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Random Vibrations of Nonlinear Continua Endowed with Fractional Derivative Elements

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Abstract

In this paper, two techniques are proposed for determining the large displacement statistics of random exciting continua endowed with fractional derivative elements: Boundary Element Method (BEM) based Monte Carlo simulation; and Statistical Linearization (SL). The techniques are applied to the problem of nonlinear beam and plate random response determination in the case of colored random external load. The BEM is implemented in conjunction with a Newmark scheme for estimating the system response in the time domain in conjunction with repeated simulations, while SL is used for estimating efficiently and directly, albeit iteratively, the response statistics.

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1. Introduction

Fractional operators have received recently considerable attention in a number of thematic domains [1]. In this regard, a notable example pertains to viscoelasticity as discussed in the works, for instance, of Nutting [2], Gemant [3], Scott-Blair and Gaffyn [4], and Bagley and Torvik [5]. The impact of fractional operators in problems involving the vibration of systems endowed with fractional derivative elements was described in two extensive review articles by Rossikhin and Shitikova [6, 7] in a deterministic setting, while analyses focusing on systems exposed to random loads are more recent. In this context, Spanos and Zeldin [8] elucidated certain pitfalls associated with the use of frequency dependent parameters in conjunction with the calculation of system response statistics. Linear systems

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were investigated by Agrawal [9], and Di Paola, Failla, et al. [10], while nonlinear systems were investigated by techniques such as SL, stochastic averaging, Weiner path integrals and harmonic wavelet based SL [11-15]. The vibration problem of continuous systems has received some attention, as well. In this context, most articles deal with the analysis of either linear or nonlinear beams [16-19], and of plates [20-22]. Despite these efforts, the need of techniques for estimating the nonlinear response of continuous systems endowed with fractional derivative elements reliably and efficiently persists. Thus, this paper describes a solution technique based on the SL technique that allows estimating efficiently the response statistics of continuous systems and, further, a strictly numerical approach based on the BEM, for conducting Monte Carlo studies. The techniques are applied to the specific problems of large beam and plate displacements.

2. Governing equations

2.1. Preliminary concepts on fractional derivatives

A critical concept underlining the definition of fractional derivative relates to the definition of fractional integral, which is obtained as the convolution of a function $w(t)$ with a power law kernel. That is,

$${}_0D_t^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{w(\tau)}{(t-\tau)^{-\alpha+1}} d\tau, \text{ for } \alpha > 0, \quad (1)$$

with $\Gamma(\alpha)$ being the gamma function [23]. Clearly, for integer values of the power law, $\alpha = n$, the gamma function renders the factorial of the integer number and, thus eq. (1) provides the classical n -fold integral. The Riemann-Liouville (RL) fractional derivative is constructed by differentiating eq. (1) m times. That is,

$${}_0^{RL}D_t^\alpha = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t \frac{w(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, \quad m-1 \leq \alpha < m. \quad (2)$$

Further, the Grünwald-Letnikov (GL) representation [24] is given by the equation

$${}_0^{GL}D_t^\alpha w(t) = \sum_{k=0}^{m-1} \frac{w^{(k)}(0)t^{k-\alpha}}{\Gamma(k+1-\alpha)} