al. 1979a,
Figure 1.9: Swift-BAT light curve showing > 50 keV flux of the giant flare from SGR 1806-20 on December 27, 2004 (taken from Palmer et al. 2005). The magnetar period of 7.6 s is easily seen in the pulsating tail. The apparent rise in the light curve to a peak at 140 s is due to the spacecraft re-orienting itself towards the SGR, improving the detection efficiency.

Mazets et al. 1979b). The second giant flare came from SGR 1900+14 in August 1998, and had similar energetics to the 1979 flare (see Hurley et al. 1999, Feroci et al. 2001). The latest giant flare, from SGR 1806-20 in December 2004, was so energetic it saturated detectors and caused measurable changes to Earth’s ionosphere (see Palmer et al. 2005, Israel et al. 2005). Its light curve above 50 keV energies is shown in Fig. 1.9.

The spectrum of the initial burst of a large flare fits to a broad, non-thermal continuum extending past 100 keV. Due to the short timescale of variability, the emission must originate near the magnetar. The model of Thompson & Duncan (1995) proposed a large-scale reconnection/interchange instability of the stellar magnetic field instigated by crustal stresses as the cause of giant flares, with the soft repeat bursts caused by cracking of the stellar crust. The hard initial spike was identified with an expanding $e^+ - e^-$ pair fireball, and the soft tail of the burst (as well as the soft repeat bursts) were identified with a cooling pair plasma trapped in the stellar magnetosphere.

In contrast, the model of Lyutikov (2006) proposes that giant flares are driven by
the dissipation of energy stored in the magnetosphere before the burst (and not in the crust), as indicated by the very short rise time scales of the flare. However, both models agree that an expanding pair fireball explains the initial spike, that the cooling of this pair plasma explains the soft tail, and that crustal fracturing can explain the post-flare activity.

1.2.4 Emission Models

Tying the simple phenomenological fits to magnetar spectra described above to a physically based model allows for a physical interpretation of the spectral properties. Such models have been developed, and contain several ingredients to explain the emission (e.g. Özel 2001a, Ho & Lai 2001, Harding & Lai 2006, Beloborodov 2009). The main ingredients are surface thermal emission with a geometrically shallow (but highly optically thick) atmosphere, and a geometrically complex magnetosphere with non-negligible plasma content. The thermal emission is the result of interior cooling produced by magnetic dissipation, as well as exterior heating by inward-falling charge from the magnetosphere.

The non-thermal soft X-ray emission (below 10 keV) is explained as repeated non-relativistic cyclotron scattering in an inner corona (e.g. Lyutikov & Gavriil 2006). Large currents in the magnetosphere provide a large optical depth to resonant Compton scattering. Photon propagation through this complicated, twisted magnetosphere requires a 3-D Monte Carlo approach, as is taken by Nobili et al. (2008) and Fernández & Thompson (2007). Such models can successfully fit the observed soft X-ray spectra, but since they rely on at most mildly relativistic particle distributions, their validity is uncertain above a few dozen keV.

The non-thermal hard X-ray emission (from 10 keV to 200-500 keV) is fit with a
much harder (flatter) power-law, indicating very drastic spectral changes in a narrow energy interval around 10 keV. This band had not been explored until the mid 2000’s, and so is a fairly recent development in the study of magnetars. Kuiper et al. (2006b) used the initial observation of these flat tails to discriminate against existing models such as the bremsstrahlung and synchrotron models of Thompson & Beloborodov (2005), and the fast-mode breakdown of Heyl & Hernquist (2005). A more successful model was presented by Baring & Harding (2007), who considered resonant cyclotron upscattering of seed photons. Unlike the process for soft X-rays, this takes advantage of the high efficiency that occurs when near the cyclotron resonance. More recently, Beloborodov (2013) developed a model for the hard X-ray spectra using a numerical radiative transfer solution for plasma flowing in a large, twisted magnetic loop. Beamed emission from relativistic electrons in this loop generates a hard power-law with emission significantly suppressed in γ-rays. This is in general agreement with observations, although the model strongly depends on the loop geometry and source orientation. It also does not self-consistently include the soft X-ray emission.

In the resonant cyclotron scattering domain, the Compton cross section is enhanced by orders of magnitude very near the cyclotron frequency – for electrons in magnetars, this is above 10 MeV in energy, and for protons this can be at 1-10 keV. It has widely been predicted that such proton cyclotron line absorption or emission features should be present in magnetar X-ray spectra (see e.g. Potekhin et al. 2016 and references therein). Excluding the result of Tiengo et al. (2013) (discussed below), no unambiguous detections of these resonant scattering features in the persistent emission of magnetars has been seen, despite extensive searches (Mereghetti 2008b). It is this resonance that drives the generation of optically thin Compton upscattering emission in the hard X-ray tails.
Detection of cyclotron line features would constitute a direct measurement of the intense magnetic fields assumed present around these objects. Such features are regularly seen in accreting X-ray binary neutron stars (Revnivtsev & Mereghetti 2015). While the electron cyclotron energy $E_{c,e} = e\hbar B/m_e c$ lies above x-rays in the (unobserved) $\sim 1$ MeV range for magnetar field strengths $\sim B_{cr}$, the proton cyclotron energy lies well within the observable x-ray range: for a surface magnetic field $B$ and gravitational redshift factor $z_G$ ($\sim 0.7 - 0.85$ at the surface), we have for the cyclotron energy (Mereghetti 2008a)

$$E_{c,p} \simeq 0.63z_g \left( \frac{B}{10^{14} \text{G}} \right) \text{ keV}. \quad (1.8)$$

Extensive searches for narrow cyclotron features in the phase-averaged persistent emission has given only negative results so far. This can be interpreted as severe line smearing caused by emission from a range of locales (and thus field strengths), reducing their potential detectability. However, some cyclotron emission and absorption features have been seen during bursts, which supports the localized magnetic loop hypothesis.

Phase-resolved (as opposed to phase-averaged) spectroscopy would be ideal for increasing detectability, though a great price is paid in the form of lower counts and thus sensitivity. Nevertheless, a phase-resolved cyclotron absorption line was recently discovered by Tiengo et al. (2013) in the transient magnetar SGR 0418+5729. As expected, it shows a strong dependence on the rotation phase, with the line energy varying between $\sim 1 - 5$ keV in a small interval of the spin period. If it is interpreted as proton cyclotron emission it supports magnetar type fields, while if it is electron cyclotron emission then it is a normal pulsar field. Electrons typically have a higher
$kT/mc^2$ ratio, so their cyclotron lines have higher Doppler broadening like those seen in PSR A 0535+26 (see Kretschmar et al. 1996). As the observed features were fairly narrow, the favored interpretation is cyclotron emission from protons trapped in a relatively small magnetic loop.

1.2.5 Polarized Transfer in Magnetars

In this thesis we model polarized Thomson scattering transport in strong magnetic fields. Scattering is important to consider in magnetars, as both the atmospheric emission and the magnetospheric flare emission must be highly optically thick to Thomson scattering. For flare emission, this is easily seen with a simple calculation of the energetics, which constrain the electron number density $n_e$. Let $\langle \gamma_e \rangle$ be the mean electron Lorentz factor, $\varepsilon_{\text{rad}}$ the radiative efficiency, and $R_c$ the column height. Then the flare luminosity can be estimated as

$$L_{X\gamma} \sim \varepsilon_{\text{rad}} \langle \gamma_e \rangle m_e c^2 \left( 4\pi n_e R_c^2 c \right).$$

Using this as an estimate for the number density, we solve the Thomson optical depth $\tau_T = \sigma_T n_e R_c$, giving

$$\tau_T \sim \frac{2 \times 10^4}{\varepsilon_{\text{rad}} \langle \gamma_e \rangle} \left( \frac{L_{X\gamma}}{10^{42}\text{erg/s}} \right) \left( \frac{10^6\text{cm}}{R_c} \right).$$

Here we have scaled the luminosity and column density by typical flare values ($L_{X\gamma} \sim 10^{42}\text{erg/s}, R_c \sim 10^6 - 10^7\text{cm}$). Since one would expect $\varepsilon_{\text{rad}} \langle \gamma_e \rangle \gtrsim 1$, we are guaranteed that $\tau_T \gg 1$.

The transport of polarized radiation in magnetized plasma has been widely studied over the last half century because of its pertinence to radio and optical astron-
1.2: MAGNETARS

omy. The well-known effect of Faraday rotation of electric field (polarization) vectors of light in dispersive media has received widespread attention due in large part to its diagnostic power on the strength of interstellar magnetic fields, usually in the range of \( \mu \text{Gauss-milliGauss} \). This is normally much more germane to radio frequency bands than for optical or X-ray photons because both the plasma frequency and the cyclotron frequency of tenuous cosmic plasmas are often well below the GHz band. This domain of interest moves to much higher frequencies when plasmas in the denser magnetospheres of neutron stars are considered.

With the establishment of magnetars as a bona fide class of neutron stars, interest in radiative processes in the super-strong magnetic regime has grown over the last two decades. These processes have an inherently anisotropic and polarization-dependent character, due to the influence of the intense magnetic field, which adds to the complexity of a description of quantum radiation interactions with relativistic electrons. Most exotic among these processes are single photon pair production \( \gamma \rightarrow e^+e^- \) (e.g. Erber 1966b) and magnetic photon splitting \( \gamma \rightarrow \gamma\gamma \) (e.g. Adler 1971), whose application to magnetars is discussed in Baring & Harding (2001).

The full quantum electrodynamic theory of Compton scattering in strong magnetic fields was addressed early on by Herold (1979) in the Thomson domain (see Canuto et al. 1971a for a classical description of the magnetic Thomson interaction), and has since been developed and refined by a number of groups (see Daugherty & Harding 1986; Bussard et al. 1986; Gonthier et al. 2000, 2014; Mushtukov et al. 2016). As magnetar atmospheres and magnetospheres are both optically thick to Thomson scattering, understanding such scattering of light by plasma in strong fields is of foremost interest to neutron star astrophysicists. This is the goal of this thesis.
Chapter 2
Polarized Radiative Transfer

Observable radiation contains information about the emitting source, as well as the medium through which it has propagated. One of the major pursuits of theoretical astrophysics is to examine how the properties of the source and intervening medium are connected to the properties of the observed radiation. The observable characteristics of radiation in a given direction are the intensity, frequency, and polarization. The subject of radiative transfer is concerned with the emission and subsequent evolution of radiation as it propagates through a medium. This can depend on various macroscopic properties of the source and medium, such as the temperature, density, velocity, electromagnetic fields, as well as atomic composition. Depending on these characteristics, certain processes will dominate in the determination of the properties of the emergent radiation.

Here we are concerned specifically with the effect of a strong magnetic field in a tenuous plasma, a situation thought to arise in the atmospheres and magnetospheres of neutron stars. We will focus almost exclusively on the polarization properties of the electron Compton scattering process in a strong magnetic field, neglecting propagation effects to help disentangle the already complex results.*

*Propagation effects are addressed in Chapter 7.
2.1 Parameterizing Polarization

The intensity and polarization of a transverse wave is described with a complex electric field 3-vector \( \vec{E} \) that is orthogonal to the direction of propagation. The wave nature of the vector is separated from the polarization vector, i.e.

\[
\vec{E}_{\text{wave}} = \vec{E} \exp\left(\vec{k} \cdot \vec{r} - \omega t\right).
\]

The magnetic field component of this wave is orthogonal to \( \vec{E} \) and \( \vec{k} \) and equal in magnitude to \( \vec{E} \); it can always be derived by forming the cross product. This polarization vector is described with 6 real components; however it is conventional to use the transversality and normalization requirements to reduce the degrees of freedom to 4. This can be accomplished via the Stokes parameters (Stokes 1851), a set of 4 numbers that can be combined into a vector \((I, Q, U, V)\). An explicit basis, or at the very least a reference direction (and rotation convention) are also required to explicitly describe this vector. The Stokes I parameter describes the intensity of the radiation; specifically, it is the radiant energy transported across an element of area, per unit area per unit solid angle per unit time. Stokes Q and U are related to the degree of linear polarization: a pure linear polarization describes radiation with its oscillating electric vector aligned in a particular direction. If \( Q/I = \pm 1 \) then the radiation is fully linearly polarized with its electric field at an angle \( 0^\circ(90^\circ) \) with respect to the reference direction, and if \( U/I = \pm 1 \) then the radiation is fully linearly polarized with its electric field at an angle \( \pm 45^\circ \). The choice of reference direction is arbitrary, and there is often a convenient choice for it. Information about linear polarization can be

\[\text{An alternative labeling convention uses } (I, Q, U, V) = (S_0, S_1, S_2, S_3).\]
2.1: \textit{PARAMETERIZING POLARIZATION}  

combined into a linear polarization degree $\Pi_{\text{lin}}$ and polarization angle $\psi$:

$$\Pi_{\text{lin}} = \sqrt{(Q/I)^2 + (U/I)^2} \quad \psi = \frac{1}{2} \arctan(U/Q). \quad (2.2)$$

The Stokes V parameter describes the degree of circular polarization: in this case the electric field vector remains constant in magnitude and swings around the wave vector $\vec{k}$ at a constant angular rate. This rate can be influenced by the dispersive properties of the medium, for example provided by the Faraday rotation effect discussed in Ch. 7. Care must be taken in defining circular polarization since there are competing phase/helicity conventions. Here we will use an \textit{increasing} phase convention, meaning that the right-hand rule is used upon the propagation vector to define the sense of rotation$^\dagger$. With this convention, $V/I = \pm 1$ corresponds to fully right-(left-) handed circularly polarized radiation. Elliptical polarization occurs when both linear and circular polarizations are nonzero, representing the most general case. For a more extensive discussion of these definitions and conventions, see Rybicki & Lightman (1979) pp.62-69.

The 3 Stokes parameters describing polarization also be combined to form an overall polarization degree $\Pi$:

$$\Pi = \sqrt{(Q/I)^2 + (U/I)^2 + (V/I)^2}, \quad 0 \leq \Pi \leq 1. \quad (2.3)$$

In the following we use the semi-classical approximation, where we assume the discrete nature of photons (i.e. $E = h\nu$) but treat the fields classically; this is valuable for its simplicity though it lacks quantum features such as eigenstate superposition and

$^\dagger$Note that some of the referenced work, importantly Whitney (1989) and related work, use the opposite convention.
tunneling. Thus we will always consider an individual photon to be fully-polarized (i.e. \( \Pi = 1 \)). For an ensemble of photons (incident on, say, a detector) polarization vectors are superimposed, forming an ensemble polarization degree that is generally nonzero and non-unity. We will refer to this ensemble quantity as the overall polarization fraction. Also the quantity \( \psi \) is often referred to as the position angle of linear polarization (or position angle for short) rather than the polarization angle.

If we consider radiation traveling along the \( z \)-axis, then the polarization vector will always be in the \( x-y \) plane, i.e. \( \vec{E} = E_x \hat{x} + E_y \hat{y} \). If we use the \( +x \) axis as the reference direction for defining Stokes parameters, we have

\[
\begin{align*}
I &= |E_x|^2 + |E_y|^2, \\
Q &= |E_x|^2 - |E_y|^2, \\
U &= 2 \text{Re} \left[ E_x E_y^* \right], \\
V &= -2 \text{Im} \left[ E_x E_y^* \right].
\end{align*}
\]

The polarization ellipse is a common way of visualizing the behavior of the electric field for a given pure polarization. Assuming a pure polarization reduces the degrees of freedom again, so that we only need 3 parameters to fully describe such an ellipse: the overall size of the ellipse (described with \( I \)), its orientation (described with an angle \( \psi \)), and the ratio of semimajor to semiminor axes (described with an angle \( \chi \)). For a photon traveling along the \( z \)-axis, the polarization ellipse is visualized in Fig. 2.1. The associated Stokes parameters \((Q,U,V)\) in terms of ellipse parameters
This formula is reminiscent of a spherical decomposition, only with the angles doubled. This was originally recognized by Poincaré (1889–1892), and we now refer to such a spherical representation of polarization as a Poincaré sphere, shown in Fig. 2.2. Note that antipodal points on this sphere correspond to orthogonal polarizations, owing to the double angled nature. The Poincaré sphere is quite useful in describing the consequences of the generalized Faraday effect (to be considered in Chapter 7) specifically because linear polarization configurations are not eigenstates of light dispersion in ionized media.
We can also relate the components of the complex polarization vector to the ellipse parameters. For the fully-polarized case (\(\Pi = 1\)) and with the polarization vector in the \(x-y\) plane as above, we are free to arbitrarily set \(\text{Im}[E_x] = 0\), reducing the parameter set to 3. We have essentially two magnitudes and a phase difference; the overall phase is unimportant. In this case, we have

\[
\begin{align*}
\text{Re}[E_x] & = \sqrt{\frac{I}{2}} \cdot \sqrt{1 + \cos 2\chi \cos 2\psi} \\
\text{Re}[E_y] & = \sqrt{\frac{I}{2}} \cdot \frac{\cos 2\chi \sin 2\psi}{\sqrt{1 + \cos 2\chi \cos 2\psi}} \\
\text{Im}[E_y] & = \sqrt{\frac{I}{2}} \cdot \frac{\sin 2\chi}{\sqrt{1 + \cos 2\chi \cos 2\psi}}
\end{align*}
\] (2.4) (2.5) (2.6)
2.1: PARAMETERIZING POLARIZATION

or, in terms of the Stokes parameters themselves:

\[
\begin{align*}
\text{Re}[E_x] &= \frac{\sqrt{I+Q}}{\sqrt{2I}} \\
\text{Re}[E_y] &= \frac{U}{\sqrt{2\sqrt{I+Q}}} \\
\text{Im}[E_y] &= \frac{V}{\sqrt{2\sqrt{I+Q}}} 
\end{align*}
\]

(2.7) (2.8) (2.9)

There also exists an alternative method of parametrizing the polarization uses a density matrix \( \vec{\rho} \) containing products of components of the polarization vector (e.g. Dolginov et al. 1995a; Landau & Lifshitz 1975). It is closely related to the Stokes parameters. In a linear basis, using the z-axis aligned radiation above, the density matrix takes the form

\[
\vec{\rho} = \begin{pmatrix}
|E_x|^2 & E_x E_y^* \\
E_x^* E_y & |E_y|^2
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
I + Q & U - iV \\
U + iV & I - Q
\end{pmatrix}.
\]

(2.10)

A basis more useful to general problems has the radiation propagating in an arbitrary direction. Using a spherical basis, propagation corresponds to the radial direction (\( \vec{k} \parallel \hat{r} \)), and the polarization vector is in a plane formed by the polar and azimuthal unit vectors, i.e.

\[
\vec{E} = E_{\theta} \hat{\theta} + E_{\phi} \hat{\phi}.
\]

(2.11)

Using the z-axis now as a reference direction for defining Stokes parameters, we have

\[
\vec{I} = (I, Q, U, V) = \left( |\vec{E}|^2, |E_{\theta}|^2 - |E_{\phi}|^2, 2\text{Re}[E_{\theta} E_{\phi}^*], -2\text{Im}[E_{\theta} E_{\phi}^*] \right).
\]

(2.12)

When the magnetic field coincides with the z-axis as well, we can modify the Stokes
parameters in a useful way in order to simplify later formulae. We combine \( I \) and \( Q \) to form two intensities for linear polarizations:

\[
I_{\parallel} = \frac{1}{2} (I + Q), \quad I_{\perp} = \frac{1}{2} (I - Q),
\]

(2.13)

Here, \( I_{\parallel} \) corresponds to polarization in the plane containing the propagation vector and magnetic field vector, while \( I_{\perp} \) corresponds to polarization perpendicular to this plane. This is relevant because an electric field perpendicular to the magnetic field will always excite cyclotron motion in a magnetized plasma, while an electric field component along the magnetic field produces unrestricted motion. In the former case, the Thomson scattering process is resonant at the cyclotron frequency, while in the latter, the scattering is more normal in that it has no such resonance. This will become apparent shortly. We combine the modified Stokes parameters into a modified Stokes vector:

\[
\vec{I} = (I_{\parallel}, I_{\perp}, U, V) = (|E_{\theta}|^2, |E_{\phi}|^2, 2\text{Re}[E_{\theta}E_{\phi}^*], -2\text{Im}[E_{\theta}E_{\phi}^*]).
\]

(2.14)

In general, the propagation direction is not along the z-axis, and so Stokes parameters have to be adapted to rotated systems of coordinates.

### 2.2 Magnetic Thomson Scattering

This Section outlines the classical derivation of magnetic Thomson scattering (see e.g. Canuto et al. 1971b; Chou 1986). It begins with an incident electromagnetic wave, whose direction of propagation is given by the unit vector \( \hat{n}_i \). The electric field of this wave is \( \vec{E}(t) = \vec{E}_i e^{i\omega t} \), where \( \vec{E}_i \) is allowed to be complex. The wave is transverse, so
\[ \vec{E}_i \cdot \hat{n}_i = 0 \] must hold true. The external magnetic field \( \vec{B} = B\hat{b} \) is separated into a magnitude \( B \) and unit vector \( \hat{b} \) encoding its direction.

The motion of an electron in this electric field and external magnetic field is determined by the Newton-Lorentz equation:

\[ m \frac{d\vec{v}(t)}{dt} = -e\vec{E}(t) - \frac{e}{c} \vec{v}(t) \times \vec{B}. \]  

Note that this applies only to non-relativistic charges, appropriate for the magnetic Thomson scattering problem. If we rewrite the acceleration as

\[ \frac{d\vec{v}(t)}{dt} = \vec{a}e^{i\omega t} = (a_x \hat{x} + a_y \hat{y} + a_z \hat{z})e^{i\omega t} \]  

then the velocity is given by \( \vec{v}(t) = \vec{a}e^{i\omega t}/i\omega \). Solving for \( \vec{a} \) in the Newton-Lorentz equation, we have

\[ \vec{a} = -\frac{e}{m} \frac{\omega^2 \vec{E}_i + i\omega \omega_B \vec{E}_i \times \hat{b} - \omega_B^2 (\vec{E}_i \cdot \hat{b})\hat{b}}{\omega^2 - \omega_B^2}, \]  

where \( \omega_B = eB/mc \) is the cyclotron frequency. Since the charge accelerates it radiates a scattered wave, and for non-relativistic motions treatment of retarded potentials is not necessary. We use the dipole radiation formula for an accelerating electron to obtain the electric field after scattering (Landau & Lifshitz 1975):

\[ \vec{E}_{\text{scat}}(\vec{r}, t) = \vec{E}_{f}e^{i\omega t} = -\frac{e}{Re^2} \hat{n}_f \times (\hat{n}_f \times \vec{a}e^{i\omega t}). \]  

where \( \hat{n}_f \) is the direction of propagation for the scattered wave. Note that \( R \) is the distance from the oscillating/radiating dipole to a point of observation. Defining the
CHAPTER 2: POLARIZED RADIATIVE TRANSFER

dimensionless frequency $\nu_a = \omega/\omega_B$, we can write $\vec{E}_f$ in a compact form:

$$\vec{E}_f = \frac{r_0}{R} \frac{1}{\nu_B^2 - 1} \hat{n}_f \times \left\{ \hat{n}_f \times \left[ \nu_a^2 \vec{E}_i + i\nu_a \vec{E}_i \times \hat{b} - (\vec{E}_i \cdot \hat{b})\hat{b} \right] \right\}. \quad (2.19)$$

where $r_0 = e^2/mc^2$ is the classical electron radius. Note that $\vec{E}_f \cdot \hat{n}_f = 0$ by construction. The scattering is elastic, so $\omega_f = \omega_i = \omega$. Expanding the vector product gives

$$\vec{E}_f = \frac{r_0}{R} \frac{1}{\nu_B^2 - 1} \left\{ (\vec{E}_i \cdot \hat{b}) \left[ \hat{b} - (\hat{b} \cdot \hat{n}_f)\hat{n}_f \right] - \nu_B^2 \left[ \vec{E}_i - (\vec{E}_i \cdot \hat{n}_f)\hat{n}_f \right] \right.$$

$$+ i\nu_a \left[ (\hat{b} \cdot \hat{n}_f)\hat{n}_f \times \vec{E}_i - (\vec{E}_i \cdot \hat{n}_f)\hat{n}_f \times \hat{b} \right] \right\}. \quad (2.20)$$

The final intensity $I_f = |\vec{E}_f|^2$ is simply related to the total and differential cross sections:

$$\sigma = \int \frac{d\sigma}{d\Omega_f} d\Omega_f, \quad \frac{d\sigma}{d\Omega_f} = R^2 \frac{|\vec{E}_f|^2}{|\vec{E}_i|^2} = R^2 \frac{I_f}{I_i}. \quad (2.21)$$

The total cross section takes a simple form:

$$\sigma = \frac{R^2}{|\vec{E}_i|^2} \int |\vec{E}_f|^2 d\Omega_f \quad (2.22)$$

$$= \frac{\sigma_T}{|\vec{E}_i|^2 (\nu_B^2 - 1)^2} \left[ (1 - 2\nu_B^2)|\vec{E}_i \cdot \hat{b}|^2 + \nu_B^2 |\vec{E}_i \times \hat{b}|^2 - 2i\nu_a^3 (\hat{b} \cdot \vec{E}_i \times \vec{E}_i^*) + \nu_a^4 |\vec{E}_i|^2 \right],$$

where $\sigma_T = 8\pi r_0^2/3$ is the ordinary Thomson total cross section.

### 2.2.1 Radiative Transfer and Polarization Matrix Form

We now consider the radiative transfer equation for pure magnetic Thomson scattering. In terms of the of the modified Stokes vector (Eq. 2.14) the transfer equation is
2.2: MAGNETIC THOMSON SCATTERING

(Whitney 1991b)

\[
-\frac{\hat{s} \cdot \nabla}{n_e \sigma} \vec{I} = \vec{I} - \vec{J}; \quad \vec{J} = \int \hat{P}. \vec{I} d\Omega,
\]

(2.23)

where \( n_e \) is the electron number density and \( \hat{s} \) is a unit vector along the line of sight (i.e. for radiation which escapes to the observer, \( \hat{s} \) corresponds to its final propagation direction). \( \sigma \) is the total cross section, detailed in Eq. 2.26 below. \( \vec{J} \) represents scattered radiation. \( \hat{P} \) is the phase matrix for magnetic Thomson scattering. This is analogous to the phase function describing the angular dependence of the scattering; when upgrading to a polarization-level description it becomes a \( 4 \times 4 \) matrix. It can be obtained by calculating scattering matrix elements connecting the final Stokes parameters \( \vec{I}_f \) and the initial Stokes parameters \( \vec{I}_i \). This can be accomplished with manipulation of the vector form in Eq. 2.19, which already encodes this information. We present details concerning this construction just below. For a complete matrix derivation see Chou (1986).

Here we describe the angular and polarization structure of the phase matrix \( \hat{P} \).

For the simple case of magnetic field along the z axis and modified Stokes vector (Eq. 2.14), and for photon initial angles \((\theta_i, \phi_i)\) and final angles \((\theta_f, \phi_f)\), we have the following relations:

\[
\hat{P} = \frac{r_0^2}{\sigma} \hat{M}; \quad \vec{I}_f = \frac{r_0^2}{R^2} \hat{M} \cdot \vec{I}_i; \quad \hat{M} = \begin{pmatrix}
M_{11} & M_{12} & M_{13} & M_{14} \\
M_{21} & M_{22} & M_{23} & M_{24} \\
M_{31} & M_{32} & M_{33} & M_{34} \\
M_{41} & M_{42} & M_{43} & M_{44}
\end{pmatrix}.
\]

(2.24)

Here, \( \sigma \) is the total cross section and \( r_0 = e^2/m_e c^2 \) is the classical electron radius.
The subscripts (1,2,3,4) refer to ($I_{||}, I_{\perp}, U, V$), respectively. The photon angles are defined by spherical decomposition of the propagation vector, e.g.,

$$\hat{k_i} = \cos \phi_i \sin \theta_i \hat{x} + \sin \phi_i \sin \theta_i \hat{y} + \cos \theta_i \hat{z}. \quad (2.25)$$

We also define the shorthand $\Delta \phi = \phi_i - \phi_f$. The scattering matrix elements $M_{ab}$ take the form

$$M_{11} = \sin^2 \theta_i \sin^2 \theta_f + \frac{1}{2} \frac{\nu_B^2}{\nu_B^2 - 1} \cos \Delta \phi \sin 2\theta_i \sin 2\theta_f$$

$$+ \left( \frac{\nu_B}{\nu_B^2 - 1} \right)^2 \cos^2 \theta_i \cos^2 \theta_f \left( \nu_B^2 \cos^2 \Delta \phi + \sin^2 \Delta \phi \right),$$

$$M_{12} = \left( \frac{\nu_B}{\nu_B^2 - 1} \right)^2 \cos^2 \theta_f \left( \cos^2 \Delta \phi + \nu_B^2 \sin^2 \Delta \phi \right),$$

$$M_{13} = -\frac{1}{2} \frac{\nu_B^2}{\nu_B^2 - 1} \left( \sin \theta_i \sin 2\theta_f \sin \Delta \phi + \cos \theta_i \cos^2 \theta_f \sin 2\Delta \phi \right),$$

$$M_{14} = -\frac{\nu_B}{\nu_B^2 - 1} \left[ \frac{\nu_B^2}{\nu_B^2 - 1} \cos \theta_i \cos^2 \theta_f + \frac{1}{2} \sin \theta_i \sin 2\theta_f \cos \Delta \phi \right],$$

$$M_{21} = \left( \frac{\nu_B}{\nu_B^2 - 1} \right)^2 \cos^2 \theta_i \left( \cos^2 \Delta \phi + \nu_B^2 \sin^2 \Delta \phi \right),$$

$$M_{22} = \left( \frac{\nu_B}{\nu_B^2 - 1} \right)^2 \left( \nu_B^2 \cos^2 \Delta \phi + \sin^2 \Delta \phi \right),$$

$$M_{23} = \frac{1}{2} \frac{\nu_B^2}{\nu_B^2 - 1} \cos \theta_i \sin 2\Delta \phi,$$

$$M_{24} = -\frac{\nu_B^3}{(\nu_B^2 - 1)^2} \cos \theta_i.$$
\[ M_{31} = \frac{\nu_B^2}{\nu_B^2 - 1} \left( \cos^2 \theta_i \cos \theta_f \sin 2\Delta\phi + \sin 2\theta_i \sin \theta_f \sin \Delta\phi \right), \]
\[ M_{32} = -\frac{\nu_B^2}{\nu_B^2 - 1} \cos \theta_f \sin 2\Delta\phi, \]
\[ M_{33} = \frac{\nu_B^2}{\nu_B^2 - 1} \left( \cos \theta_i \cos \theta_f \cos 2\Delta\phi + \sin \theta_i \sin \theta_f \cos \Delta\phi \right), \]
\[ M_{34} = -\frac{\nu_B}{\nu_B^2 - 1} \sin \theta_i \sin \theta_f \sin \Delta\phi, \]
\[ M_{41} = -\frac{\nu_B}{\nu_B^2 - 1} \left( \frac{2\nu_B^3}{\nu_B^2 - 1} \cos \theta_i \cos \theta_f \sin 2\theta_i \sin \theta_f \cos \Delta\phi \right), \]
\[ M_{42} = -\frac{2\nu_B^3}{(\nu_B^2 - 1)^2} \cos \theta_f, \]
\[ M_{43} = \frac{\nu_B}{\nu_B^2 - 1} \sin \theta_i \sin \theta_f \sin \Delta\phi, \]
\[ M_{44} = \frac{\nu_B^2}{\nu_B^2 - 1} \left( \frac{\nu_B^2 + 1}{\nu_B^2 - 1} \cos \theta_i \cos \theta_f \sin \theta_i \sin \theta_f \cos \Delta\phi \right). \]

The total cross section \( \sigma \) in this system takes the form

\[ \sigma = \frac{\sigma_1 I_\parallel + \sigma_2 I_\perp + \sigma_3 U + \sigma_4 V}{I_\parallel + I_\perp}, \]

where

\[ \sigma_1 = \sigma_T \left[ \sin^2 \theta_i + \frac{\nu_B^4 + \nu_B^2}{(\nu_B^2 - 1)^2} \cos^2 \theta_i \right], \]
\[ \sigma_2 = \sigma_T \frac{\nu_B^4 + \nu_B^2}{(\nu_B^2 - 1)^2}, \]
\[ \sigma_3 = 0, \]
\[ \sigma_4 = -2\sigma_T \frac{\nu_B^3}{(\nu_B^2 - 1)^2} \cos \theta_i. \]

Note the result is independent of azimuthal angles \( \phi_i \) and \( \phi_f \) around the field direction.
The formula for $\hat{M}$ and $\sigma$ are equivalent to those presented by Whitney (1991b), with the correction of a typo in Whitney’s $M_{11}$ as well as a change in phase convention for the Stokes $V$ parameter; (matrix elements here incur a minus sign relative to those in Whitney for each appearance of the subscript 4). This formula, as well as the vector form (Eq. 2.19) agrees with the quantum field theory derivation of magnetic Compton scattering when the classical limit is taken (see e.g. Herold 1979; Melrose & Parle 1983).

The phase matrix is normalized such that

$$\frac{1}{I_i} \int (\vec{P} \cdot \vec{I}_i) I d\Omega_f = \frac{1}{\sigma} \int \frac{d\sigma}{d\Omega_f} d\Omega_f = 1,$$

(2.28)

where $(\ldots)_f$ represents the $I$ component of the Stokes vector ($I = I_\parallel + I_\perp$). For a region where the electron density and magnetic field are constant, the solution to the transfer equation is

$$\vec{I} = \vec{I}_0 e^{-\tau} + (1 - e^{-\tau}) \int \vec{P} \cdot \vec{I} d\Omega.$$

(2.29)

The first term on the right represents radiation that transmits unscattered, while the integral term gives the scattered contribution. The optical depth follows the usual relation:

$$\tau = n_e \sigma s,$$

(2.30)

where $s$ is the pathlength traversed by the light, and $n_e$ is the number density of the electrons.
2.2.2 Total Cross Section

It is useful to examine some properties of the total cross section, as it is closely related to the optical depth. For unpolarized radiation we have \((I_||, I_\perp, U, V) = (I/2, I/2, 0, 0)\). The total cross section for unpolarized radiation (integrated over final angles) is then

\[
\sigma_{\text{unpol}} = \frac{\sigma_T}{2} \left[ \sin^2 \theta_i + \frac{\nu_B^4 + \nu_B^2}{(\nu_B^2 - 1)^2} (1 + \cos^2 \theta_i) \right].
\] (2.31)

We can re-express the frequency dependence in terms of \(\omega\) and \(\omega_B\) to make clear the resonant nature of the cross section:

\[
\sigma_{\text{unpol}} = \frac{\sigma_T}{2} \left\{ \sin^2 \theta_i + \frac{1}{2} \left[ \frac{\omega^2}{(\omega_B - \omega)^2} + \frac{\omega^2}{(\omega_B + \omega)^2} \right] (1 + \cos^2 \theta_i) \right\} ;
\] (2.32)

(Herold 1979; Whitney 1991b). This unpolarized magnetic Thomson total cross section is shown (for radiation traveling along the magnetic field) in Fig. 2.3, and is compared to the ordinary Thomson cross section \(\sigma_T = 8\pi r_0^2/3\) (\(r_0 = e^2/m_e c^2\) is the classical electron radius). This allows us to observe the major features of this cross section. Near the resonance it is sharply peaked and formally divergent at \(\omega = \omega_B\), while the two cross sections coincide at \(\omega/\omega_B = 1/\sqrt{3}\). Below this value it is sharply dropping as a function of frequency.

We can also understand this cross section in terms of the normal modes for propagation in a magnetized plasma (described further in Ch. 7). These are the polarization eigenmodes of the dielectric tensor, and for the magnetic case they are non-degenerate, representing the anisotropy introduced by the magnetic field. As usual, they are dependent on the strength of the field and its direction with respect to propagation. For a system where \(\vec{B} || \hat{z}\), with angle \(\theta_i\) between the propagation direction and the
Figure 2.3: The magnetic Thomson cross section for unpolarized light as a function of frequency, for two different angles with respect to the magnetic field. The unmagnetized cross section is shown as well. Near the resonance, the cross section is sharply peaked. The different cross sections coincide at $\omega/\omega_B = 1/\sqrt{3}$. Below this value the magnetic cross section is sharply dropping as a function of frequency.

magnetic field, the normalized modified Stokes vectors for the modes are

$$ (I_\parallel, I_\perp, U, V)_\pm = \frac{1}{K_\pm^2 + 1} \left( K_\pm^2, 1, 0, 2K_\pm \right), \quad (2.33) $$

where

$$ K_\pm = \beta_0 \pm \sqrt{\beta_0^2 + 1}, \quad \beta_0 = \frac{1}{2\nu_0} \sin \theta \tan \theta_i; \quad (2.34) $$

(see e.g. Eq. 7.23, Eq. 7.25, Eq. 7.29). Note that the normal modes incur a circularity associated with the gyration of electrons in the magnetic field. Accordingly, the $\pm$ sign denotes the helicity of the electromagnetic eigenmode. The important feature of the normal modes is that only one of them is affected by the cyclotron resonance,
while the other is not. While it is difficult to see from the full form of the modes’ cross sections (i.e. by inserting the Stokes vector into the cross section formula), it can easily seen by plotting the cross sections for each mode (or helicity state ± in Eq. 2.33) for the simple case $\theta_i = 0$ (i.e., light initially travelling along the magnetic field):

$$\sigma_{\pm}(\theta_i \to 0) = \frac{\nu_B^2}{(\nu_B \pm 1)^2} = \frac{\omega^2}{(\omega \pm \omega_B)^2}. \quad (2.35)$$

This is exhibited in Fig. 2.4. When $\theta_i$ is near $90^\circ$, the nature of the cross sections closely resembles the previous plot (Fig. 2.3), i.e. the mode unaffected by the resonance is nearly at the ordinary Thomson value. In the literature, terminology is often borrowed from studies of birefringent crystals to classify the two modes: the mode resembling the ordinary case is called the “ordinary mode”, and the mode affected by the cyclotron resonance is called the “extraordinary mode”. For the modes defined above, this corresponds to the $+$ and $-$ mode respectively for $0 \geq \theta_i \geq 90^\circ$, while the opposite is true for $90 \geq \theta_i \geq 180^\circ$.

It must be noted that while the cyclotron resonance in the magnetic Thomson cross section is formally divergent in a classical exposition, photons precisely at $\omega = \omega_B$ are not treated in this thesis. At neighboring frequencies in the wings of the resonance, diffusive transport through scattering is tracked in the simulations outlined in Ch. 3. Yet this can quickly become computationally prohibitive when $\omega$ becomes very close to $\omega_B$, a circumstance avoided in the results presented herein. In future developments of the simulation in which small energy exchanges between photons and electrons will be treated, the cyclotron resonance will be encountered sporadically in the transport process. In such circumstances, the standard protocol is to incorporate a small Breit-Wigner width $\Gamma$, via a complex frequency $i\Gamma$, to establish a Lorentz
Figure 2.4: The magnetic Thomson cross section for normal modes traveling along or perpendicular to the magnetic field, as a function of frequency. Only the extraordinary mode (blue, -) is affected by the cyclotron resonance.

profile for the resonance (e.g. see Harding & Daugherty 1991). This is an inherently quantum inclusion in that the width represents the inverse lifetime for the decay of an excited intermediate (virtual) electron state in QED due to radiative cyclotron transitions. Thus, the divergence is truncated to finite values that are of the order of $(\omega_b/\Gamma)^2$ larger than the Thomson cross section when the magnetic field is well below the quantum critical value; details can be found in the papers by Baring et al. (2011) and Gonthier et al. (2014).

While we can learn some useful information by looking at the total cross section, it is difficult to directly investigate the nature of the differential cross section, as it depends on the frequency, the initial polarization and direction, as well as the final direction. This leads to an 8-dimensional phase space, which is difficult to
2.3: POLARIZATION OF MULTIPLY SCATTERED RADIATION

represent in figures. Instead, we will be able to glean information about the nature
of this scattering process through some basic examples produced using the Monte
Carlo code in the next chapter. Regarding magnetars, the intrinsic brightness of
magnetar flares leads to inferences of high Thomson opacity. This high opacity will
be further enhanced for magnetic Thomson scattering if the emission frequency and
local magnetic field are such that \( \omega \approx \omega_B \), meaning we are on the cyclotron resonance.
This is due to the resonant cross section (i.e. Eq. 2.3). Figure 2.5 shows how for a
typical magnetar, this resonance condition is met in the nearby magnetosphere for
X-ray frequencies typical of magnetar surface emission.

2.3 Polarization of multiply scattered radiation

To understand the evolution of intensity and polarization under magnetic Thomson
scattering, we show the result a large number of scatterings as a function of frequency
and angle \( \theta \) with respect to the magnetic field. This is an unphysical example in that
it does not treat the escape or exit conditions from a scattering region, but is useful
towards understanding the properties of the differential cross section.

We use an isotropic injection (as described in Eq. 3.2) of an ensemble of initially
unpolarized photons uniformly distributed in \( \log(\omega/\omega_B) \), and calculate the evolution
of the angular distribution and polarization properties after a large number of scat-
terings. Specifically, we use a mix of photons fully linearly polarized (corresponding
to either \( I_\parallel = 1 \) or \( I_\perp = 1 \)) such that the net polarization is zero, and record their
final direction and polarization after exactly \( N_{\text{scat}} = 10^4 \) scatterings. This protocol
is obviously unphysical, as this is not associated with any optical depth or proba-
bility for scattering (it specifically misses the polarization and frequency dependence
of the optical depth which allows some photons to escape without scattering). How-
CHAPTER 2: POLARIZED RADIATIVE TRANSFER

Figure 2.5: The cyclotron “resonosphere”: A diagram showing a magnetar of radius 10 km with equatorial field equal the quantum critical field $B_{cr} = 4.4 \times 10^{14}$ Gauss. Black curves give the location where the labeled frequency is on the cyclotron resonance ($\omega = \omega_B$). The innermost black curve touching the equatorial surface is at 511 keV. Red curves trace the field structure, assumed dipolar.

However, it gives useful information about preferential scattering directions, as well as the polarization properties. The result is shown in Fig. 2.6.

When $\omega < \omega_B/\sqrt{3}$ radiation is fully linearly polarized, and scattering into an-
gles $\cos \theta \simeq \pm 1$, i.e. along the field, is heavily suppressed. Perpendicular to the field ($\cos \theta = 0$) it is enhanced. Near $\omega = \omega_B$ radiation is fully polarized and elliptical in general, being circular along the field and linear perpendicular to it. Scattering along the field is now enhanced. For large $\omega/\omega_B$ scattering is isotropic and has small circular polarization, reminiscent of ordinary Faraday rotation (and in the same regime). The detailed transfer simulations discussed in Chapters 4, 5 and 6 are complicated convolutions of the information presented here, with the output polarization measures being weighted by the angular and polarization dependence present in the escape probability.
Figure 2.6: Angular distribution and polarization properties of multiply scattered radiation. An isotropic and unpolarized distribution, uniformly distributed in \( \log_{10}(\omega/\omega_B) \) was evolved for exactly \( 10^4 \) scattering events for each particle. The final angular distribution (normalized to unit range) and polarization fraction according to the color legend on the upper right are shown in the top row. Stokes Q and V are shown in the bottom row (Stokes U is negligible in this geometry), with the colors corresponding to the legend scale on the lower right. For \( \omega < \omega_B/\sqrt{3} \), radiation is fully linearly polarized, and scattering into angles along the magnetic field (\( \cos \theta = \pm 1 \)), is heavily suppressed, while perpendicular to the field (\( \cos \theta = 0 \)) it is enhanced. Near the resonance (\( \omega = \omega_B \)), radiation is fully polarized and elliptical in general (with helicity being strongly correlated with angle \( \theta \)), and scattering along the field is now enhanced. For large \( \omega/\omega_B \) scattering is isotropic and possesses a small circular polarization.
CHAPTER 3
Monte Carlo Simulation

The Monte Carlo method is a useful tool for scattering physics, especially so for the multidimensional transfer that occurs when analyzing radiative transfer in stratified media on the polarization level. The cross sections are somewhat complicated functions of angle and polarization, and scattering integrals are often analytically intractable. This renders integro-differential equation methods like Feautrier techniques CPU intensive. In contrast, the computational demands of the Monte Carlo method are modest in the current age of modern computing, and simulations generally require no more than hours to run on a single computer with a multi-core CPU.

In the simulation, we use Monte Carlo probabilistic methods to simulate the transport of individual photons through a scattering atmosphere. Each photon’s properties, such as frequency, polarization vector, and propagation vector, are tracked in flat spacetime (general relativistic influences are neglected as they don’t alter the character of the radiation transfer results on atmospheric scales). Photons are allowed to exit via some conditional requirement simulating, e.g., a given optical depth, after which relevant observable properties are tabulated such as its final propagation direction, intensity, and polarization. Other non-observable parameters such as the number of scatters each particle undergoes before exiting can also be recorded as a diagnostic that can aid code validation.

All that is required to solve the transfer problem using Monte Carlo methods is a description of all sources of radiation and a description of all relevant interactions.
The bulk of this thesis (specifically Chapter 4) is concerned only with a single source incident at the bottom of an atmospheric layer, and a single class of electron/photon interaction: magnetic Thomson scattering off electrons. First we will review the methods used in the transfer model, then we present basic results to better understand the scattering kernel. Models tuned to replicate astrophysical objects are reserved for Chapter 4 and Ch.5.

For this research project, I have written a Monte Carlo code from scratch. It is implemented in C/C++ to calculate the radiative transfer of polarized photons due to magnetic Thomson scattering off electrons in a strong magnetic field. This utilizes random number generation to create the initial particle energy and angular distributions, as well as to sample the differential cross section for scattering on the polarization level. For the presentation in Chapter 4, we will typically be modeling transfer through an atmospheric slab of a given optical depth. This section reviews the necessary concepts: how to generate initial particle injections at the bottom of a slab, how to utilize Monte Carlo sampling in the scattering kernel, how to record the properties upon exiting etc. Finally, we summarize how the overall algorithm is structured.

### 3.1 Intensity

Consider the amount of energy $dE_\nu$ contained in radiation in a frequency interval $(\nu, \nu + d\nu)$ passing across a surface of area $dA$ in time $dt$, confined to a solid angle $d\Omega$ about the angle $\Theta$ with respect to the surface normal. As described by Chandrasekhar (1960), this takes the form

$$dE_\nu = I_\nu \cos \Theta d\nu dA d\Omega dt. \quad (3.1)$$
3.1: INTENSITY

We will ignore the time and positional dependence, which are absent in the present problem. We first consider the monochromatic case so that we can ignore the frequency dependence as well. For a surface normal oriented along \(+z\), we have for the intensity

\[
I = \frac{dE}{\cos \theta d\Omega} = \frac{dE}{\cos \theta \sin \phi d\phi d\theta}
\]  

(3.2)

where \(\theta\) and \(\phi\) are the usual polar and azimuthal angles, respectively. Note that the angular dependence will be different depending on the orientation of the surface; it is only in this simple case that \(\Theta = \theta\). If the surface normal is \(\hat{n}\) and the direction of interest is \(\hat{r}\), we instead have

\[
\Theta = \arccos(\hat{n} \cdot \hat{r}).
\]  

(3.3)

The initial injection of photons at the bottom of an atmospheric layer will usually be distributed isotropically in intensity, i.e. \(I = \text{(const.)}\). For a given frequency, and for \(\hat{n} = \hat{z}\) as above, the intensity then obeys

\[
I = \text{(const.)} = \frac{h\nu dN}{\cos \theta \sin \phi d\phi d\theta}
\]  

(3.4)

where \(dN/d\theta d\phi\) is the differential number of photons in an interval \((\theta, \theta + d\theta)\) in polar angle and \((\phi, \phi + d\phi)\) in azimuthal angle. Thus, the photon distribution has the following angular dependence:

\[
\frac{dN}{d\theta d\phi} = f(\theta, \phi) \propto \cos \theta \sin \theta.
\]  

(3.5)

Integrated over all angles, this should produce the total number of photons passing
through the surface:

\[ \int_0^{\pi/2} \int_0^{2\pi} f(\theta, \phi) d\theta d\phi = N, \]  

(3.6)

where the integration over \( \theta \) stops at \( \pi/2 \) because we are considering only the radiation emerging from the bottom of the surface, i.e., into a hemisphere. This function \( f(\theta, \phi) \) represents the number density function for initial angular distribution. A description of how it is sampled in the code is provided in the next subsection.

### 3.2 Sampling

To sample the injection distribution using Monte Carlo methods, we use the fundamental principle of Monte Carlo. This simply requires that the cumulative distribution function be analytically invertible. To define this, we first need the probability density function \( p(x) \), which itself represents the likelihood of finding a random variable at a given value of \( x \). \( p(x) \) is normalized:

\[ \int_{x_1}^{x_2} p(x) dx = 1, \]  

(3.7)

i.e. there is a 100% probability that the random variable lies in the range \( (x_1, x_2) \).

The cumulative distribution function \( P(x) \) is closely related:

\[ P(x) = \int_{x_1}^{x} p(x') dx'. \]  

(3.8)

This grows monotonically from 0 at \( x = x_1 \) to 1 at \( x = x_2 \).

Next, we need a uniform random variate in the range \([0, 1]\); call it \( \xi \).\(^*\) To generate a

\(^*\)Each time \( \xi \) appears in a formula, we assume it must be randomly generated independently of its other occurrences. We will use subscripts where necessary to help clarify that these are independent invocations of the random variate.
random variable $x$ that follows the given probability density function $p(x)$, we simply solve the equation $\xi = P(x)$ for $x$, i.e.

$$x = P^{-1}(\xi). \quad (3.9)$$

This is essentially a statement of the fundamental principle of Monte Carlo.

Consider an individual photon in the simulation; we need to generate its initial propagation direction $\hat{n}_i$. This is accomplished by first generating a polar and azimuthal angle, then using the usual formula:

$$\hat{n}_i = (\sin \theta_i \cos \phi_i, \sin \theta_i \sin \phi_i, \cos \theta_i). \quad (3.10)$$

To sample an isotropic distribution in $\theta, \phi$, we have

$$\theta = \arccos (2\xi_\theta - 1), \quad \phi = 2\pi \xi_\phi. \quad (3.11)$$

However, this is not the formula we require to sample an initial injection, which is an isotropic flux of intensity through a surface. This is because it needs to be weighted by a flux factor that reflects the fact that only particles passing through the surface contribute. If we consider the isotropic intensity passing upwards through a surface with normal direction $\hat{z}$, we have instead

$$\theta = \frac{1}{2} \arccos (2\xi_\theta - 1), \quad \phi = 2\pi \xi_\phi. \quad (3.12)$$

To sample a similar flux through an arbitrarily oriented surface, it is simplest to first sample the above case for normal direction $\hat{z}$, then rotate $\hat{n}_i$ accordingly. For a
surface normal \( \hat{n}_s \) described by

\[
\hat{n}_s = (\sin \theta_s \cos \phi_s, \sin \theta_s \sin \phi_s, \cos \theta_s),
\]

(3.13)

the rotation and the corresponding matrix is described by

\[
\hat{n} = \overrightarrow{R} \cdot \hat{n}_i, \quad \overrightarrow{R} = \begin{pmatrix}
\cos \theta_s \cos \phi_s & -\sin \phi_s & \sin \theta_s \cos \phi_s \\
\cos \theta_s \sin \phi_s & \cos \phi_s & \sin \theta_s \sin \phi_s \\
-\sin \theta_s & 0 & \cos \theta_s
\end{pmatrix}.
\]

(3.14)

This is routinely extendable to treat anisotropic distributions of a specified form.

We also use the fundamental principle to generate the distance travelled by a photon before it undergoes a scattering event. This is a fairly standard procedure; one only needs to know the total cross section and number density of scatterers. For our case the total cross section \( \sigma \) is a function of polarization and direction, but these are constant for a given photon as it propagates in between scatterings (neglecting propagation effects that are the subject of Ch. 7). The distance \( s \) that a photon travels along its propagation direction before scattering due to electrons of number density \( n_e \) is sampled according to the formula

\[
\xi = \exp (-n_e \sigma s).
\]

(3.15)

This is Poisson statistics. In other words,

\[
s = -\frac{\log \xi}{n_e \sigma}.
\]

(3.16)

If the photon was initially at position \( \vec{r}_i \) with propagation vector \( \hat{n}_i \), then the position
at which it scatters $\vec{r}_f$ is easily described:

$$\vec{r}_f = \vec{r}_i + s\hat{n}_i.$$  \hfill (3.17)

Once a scattering is determined to occur, the differential cross section must also be sampled probabilistically in order to determine the new propagation direction and polarization after scattering. It is also possible to determine the distance travelled through a region of non-uniform field and density; this method is described in the appendix section A.2.

To sample the polarization-dependent differential cross section, the accept-reject method is used. The accept-reject method in its simplest form draws a box enclosing the (1D) probability distribution $p(x)$, and “throws darts” at it using two uniform random variates to describe the $x, y$ position. If this value lies within the function being sampled (i.e. the area under the curve), it is considered a “hit” and the value is accepted. If it lies above it, the value is rejected and the process must repeat. The rejection ratio will approach the ratio of these areas for a large number of attempts. This can be improved by changing the boundary from a simple box to a more optimized bounding function. For our case we have a 2D probability distribution but the concept is the same.

To accomplish this sampling, first a candidate final direction $\hat{n}_f$ is sampled from an isotropic distribution:

$$\hat{n}_f = (\sin \theta_f \cos \phi_f, \sin \theta_f \sin \phi_f, \cos \theta_f), \quad (3.18)$$

$$\theta_f = \arccos (2\xi_{\theta_f} - 1), \quad \phi_f = 2\pi \xi_{\phi_f}. \quad (3.19)$$
This is the direction the photon will be traveling after scattering if the rejection test is passed. A uniform random variate $\xi$ is sampled. If $\xi < (1/\sigma)d\sigma/d\Omega_f(\hat{n}_f)$, the final direction is accepted; otherwise we generate a new candidate final direction and the process repeats. Note that both $\sigma$ and $d\sigma/d\Omega$ are dependent on the initial polarization state and propagation direction. The total cross section $\sigma$ plays the role of the optimized bounding function; it produces a rejection rate of $1/3$ on average, which can be proven by numerical integration or shown empirically. The final polarization vector $\vec{E}_f$ is then easily calculated from the formula given in Eq. 2.19:

$$\vec{E}_f \propto \hat{n}_f \times \left\{ \hat{n}_f \times \left[ \nu_n \vec{E}_i + i\nu_n \vec{E}_i \times \hat{b} - (\vec{E}_i \cdot \hat{b}) \hat{b} \right] \right\}. \tag{3.20}$$

Since $\vec{E}_f$ is treated as a polarization vector, its magnitude is determined by normalization requirements.

To simulate an atmospheric slab of a given temperature, we would naturally want to sample a thermal distribution of photons; in other words, we want to sample a zero-mass Bose-Einstein number distribution, i.e. a Planck function:

$$n_\gamma \propto \frac{\omega^2}{e^{\hbar \omega/kT} - 1}. \tag{3.21}$$

If we want sample this distribution using Monte Carlo methods, this can be accomplished using a so-called table method for sampling. The table method allows us to use the fundamental principle of Monte Carlo even when the cumulative distribution function is not analytically invertible. It works by discretizing the probability density, making integration and inversion trivial, but requiring extra computational power to perform the integration and extra memory to store a table of numerical values. It can be much faster than the accept-reject method, especially when the bounding function
3.2: SAMPLING

is computationally expensive or not well optimized. However, it is only efficient if the same table can be used numerous times without needing to be re-generated (e.g. if the function to be sampled changes).

To utilize the table method, we must slice the function to be sampled into a number of discrete patches, similar to the binning described in the next subsection. This is convenient as we can match the width of the slices to the width of the frequency bins, which makes the discretized nature of this sampling method indiscernible from other more CPU-intensive methods.

Say we have a distribution function $p(\nu)$ and we want to sample a value $x$ from it. First the function is cut into $N$ discrete slices of constant width $\Delta\nu$. The $i$-th slice is associated with a range of values going from $\nu_i$ to $\nu_{i+1}$, where $i$ goes from 0 to $N - 1$. It is important that this range encompass the full area under the curve of the function. We form the cumulative distribution function by numerical integration using the slices as the discretized numerical grid; the cumulative distribution function value associated with the $i$-th slice is $P_i$. We form the value

$$F_j = \frac{\sum_{i=0}^{j} P_i}{\sum_{i=0}^{N-1} P_i}.$$  \hfill (3.22)

This is easily seen to be normalized, so $F_{N-1} = 1$. We now generate a uniform random variate $\xi$; there must be some slice $j$ such that $F_j \leq \xi < F_{j+1}$. This tells us that the value to sample falls in the range described by this slice, and we generate it uniformly within that region, i.e.

$$x = \nu_i + \xi \Delta\nu.$$  \hfill (3.23)

As long as $\Delta\nu$ is chosen small enough, this is an effective protocol for sampling the Planck spectrum in the bulk of its frequency range. However, directly sampling this
distribution can have drawbacks; namely it can suffer from poor number statistics in the tails of the distribution. This can be remedied by applying a method similar to particle splitting: we inject a quasi-uniform frequency distribution of photons, but alter their weight so as to conform to the desired frequency distribution, i.e., photons with $h\nu \ll kT$ or $h\nu \gg kT$ possess small weights, much inferior to unity. This is important when exploring frequency-integrated values for polarization signals as would be measured from magnetars by future X-ray polarimeters. We investigate such a case in Chapter 4.

3.3 Binning

To calculate the intensity emerging from the atmosphere, we start with the natural quantity in the code: the number of photons exiting a given angular bin. This angular binning accomplished in the code by splitting the relevant quantity ($\theta$ or $\mu = \cos \theta$, and $\phi$) into a number of equally spaced bins ($N_\theta$, $N_\mu$, $N_\phi$). The width of each bin ($\Delta \theta$, $\Delta \mu$, $\Delta \phi$) depends on the total number of bins and the range of values over which that quantity can take. For example,

$$\Delta \theta = \frac{\pi}{N_\theta} \quad \text{or} \quad \Delta \mu = \frac{2}{N_\mu} \quad \Delta \phi = \frac{2\pi}{N_\phi}.$$  \hspace{1cm} (3.24)

The average value in each bin ($\theta_i$, $\mu_i$, $\phi_i$) is easily defined:

$$\theta_i = \Delta \theta \left( i + \frac{1}{2} \right) \quad \text{or} \quad \mu_i = \Delta \mu \left( i + \frac{1}{2} \right), \quad \phi_i = \Delta \phi \left( i + \frac{1}{2} \right)$$ \hspace{1cm} (3.25)

Here, $i$ goes from 0 to $N - 1$, just like a loop variable.

First, consider the monochromatic case. If $N_{ij}$ is the number of photons exiting a
3.3: BINNING

surface with $\hat{n} = \hat{z}$, into angular bins $\theta_i$ and $\phi_j$, we have for $I_{ij}$ (the intensity in each bin)

$$I_{ij} = \frac{h \nu N_{ij}}{\cos \theta_i \sin \theta_i \Delta \theta \Delta \phi} \propto \frac{N_{ij}}{\cos \theta_i \sin \theta_i}.$$  (3.26)

The proportionality here reflects the fact that $\nu$ and the bin sizes are constant, though the full equality is needed to preserve normalization between runs of differing bin sizes.

If we bin in $\mu$ instead of $\theta$, we have

$$I_{ij} = \frac{h \nu N_{ij}}{\mu_i \Delta \mu \Delta \phi} \propto \frac{N_{ij}}{\mu_i}.$$  (3.27)

The intensity is usually normalized to the incident flux, which puts a factor of the total number of injected photons $N_{tot}$ in the denominator.

For binning upon exit from an arbitrarily oriented surface, we must use a more general formula for the intensity:

$$I_{ij} = \frac{h \nu N_{ij}}{\cos \Theta \sin \theta_i \Delta \theta \Delta \phi} \propto \frac{N_{ij}}{\mu_i}.$$  (3.28)

where $\Theta = \arccos(\hat{k} \cdot \hat{n})$; $\hat{n}$ is the surface normal, and $\hat{k}$ is the propagation vector.

We can also consider non-monochromatic injections; in this case the frequency dependence must be included. Considering it separately from the angular dependence, it is quite simple: for a number of photons $N_k$ in a given frequency bin $\nu_k$ of width $\Delta \nu$, the intensity is

$$I_k = \frac{h \nu_k N_k}{\Delta \nu} \propto \nu_k N_k.$$  (3.29)
CHAPTER 3: MONTE CARLO SIMULATION

3.4 Algorithm for an Atmospheric Slab

The protocol outlined in this subsection is principally applied to the X-ray emission from magnetar surfaces, though it is routinely adapted to the magnetospheric considerations of Chapter 6. The algorithm to model transfer through an atmospheric slab begins with input parameters; they are:

- orientation of the slab normal
- orientation and strength of the magnetic field (assumed constant and uniform)$^\dagger$
- frequency for monochromatic injection, or temperature for thermal injection
- Thomson optical depth $\tau_T$, or the equivalent depth $d$ and number density $n_e$
- Total number of photons $N_\gamma$ to be initially injected

Once these input quantities are decided, the algorithm can proceed. Note that algorithm operates on a photon-by-photon basis; this is highly parallelizable, meaning multiple threads can calculate the transfer on a photon-by-photon basis with minimal modification to the code. As such, the code is run in parallelized form. For purposes of this discussion we will assume an atmospheric slab with normal direction $\hat{z}$, the geometry of which is shown in Fig. 3.1. We will describe the algorithm’s process calculating the transfer of a single photon.

First, the initial properties of the photon are generated based on the injection distribution. For the angular distribution it is the isotropic flux passing through a slab, as described by Eq. 3.2. For physical models the frequency distribution is either monochromatic or generated from a thermal distribution. Sometimes a distribution

$^\dagger$For handling non-uniform fields and/or number densities, see appendix section A.2.
3.4: ALGORITHM FOR AN ATMOSPHERIC SLAB

uniform in frequency (or log frequency) is used to analyze the effects of scattering; this is very simple to sample.

Figure 3.1: Simulation geometry for transfer through an atmospheric slab with normal direction \( \hat{z} \) and magnetic field in the \( x-z \) plane. The grey trajectory shows a photon which scatters 8 times before exiting the top of the slab, reaching the observer.

Spatially, the photon is put at the bottom of the slab, i.e., \( z = 0 \). Each photon is fully polarized, but the initial injection has zero average polarization. This presumes that the injection altitude is deep in the slab; adaptation to treat polarized injections is routine and sometime useful for simulation efficiency. We can establish the zero polarization configuration by isotropically sampling a polarization from the Poincaré sphere. Alternatively, we can replicate the method of Whitney (1989), which puts photons initially into one of the two independent linear polarizations, i.e., \( (I_{||}, I_{\perp}) = (I, 0) \) or \( (0, I) \). Another alternative is to put photons initially into one of the two polarization eigenmodes described in Chapter 7. As previously mentioned, the code uses complex polarization vectors rather than modified Stokes vectors to describe
polarization states. We keep these vectors normalized so that the magnitude reflects the statistical weight assigned via the frequency distribution.

With the initial properties established, the algorithm proceeds to calculate scattering using the following loop. First the distance to the next scatter is determined, then the photon moves to that position. The exit conditions are then tested; if no exit occurs, the scattering kernel is invoked. The new direction and polarization is determined, and this process repeats until an exit condition is reached.

Physical exit conditions can be divided into 2 types. If the photon’s $z$-position is greater than $d$ it escapes the top of the slab; it is able to reach the observer shown in Fig.3.1, and is recorded. If the photon’s $z$-position is less than zero, it escapes the bottom of the slab. This bottom-going intensity can be recorded if desired, but will not contribute to intensity reaching the observer; it is effectively absorbed. This in turn causes the observed intensity (which is normalized to the initial intensity) to be lowered accordingly. This is of course what is expected from basic radiative transfer; for higher optical depths this effect is stronger.

Sometimes an additional exit condition is useful in practice: when a particle has scattered a large number of times, far in excess of what would be expected given the optical depth, we can consider it effectively absorbed. Such particles when present in the code are the consequence of finite statistics. This exit condition can save on runtime by avoiding these relatively rare occurrences which are statistically insignificant to the results. For example, at frequencies near the cyclotron resonance this becomes much more likely, and so is a real possibility if the Thomson optical depth is not very small. A histogram for all photons showing the number scattering events can be examined to confirm that we are not unduly truncating the high-scattering tail of the distribution.
Chapter 4
Atmospheric Transfer: Basic Results

Here we present a variety of results for transfer through atmospheric slabs. Throughout this chapter we set the field along the $z$-axis; this often produces a symmetry that lets us average over the azimuthal angle. First we show the simplest case, of transfer through a single slab with the slab normal parallel to the magnetic field direction. This for the three different initial polarization scenarios earlier described: isotropic Poincaré sphere sampling, Whitney’s independent linear polarizations, and polarization eigenmodes. Though each photon is fully polarized, the initial injection has zero net polarization. We show the resulting intensity and polarization properties as a function of frequency and observer angle for a slab of constant Thomson optical depth for these three cases.

We also show a series of results for monochromatic injections through $z$-normal slabs of a range of constant magnetic optical depths, a situation which replicates earlier results from Whitney (1989). Next, we address the intermediate case of oblique field orientation. Finally, we show equivalent results for transfer through $x$-normal slabs of a range of constant magnetic optical depths.

4.1 Parallel Field

Here we present results for a plane atmosphere of a given Thomson optical depth due to magnetic Thomson scattering, with the magnetic field and surface normal parallel
and oriented along the z-axis. This situation is analogous to results presented in Whitney (1991), allowing us to perform an important consistency check.

The simulation geometry is that shown in Fig. 3.1, with the magnetic field aligned with the normal so that $\theta_B = 0$. The initial injection is that of an isotropic flux through the bottom boundary, as described in Eq. 3.2. We first show results over a wide frequency range using an injection uniformly distributed in $\log_{10}(\omega/\omega_B)$; this is done for the regime of high Thomson optical depth $\tau_T = \sigma_T n_e d$. Note that the actual optical depth felt by a photon depends on its frequency, direction, and polarization; most importantly it is a rapidly falling function of frequency for $\omega \lesssim \omega_B/\sqrt{3}$. Figs. 4.1a-c show the case for a Thomson optical depth of $\tau_T = 20$ for different polarization injections, plotted as functions of of the observer’s viewing angle relative to the zenith.

From Fig. 4.1a-c we see, as expected, that the intensity is highest for low frequencies and is nearly zero in the resonance. This reflects the properties of the total cross section (Eq. 2.31); when it is large the mean free path becomes small, and most of the injected photons escape through the bottom boundary rather than emerge at the top of the slab. This is how the Monte Carlo simulation produces effective attenuation (i.e., loss) due to scattering. The polarization behavior near the resonance is very similar to those described in Fig. 2.6; the cross section is optically thick to all polarizations so the radiation is multiply scattered. This behavior occurs for frequencies $\omega > \omega_B/\sqrt{3}$. The circular polarization is high in the resonance when looking tangent to the slab.

At $\omega = \omega_B/\sqrt{3}$ the magnetized and ordinary Thomson cross sections are equal, as seen in Fig. 2.3. As we go lower in frequency from $\omega = \omega_B/\sqrt{3}$, an increasing amount of photons pass without scattering, although the scattered population tends
Figure 4.1a: Intensity and polarization properties of radiation exiting a slab of Thomson optical depth $\tau_T = 20$, as functions of the observer zenith angle $\theta_{\text{obs}}$. The magnetic field and slab normal are both aligned in the $\hat{z}$ direction. The initial polarization of photons is sampled isotropically from the Poincaré sphere. The dip in intensity at $\omega = \omega_B$ is caused by the resonant cross section. For $\omega > \omega_B/\sqrt{3}$ photons are multiply scattered and the polarization properties are the same as that described Fig. 2.6. For $\omega < \omega_B/\sqrt{3}$ and along the field (bottom right of each figure) there is an excess caused by net unpolarized radiation leaving the system unscattered. This causes the increase in intensity and decrease in polarization in that region. The effect increases with decreasing $\omega$. 
Figure 4.1b: Here we have the same slab properties ($\tau_T = 20, \vec{B} \parallel \hat{n} \parallel \hat{z}$), but the photons are injected with a mix of polarization eigenmodes (which are dependent on both $\theta$ and $\omega/\omega_B$). At the resonance the intensity is relatively high compared to the other two polarization injections because the ordinary mode photons have a lower opacity in this regime; they are the only polarization state that has this feature. At lower frequencies a comparative intensity excess is generated, this time caused by extraordinary mode photons leaving the system unscattered. The excess is more uniform in angle than Fig. 4.1a though still focused along the field, and the polarization properties become dominated by the strong linear polarization ($Q = -1$) and weaker circular polarization associated with this mode.
4.1: PARALLEL FIELD

Figure 4.1c: We again have the same slab properties ($\tau = 20$, $\vec{B} \parallel \vec{n} \parallel \vec{z}$), but the photons are injected with a mix of linear polarizations ($I_\parallel$ or $I_\perp$), following Whitney (1991a). At the resonance the intensity is very low, as both of these polarization states are optically thick. At lower frequencies we again see an intensity excess caused by increasing amounts of $I_\perp$ leaving the system unscattered. The small circular polarization in at low frequencies is now absent.
towards $Q = +1$ as previously described in Chapter 2. For the isotropic Poincaré injection the unscattered population is net unpolarized, lowering the overall polarization in this regime. For the linear injection this produces large linear polarizations of $Q = -1$ in this regime. For the eigenmode injection it is the extraordinary mode that is escaping, and it is nearly linear ($Q = -1$) though it also has a circular component when traveling along the field. We also see an angular dependence in intensity at low frequency: it is peaked along the field direction. This is also an effect from the total cross section; these increasing intensities represent an increasing number photons that escape unscattered. As such, this effect is sensitive to the initial polarization injection. The Stokes parameters see some variance with respect to $\theta_{\text{obs}}$; linear polarization is highest near $\theta_{\text{obs}} = 90^\circ$ $(\cos \theta_{\text{obs}}=0)$, while it is minimal near $\theta_{\text{obs}} = 0^\circ$ $(\cos \theta_{\text{obs}}=1)$. The opposite is true for circular polarization. This is expected behavior: transfer along the field cannot discern between linear polarizations, but one helicity of circular polarization transfers more readily than the other due to cyclotron excitation. Similarly, transfer perpendicular to the field cannot discern between circular polarizations, but one linear polarization transfers more readily than the other.

Figs. 4.2a-i shows the intensity and polarization of radiation emergent from a plane atmosphere with $B\parallel z$, for monochromatic injections with frequencies defined relative to the cyclotron frequency $\omega_B$, and for different observer angles with respect to the magnetic field (here parallel to the slab normal). Only Stokes $Q$ and $V$ are depicted since Stokes $U$ is zero by our choice of reference for defining Stokes parameters.*

Since the optical depth is strongly dependent on the ratio $\omega/\omega_B$ in the highly magnetic

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*Because we have a constant, uniform magnetic field, we can define the reference direction such that Stokes $Q = I_\parallel - I_\perp$. With this reference, Stokes $U$ will be zero by definition. However, when summing over regions of different magnetic fields this choice is no longer possible, and both $Q$ and $U$ will be nonzero.
regime \((\omega < \omega_B)\), it is sensible to use the frequency-dependent total cross section when labeling the optical depth, as opposed to the Thomson value. For net unpolarized radiation traveling a distance \(d\) we can use Eq. 2.31 for the total cross section. Along the magnetic field we have an optical depth

\[
\tau_\parallel = n_e d \times \sigma_{\text{unpol}}(\theta_i \to 0) = \tau_T \times \frac{\nu_B^4 + \nu_B^2}{(\nu_B^2 - 1)^2}, \tag{4.1}
\]

while for radiation traveling perpendicular to the field we have

\[
\tau_\perp = n_e d \times \sigma_{\text{unpol}}(\theta_i \to 90^\circ) = \tau_T \times \frac{1}{2} \left[ 1 + \frac{\nu_B^4 + \nu_B^2}{(\nu_B^2 - 1)^2} \right], \tag{4.2}
\]

where \(\nu_B = \omega/\omega_B\). The cross sections corresponding to these optical depths (Eq. 2.31) can be seen in Fig. 2.3. Using the optical depth described in Eq. 4.1 allows us to compare directly with results from Whitney (1989), as this is the optical depth she employed to produce those results. This produces the same effective optical depth for different frequencies, in contrast to using a constant Thomson optical depth as in the previous plots. Thus, as the frequency ratio \(\nu_B\) is varied through the figure sequence, the Thomson optical depths span a wide range of values. This choice elucidates the polarized transport properties, though it must be remembered that it renders the slab thickness frequency-dependent. We also use the linear polarization injection so that we may compare directly with Whitney’s results. These plots more accurately show the angular dependence of the intensity and polarization emerging from such atmospheres, as well as how that angular dependence changes with frequency. Note that only the high optical depth curves (blue/purple Figs. 4.2a-i) approximate conditions for polarized radiation emergent from self-consistent atmospheres.

At frequencies much higher than the cyclotron frequency \((\omega \gg \omega_B)\), the behavior
\[ \hat{n}_{\text{slab}} \| \hat{z}, \quad \omega/\omega_B = 10 \]

![Intensity and polarization plots](image)

Figure 4.2a: Intensity and polarization of radiation emergent from a plane atmosphere of a given optical depth. 10 contours are shown, corresponding to different optical depths ranging from $\tau_\parallel = 1$ (red) to $\tau_\parallel = 10$ (violet), in unit increments. An isotropic, monochromatic injection is used, and $\hat{n}_{\text{slab}} \| \hat{z}$, i.e. the geometry seen in Fig. 3.1. Each page shows a different value for $\omega/\omega_B$. Black squares show results from Figure 13f of Whitney (1989) for $\tau_\parallel = 7$, which are in good agreement with this work (blue-grey curves). On this page, $\omega/\omega_B = 10$, and the behavior is approaching the limit to that of an unmagnetized plasma, with the magnetic field inducing only a small circular polarization.
Figure 4.2b: Here $\omega/\omega_B = 2$ for an identical slab set-up to Fig. 4.2a, and the the histograms color-coded as in that figure. The polarization fraction is growing at $\omega$ approaches $\omega_B$, and the character of the polarization is elliptical in general, with circular polarization dominating along the field ($\theta_{obs} = 0$) and linear polarization dominating perpendicular to the field ($\theta_{obs} = 90^\circ$). Black squares show equivalent results from Figure 13h of Whitney (1989) for $\tau_{||} = 7$. 

$\hat{n}_{slab} \parallel \hat{z}$, $\omega/\omega_B = 2$
Figure 4.2c: Here \( \omega/\omega_B = 1.2 \), and the magnetic optical depth is significantly overestimating the true optical depth for scattering, as particles are preferentially scattering into angles near \( \theta_{\text{obs}} = 90^\circ \) where the optical depth is lower. Thus, more radiation is transferred along this direction, and the effective optical depth (i.e. Eq. 4.1) is lowered. Thus even for \( \tau_\parallel = 10 \) the properties of the emergent radiation have not quite converged on their optically thick forms. Black squares show equivalent results from Figure 13j of Whitney (1989) for \( \tau_\parallel = 7 \).
\( \hat{n}_{\text{slab}} \parallel \hat{z}, \ \omega/\omega_B = 0.99 \)

Figure 4.2d: \( \omega/\omega_B = 0.99 \), almost directly on the resonance. There is significant excess perpendicular to the field, particularly in the linear polarization parallel to the field, where the cross section is reduced to the ordinary Thomson value. As this causes Stokes Q to pass through zero, the overall polarization fraction is profoundly reduced at these angles, a distinctive feature that can be probed with phase-resolved observations in rotating magnetars. Whitney (1989) did not treat this resonant domain.
Figure 4.2e: $\omega/\omega_B = 0.8$; we are approaching the magnetically dominated regime. The properties here resemble that of Fig. 4.2c, i.e. the properties are similar for a region of frequencies on either side of the resonance. Black squares show equivalent results from Figure 13k of Whitney (1989) for $\tau || = 7$; there is a small discrepancy between Whitney’s data and ours for Stokes $Q$. 

$\hat{n}_{\text{slab}} \hat{z}$, $\omega/\omega_B = 0.8$
\[ \hat{n}_{\text{slab}} \parallel \hat{z}, \quad \omega / \omega_B = 1 / \sqrt{3} \approx 0.577 \]

Figure 4.2f: \( \omega / \omega_B = 1 / \sqrt{3} \), the value at which the unpolarized magnetic cross section along the field equals the ordinary cross section. As \( \omega / \omega_B \) is reduced from its value in Fig. 4.2e, linear polarization is reduced (i.e. Stokes Q is increasing from negative values. Stokes Q will continue to increase as the frequency ratio is lowered, sweeping through zero at roughly \( \omega / \omega_B \approx 0.58 \) and becoming positive. The intensity here resembles the unmagnetized limit. Black squares show equivalent results from Figure 13m of Whitney (1989) for \( \tau_\parallel = 7 \).
Figure 4.2g: $\omega / \omega_B = 0.5$; in this regime circular polarization is reducing, while Stokes Q is increasing. This new behavior occurs because we are in a regime where the total cross section for each normal mode is comparable, but scattering into the ordinary mode ($Q = +1$) is favored, as can be seen in Fig.2.5. The peak in intensity is moving towards smaller angles. Black squares show equivalent results from Figure 13q of Whitney (1989) for $\tau_\parallel = 7$. 
Figure 4.2h: $\omega/\omega_B = 0.25$; here we start seeing the main features of the highly magnetized regime. Namely, the intensity is now peaking near $\theta_{obs} = 0$, where there is a small circular polarization. This is because the opacity for circular polarization along the field is vastly reduced, as the photons’ electric field circulates in the same way as cyclotron motion. Elsewhere the polarization is almost entirely linear. Black squares show equivalent results from Figure 13t of Whitney (1989) for $\tau_\parallel = 7$. 

\begin{equation}
\hat{n}_{\text{slab}} || \hat{z}, \quad \omega/\omega_B = 0.25
\end{equation}
\[ \hat{n}_{\text{slab}} \parallel \hat{z}, \ \omega / \omega_B = 0.1 \]

![Graph of intensity, Pol. fraction, and Stokes Q, V vs. \( \theta_{\text{obs}} \)]

Figure 4.2i: \( \omega / \omega_B = 0.1 \); here we are fully in the highly magnetized regime. The features seen in the last panel are now even more pronounced. There is no data for this deeply magnetized regime from Whitney (1989).
limits to that of an unmagnetized plasma. As $\omega/\omega_B$ is lowered towards unity, a growing circular polarization appears, dominating at lower values of $\theta_{\text{obs}}$. This is joined by a linear polarization, which dominates at larger values of $\theta_{\text{obs}}$. These trends continue to the resonance at $\omega = \omega_B$. In the range $\omega_B > \omega > \omega_B/\sqrt{3}$, linear polarization is lowering in magnitude.

At $\omega/\omega_B = 1/\sqrt{3}$, the magnetic and unmagnetized cross sections are equal; above this value the magnetic cross section grows larger, while below it the cross section shrinks. Near this point the linear polarization passes through zero, and below $\omega/\omega_B = 1/\sqrt{3}$ the linear polarization grows in the opposite direction, while the circular polarization shrinks. At frequencies much lower than the cyclotron frequency ($\omega \ll \omega_B$), the radiation is almost entirely linearly polarized. In this strongly magnetic regime, intensity is peaked around the direction of the magnetic field as well. At all frequencies there are two competing transfer effects near $\theta_{\text{obs}} = 90^\circ$: one is due to radiation leaving the slab at a shallow angle, and one is due to the directional dependence of the magnetic field; these effects will be disentangled for different slab geometries. A number of these characteristics can potentially provide useful diagnostic tools in phase-resolved polarimetric observations of neutron stars. Most striking among these is the rapid change in polarization degree (and associated position angle) in the resonant case, which could prove interesting for pulsars of lower magnetizations.

## 4.2 Oblique Field

The case of an aligned slab normal and magnetic field has the advantage of symmetry that lets us average over the azimuthal direction. However, in reality this symmetry usually does not exist; we must handle arbitrarily oriented slabs and magnetic fields in order to treat regions in atmospheres and magnetospheres that are not above the
CHAPTER 4: ATMOSPHERIC TRANSFER: BASIC RESULTS

Figure 4.3: Simulation geometry for transfer through an atmospheric slab of depth $d$ with magnetic field direction $\hat{z}$. The slab normal (shown in blue) is tilted into an arbitrary direction, and is described by spherical angles $\theta_s$ and $\phi_s$. Radiation is incident on the bottom of the slab as usual, and radiation emerging from the top is now recorded in spherical angles as well ($\theta_{\text{obs}}$ and $\phi_{\text{obs}}$).

magnetic pole. Here we take an informative look at what happens when we rotate the slab normal, keeping the magnetic field fixed in the $\hat{z}$ direction; this geometry is shown in Fig. 4.3. We must now track the full $\theta$ and $\phi$ dependence of the outgoing radiation. A particular consequence of this generalized configuration is that linear polarization and circular polarization characteristics are not correlated with slab geometry and so become convolved in the output emission.

As a simple case, we show in Fig. 4.4 the radiation properties as a function of observer angle in $(\theta_{\text{obs}}, \phi_{\text{obs}})$ phase space for a monochromatic injection at $\omega/\omega_B =$
0.99, giving a correspondence to the plots in Fig. 4.2d above and Fig. 4.7d below. The Thomson optical depth $\tau \approx 4 \times 10^{-3}$ corresponding to $\tau_\perp = 10$ in Eq. 4.1. A net unpolarized injection of linearly polarized radiation is used. Three cases are shown: $\theta_s = 0^\circ, 45^\circ, 90^\circ$, the first of which has already been discussed. This gives an idea of the volume of $(\theta, \phi)$ phase space that is occupied by radiation leaving the top such a slab, as well as the angular dependence. For any orientation the radiation emergent from one side of a slab will take up 50% of the $(\theta_{\text{obs}}, \phi_{\text{obs}})$ phase space.

We set $\phi_s = 180^\circ$ for nonzero $\theta_s$ in order to bring the result into a clearer form.

For $\theta_s = 0$, the outgoing radiation fills the upper hemisphere, i.e. $0^\circ \geq \theta_{\text{obs}} \geq 90^\circ$, $0^\circ \geq \phi_{\text{obs}} \geq 360^\circ$. As expected, there is no dependence on $\phi$. For $\theta_s = 45^\circ$, the outgoing radiation has a more complicated dependence on observer angle. When $\theta_s = 90^\circ$, it fills a region $0^\circ \geq \theta_{\text{obs}} \geq 180^\circ$, $90^\circ \geq \phi_{\text{obs}} \geq 270^\circ$, a simpler dependence than for intermediate angles. The emergent properties do have $\phi$ dependence in this case; this dependent would be absent only of an axisymmetric exit condition, i.e. escape out the sides of a cylinder. For nonzero $\theta_s$ angles, there is a feature centered at $\theta_{\text{obs}} = 90^\circ$, $\phi_{\text{obs}} = 180^\circ$ this is generated by unscattered photons linearly polarized along the field direction; these photons are in the ordinary mode at this angle, which is extremely optically thin. Thus the intensity is un-attenuated, the linear polarization in the ordinary mode is high, and the circular polarization is minimal (due to viewing perpendicular to the field direction).

### 4.3 Perpendicular Field

Here we describe transfer through an atmospheric slab with the magnetic field perpendicular to the slab normal. This corresponds to one of the cases highlighted in Section 4.2, and is appropriate for atmospheres proximate to the magnetic equator.
Figure 4.4: Angular maps in $(\phi_{\text{obs}}, \theta_{\text{obs}})$ showing logarithmic intensity (top row), Stokes Q (middle row), and Stokes V (bottom row) for radiation emerging from the top of 3 slabs with different magnetic orientations. The color coding is described in the two legends on the right. The magnetic field is along $\hat{z}$ and the slab normal is tilted at an angle $0^\circ$ (left column), $45^\circ$ (middle column), $90^\circ$ (right column) towards the $-x$ direction. Here, $\omega/\omega_B = 0.99$ and giving a correspondence to the plots in Fig. 4.2d and Fig. 4.7d. For $0^\circ$ we see there is no $\phi_{\text{obs}}$ dependence as expected. For the $90^\circ$ case we see that there is a weak $\phi_{\text{obs}}$ dependence in intensity and Stokes parameters introduced by the exit boundary; it would be absent for an axisymmetric exit condition.
geometry possesses a weak azimuthal symmetry with respect to the field direction, which sufficiently justifies averaging over the azimuthal angle.

This is especially nice as we can capture the difference in circular polarization going along or against the field. Note however that all radiation reaching the observer has positive \( x \) component. We first examine results over a wide frequency range using an injection uniformly distributed in \( \log_{10}(\omega/\omega_B) \), analogous to the cases shown in Figs.4.1a-c but using the new geometry. One notable difference in this geometry is that we can view radiation traveling along the field at \( \cos \theta_{\text{obs}} = 1 \) versus radiation traveling in the opposite direction, \( \cos \theta_{\text{obs}} = -1 \). As expected, these different directions correspond to opposing helicities for the circular polarizations.

Figs.4.6a-c shows the case for a Thomson optical depth of \( \tau = 20 \) for the three different prescriptions for polarization injection. We see largely the same trends seen for the \( z \)-oriented slab. The opacity near the resonance is still small, but higher
than the field-aligned case. This is because of decreased opacity to photons linearly polarized along the field. The circular polarization is anti-symmetric in $\cos \theta_{\text{obs}}$, since opposing helicities are favored for transfer parallel versus anti-parallel to the field, as seen in the previous section. The linear polarization shows trends very similar to the case shown in Sec. 4.1.

Figs. 4.7a-h are analogous to the case of Figs. 4.2a-i but for the new geometry; they show the intensity and polarization of radiation emergent from a plane atmosphere with $B_{\parallel} z$ and normal parallel to $x$, as shown in Fig. 4.5, for monochromatic injections with frequencies defined relative to the cyclotron frequency $\omega_B$. The initial injection is of the linear type in a similar fashion. This case, corresponding to an atmosphere (or perhaps a magnetosphere) at the magnetic equator, was not explored by Whitney, and so there are no comparison data points on the plots. We also note that this equatorial case is not commonly addressed in atmosphere models in the literature.

We see very similar trends to the previous geometry, but this geometry opens up $90^\circ < \theta_{\text{obs}} < 180^\circ$ as well, allowing us to clearly see effects happening at $\theta_{\text{obs}} = 90^\circ$. The expected (anti-)symmetry of Stokes $V$ about $\theta_{\text{obs}} = 90^\circ$ is clearly seen, as is the expected symmetry of the Stokes $Q$ about this value. Near the resonance we see the domination of $Q = 1$ at $\theta_{\text{obs}} = 90^\circ$; this is the ordinary mode at this angle, and it has far lower opacity than the other polarizations, passing unscattered. This leads to increases in intensity and Stokes $Q$ near this angle. Below $\omega = \omega_B/\sqrt{3}$ we see the $Q = -1$ begin to dominate. As before, we see intensity enhancement along the field in the highly magnetic regime.

Both modes in this regime are largely linear for all angles, and the lower opacity of the extraordinary mode causes this population to pass unscattered, decreasing Stokes $Q$. However, radiation that does scatter will be preferentially polarized with $Q = +1$. 
4.3: PERPENDICULAR FIELD

Figure 4.6a: Intensity and polarization properties of radiation exiting a slab of Thomson optical depth $\tau = 20$, as functions of the observer zenith angle $\theta_{\text{obs}}$. The slab and magnetic field direction are perpendicular with the field in the $\hat{z}$ direction and the slab normal in the $\hat{x}$ direction. The initial polarization of photons is sampled isotropically from the Poincaré sphere, as in Fig. 6. The legends on the right define the color scale.
Figure 4.6b: Intensity and polarization properties of radiation exiting a slab of Thomson optical depth $\tau = 20$. The slab and magnetic field direction are perpendicular with the field in the $\hat{z}$ direction and the slab normal in the $\hat{x}$ direction. The photons are injected with a mix of polarization eigenmodes (which are dependent of both $\theta_{\text{obs}}$ and $\omega/\omega_{B}$).
4.3: PERPENDICULAR FIELD

Figure 4.6c: Intensity and polarization properties of radiation exiting a slab of Thomson optical depth $\tau = 20$. The slab and magnetic field direction are perpendicular with the field in the $\hat{z}$ direction and the slab normal in the $\hat{x}$ direction. The photons are injected with a mix of linear polarizations ($I_\parallel$ or $I_\perp$).
\[ \hat{n}_{\text{slab}} \parallel \hat{x}, \ \omega / \omega_B = 10 \]

Figure 4.7a: Intensity and polarization of radiation emergent from a plane atmosphere of a given optical depth. 10 contours are shown, corresponding to different optical depths ranging from \( \tau_\perp = 1 \) (red) to \( \tau_\perp = 10 \) (violet), in unit increments. An isotropic, monochromatic injection is used, and \( \hat{n}_{\text{slab}} \parallel \hat{x} \). This geometry allows us to average over the azimuthal angle as before, and there is now the full 180\(^\circ\) of polar angle, allowing us to see the behavior parallel and anti-parallel to the field. We use the linear polarization injection on these plots. On this page \( \omega / \omega_B = 10 \), and the behavior is approaching the limit to that of an unmagnetized plasma, with the magnetic field inducing circular polarization whose sign depends on the orientation with respect to the field.
4.3: PERPENDICULAR FIELD

\( \hat{n}_{\text{slab}} \parallel \hat{x}, \quad \omega / \omega_B = 2 \)

![Intensity, Pol. fraction, Stokes Q, Stokes V graphs]

Figure 4.7b: Here \( \omega / \omega_B = 2 \). The polarization fraction is growing at \( \omega \) approaches \( \omega_B \), and the character of the polarization is elliptical in general, with circular polarization dominating along the field (\( \theta_{\text{obs}} = 0, 180^\circ \)) and linear polarization dominating perpendicular to the field (\( \theta_{\text{obs}} = 90^\circ \)). The intensity is developing a peak perpendicular to the field.
\[ \hat{n}_{\text{slab}} \parallel \hat{x}, \quad \omega / \omega_B = 1.2 \]

Figure 4.7c: Here \( \omega / \omega_B = 1.2 \); particles are passing unscattered at angles near \( \theta_{\text{obs}} = 90^\circ \) where the optical depth is lower for the linear modes with \( Q = +1 \). This generates the excess in intensity and the effect on the linear polarization seen at this angle. Scattered radiation is elliptical in general; it possesses \( Q = -1 \) near \( 90^\circ \) and large circular polarization along the field.
\[ \hat{n}_{\text{slab}} \parallel \hat{x}, \quad \omega / \omega_B = 0.99 \]

Figure 4.7d: $\omega / \omega_B = 0.99$, almost directly on the resonance. The unscattered population mentioned in the previous figure is now even more pronounced near $90^\circ$, while the properties of the scattered radiation remains the same.
$\hat{n}_{\text{slab}} \parallel \hat{x}$, $\omega/\omega_B = 0.8$

Figure 4.7e: $\omega/\omega_B = 0.8$; we are approaching the magnetically dominated regime. The properties here resemble that of Fig. 4.7c, i.e. the properties are similar for a region of frequencies on either side of the resonance.
Figure 4.7f: \( \omega / \omega_B = 1 / \sqrt{3} \), the value at which the unpolarized magnetic cross section along the field equals the ordinary cross section. Stokes Q is increasing from negative values (linear polarization is reducing). For lower frequencies the contribution of the scattered radiation will increase Stokes Q, bringing it towards +1. Competing with this is the unscattered population, which is becoming increasingly optically thin as we go into lower frequencies. With our definition of optical depth (Eq. 4.1), it is the unscattered population that will win out.
\[ \hat{n}_{\text{slab}} \| \hat{x}, \quad \omega / \omega_B = 0.2 \]

Figure 4.7g: $\omega / \omega_B = 0.2$; Stokes $Q$ has grown and circular polarization is very small except along the field. Multiply scattered radiation generates the intensity excesses along the field, as well as the circular and the $Q = +1$ polarization. Near $90^\circ$ radiation with $Q = -1$ is escaping unscattered, with the corresponding effect seen in intensity.
\[ \hat{n}_{\text{slab}}||\hat{x}, \quad \omega/\omega_B = 0.1 \]

Figure 4.7h: \( \omega/\omega_B = 0.1 \); here we are deeper in the highly magnetized regime. Now the unscattered radiation is dominating, producing linearly polarized radiation with negative Q at all angles.
For extremely high Thomson optical depths the scattered radiation will dominate the polarization in this regime, while for modestly high Thomson optical depths the extraordinary mode will escape unscattered and dominate. Because $\tau_\perp$ averaged over linear polarizations, it overestimates the true optical depth.

For some astrophysical context, we can imagine what the time trace of soft X-ray polarization properties would be for a spinning magnetar. This amounts to a sinusoidal oscillation in time of $\theta_{\text{obs}}$, in the highly magnetized regime ($\omega \ll \omega_B$). The perpendicular field plots do not exhibit interesting variations in polarization properties in this regime, while there are significant variations in the parallel field plots (e.g. Figs. 4.1a-c). This suggests that the diagnostic power of X-ray polarimetry would be maximized for these geometries. We can also examine Fig. 4.7h, which shows $\omega/\omega_B = 0.1$ for the perpendicular field case. Here, the linear polarization degree ranges from $\sim 0$ to 75\% between $0^\circ < \theta_{\text{obs}} < 60^\circ$, which would provide good diagnostic power to a phase-resolved X-ray polarization measurements in magnetars. In particular, these diagnostics afford determinations of the inclination angle between the magnetic dipole and rotation axes.

Perhaps the richest array of behaviors is afforded by frequencies near the cyclotron resonance, which aids the diagnostic potential of X-ray polarization measurements. Therefore, our model could also have applicability and utility for accreting X-ray pulsars, where photon energies in the 10-30 keV range sample the cyclotron resonance. Resonant cyclotron transport in the accretion columns of such sources has been handled previously by Araya & Harding (1999); they do not track polarization, but our model can facilitate such an endeavor.
CHAPTER 5

Magnetar Atmospheric Emission

To connect to X-ray observations of magnetar atmospheric emission, it is imperative to integrate our slab models over an entire neutron star surface, complete with an array of magnetic field zenith angles. All confirmed magnetars exhibit pulsed, hot quasi-thermal emission in soft X-rays, with temperatures around 0.3-0.7 keV (e.g., see Perna et al. 2001; Olausen & Kaspi 2014a). Because of its quasi-thermal spectrum, this emission is believed to be generated at the atmospheric surface of the neutron star. These type of spectra can often be fit by a sum of two blackbodies, and the cooler blackbody component greatly dominates over the hotter blackbody in terms of luminosity fraction.

From this we can infer that the hot blackbody component is associated with a small hotspot, while the cool blackbody is associated with a much larger portion of the neutron star surface. It is reasonable to assume that the hot components are centered on the magnetic poles, since magnetospheric plasma traveling along field lines can facilitate heat transport from crustal layers to the surface due to enhanced thermal conductivities along field lines.

Thus we have developed a model of all surface emission coming from the visible surface of the star, including the magnetic dipole field, and featuring regions of two distinct temperatures $T_{\text{hot}}$ and $T_{\text{cold}}$. The geometry of this model shown in Fig. 5.1. To model an object whose spectrum is fit by a sum-of-two-blackbodies, we use the relative weighting of those blackbodies to estimate the relative size of the hot and
cold regions in accordance with the Stefan-Boltzmann law. We also show results for a single temperature atmosphere (at temperature $T_{\text{cold}}$) in order to compare with the two temperature results. This model presumes that light travels in straight lines to an observer, and therefore neglects the light bending effects of general relativity that are significant near the stellar surface. Treating such gravitational influences will probably change some of the details of the results presented, but the overall generic character will remain robust.

The surface of the star is sliced into a spherical grid aligned with the magnetic field, and each grid zone then treated using the slab geometry, with the slab normal and magnetic field being calculated based on the mean value of $\phi$ and $\cos \theta$ within each zone. The final emission pattern is produced by summing the emission and Stokes parameters for a particular instantaneous observer direction from each grid zone, weighted by the area of each zone. For example, a grid zone bounded by $\phi_1$, $\phi_2$, $\theta_1$, and $\theta_2$, has area $A = |(\phi_1 - \phi_2)(\cos \theta_1 - \cos \theta_2)|$ and mean values $\phi_{\text{avg}} = (\phi_1 + \phi_2)/2$ and $\theta_{\text{avg}} = \arccos((\cos \theta_1 + \cos \theta_2)/2)$. Note that slicing uniformly in $(\phi, \cos \theta)$ creates equal area grid zones. We typically use a $15 \times 15$ grid sliced in this way, which is a sufficient approximation for our purposes. We do not consider the heat transfer between the two regions; a more realistic atmosphere would have a range of temperatures instead of two distinct regions. We plan to analyze such a case in the future, but it is not included in this thesis.

In the first section, we consider an observer whose viewing angle is specified relative to the magnetic moment vector (i.e. using spherical angles with $\vec{m}$ as the polar axis). The resulting azimuthal symmetry allows us to show the emission and polarization properties as a function of polar angle and frequency as in Chapter 4, and also identify the magnetic colatitude dependence of polarization properties. We present results
for a sub-magnetar field strength so that we can view the interesting effects of the resonance within the frequency range of the thermal distributions. We note that this can be applicable to some of the isolated neutron stars with supersecond periods, and should be similar to results applicable for the low-field magnetar SGR 0418+5729, whose surface polar magnetic field strength is $1.2 \times 10^{13}$ Gauss.

In the next section we consider a more physical model of a magnetar, where the observer’s viewing direction is specified relative to the spin axis. We can then think of an observer at a fixed polar angle $\theta_{\text{obs}}$, and by increasing $\phi_{\text{obs}}$ we effectively rotate the star from the observer’s frame. We show angular maps in $\theta_{\text{obs}}$ and $\phi_{\text{obs}}$, as well as phase-resolved polarization curves, for $\alpha = 45^\circ, 90^\circ$. Rather than the full
frequency dependence, we consider the frequency-integrated emission in the 2-8 keV frequency band. This is done to emulate the expected sensitivity of the Imaging X-ray Polarimetry Explorer* (IXPE) (Weisskopf et al. 2016), which was recently selected as a future NASA Small Explorer Mission.

To decide the model parameters, we use observations of the magnetar CXOU J010043.1-721134 as presented by Tiengo et al. (2008). Its spectrum is shown in Fig. 5.2; it is fit by two blackbodies of temperature $T_{\text{cool}} \approx 0.3$ keV and $T_{\text{hot}} \approx 0.7$ keV. Its field strength as estimated by spin-down measurements is $B_p \approx 3.9 \times 10^{14}$ Gauss (i.e., $B \sim 9B_{\text{cr}}$). Comparing the luminosity fractions and temperatures of the

\*See the IXPE website at \url{https://wwwastro.msfc.nasa.gov/ixpe/}
two-blackbody fit, we can estimate that the size of the emission areas $A_{\text{hot}}$ and $A_{\text{cool}}$ are in the ratio $\sim 1/40$ using the Stefan-Boltzmann law: $L_\gamma \propto T^4 A$. We can then calculate the area of the hot polar regions required in order to match this ratio; this corresponds to an angular diameter of $\theta_{\text{cap}} \simeq 12.7^\circ$ for these regions. The remaining parameter to decide is the optical depth of the atmosphere: we arbitrarily choose a Thomson optical depth of $\tau = 3$. There is no need to set the spatial depth or number density of scatterers, as long as we assume the atmosphere is thin enough for the slab geometry to apply locally.

5.1 Stationary Observers in the Magnetar Frame

In this section, the definition of observer angle is aligned with the magnetic moment, so that $\theta_{\text{obs}}$ is essentially giving its angle with respect to the magnetic moment. The resulting azimuthal symmetry allows us to eliminate $\phi_{\text{obs}}$ as a variable. We can then look at the frequency and polar angle dependence of the emergent radiation. We use the temperature and area values determined from CXOU J010043.1-721134 ($T_{\text{cool}} = 0.3 \text{ keV}, T_{\text{hot}} = 0.7 \text{ keV}, \theta_{\text{cap}} = 12.7^\circ$), but use a sub-magnetar field strength $B_p = 8.6 \times 10^{11} \text{ Gauss}$, corresponding to an electron cyclotron frequency of 10 keV at the magnetic pole. This weaker field strength lets us look at the emission properties in the various regimes of $\omega/\omega_B$, and in particular, display the interesting polarization properties associated with proximity to the cyclotron resonance.

Using this view, we first investigate emission from a spherical atmosphere at a single temperature $T = T_{\text{cool}} = 0.3 \text{ keV}$; this is shown in Fig.5.3. We see the usual diminished intensity near the resonance due to the increased optical depth in its vicinity. This feature is now dependent on the observer viewing angle, as the field strength is different in different parts of the atmosphere. Compared to results from
Chapter 4, the polarization levels are much lower, realizing a maximum level of around 40%. This is due to a depolarizing effect caused by radiation emerging from areas of different magnetic field, but with the same temperature (and thus comparable in intensity). There is also a modestly enhanced circular polarization signal at $\theta_{\text{obs}} = 0^\circ, 180^\circ$, where the observer is looking down a magnetic pole.

Next we investigate the two-blackbody model, with $T_{\text{cool}} \approx 0.3$ keV and $T_{\text{hot}} \approx 0.7$ keV; this is shown in Fig. 5.4. The resulting properties are dominated by the hot component at higher frequencies, i.e. $\geq 2$ keV; it is at a higher intensity and is coming from a much smaller region of the total surface. Furthermore, the field direction and strength is similar at each pole, leading to a reinforcement rather than a cancellation of the emergent radiation’s polarization properties in this frequency range. Note that at lower frequencies ($< 2$ keV), the cool "equatorial" blackbody instead dominates the polarization characteristics. Looking at Stokes V, we can see the expected antisymmetry of the circular polarization with respect to $\cos \theta_{\text{obs}}$; the circular polarization can also be quite large near the resonance for certain viewing angles. The character of Stokes Q is rather like that seen in Chapter 4, as most of the emission is coming from the hot polar regions where the field are alike.

From these results, we can conclude that when the hot component is dominating the overall emission (above 2 keV), we will see much larger polarization signals, especially near the cyclotron frequency where the total intensity is suppressed somewhat. However, the depolarizing effect seen in the single temperature result is not 100% efficient, as there is still a frequency-dependent polarization signal. The intensity is much more uniform as a function of angle for the single temperature case as well. All of these characteristics are naturally expected to provide discrimination between the isothermal and two-temperature constructions.
### 5.1: STATIONARY OBSERVERS IN THE MAGNETAR FRAME

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Figure 5.3: Intensity and polarization emergent from a stellar atmosphere described by a single temperature blackbody at $T = T_{\text{cold}} = 0.3$ keV, as a function of observer polar angle and frequency. The frequency range is given logarithmically in keV, and so ranges from 1 to 100 keV. Here the intensity is normalized to its initial pure blackbody value at each frequency. The curved blue feature in intensity reflects the increased optical depth near the resonance; it is naturally at lower frequency near the equator ($\cos \theta_{\text{obs}} = 0$) where the field is lower. The polarization characteristics are much lower than for the individual slabs of Chapter 4, because radiation is originating from regions of differing magnetic field but comparable intensity, producing a depolarizing effect.
Figure 5.4: Intensity and polarization emergent from a stellar atmosphere described by the two blackbody model ($T_{\text{cool}} = 0.3$ keV, $T_{\text{hot}} = 0.7$ keV), as a function of polar angle and frequency. Here the intensity is normalized to the initial two blackbody value (superposition of two Planck forms) at each frequency. The blue feature in intensity is no longer curved, as emission is dominated by the hot component at these frequencies, and thus samples only the polar field strength. Polarization properties are also dominated by the hot component, reaching as high as unity near the resonance.
5.2 Views for an Inclined, Rotating Star

In this section, the observer angle is aligned with the spin axis, as opposed to being specified in the instantaneous rest frame of the spinning star relative to magnetic coordinates, with $\theta_{\text{obs}}$ being effectively the inclination angle of the rotating star as seen by a distant, static observer. By increasing $\phi_{\text{obs}}$ the observer location moves around the star, which means that $\phi_{\text{obs}}$ effectively corresponds to a rotational phase. Thus, we are able to analyze the properties of a spinning star simply by considering the emission in all directions using these angles. Here we use the inferred field strength for CXOU J010043.1-721134, $B_p = 3.9 \times 10^{14}$ Gauss, corresponding to $\omega_B = 4.5$ MeV.

As mentioned above, we now consider the integrated flux in the band $2 - 8$ keV to replicate the frequency response for the IXPE X-ray polarimeter. This means all emission is firmly in the regime $\omega \ll \omega_B$.

We first look at the case where $\alpha = 45^\circ$; this means that the two magnetic poles are located at $(\cos \theta_{\text{obs}}, \phi_{\text{obs}})$ values of $(\cos 45^\circ, 0)$ and $(\cos 135^\circ, \pi)$. The single-temperature case is shown in Fig. 5.5, with corresponding phase-resolved polarization curves shown in Fig. 5.6. The intensity is very uniform (to within 10%), meaning the intensity transfer in the atmosphere is not hugely affected by the variance in magnetic strength and inclination. The polarization is purely linear in this frequency regime, with the linear polarization degree peaked at $\sim 10\%$ at angles perpendicular to the magnetic moment vector $\vec{m}$. This is because polar and equatorial regions, which have their magnetic fields parallel or anti-parallel to $\vec{m}$, all favor the same linear polarization mode. Also, because our view (and Stokes reference vector) is not aligned with $\vec{m}$, Stokes U is nonzero in general. The position angle sweeps through one phase for views near the magnetic poles, while it sweeps through two phases for perpendicular views.
Figure 5.5: Intensity and polarization properties of radiation emergent from a stellar surface at a single temperature, as a function of $\cos \theta_{\text{obs}}$ and $\phi_{\text{obs}}$. The angle between the magnetic moment vector and the spin axis is $\alpha = 45^\circ$. The intensity is fairly uniform, and the polarization is purely linear, with a low peak polarization of about 10%. Also shown is the linear polarization degree $\Pi_{\text{lin}}$ and the position angle of polarization $\psi$ (in radians).
Figure 5.6: Phase-resolved polarization curves for the single-temperature case, with $\alpha = 45^\circ$. This plot shows the degree of linear polarization $\Pi_{\text{lin}}$ as a function of rotation phase, which is equivalent to $\phi_{\text{obs}}$. Note the integrated range of frequencies is such that $\omega \ll \omega_B$. A selection of viewer orientations $\theta_{\text{obs}}$ is shown.
CHAPTER 5: MAGNETAR ATMOSPHERIC EMISSION

Emergent Intensity Stokes V

Stokes Q Stokes U

Position Angle Linear Polarization Degree

Figure 5.7: Intensity and polarization properties of radiation emergent from a stellar surface for the two-temperature case, as a function of \( \cos(\theta_{\text{obs}}) \) and \( \phi_{\text{obs}} \). The angle between the magnetic moment and the spin axis is \( \alpha = 45^\circ \). The intensity is now highly angle-dependent, and the polarization is purely linear, with a higher peak polarization of about 40%.
5.2: VIEWS FOR AN INCLINED, ROTATING STAR

Figure 5.8: Phase-resolved polarization curves for the two-temperature case, with $\alpha = 45^\circ$. This plot shows the degree of linear polarization $\Pi_{\text{lin}}$ as a function of rotation phase, which is equivalent to $\phi_{\text{obs}}$. Note the integrated range of frequencies is such that $\omega \ll \omega_B$. A selection of viewer orientations $\theta_{\text{obs}}$ is shown.
The two-temperature case is shown in Fig. 5.7, with corresponding phase-resolved polarization curves shown in Fig. 5.8 and phase-resolved position angle curves shown in Fig. 5.9. The intensity now varies by roughly an order of magnitude; it is greatest near the hot polar regions and smallest along the equator. The linear polarization degree is larger, having a maximum of $\sim 40\%$, with a more complicated angular dependence. There is an equatorial band seen in Stokes Q and U; this is due to the suppression of hot polar emission at these angles, which lets the cool equatorial region shine through. This case produces much more complicated polarization curves; namely the appearance of strong, higher order Fourier components to the light curve, where two disparate zones contribute polarized pulsation signals that are out of phase. However, the position angle possesses essentially the same characteristics as the previous case, since it is a tracer only for the local magnetic field.

Next we examine the case where $\alpha = 90^\circ$; the two magnetic poles are located at $(\cos \theta_{\text{obs}}, \phi_{\text{obs}})$ values of $(0, 0)$ and $(0, \pi)$. The single-temperature case is shown in Fig. 5.10, with corresponding phase-resolved polarization curves shown in Fig. 5.11. The two-temperature case is shown in Fig. 5.12, with corresponding phase-resolved polarization curves shown in Fig. 5.13. These two cases share all the same qualities of the previous two, with the only change being in the angle $\alpha$. Thus we see the same trends in intensity, polarization degree and position angle with respect to $\vec{m}$, but the change in $\alpha$ has significantly altered the location of the peaks, valleys and zeroes of the various polarization quantities. The difference is most distinct when looking at the phase-resolved polarization curves. Thus we see how our model can be a powerful tool for diagnosing the inclination angle.

It is important to mention the recent result of Mignani et al. (2017), who looked at the first optical polarimetry measurement of the isolated neutron star RX J1856.5-
Figure 5.9: Phase-resolved position angle curves for the two-temperature case, with $\alpha = 45^\circ$. This plot shows the position angle $\psi$ in radians as a function of rotation phase, which is equivalent to $\phi_{\text{obs}}$. A range of viewer orientations $\theta_{\text{obs}}$ is shown. Note that $2\psi$ is periodic in the usual sense, so that e.g. $\psi + \pi = \psi$. Note that when $\psi = \pm \pi/2$, i.e. at $\theta_{\text{obs}} = 0, \pi, 2\pi$, the magnetic pole is facing the observer.
Figure 5.10: Intensity and polarization properties of radiation emergent from a stellar surface for the single-temperature case, as a function of $\cos \theta \text{obs}$ and $\phi \text{obs}$. The angle between the magnetic moment and the spin axis is $\alpha = 90^\circ$. The intensity is fairly uniform, and the polarization is purely linear, with a low peak polarization of about 10%.
Figure 5.11: Phase-resolved polarization curves for the single-temperature case, with $\alpha = 90^\circ$. This plot shows the degree of linear polarization $\Pi_{\text{lin}}$ as a function of rotation phase, which is equivalent to $\phi_{\text{obs}}$. A range of viewer orientations $\theta_{\text{obs}}$ is shown.
Figure 5.12: Intensity and polarization properties of radiation emergent from a stellar surface for the two-temperature case, as a function of $\cos \theta_{\text{obs}}$ and $\phi_{\text{obs}}$. The angle between the magnetic moment and the spin axis is $\alpha = 90^\circ$. The intensity is now highly angle-dependent, and the polarization is purely linear, with a higher peak polarization of about 40%.
Figure 5.13: Phase-resolved polarization curves for the two-temperature case, with $\alpha = 90^\circ$. This plot shows the degree of linear polarization $\Pi_{\text{lin}}$ as a function of rotation phase, which is equivalent to $\phi_{\text{obs}}$. A range of viewer orientations $\theta_{\text{obs}}$ is shown.
3754, a neutron star with a magnetic field of roughly \( \sim 10^{13} \) Gauss that exhibits very weak pulsation. Using observations from the Very Large Telescope, they found a polarization degree of \( \sim 15\% \) in the optical V-band (i.e. what would correspond to deep in the Rayleigh-Jeans tail of a Planck spectrum), and sought to fit their observables such as polarization degree, pulse fraction, etc., to a polarized emission model. They considered two models, both with surface emission taken to be fully polarized in the extraordinary mode, and considered either free propagation without any dispersion modifications, or a model which considered the propagation influenced by strong-field QED vacuum polarization effects.

In QED, the vacuum is polarized in a birefringent manner when in the presence of a strong external field. This is due to the fact that the probability of production of virtual e+e- pairs is dependent on the polarization state of a photon moving through the medium. Accordingly photons of different polarizations travel at slightly different speeds in the magnetic vacuum. This leads to rotation of the electric field vectors of electromagnetic waves in a manner similar to Faraday rotation in magnetized plasma. Such birefringence of the magnetized vacuum is a fundamental prediction of QED that has not yet be unequivocally demonstrated in the laboratory.

Since only the models of Mignani et al. (2017) which included vacuum effects were able to reproduce the observed polarization, they claimed this discovery to be the first observational evidence of QED vacuum polarization effects. We have seen here that, at least in the soft X-ray band, models of polarized emission can produce comparable polarization degrees, i.e. 15% and even higher, when integrating over an entire surface, \textit{without} including vacuum effects. This is due in part to our detailed treatment of radiative transfer in atmospheres of different field zenith angles and field strengths, and in part due to the prominence of contributions from the hot spot
to emission above 2 keV. We anticipate that similarly high degrees of polarization will result at optical frequencies in our model when the cool equatorial component dominates, though the detailed values of the Q, U and V Stokes parameters and position angles will change from the results presented here. With the advent of X-ray polarimetry on the horizon, we can expect that there will be much to say about this topic in the near future.
Chapter 6
Magnetospheric Transfer: Magnetar Flares

Magnetars are also seen to produce outbursts of emission attributed to flare activity in the neutron star magnetosphere. Emission extends to a few hundred keV, past which detectable emission is negligible. Broadband observations of magnetar flares such as those seen in SGR J1550-5418 by Fermi-GBM, Swift and other instruments are compatible with a spectral fit that consists of a sum of blackbodies, somewhat similarly to the surface emission but with higher temperatures (Lin et al. 2012). This flare emission is too hot to be attributed to the stellar atmosphere.

Here we explore a toy model of magnetar flare emission: a sum of emission over two areas of different temperature, opacity, and magnetic field. The geometry of this model is visualized in Fig. 6.1. We assume that heat is transferred from the crustal layers through the surface, and deposited in the inner magnetosphere; because conductivity is far superior along the magnetic field than perpendicular to it, this produces hot regions near the magnetic poles. Adiabatic cooling then yields outer layers at high altitudes of lower temperature. Thus we reason that the hot component is close to the surface and near the pole, while the cool component we associate with a thinner region further out in the magnetosphere, at roughly $r = 7R_{\text{NS}}$. To model this two-zone emission, we treat the transfer in each region separately using the slab geometry, then sum them. To a first approximation, this will define the overall polarization characteristics.
CHAPTER 6: MAGNETOSPHERIC TRANSFER: MAGNETAR FLARES

Figure 6.1: Geometry of a toy model of magnetar flare emission. Transfer is treated using a thermal injection of frequencies through a slab of the prescribed optical depth and magnetic field. Two such regions are used with different temperatures, optical depth and magnetic field; The emergent radiation is then summed. Two perspectives are modeled: the polar view is aligned with the magnetic pole, and the equatorial view is at 90° to this. The orientation of the slab normals differ with the two perspectives: they are aligned with the viewing direction of each view. The fixed polarization reference is perpendicular to the viewing direction and aligned with the perpendicular field (cold region for polar view, hot region for equatorial).

Specifically, we inject thermal distributions (Planck spectra), which scatters through a slab of a given Thomson optical depth and magnetic field. For the hot region we choose $T = 15$ keV, $\tau = 50$, and a strong magnetar-level magnetic field of $B = 5B_{cr}$, putting the cyclotron frequency at $\omega_B \simeq 2.5$ MeV. For the cool region, we have $T = 5$ keV, $\tau = 20$, and a field strength reduced by the distance cubed, $B = (5/7^3)B_{cr}$, giving $\omega_B \simeq 7$ keV. The field direction in this region is set to be orthogonal to that of the hotter component. We choose these temperatures and field strengths to roughly coincide with those inferred from observations of SGR J1550-5418 described by Lin.
et al. (2012).

Two different observer perspectives of the model are presented: the “polar view” has the observed looking down the pole from above, and the transfer in each region is treated as that of a slab with its normal oriented in this polar direction, as shown in Fig. 6.1. The “equatorial view” has the observer at 90° to the polar view, looking down the the magnetic field of the cooler region. For this view the slabs have their normals oriented towards the new observer position. All emergent radiation within 15° of the observer is considered detected. In other words, a patch of solid angle constituting a circle of angular radius 15°. These two perspectives represent data an observer might see for two different magnetar pulse phases.

An important difference between results here and those presented in Ch.3 involves the orientation of the reference direction for defining polarization. In Ch.3, the reference direction is always in the plane of the magnetic field and the propagation direction\(^*\). This has several advantages, such as allowing for Stokes Q to be interpreted in terms of parallel and perpendicular polarizations. However, it is only defined for a homogeneous magnetic field. When summing over emission from regions of different magnetic field, an arbitrary reference direction must be chosen.

This has a few consequences: Stokes U will no longer be zero in general, though it will be if the reference direction coincides with the polarization angle defined in Eq. 2.1. Also, linear polarizations possessed by photons traveling nearly parallel to the field are not detectable in this case: these contributions cancel for a fixed reference instead of adding up as they do for a reference tied to the field. Thus the linear polarization will be a maximum when looking orthogonal to the local field, while

---

*For angular binning, the propagation direction roughly coincides with the average angle in each bin.
the circular polarization is maximized when looking parallel to the field. Due to this dichotomy, we choose the reference direction to coincide with the orthogonal field direction as indicated in the figure; thus Stokes U will be zero for each viewing perspective.

The results are shown in Fig. 6.2 and Fig. 6.3. From the emergent flux we see a notched feature in the cooler component due to the high scattering cross section at the cyclotron resonance, which appears near the flux peak of the cool Planck component. Flux in both cases is attenuated compared to pure blackbody emission just as a consequence of the high opacities involved. The information contained in Fig. 4.6b gives an understanding of the polarization properties for these cases. We see a notched feature in the cool component at the cyclotron frequency; this is from attenuation due to the high optical depth for photons near the resonance. Similar features are seen in the atmosphere models of Ho & Lai (2001) and Özel (2001a).

In the polar view, the hot region is viewed along the field; linear polarization is zero and there is a small circular polarization. As all frequencies are well below the cyclotron frequency, this is analogous to the bottom right area of the color plots in Fig. 4.6b. Meanwhile, the cool region is viewed orthogonal to the field; circular polarization is zero and there is significant linear polarization. The frequency dependence of the linear polarization also follows the trend shown in Fig. 4.6b or Fig. 4.1b, but since the optical depth is lower, we see less scattered polarization with $Q = +1$ exiting.

In the equatorial view, the hot region is viewed perpendicular to the field while the cool region is viewed along it; the linear and circular polarizations in these respective regions are thus high. For the hot region $\omega / \omega_B$ is small enough that we are optically thin to $Q = -1$ polarized photons. This polarization state dominates the hot emission
and sets the overall trend in polarization fraction. The cyclotron notch seen in the cool component is broader for the polar view compared to the equatorial view. This is because photons are preferentially scattered towards/against the magnetic field rather than perpendicular to it. Since the polar view has the cool component’s field perpendicular to the view, this increases the effective optical depth near the resonance, producing a broader attenuation.

These two views can give us the same overall spectral intensities, of which there are many published observations. However, the polarization signatures clearly tell us what our viewing perspective is and how the magnetic geometry is inclined relative to the rotation axis; this helps motivate the science agenda for future hard X-ray polarimeters. The next logical next step for our model is to treat a summation over non-uniform field geometries, for which my code is naturally designed. We will also in the future treat electron-photon energy exchange to explore altitudinal cooling and Doppler smearing of the cyclotron lines.
Figure 6.2: Flare model polar view. Note that the sum of the Stokes parameters for each component is weighted by the frequency-dependent intensity of each. Thus, the low polarization fraction of the hot component causes the overall polarization fraction to be low. The emission properties of each region can be understood in terms of the earlier frequency-dependent Stokes parameter profiles for slabs seen in Chapter 4 (e.g. Fig. 4.1).
Figure 6.3: Flare model equatorial view. The high polarization fraction of the hot component causes the overall polarization fraction to be high.
CHAPTER 7

Generalized Faraday effect

The practical determination of the rate of Compton scattering depends on the polarization configuration of incoming photons. This in turn is sensitive to the details of radiation dispersion and transport in hot plasmaspheres near neutron stars. In this chapter we explore some of the issues and nuances of such birefringent dispersion in strongly-magnetized plasmas that can profoundly influence the determination of scattering probabilities both in uniform and non-uniform field configurations.

Many authors have addressed the radiative transfer of polarized radiation in an anisotropic, dispersive medium using the normal mode description (e.g., see Dolginov et al. 1995b). For example, Gnedin & Sunyaev (1974) studied x-ray emission in near-surface accretion columns at the poles of accreting neutron stars. Pertinent to the higher field domain of magnetars, Özel (2001b); Lai & Ho (2003) studied radiative transfer in strongly magnetized plasmas using this formalism, focusing on the effects of vacuum polarization (see Meszaros & Ventura 1979), birefringent dispersion of the quantum vacuum due to the presence of such an intense magnetic field.

Heyl & Shaviv (2000) introduced a formalism for the adiabatic evolution of pure modes, and predicted a high level of polarization of the thermal radiation from the surfaces of highly-magnetized neutron stars (Heyl & Shaviv 2002). Their formalism was adopted by Lai & Ho (2002), who studied adiabatic “mode conversion” of X-rays in an electron-ion gas due to the vacuum resonance, a domain of strong anomalous dispersion that occurs because of the competing refractive index properties of the
quantum magnetic vacuum polarization and the classical magnetized plasma domains. When photons transit this resonance “boundary” in propagating from locales of higher to lower gas density, the polarization states switch from being approximate eigenstates of the magnetic plasma, to being those pertinent to the magnetized vacuum.

Here we address the the radiative transfer of polarized radiation in the case where the assumption of large Faraday depolarization is not valid. This is a regime where a modest amount of polarization change occurs between scatterings, and requires a more sophisticated treatment of the evolution of the four photon polarization parameters during propagation. We still rely on the assumption that the modes are orthogonal and transverse. Our considerations are restricted to treating just the physics of classical magnetized plasma, and thereby are most essential for zones deep inside neutron star atmospheres, and also in the dense and dynamic plasma environments expected in magnetar flares.

Finite electron temperature modifications are included in our analysis, since warm plasma is anticipated in accretion columns and in the magnetosphere. The main emphasis will focus on magnetar field strengths, where the proton cyclotron resonance lies in the low energy X-ray window; there is an angle-dependent domain at energies near this resonance where the modes are transverse and orthogonal, but for which the large Faraday depolarization assumption breaks down. Results for X-ray transfer near the proton cyclotron resonance in magnetized warm 1D plasmas, as it applies to magnetar environments, are presented.

The prospects of space-based X-ray polarimeters in the near future presses the need to carefully re-analyze the theory of polarization-dependent dispersion and radiation transfer, to better understand the predictions and limitations of previous work, as well as to generate a formalism conducive to numerical Monte Carlo radiative
transfer simulations. This motivates the study presented here.

The layout of the chapter is as follows. In section 7.1, we present the radiative transfer formalism, giving relevant dielectric tensors, normal modes, and refractive indices for a warm plasma of a given species with a 1-D Maxwellian particle distribution, i.e. a classical magnetized plasma. We also address the generalized Faraday effect in domain where depolarization is not prolific. Section 7.2 discusses its astrophysical applications to magnetars.

7.1 Dispersion and Electromagnetic Eigenmodes

7.1.1 Normal Mode Approximation

The description of light polarization in dispersive media involves four quantities: these could be the Stokes parameters or equivalently the elements of the $2 \times 2$ polarization density matrix (see Ch. 2). In general, a photon’s polarization configuration changes as it traverses a magnetized plasma. In a uniform magnetic field, this establishes extensions of the well-known Faraday effect, causing a periodic change in polarization. It is most easily described using the Poincaré sphere (see Walker 1954), a unit sphere in the phase space of Stokes $Q,U,V$ parameters: it is a rotation on the Poincaré sphere akin to precession. In non-uniform field geometries, the polarization description changes continually during light propagation, so that since cross sections for Compton scattering, pair creation and photon splitting are also strongly polarization dependent, models for radiative processes in neutron stars must incorporate the nuances of polarization variations along photon paths.

Gnedin & Pavlov (1974) realized that under certain domains for photon frequency, plasma density and magnetic field strength, the whole transfer problem can be reduced
to the transfer of two quantities: the **normal mode intensities**. This is termed the “normal mode description,” and is a popular and convenient invocation in many astrophysical models. Its validity relies on two assumptions: (1) the normal modes must be approximately transverse and orthogonal, and (2) photons undergo many rotations in between scatterings, also known as large Faraday depolarization. To quantify this, assumption (1) demands

\[ |\hat{e}_1 \cdot \hat{e}_2| \ll 1 \quad , \quad |\hat{e}_i \cdot \hat{k}| \ll 1 \quad , \quad (7.1) \]

where \( \hat{e}_i \) is the normalized, complex electric field polarization vector for mode \( i \), and \( \hat{k} \) is the normalized propagation vector, i.e. scaled photon momentum. The first equation establishes the orthonormality of the modes, and the second the transversality. Under these assumptions, we can use the geometric optics theorem to establish the criterion for large Faraday depolarization (e.g. see Eq.3.6.1 of Mészáros 1992):

\[ |\text{Re}(N_1 - N_2)| \gg |\text{Im}(N_1 + N_2)| \quad , \quad (7.2) \]

where \( N_i \) is the (complex) index of refraction for mode \( i \). The left hand side of this constraint is associated with the strength of the generalized Faraday effect, and the right hand side is associated with the average mean free path. For radiative transfer in a tenuous magnetized plasma, there is a wide range of parameter space where these two assumptions hold, and the normal mode description is valid. However, there are domains of importance, in particular near the electron cyclotron resonance, where one or both of these assumptions break down, and the normal mode description needs revision: this is the thrust of the offering of our work here.
7.1.2 Dielectric Tensor of a Hot Magnetized Plasma

We begin with the wave equation for an arbitrary wave in a magnetized medium. We initially neglect vacuum effects, so we have a linear relationship between $\vec{E}$ (applied field) and $\vec{D}$ (induced field): $D_i = \epsilon_{ij} E_j$. Here, $\epsilon_{ij}$ is the dielectric tensor. For a plane wave of frequency $\omega$, wave vector $\vec{k}$, electric field $\vec{E}$, and index of refraction $N$, the wave equation reads

$$N^2 \left( \hat{k}_i \hat{k}_j - \delta_{ij} \right) + \epsilon_{ij} E_j = 0, \quad (7.3)$$

where $\delta_{ij}$ is the Kronecker delta, and $\hat{k} = \vec{k} / k$ (Chandrasekhar, 1960).

To help simplify the formulae in the following section, we consider only a single species of particle and repress the species index. We will later combine the relevant results to form the tensor for a magnetized hydrogenic plasma. That said, we seek the contribution to the dielectric tensor for a particle distribution function that is the product of a 2-D Maxwellian in the field-perpendicular directions, of characteristic speed $v_\perp = \sqrt{2T_\perp / m}$, and a 1-D Maxwellian along the field with drift speed $v_0$ and characteristic speed $v_z = \sqrt{2T_\parallel / m}$; such a tensor is given by Swanson (2008). In a coordinate system with $\vec{B}\parallel \hat{z}$ and $\vec{k} = (k_\perp, 0, k_z)^*$, we have

$$\epsilon_{ij} = \delta_{ij} + \chi_{ij}, \quad \chi_{ij} = \begin{bmatrix} \chi_{xx} & \chi_{xy} & \chi_{xz} \\ -\chi_{xy} & \chi_{yy} & \chi_{yz} \\ \chi_{xz} & -\chi_{yz} & \chi_{zz} \end{bmatrix}, \quad (7.4)$$

$\*We can always find this coordinate system through a simple rotation.
CHAPTER 7: GENERALIZED FARADAY EFFECT

\begin{align}
\chi_{xx} &= \frac{\omega_p^2 e^{-\lambda}}{\omega k_z v_z} \sum_{n=-\infty}^{\infty} \frac{n^2 I_n}{\lambda} \left[ \left( 1 - \frac{k_z v_0}{\omega} \right) Z(\zeta_n) + \frac{k_z v_z}{\omega} \left( 1 - \frac{T_{\perp}}{T_\parallel} \right) \frac{Z'(\zeta_n)}{2} \right], \quad (7.5) \\
\chi_{yy} &= \frac{\omega_p^2 e^{-\lambda}}{\omega k_z v_z} \sum_{n=-\infty}^{\infty} \frac{n^2 I_n}{\lambda} \left[ \left( 1 - \frac{k_z v_0}{\omega} \right) Z(\zeta_n) + \frac{k_z v_z}{\omega} \left( 1 - \frac{T_{\perp}}{T_\parallel} \right) \frac{Z'(\zeta_n)}{2} \right] \\
\chi_{xy} &= i \frac{q}{|q|} \frac{\omega_p^2 e^{-\lambda}}{\omega k_z v_z} \sum_{n=-\infty}^{\infty} n (I_n - I'_n) \left[ \left( 1 - \frac{k_z v_0}{\omega} \right) Z(\zeta_n) + \frac{k_z v_z}{\omega} \left( 1 - \frac{T_{\perp}}{T_\parallel} \right) \frac{Z'(\zeta_n)}{2} \right], \\
\chi_{xz} &= \frac{k_1 \omega^2 e^{-\lambda}}{k_z \omega_B} \sum_{n=-\infty}^{\infty} \frac{n I_n}{\lambda} \left\{ \frac{n \omega_B v_0}{\omega v_z} Z(\zeta_n) + \left[ \frac{T_{\perp}}{T_\parallel} - \frac{n \omega_B}{\omega} \left( 1 - \frac{T_{\perp}}{T_\parallel} \right) \right] \frac{Z'(\zeta_n)}{2} \right\}, \\
\chi_{yz} &= -i \frac{q}{|q|} \frac{k_1 \omega^2 e^{-\lambda}}{k_z \omega_B} \sum_{n=-\infty}^{\infty} (I_n - I'_n) \left\{ \frac{n \omega_B v_0}{\omega v_z} Z(\zeta_n) + \left[ \frac{T_{\perp}}{T_\parallel} - \frac{n \omega_B}{\omega} \left( 1 - \frac{T_{\perp}}{T_\parallel} \right) \right] \frac{Z'(\zeta_n)}{2} \right\}, \\
\chi_{zz} &= -\frac{\omega_p^2 e^{-\lambda}}{\omega k_z v_z} \sum_{n=-\infty}^{\infty} I_n \left( \frac{\omega + n \omega_B}{k_z v_z} \right) \times \left\{ \left[ 1 + \frac{n \omega_B}{\omega} \left( 1 - \frac{T_{\perp}}{T_\parallel} \right) \right] Z'(\zeta_n) + \frac{2 n \omega_B T_{\perp} v_0}{\omega T_{\parallel} v_z} \left[ Z(\zeta_n) + \frac{k_z v_z}{\omega + n \omega_B} \right] \right\}, \\
\lambda &= \frac{k_1^2 v_0^2}{2 \omega_B^2}, \quad \zeta_n = \frac{\omega + n \omega_B - k_z v_0}{k_z v_z}.
\end{align}

where \( \omega_B = |q| B/mc \) is the cyclotron frequency, \( \omega_p = \sqrt{4\pi n_0 e^2/m} \) is the plasma frequency, \( I_n \) is the Bessel I function with argument \( \lambda \), and \( Z \) is the plasma dispersion function (see Fried & Conte 1961):

\begin{equation}
Z(\zeta) = \pi^{-1/2} \int_{-\infty}^{\infty} e^{-x^2} dx = 2ie^{-\zeta^2} \int_{-\infty}^{\zeta} e^{-t} dt
\end{equation}

\( Z \) will be complex in general; its real part contributes to the Hermitian part of the dielectric tensor describing propagation effects, and its imaginary part contributes to
the anti-Hermitian part of the dielectric tensor describing absorption effects. The real and imaginary parts of the plasma dispersion function are:

\[
\begin{align*}
\text{Re}[Z(\xi)] &= -2F(\xi), \\
\text{Im}[Z(\xi)] &= \sqrt{\pi}e^{-\xi^2}, \\
\text{Re}[Z'(\xi)] &= 2(2\xi F(\xi) - 1), \\
\text{Im}[Z'(\xi)] &= -2\sqrt{\pi}\xi e^{-\xi^2},
\end{align*}
\]  

(7.7)

where \( F(x) = \exp(-x^2) \int_0^x \exp(y^2)dy \) is the so-called Dawson integral, and is easily evaluated using numerical software. Note that \( Z'(\zeta) = -2(1+\zeta Z(\zeta)) \) is the derivative of the plasma dispersion function.

We seek the dielectric tensor which has a temperature along the field only, and no drift. In this case, \( T_\perp = 0 \), \( v_0 = 0 \), and \( T_\parallel = T \neq 0 \). Note that only the \( n = 0, \pm 1 \) terms contribute to the sum. The result is the dielectric tensor for a 1-D Maxwellian; the components of its associated mobility tensor \( \chi \) are

\[
\begin{align*}
\chi_{xx} &= \chi_{yy} = -\frac{w_p^2}{k_z^2} \left( 1 + \frac{1}{2} w_B \zeta_0 [Z(\zeta_{+1}) - Z(\zeta_{-1})] \right), \\
\chi_{xy} &= -\frac{i}{2} \frac{w_p^2}{k_z} w_B \zeta_0 [Z(\zeta_{+1}) + Z(\zeta_{-1})], \\
\chi_{xz} &= \frac{w_p^2 k_{+1}}{k_z} \left( 1 + \frac{1}{2} \zeta_{+1} Z(\zeta_{+1}) + \zeta_{-1} Z(\zeta_{-1}) \right), \\
\chi_{yz} &= -\frac{i}{2} \frac{w_p^2}{k_z} \frac{k_{+1}}{k_z} \left[ \zeta_{+1} Z(\zeta_{+1}) - \zeta_{-1} Z(\zeta_{-1}) \right], \\
\chi_{zz} &= w_p^2 \left\{ 2\zeta_0^2 [1 + \zeta_0 Z(\zeta_0)] - \frac{k_{+1}^2}{k_z^2} \left( 1 + \frac{1}{2\zeta_0 w_B} \left[ \zeta_{+1}^2 Z(\zeta_{+1} - \zeta_{-1}^2 Z(\zeta_{-1})) \right] \right) \right\},
\end{align*}
\]  

(7.8)

\[
\zeta_0 = \frac{\omega}{k_z v_z}, \quad \zeta_{\pm 1} = \frac{\omega \pm \omega_B}{k_z v_z} = \zeta_0 (1 \pm w_B).
\]
\[ w_p = \frac{\omega_p}{\omega} = \sqrt{\frac{4\pi n_0 q^2}{m\omega^2}}, \quad w_B = \frac{\omega_B}{\omega} = \frac{|q| B}{mc\omega} = \nu_b^{-1}. \]

For low temperatures such that \( \zeta_0 \gg 1 \), we can use an asymptotic expansion for large argument in the plasma dispersion function, except when \( n = -1 \) where expansion is invalid due to the resonant numerator. This gives

\[
\begin{align*}
\chi_{xx} &= \chi_{yy} = -w_p^2 \left( 1 - \frac{1}{2} w_B \left[ \frac{1}{1 + w_B} + \zeta_0 Z(\xi) \right] \right), \\
\chi_{xy} &= \frac{i}{2} w_p^2 w_B \left[ \frac{1}{1 + w_B} - \zeta_0 Z(\xi) \right], \\
\chi_{xz} &= \frac{1}{2} \frac{w_p^2}{k_z} \left[ 1 + \xi Z(\xi) \right], \\
\chi_{yz} &= -\frac{i}{2} \frac{w_p^2}{k_z} \left[ 1 + \xi Z(\xi) \right], \\
\chi_{zz} &= -w_p^2 \left( 1 - \frac{1}{2} \frac{k_z^2}{k_z^2} \frac{1 - w_B}{w_B} \left[ 1 + \xi Z(\xi) \right] \right),
\end{align*}
\]

where \( \xi = \zeta^{-1} \). In the limit of zero temperature, this reproduces the dielectric tensor for a cold magnetized plasma, a well-known result:

\[
\begin{align*}
\chi_{xx} = \chi_{yz} = 0, \quad \chi_{zz} = -w_p^2, \\
\chi_{xx} = \chi_{yy} = -\frac{w_p^2}{1 - w_B^2}, \quad \chi_{xy} = \frac{i}{2} \frac{w_p^2 w_B}{1 - w_B^2}.
\end{align*}
\]

We can separate the mobility tensor into Hermitian and anti-Hermitian parts. We can also isolate the dependence on \( w_p \), i.e.

\[
\chi_{ij} = \chi_{ij}^{(H)} + \chi_{ij}^{(AH)},
\]
where

\[
\chi_{ij}^{(H)} = w_p^2 \begin{bmatrix}
    f_{xx} & i f_{xy} & f_{xz} \\
    -i f_{xy} & f_{yy} & i f_{yz} \\
    f_{xz} & -i f_{yz} & f_{zz}
\end{bmatrix}, \quad \chi_{ij}^{(AH)} = w_p^2 \begin{bmatrix}
    i g_{xx} & g_{xy} & i g_{xz} \\
    -g_{xy} & g_{yy} & g_{yz} \\
    i g_{xz} & -g_{yz} & i g_{zz}
\end{bmatrix}.
\] (7.11)

Figure 7.1: \( f \) and \( g \) functions plots for the case \( k_\perp = k_z \). The xx, xy, xz, zz components are shown in figures a, b, c, d, respectively. The black(gray) curves show the value of the \( f(g) \) function for \( v_z = 0.16c \), the dashed curves show the cold value of the \( f \) function. Note in (d) the \( f \) functions have been shifted by +1.
The components associated with the Hermitian part are

\[
\begin{align*}
f_{xx} &= f_{yy} = -1 + \frac{1}{2} w_B \left[ \frac{1}{1 + w_B} - 2 \zeta_0 F(\xi) \right], \\
f_{xy} &= \frac{1}{2} w_B \left[ \frac{1}{1 + w_B} + 2 \zeta_0 F(\xi) \right], \\
f_{xz} &= f_{yz} = \frac{1}{2} k_{\perp}^2 \frac{1 - w_B}{w_B} [1 - 2 \xi F(\xi)], \\
f_{zz} &= -1 + \frac{1}{2} k_z^2 \frac{1 - w_B}{w_B} [1 - 2 \xi F(\xi)],
\end{align*}
\]

(7.12)

The components associated with the anti-Hermitian part are

\[
\begin{align*}
g_{xx} &= g_{yy} = -g_{xy} = \frac{1}{2} w_B \zeta_0 \sqrt{\pi} e^{-\xi^2}, \\
g_{xz} &= -g_{yz} = \frac{1}{2} k_{z} \frac{1 - w_B}{w_B} \sqrt{\pi} \xi e^{-\xi^2}, \\
g_{zz} &= \frac{1}{2} k_z^2 \frac{1 - w_B}{w_B} \sqrt{\pi} \xi e^{-\xi^2}.
\end{align*}
\]

(7.13)

The anti-Hermitian part of the dielectric tensor contains the physics of scattering/absorption. The Hermitian part describes propagation effects such as dichroism and birefringence, here corresponding to the generalized Faraday effect (Melrose, 2012). Figure 1 shows examples of these functions.

7.1.3 Dispersion in a Hot Magnetized Plasma

Consider a coordinate system \((x, y, z)\), where \(\vec{k}\parallel \hat{z}\) and \(\vec{B}_0 = B_0 (\hat{z} \cos \theta - \hat{x} \sin \theta)\). We can construct a basis from \(\vec{k}\) and \(\vec{B}\) as follows:

\[
\begin{align*}
\hat{z} &= \frac{\vec{k}}{k}, \\
\hat{y} &= \frac{\vec{k} \times \vec{B}}{|\vec{k} \times \vec{B}|}, \\
\hat{x} &= \hat{y} \times \hat{z}, \\
\cos \theta &= \frac{\vec{k} \cdot \vec{B}}{k B}.
\end{align*}
\]

(7.14)
The dielectric tensor is now \( \epsilon'_{ij} = \delta_{ij} + \chi'_{ij} \), where

\[
\begin{align*}
\chi'_{xx} &= \chi_{xx} \cos^2 \theta + \chi_{zz} \sin^2 \theta - \chi_{xz} \sin 2\theta, \\
\chi'_{yy} &= \chi_{yy}, \\
\chi'_{xy} &= \chi_{xy} \cos \theta + \chi_{yz} \sin \theta, \\
\chi'_{xz} &= \chi_{xz} \cos 2\theta + \frac{\chi_{xx} - \chi_{zz}}{2} \sin 2\theta, \\
\chi'_{yz} &= -\chi_{xy} \sin \theta + \chi_{yz} \cos \theta, \\
\chi'_{zz} &= \chi_{xx} \sin^2 \theta + \chi_{zz} \cos^2 \theta + \chi_{xz} \sin 2\theta,
\end{align*}
\] (7.15)

and the wave equation now reads

\[
\left[ N^2 \left( \hat{k}_i \hat{k}_j - \delta_{ij} \right) + \epsilon'_{ij} \right] E_j = \begin{bmatrix} \epsilon'_{xx} - N^2 & \epsilon'_{xy} & \epsilon'_{xz} \\ -\epsilon'_{xy} & \epsilon'_{yy} - N^2 & \epsilon'_{yz} \\ \epsilon'_{xz} & -\epsilon'_{yz} & \epsilon'_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = 0. \tag{7.16}
\]

The third line of this matrix equation can be used to solve for \( E_z \) (Mészáros, 1992):

\[
E_z = \frac{-\epsilon'_{xz} E_x + \epsilon'_{yz} E_y}{\epsilon'_{zz}}, \tag{7.17}
\]

while the remaining two equations can be written compactly:

\[
\begin{bmatrix} \eta_{xx} - N^2 & i\eta_{xy} \\ -i\eta_{xy} & \eta_{yy} - N^2 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} = 0 \tag{7.18}
\]

where

\[
\eta_{xx} = \epsilon'_{xx} - \frac{(\epsilon'_{zz})^2}{\epsilon'_{zz}}, \quad \eta_{yy} = \epsilon'_{yy} + \frac{(\epsilon'_{yz})^2}{\epsilon'_{zz}}, \quad i\eta_{xy} = \epsilon'_{xy} + \frac{\epsilon'_{xz} \epsilon'_{yz}}{\epsilon'_{zz}}. \tag{7.19}
\]
Solving the above system of equations, we find the mode ellipticity \( iK_\pm \equiv \frac{E_x}{E_y} \), defined in terms of the parameter \( \beta \):

\[
iK_\pm \equiv \frac{E_x}{E_y} = i \left( \beta \pm \sqrt{\beta^2 + 1} \right), \quad \beta = \frac{\eta_{xx} - \eta_{yy}}{2 \eta_{xy}} = \frac{1}{2i} \frac{(\epsilon'_{yy} - \epsilon'_{xx})\epsilon'_{zz} + (\epsilon'_{xx})^2 + (\epsilon'_{yz})^2}{\epsilon'_{xx} \epsilon'_{yy} + \epsilon'_{xy} \epsilon'_{zz}}
\]

\[\text{(7.20)}\]

\[
\beta = \frac{1}{2i} \frac{(\epsilon_{xx}^2 + \epsilon_{yy}^2 - \epsilon_{xx} \epsilon_{zz}) \sin^2 \theta + \epsilon_{yz}^2 \cos^2 \theta + (\epsilon_{xx} \epsilon_{yy} - \epsilon_{yz} \epsilon_{xy}) \sin 2\theta + \epsilon_{xz}^2}{(\epsilon_{xy} \epsilon_{zz} + \epsilon_{xx} \epsilon_{yz}) \cos \theta + (\epsilon_{xy} \epsilon_{xz} + \epsilon_{xx} \epsilon_{yz}) \sin \theta}
\]

\[\text{(7.21)}\]

The index of refraction \( N = n + i\kappa \) is given by

\[
N_{\pm}^2 = \eta_{xx} + \frac{\eta_{xy}}{K_\pm} = \eta_{yy} + \eta_{xy} K_\pm
\]

\[\text{(7.22)}\]

We will neglect scattering/absorption at this point, i.e. we are considering only the Hermitian part. In this case \( \beta, K \) and \( N \) are then real; \( \kappa = 0 \). For our purposes, \( w_p \ll 1; \) to lowest order in \( w_p, E_z \sim 0 \); the eigenvectors are transverse:

\[
\hat{e}_\pm = \frac{1}{\sqrt{|E_x|^2 + |E_y|^2}} (E_x, E_y, 0) = \frac{1}{\sqrt{K_{\pm}^2 + 1}} (iK_\pm, 1, 0). \]

\[\text{(7.23)}\]

Other formulae are simplified:

\[
\eta_{xx} \rightarrow \eta_{0xx} = 1 + w_p^2 \left( f_{xx} \cos^2 \theta + f_{zz} \sin^2 \theta - f_{xz} \sin 2\theta \right)
\]

\[
\eta_{yy} \rightarrow \eta_{0yy} = 1 + w_p^2 f_{yy}
\]

\[
\eta_{xy} \rightarrow \eta_{0xy} = w_p^2 (f_{xy} \cos \theta + f_{yz} \sin \theta)
\]
The eigenvectors are orthogonal and describe elliptical polarizations in general. Also note $K_{\pm}K_{\mp} = -1$. The $+/ -$ mode possesses a left/right-handed component of circular polarization; the magnitude of this component depends on the field strength and angle. Because of this, $+$ or $-$ modes that propagate against ($\theta > \pi/2$) or towards ($\theta < \pi/2$) the direction of the magnetic field, respectively, will be able to resonate while those propagating in the opposite sense will not. It is common for authors to designate modes in terms of resonant/non-resonant, usually called extraordinary(X)/ordinary(O).

We take the index of refraction to first order in $w_p^2$:

$$n_{\pm} = 1 + \frac{1}{2} [\eta_{yy} - 1 + \eta_{xy} \pm K_{0}] + O(w_p^4).$$

(7.26)

$$n_{\pm} = 1 + \frac{1}{2} w_p^2 [f_{yy} + (f_{xy} \cos \theta + f_{yz} \sin \theta) K_{0}] + O(w_p^4).$$

(7.27)

The difference in indices is

$$\Delta n = n_+ - n_- = \eta_{xy} \sqrt{\beta_0^2 + 1} = w_p^2 (f_{xy} \cos \theta + f_{yz} \sin \theta) \sqrt{\beta_0^2 + 1}$$

(7.28)

The quantity $\Delta n$ characterizes the strength of the generalized Faraday effect.
For a cold, magnetized plasma, we have

$$\beta_0 = w_B \sin \theta \tan \theta / 2, \quad \eta_{\text{xy}} = w_p^2 w_B \cos \theta / (1 - w_B^2),$$  \hspace{1cm} \text{(7.29)}$$

so

$$\Delta n = \frac{w_p^2 w_B}{1 - w_B^2} \cos \theta \sqrt{1 + w_B^2 \sin^2 \theta \tan^2 \theta / 4} \hspace{1cm} \text{(7.30)}$$

If in this cold, magnetized plasma we have $w_B \ll 1$, then the modes are circular ($K_\pm = \pm 1$), and

$$\Delta n = w_p^2 w_B \cos \theta \hspace{1cm} \text{(7.31)}$$

which is the well-known version of the Faraday effect used in radio astronomy.

The cold, magnetized plasma discussed above has a real refractive index; this is capturing only the propagation effects. However, by introducing a radiative damping term, we incur an imaginary part to the index, which corresponds to magnetic Thomson scattering. This involves the replacements (see Mészáros 1992)

$$w_p^2 \rightarrow w_p^2(1 + i\gamma)^{-1}, \quad \omega_B \rightarrow \omega_B(1 + i\gamma)^{-1},$$  \hspace{1cm} \text{(7.32)}$$

Where $\gamma = 2e^2 \omega / (3mc^3)$ is the radiative damping rate. To lowest order in $\gamma$ and $w_p$, the index is given by

$$N_\pm = 1 + \frac{w_p^2}{2} \left[ \frac{1 - w_B^2 \sin^2 \theta / 2}{w_B^2 - 1} \pm A \right.$$

$$\left. + \frac{i\gamma}{2} \left( C \pm \frac{w_B^2 (8 \cos^2 \theta + w_B^2 (3 - w_B^2) \sin^4 \theta)}{2A(w_B^2 - 1)^3} \right) \right],$$  \hspace{1cm} \text{(7.33)}$$

where $A = \Delta n / 2w_p^2$ is related to the generalized Faraday rate and $C = \sigma_{\text{unpol}} / 2\sigma_T$ is
related to the magnetic Thomson scattering cross section for unpolarized radiation: (Whitney 1991a)

\[
\sigma_{\text{unpol}} = \frac{\sigma_+ + \sigma_-}{2} = \frac{1}{2} \sigma_T \left[ \sin^2 \theta + \frac{1}{2} (1 + \cos^2 \theta) \left( \frac{1}{(w_B - 1)^2} + \frac{1}{(w_B + 1)} \right) \right]
\]

(7.34)

The optical theorem then gives the scattering cross section for each mode:

\[
\sigma_\pm = \frac{2 \omega}{c n_0} \text{Im}(N_\pm)
\]

(7.35)

Quantum absorption processes such as cyclotron, pair processes, and splitting are not included; a quantum description would include them. Regarding cyclotron processes, the scattering intermediate state can of course be in the n=1 Landau level (cyclotron fundamental), though one-vertex processes aren’t considered. Note that for a warm plasma an imaginary part to the index appears even in the absence of radiative damping. This corresponds to purely thermal effects.

7.1.4 Birefringence in a Magnetized Plasma:

The Generalized Faraday Effect

Consider an plane monochromatic wave of arbitrary polarization with \( \vec{k} \parallel \hat{z} \) as before, initially at \( z = 0 \). We can cast its electric field in terms of the eigenmodes:

\[
\vec{E} = A_+ e^{i\phi_+} \hat{e}_+ + A_- e^{i\phi_-} \hat{e}_-,
\]

(7.36)
where $A_{\pm}$ are complex amplitudes, $\phi_{\pm} = \int_0^z k_{\pm} dz$, and $k_{\pm} = \omega n_{\pm}/c$ (we neglect absorption here). The eigenmodes can be brought into a simpler form:

$$
\hat{e}_{\pm} = (i \sin \xi, \cos \xi, 0), \quad \hat{e}_{-} = (i \cos \xi, -\sin \xi, 0),
$$

(7.37)

where the angle $\xi$ (not to be confused with the $\xi$ defined above) is given by: (Lai & Ho 2002)

$$
\tan 2\xi = \frac{1}{\beta}, \quad \tan \xi = -\varepsilon = \frac{1}{\varepsilon_{\pm}}.
$$

(7.38)

As the wave propagates in the medium, the phase lag between the two modes produces a change in the polarization that would not be present if the modes were degenerate; this is the so called generalized Faraday effect (Melrose, 2013). It is an effect present even for pure propagation; thus we consider here this case, with no absorption or scattering. The regular version of the Faraday effect takes place when eigenmodes are circularly polarized; it produces a rotation of the plane of linear polarization. Any degree of circular polarization is preserved.

The polarization of a transverse wave can be fully accounted for by Stokes parameters $(I, Q, U, V)$ up to an overall phase factor, as long as we provide a reference direction on the transverse plane. As a reminder, $I$ is the intensity, $Q$ is the degree of linear polarization in directions $0^\circ$ or $90^\circ$ to the reference direction, $U$ is the degree of linear polarization in directions $\pm45^\circ$ to the reference direction, and $Q$ is the degree of circular polarization. They satisfy $Q^2 + U^2 + V^2 = p^2 I^2$, where $p$ is the degree of polarization. Waves will be fully polarized for our purposes, so $p = 1$.

$Q$, $U$, and $V$ can be positive or negative, where the sense depends on arbitrary conventions. In this work, in a basis where $\hat{k} = \hat{z}$, the reference direction will be $\hat{x}$. Then positive(negative) $Q$ corresponds to polarization along the $x(y)$ axis, posi-
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Negative $Q$ means polarization along directions parallel to to $\hat{x} \pm \hat{y}$, and positive negative $V$ gives left-(right-)handed polarization, i.e. polarization along directions parallel to to $\hat{x} \pm i\hat{y}$. Note that the Stokes vectors for the eigenstates are always antipodal points on the Poincaré sphere (Melrose, 2012):

$$\vec{S}_\pm = \mp (\cos 2\xi, 0, \sin 2\xi). \quad (7.39)$$

In terms of the Poincaré sphere, the generalized Faraday effect produces a precession of the polarization about the axis defined by the eigenstates' antipodal points. Polarization along the eigenmodes is preserved, while polarization orthogonal to it (on the Poincaré sphere) is rotated at a constant rate: the generalized Faraday rotation rate. We can relate initial and final Stokes parameters for propagation to describe it:

$$\begin{bmatrix} Q \\ U \\ V \end{bmatrix} = \begin{bmatrix} \cos^2 2\xi + \sin^2 2\xi \cos \varphi & -\sin 2\xi \sin \varphi & \sin 4\xi \sin^2 \frac{\varphi}{2} \\ \sin 2\xi \sin \varphi & \cos \varphi & -\cos 2\xi \sin \varphi \\ \sin 4\xi \sin^2 \frac{\varphi}{2} & \cos 2\xi \sin \varphi & \cos^2 2\xi \cos \varphi + \sin^2 2\xi \end{bmatrix} \begin{bmatrix} Q_0 \\ U_0 \\ V_0 \end{bmatrix} \quad (7.40)$$

where $\vec{S} = (Q, U, V)$ is a polarization state and $\varphi = \phi_+ - \phi_- = (\omega/c) \int_0^z \Delta n \, dz$ quantifies the magnitude of the generalized Faraday effect. If we rotate the Stokes vector space, the formula becomes much simpler. We introduce a modified Stokes vector $\vec{S}' = (W, U, X)$, where

$$W = Q \sin 2\xi - V \cos 2\xi, \quad X = -Q \cos 2\xi - V \sin 2\xi. \quad (7.41)$$
In terms of $S^\theta$, we have

\[
\begin{bmatrix}
W \\
U \\
X
\end{bmatrix} =
\begin{bmatrix}
\cos \varphi & -\sin \varphi & 0 \\
\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
W_0 \\
U_0 \\
X_0
\end{bmatrix}.
\] (7.42)

We perform a spherical decomposition to characterize the effect in terms of angles:

\[
\vec{S}^\theta = (W, U, X) = (\cos 2\psi \cos 2\chi, \sin 2\psi \cos 2\chi, \sin 2\chi).
\] (7.43)

Here, $\tan 2\psi = U/W$ and $\tan 2\chi = X/\sqrt{U^2 + W^2}$. In terms of these angles, the effect is to change the angle $\psi$:

\[
\chi = \chi_0, \quad \psi = \psi_0 + \Delta \psi, \quad \Delta \psi = \frac{\dot{\psi}}{2} = \frac{\omega}{2c} \int_0^z \Delta n \, dz.
\] (7.44)

$\chi$ gives the amount of polarization along the eigenmodes, which is preserved. Polarization orthogonal to it (on the Poincaré sphere) is rotated at a constant rate $\Delta \psi$.

### 7.2 Application to Magnetars

For X-ray transfer in neutron stars, we consider a hydrogenic plasma; the mobility tensor is a superposition of that for each species. The electron cyclotron frequency $\omega_{B,e} \gg \omega$, so we can use the cold magnetized plasma mobility tensor (Eq.7) for simplicity, since warm plasma effects are concentrated around the resonance. Subscript $e, i$ will refer to electron, ion(proton) species, respectively. The proton cyclotron resonance will be in X-rays, so the full warm 1-D mobility tensor (Eq.6) will be used to
describe this component.

There are two assumptions that go into the “normal mode description”. Assumption (1) requires orthonormality and transversality:

\[ |\hat{e}_1 \cdot \hat{e}_2| \ll 1, \quad |\hat{e}_i \cdot \hat{k}| \ll 1, \quad (7.45) \]

Assumption (2) requires large Faraday depolarization:

\[ |\text{Re}(N_1 - N_2)| \gg |\text{Im}(N_1 + N_2)| \quad (7.46) \]

For an ion temperature \(\sim 50 \text{ keV} \), \(B = 2 \times 10^{14} \text{ Gauss} \) \((\omega_{c,i} \sim 1 \text{ keV})\), and \(\omega_{p,i}/\omega_{c,i} = 10^{-4}\), either or both of these assumptions can break down near \(\omega_{c,i}\), as shown in Figure 2. When Faraday depolarization is not large, the transfer problem becomes significantly more complicated. The polarization then evolves at a rate comparable to the mean free path for scattering. Monte Carlo methods can be used to deal with these cross sections that vary along the photon path.
Figure 7.2: For certain combinations of angle and frequency, either or both assumptions required for the “normal mode description” can fail. This plot shows the extent of such regions as a function of frequency and angle, for a plasma consisting of cold electrons and warm protons of thermal velocity $v_z$ (nonrelativistic). In the red outlined region consisting of four teardrops, orthonormality of the normal modes (assumption 1) is broken. In the blue outlined hourglass region, faraday depolarization is large. Red/blue shading corresponds to the regions where assumption 1(2) breaks down. In the purple shaded region, both assumptions break down. Our 4-parameter description extends the region of applicability to the blue shaded region.
In this thesis, magnetized Thomson scattering on the polarization level has been thoroughly analyzed. As background, we have reviewed the relevant astrophysical sources that this thesis pertains to, as well as the physics of polarization and magnetized Thomson scattering. We have developed and vetted a simulation code to model the effect of this process on radiation transfer, handling the basic case of transfer through an atmospheric slab. This core code is then embedded into a model of emission from the full surface of a magnetar in soft X-rays, as well as a model of the flare emission seen in hard X-rays. This is the first treatment on the polarization level of magnetar surface emission under magnetized Thomson scattering that routinely captures all possible zenith angles for the local magnetic field, and therefore can be applied to every point on the neutron star surface. In addition, we have taken a new look at plasma birefringence and the generalized Faraday effect it induces, analyzing its consequences on polarized radiative transfer in warm, magnetized media. Finally, in anticipation of future work, we have developed the methods needed to include in the simulation more complicated field geometries as well as additional photon processes such as inverse Compton scattering.

With the age of X-ray polarimetry quick on the horizon, there is no better time to produce X-ray polarization predictions. From the atmospheric results, we have seen that localized patches of magnetized atmosphere put off strong polarization signals for a large range of frequencies and magnetic field orientations. Yet when integrating over
an entire thermalized atmosphere, there is a strong depolarizing effect that occurs, mainly due to the different magnetic field directions sampled. Even in the presence of this depolarizing effect, the field directions near the poles and equators are on the same axis, so their linear polarizations are able to survive the integration process and produce noticeable polarization signals, as high as 15%. However, it is the presence of localized hot spots, either on the surface or in the magnetosphere during flare activity, that induces the strongest polarization signals (> 45%) in frequency bands where the hot emission dominates. The higher polarization in this case is due to the majority of photons originating from the hot region, which is spatially localized. We have seen from our results that, at least in the soft X-ray band, models of polarized emission can produce significant polarization degrees without including vacuum birefringence effects in photon propagation through the magnetosphere: these help facilitate high degrees of polarization from entire stellar surfaces. Thus, the claims that high degrees of polarization seen in optical for RX J1856.5-3754 serve as evidence for the action of vacuum birefringence may be premature.

The core code used in this thesis is a fairly simple implementation of magnetized Thomson scattering in atmospheric slabs of a prescribed (constant) magnetic field. In the desire to go beyond this, we have developed additional components described mostly in the appendix. In the future we will be able to more easily handle transfer in regions of changing magnetic field, such as in large optically thick regions of the outer magnetosphere. We will include the effect of bulk motion and nonzero temperature for the electrons, which will introduce energy exchange between the photon and electron populations and thereby effect the spectral formation. We have developed the necessary tools to examine this same process in pulsar magnetospheres at higher energies.
Regarding the generalized Faraday effect, we have seen that the inclusion of this effect is critical towards the study of polarized radiative transfer. However, we cannot always assume we are in a regime where this effect dominates; when the effect is only modest, it must be included specifically. We have developed a method to handle this effect in the simulation, and such an inclusion would be complementary to the method of adiabatic normal mode transport currently used by several groups that have developed complicated atmosphere models.
Appendix A

A.1 Optical Depth in a Non-Uniform Field

Here we briefly outline a method for handling optical depth determinations in the Monte Carlo model for a non-uniform field. This is preliminary work, and is an alternative to other methods such as stacking atmospheric slabs of differing magnetic field and/or density (but constant within each slab). Results presented in this thesis have used the assumption of uniform field within each slab. The protocols outlined here will help facilitate more developed modeling of the resonant Compton transport in flares in the magnetosphere.

A photon in the model framework has a unit vector in the propagation direction $\hat{n}_i$, a polarization vector $\vec{E}_i$, and a frequency $\omega$. It is located at the point $\vec{r}$ at time $t$. A single Runge-Kutta (RK) step calculates $\tau_0 = n_e \sigma_0 h$, which is the optical depth between $\vec{r}$ and $\vec{r} + h\hat{n}_i$. The distance $h$ is the initial step size of the RK integration, generally being chosen to be small, i.e. $h \ll R_{ns}$, and $\sigma_0$ comes from the RK integration, which will vary depending on the choice of RK algorithm. We use the RK method of Bogacki & Shampine (1989), which gives

$$\sigma_0 = \frac{1}{9} \left[ 2\sigma(\vec{r}) + 3\sigma(\vec{r} + h\hat{n}_i/2) + 4\sigma(\vec{r} + 3h\hat{n}_i/4) \right], \quad (A.1)$$

as well as a higher order method that is used to estimate error and adapt the step
size,

\[
\sigma'_0 = \frac{1}{24}[7\sigma(\vec{r}) + 6\sigma(\vec{r} + h\hat{n}_i/2) + 8\sigma(\vec{r} + 3h\hat{n}_i/4) + 3\sigma(\vec{r} + h\hat{n}_i)].
\]  \hfill (A.2)

Note that we have repressed the polarization and frequency dependence, \(\sigma = \sigma(\vec{r}, E_i, \omega)\).

A uniform random variate \(x\) in the range \([0, 1]\) is sampled. If \(0 < \log x < \tau_0\), the photon is determined to scatter within the distance \(h\); the exact location of scattering is

\[
s = -h \frac{\log x}{\tau_0}. \hfill (A.3)
\]

Otherwise, we advance the RK integration another step \(h'\) (which may be of a different size from the initial step), obtaining \(\tau_1\): the optical depth between \(\vec{r} + h\hat{n}_i\) and \(\vec{r} + (h + h')\hat{n}_i\). If \(\tau_0 < -\log x < \tau_0 + \tau_1\), the photon is determined to scatter within the distance \(h\); the exact location of scattering is

\[
s = h - h' \frac{\tau_0 + \log x}{\tau_0 + \tau_1}. \hfill (A.4)
\]

The algorithm proceeds in this way until the scattering distance \(s\) is identified; The photon is then advanced to \(\vec{r} + s\hat{n}_i\), and the scattering algorithm is invoked, with the local field at \(\vec{r} + s\hat{n}_i\) being used in the scattering algorithm.

\section*{A.2 Compton Scattering with Hot Electrons}

The inferred temperature from sources considered in this thesis can imply that portions of the electron population are in the mildly relativistic regime, where the Lorentz factor \(\gamma = (1 - v^2/c^2)^{-1/2}\) has departed from unity. In atmospheres we are at the lower threshold where \(\gamma - 1 \sim 10^{-3}\), while in magnetosphere flare work \(\gamma\) can be as high as
\( A.2: \) COMPTON SCATTERING WITH HOT ELECTRONS

\[ \sim 1.5; \text{ in this regime we must include the effect of Inverse Compton scattering.} \]

Inverse Compton scattering amounts to a Lorentz boost, a Thomson scattering, and an inverse Lorentz boost. The scattering itself is coherent and elastic, but the asymmetry of the boosts leads to a gain in energy and momentum for the photon. The energy exchange between a photon and electron population due to inverse Compton scattering, known as Comptonization, produces a non-thermal power-law tail to an initially thermal distribution of photons. This is the most likely candidate process for generating these power-law distributions. Spectral investigation of magnetar flares is an active topic, and their intrinsic brightness leads to inferences of high Thomson opacity, so that it is expected that Comptonized emission is present.

In a neutron star atmosphere or magnetosphere, the electron velocity distribution is approximately one-dimensional; the electrons are only free to move along the direction of the field. We will assume a thermal distribution, i.e. a 1-D Maxwellian, for this velocity distribution. Bulk motion can also be added, but we will not consider it here as it also alters the determination of the total cross section for scattering.

The energy, propagation direction, and polarization all change in a Lorentz boost. The wave 4-vector transformation accounts for the change in energy and direction:

\[
\omega' = \omega \gamma \left(1 - \beta \cdot \hat{n}\right), \quad \hat{n}' = \frac{\hat{n} + \beta \left[(\gamma - 1)(\beta \cdot \hat{n})\beta^{-2} - \gamma\right]}{\gamma \left(1 - \beta \cdot \hat{n}\right)}
\]  

(A.5)

where \( \vec{v} = \beta c \) is the boost velocity, and primes here refer to the boosted frame. The transformation formula for \( \vec{n}' \) is a vector form of the usual aberration formula, and can be derived from the wave 4-vector transformation or velocity considerations.
The electric field transformation accounts for the change in polarization:

\[
\vec{E}' = \gamma (\vec{E} + \vec{\beta} \times \vec{B}) - \frac{\gamma^2}{\gamma + 1} \vec{\beta} (\vec{\beta} \cdot \vec{E}).
\] (A.6)

The magnetic field of a transverse EM wave is given by \( \vec{B} = \hat{n} \times \vec{E} \), which simplifies the formula:

\[
\vec{E}' = \gamma \left[ \vec{E} \left( 1 - \vec{\beta} \cdot \hat{n} \right) + \left( \hat{n} - \frac{\gamma}{\gamma + 1} \vec{\beta} \right) (\vec{\beta} \cdot \vec{E}) \right].
\] (A.7)

For the magnitude, we have

\[
|\vec{E}'|^2 = |\vec{E}|^2 \gamma^2 \left( 1 - \vec{\beta} \cdot \hat{n} \right)^2.
\] (A.8)

We can summarize the process of inverse Compton scattering as would be implemented in code. First a scattering is determined to occur in the usual way (we neglect bulk motion which would alter this). Before scattering, we have a photon with energy \( \omega \), propagation direction \( \hat{n} \), and polarization vector \( \vec{E} \). The electron velocity distribution is sampled and a corresponding boost vector \( \vec{\beta} \) is obtained. We transform into the electron rest frame; here the photon properties are \( \omega', \hat{n}', \) and \( \vec{E}' \), as given above. The magnetic Thomson differential cross section is sampled to simulate the scattering, changing the direction and polarization to \( \hat{n}'_s \) and \( \vec{E}'_s \), while the energy \( \omega'_s = \omega' \) remains the same. Also note that since we transform along the magnetic field direction, its magnitude and direction remain unchanged by the transformation. We then transform back to the lab frame, giving the final scattered quantities \( \omega_s, \hat{n}_s, \vec{E}_s \).

The energy is altered due to the change in \( \hat{n} \) caused by scattering that occurs in
between the boosts. Explicitly, we have

\[ \omega' = \omega' \gamma \left( 1 + \vec{\beta} \cdot \vec{n}' \right) = \omega \gamma^2 \left( 1 - \vec{\beta} \cdot \vec{n} \right) \left( 1 + \vec{\beta} \cdot \vec{n}' \right). \]  

(A.9)

For a perfect head-on/trailing collision, the energy increases/decreases by $4\gamma^2$, respectively.

Note also that the electric field will lose its normalization after scattering, as its magnitude reflects the energy content. This can easily be accounted for by renormalizing before taking any statistics.

**A.3 Rotating Neutron Stars**

In this section we present some general results that are applicable to the study of hard X-ray and $\gamma$-ray emission in pulsars. It can also be pertinent to photon attenuation calculations in magnetars. First, we derive the emission angle of curvature radiation as would be produced by highly relativistic gap-accelerated electrons in a pulsar magnetosphere. Such curvature radiation is an integral part of the modern understanding of high energy pulsar emission. Next we derive the analytic solution to the polar cap region.

**A.3.1 Field Structure**

We consider the neutron star magnetic field to be that of an oblique rotating magnetic dipole. The dipole rotates around the $z$-axis with angular frequency $\Omega$, and has inclination angle $\alpha$ with respect to $z$. The magnetic moment is

\[ \vec{\mu}(t) = \mu \left( \sin \alpha \cos \Omega t \hat{x} + \sin \alpha \sin \Omega t \hat{y} + \cos \alpha \hat{z} \right) \]  

(A.10)
For this oblique rotating magnetic dipole, the vacuum electromagnetic fields are

\[
\vec{E}_v = \hat{r} \times \left( \frac{\dot{\vec{\mu}}}{cr^2} + \frac{\vec{\mu}}{c^2 r} \right), \quad \vec{B}_v = \hat{r} \cdot \left[ 3 \frac{\vec{\mu}}{r^3} + 3 \frac{\dot{\vec{\mu}}}{cr^2} + \frac{\ddot{\vec{\mu}}}{c^2 r} \right] - \left[ \frac{\vec{\mu}}{r^3} + \frac{\dot{\vec{\mu}}}{cr^2} + \frac{\ddot{\vec{\mu}}}{c^2 r} \right],
\]

(A.11)

where these formulae must be evaluated at the retarded time \( t_r = t - r/c \). The subscript \( v \) indicates that they are vacuum fields.

The magnetic field takes the following form:

\[
\vec{B}_v = \frac{\mu}{r^3} \left( b_r \hat{r} + b_\phi \hat{\phi} + b_\theta \hat{\theta} \right),
\]

(A.12)

where

\[
\begin{align*}
  b_r &= 2 \left[ \cos \alpha \cos \theta + \left( \cos \lambda + r_n \sin \lambda \right) \sin \alpha \sin \theta \right], \\
  b_\phi &= - \left[ r_n \cos \lambda + (r_n^2 - 1) \sin \lambda \right] \sin \alpha, \\
  b_\theta &= \cos \alpha \sin \theta + \left[ (r_n^2 - 1) \cos \lambda - r_n \sin \lambda \right] \sin \alpha \cos \theta,
\end{align*}
\]

(A.13-15)

and

\[
\lambda = r_n + \phi - \Omega t, \quad r_n = r/R_{lc}, \quad R_{lc} = c/\Omega.
\]

(A.16)

This is the so-called Deutsch solution (Deutsch 1955, Dyks & Harding 2004). It is visualized in Figure A.1.

We now assume that the magnetosphere is filled with plasma, and that it is in rigid co-rotation with the star up until the light cylinder radius \( R_{lc} \), where co-rotation reaches the speed of light. We neglect the effect the presence of plasma will have on the magnetic field; we will assume it takes the same form as the vacuum case, \( \vec{B}_v \).
Figure A.1: Visualization of the magnetic field of a rotating dipole. A line integral convolution is performed using Mathematica for the function in Eq. A.12; this is done in the \( y = 0 \) plane at \( t = 0 \) for \( \alpha = 45^\circ \). Only the \( x \) and \( z \) components of the field are shown. Axes are scaled by the light cylinder radius, and so we see mostly the mid-to far-field regime. The dipolar structure at low altitudes is apparent.

These assumptions force the electric field to take the following form:

\[
\vec{E}_c = \vec{B}_v \times \vec{v}/c, \quad \vec{v} = \vec{\Omega} \times \vec{r}.
\]  

(A.17)
The subscript $c$ here indicates a co-rotational drift field.*

In our coordinate system, $\tilde{\Omega} = \Omega \hat{z}$, so $\tilde{\beta} = \tilde{v}/c$ takes the form

$$\tilde{\beta} = \beta \hat{\phi}, \quad \beta = r_n \sin \theta.$$  \hspace{1cm} (A.18)

We can define a Lorentz transformation at each point within the light cylinder that corresponds to an instantaneous boost to the co-rotating frame, which is also the plasma rest frame.

$$\bar{E}' = \gamma (\bar{E}_c + \tilde{\beta} \times \bar{B}_v) - \frac{\gamma^2}{\gamma + 1} \tilde{\beta} (\bar{\beta} \cdot \bar{E}_c) \equiv 0,$$  \hspace{1cm} (A.19)

$$\bar{B}' = \gamma (\bar{B}_v - \tilde{\beta} \times \bar{E}_c) - \frac{\gamma^2}{\gamma + 1} \tilde{\beta} (\bar{\beta} \cdot \bar{B}_v).$$  \hspace{1cm} (A.20)

Where $\gamma = (1 - \beta^2)^{-1/2}$ as usual. The magnetic field takes the following form:

$$\bar{B}' = \frac{\mu}{r^3} \left( \frac{b_r \hat{r} + b_\theta \hat{\theta}}{\gamma} + b_\phi \hat{\phi} \right),$$  \hspace{1cm} (A.21)

where the $b$ functions are the same as above.

A.3.2 Polar Caps

The polar cap of a neutron star is the collection of all magnetic field lines which pass through both the neutron star surface and the light cylinder. As the name suggests, the region is a cap shape centered on each magnetic pole; field lines closer to the equator will form closed loops within the light cylinder radius. Numerically, it is easy enough to calculate if the field is known: simply follow the field lines by numerical

---

*For the electric field to have this form, it must be the sum of the plasma-produced electric field and the vacuum electric field, i.e. $\bar{E} \equiv \bar{E}_c = \bar{E}_p + \bar{E}_v$, where $\bar{E}_p \equiv \bar{E}_c - \bar{E}_v$ is the electric field produced by the plasma. In the co-rotating frame, $\bar{E}' = 0$, so that $\bar{E}_p' + \bar{E}_v' = 0.$
A.3: ROTATING NEUTRON STARS

Figure A.2: Cross sections of the open field line volume threading a sphere of radius $r < R_{lc}$. Plots show a top-down view of the magnetic pole, located at the center of each plot $(x_m, y_m) = (0, 0)$, in the so-called magnetic coordinates, defined below. The circles shown in grey have radius $rr_n^{1/2}$, where $r_n = r/R_{lc}$; this corresponds to the polar cap of an aligned rotator in either the static or Deutsch solution. The blue curves show the polar cap of the Deutsch solution, while red curves show that of the static dipole. The properties of the oblique rotator are $P = 1\text{s}$, $r = 10^{-2}R_{lc}$, $\alpha = 45^\circ$, $90^\circ$. The field lines corresponding to the notched feature in the $\alpha = 45^\circ$ case are due to rotational sweepback of the field lines: the larger radius section of the notch corresponds to field lines tangent to the cylinder roughly along the cylinder axis, while the inner part of the notch corresponds to swept back field lines tangent to the cylinder that are perpendicular to the cylinder axis. For $\alpha = 90^\circ$ this same effect produces the asymmetry when compared to the static dipole. These results are equivalent to those shown in Dyks & Harding (2004).

integration until the boundary is identified. This is the necessary strategy for complex fields such as the Deutsch solution above. Examples of polar caps for the Deutsch and static solutions are shown in Fig. A.2; these can be compared to figures from Dyks & Harding (2004), with which there is no discrepancy.† However, for a static dipole it can be analytically obtained; we present its derivation here.

Consider a magnetic dipole that rotates around the z-axis with angular frequency

†See Dyks & Harding (2004) for a detailed review of rotational sweepback and its effect on polar caps.
The magnetic moment is

$$\vec{\mu}(t) = \mu (\sin \alpha \cos \Omega t \hat{x} + \sin \alpha \sin \Omega t \hat{y} + \cos \alpha \hat{z}).$$  \hfill (A.22)

The vacuum magnetostatic field is

$$\vec{B}_s = \frac{3(\vec{\mu} \cdot \hat{r}) \hat{r} - \vec{\mu}}{r^3}. \hfill (A.23)$$

The subscript \( s \) indicates 'static'.

The magnetic field takes the following form:

$$\vec{B}_v = \frac{\mu}{r^3} \left( 2 \cos \theta_m \hat{r}_m + \sin \theta_m \hat{\theta}_m \right), \hfill (A.24)$$

where

$$\hat{r}_m = \sin \theta_m \cos \phi_m \hat{x}_m + \sin \theta_m \sin \phi_m \hat{y}_m + \cos \theta_m \hat{z}_m, \hfill (A.25)$$

$$\hat{\phi}_m = \cos \theta_m \cos \phi_m \hat{x}_m + \cos \theta_m \sin \phi_m \hat{y}_m - \sin \theta_m \hat{z}_m, \hfill (A.26)$$

$$\hat{\theta}_m = -\sin \phi_m \hat{x}_m + \cos \phi_m \hat{y}_m, \hfill (A.27)$$

and

$$\hat{x}_m = \cos \alpha \cos \Omega t \hat{x} + \cos \alpha \sin \Omega t \hat{y} - \sin \alpha \hat{z}, \hfill (A.28)$$

$$\hat{y}_m = -\sin \Omega t \hat{x} + \cos \Omega t \hat{y}, \hfill (A.29)$$

$$\hat{z}_m = \sin \alpha \cos \Omega t \hat{x} + \sin \alpha \sin \Omega t \hat{y} + \cos \alpha \hat{z}. \hfill (A.30)$$

The subscript \( m \) indicates 'magnetic'. \( \{\hat{x}_m, \hat{y}_m, \hat{z}_m\} \) is a Cartesian basis co-rotating...

\( \Omega \), and has inclination angle \( \alpha \) with respect to \( z \).
with the dipole, and \(\{r, \phi_m, \theta_m\}\) are 'magnetic coordinates': spherical coordinates on this basis. Note that \(\vec{\mu} = \mu \hat{z}_m\).

We seek the field lines tangent to the light cylinder surface defined by \(x^2 + y^2 = R_{lc}^2\), where \(R_{lc} = c/\Omega\). Using \(\vec{r} = r\hat{r}_m, x = \vec{r} \cdot \hat{x}\), etc., we can define \(r_t\), the radial distance to any point on the light cylinder, in terms of magnetic coordinates:

\[
r_t = \frac{R_{lc}}{\sqrt{(\sin \alpha \cos \theta_m + \cos \alpha \sin \theta_m \cos \phi_m)^2 + \sin^2 \theta_m \sin^2 \phi_m}} \tag{A.31}
\]

The tangent condition can be expressed as \(\vec{B}_s \cdot \vec{s} = 0\), where \(\vec{s}\) points in the cylindrical radial direction, i.e. \(x\hat{x} + y\hat{y}\). With some simplification, this leads to a cubic equation in \(\tan \theta_m\):

\[
0 = 2\sin^2 \alpha + \sin \alpha \cos \alpha \cos \phi_m \left(5 \tan \theta_m - \tan^3 \theta_m \right) + \left(3 \cos^2 \alpha \cos^2 \phi_m - \sin^2 \alpha + 3 \sin^2 \phi_m \right) \tan^2 \theta_m. \tag{A.32}
\]

The real solution of this cubic we will call \(\theta_l\): the \(\theta_m\) angle of the last open field lines on the light cylinder.

What we ultimately want is the sine of the \(\theta_m\) angle of the last open field lines on the stellar surface \(R_{ns}\). The field lines for a static dipole conserve the quantity \(r/\sin^2 \theta_m\), so we can relate two points on a field line:

\[
\frac{R_{ns}}{\sin^2 \theta_{cap}} = \frac{r_t}{\sin^2 \theta_l}. \tag{A.33}
\]

The final result is

\[
\sin \theta_{cap} = \sqrt{\frac{R_{ns}}{R_{lc}}} \sin \theta_l \left[\left(\sin \alpha \cos \theta_l + \cos \alpha \sin \theta_l \cos \phi_m\right)^2 + \sin^2 \theta_l \sin^2 \phi_m\right]^{1/4}. \tag{A.34}
\]
We can compare this result with the numerically obtained result for the Deutsch solution; this is shown in Fig. A.2.

A.3.3 Photon Aberration

We are concerned with the direction of emission of curvature photons, or those that are the product of inverse Compton scattering, due to particles traversing the magnetic field described in Sec. A.3.1. Curvature photons are emitted tangent to the particle trajectory, and we assume that the particle trajectory is tangent to the magnetic field in the plasma rest frame (co-rotating frame). This assumption is valid if the magnetic field is strong enough: the transverse momentum are then quantized as Landau levels, and spontaneous decay to the ground level will be increasingly rapid for increasing field strength. The particle is effectively a “bead on a (magnetic field line) wire”.

Thus, the photon propagation direction is

\[ \hat{k}' = \eta \frac{\vec{B}'}{|\vec{B}'|}, \]  

(A.35)

where \( \eta = \pm 1 \). The choice of \( \eta \) decides whether the photon propagates parallel or antiparallel to the field.

Finally we must transform back to the unprimed (stationary observer) frame. Let \( \vec{u} = c \hat{k}' \) be the photon’s vector velocity in the co-rotating frame. Let

\[ \vec{u}_\parallel = (\vec{u} \cdot \hat{\beta}) \hat{\beta} = \eta c \frac{b_\phi \hat{\phi}}{b'}; \]  

(A.36)

and

\[ \vec{u}_\perp = \vec{u} - \vec{u}_\parallel = \eta c \frac{b_r \hat{r} + b_\theta \hat{\theta}}{\gamma b'}; \]  

(A.37)
where

\[ b' = \sqrt{\gamma^{-2}(b_1^2 + b_0^2) + b_\phi^2}. \] (A.38)

The vector velocity in the stationary frame, \(\vec{s}'\), is then

\[ \vec{s}' = \vec{v} + \vec{u}_\parallel + \gamma^{-1}\vec{u}_\perp \]
\[ 1 + \vec{v} \cdot \vec{u}/c^2 \] (A.39)

or

\[ \vec{s}' = c\hat{s}; \quad \hat{s} = s_r\hat{r} + s_\phi\hat{\phi} + s_\theta\hat{\theta} \] (A.40)

where

\[ s_r = \frac{\eta b_r}{\gamma^2(b' + \beta\eta b_\phi)}, \quad s_\phi = \frac{\beta b' + \eta b_\phi}{b' + \beta\eta b_\phi}, \quad s_\theta = \frac{\eta b_\theta}{\gamma^2(b' + \beta\eta b_\phi)}. \] (A.41)

When \(\beta \to 0\) and \(\gamma \to 1\), these aberration formula reduce to the familiar static dipole field result. The Doppler shift is given by

\[ \frac{\omega}{\omega'} = \gamma (1 + \beta\eta b_\phi/b') \] (A.42)

Thus, given an emission point \(\vec{r}\) of a curvature photon emitted with energy \(w'\) in the plasma rest frame, the photon will have propagation direction \(\hat{s}\) and energy \(w\) as seen in the observer frame.
Bibliography

Adler, S. L. 1971, Annals of Physics, 67, 599
Baade, W., & Zwicky, F. 1934, Phys. Rev., 46, 76


—. 1971b, Phys. Rev. D, 3, 2303

Chandrasekhar, S. 1960, Radiative transfer


Deutsch, A. J. 1955, Annales d’Astrophysique, 18, 1


Dolginov, A. Z., Gnedin, Y. N., & Silant’ev, N. A. 1995a, Propagation and polarisation of radiation in cosmic media

—. 1995b, Propagation and polarisation of radiation in cosmic media


Erber, T. 1966a, Reviews of Modern Physics, 38, 626

—. 1966b, Reviews of Modern Physics, 38, 626


Herold, H. 1979, Phys. Rev. D, 19, 2868


—. 1998b, Nature, 393, 235


Lorimer, D. R., & Kramer, M. 2004, Handbook of Pulsar Astronomy


Mészáros, P. 1992, High-energy radiation from magnetized neutron stars.


—. 2014b, ApJS, 212, 6


—. 1968, Nature, 219, 145


Rea, N., & Esposito, P. 2011, Astrophysics and Space Science Proceedings, 21, 247


Rybicki, G. B., & Lightman, A. P. 1979, Radiative processes in astrophysics

Shapiro, S. L., & Teukolsky, S. A. 1983, Black holes, white dwarfs, and neutron stars: The physics of compact objects
Stokes, G. G. 1851, Transactions of the Cambridge Philosophical Society, 9, 399


Walker, M. J. 1954, American Journal of Physics, 22, 170


  Telescopes and Instrumentation 2016: Ultraviolet to Gamma Ray, 990517


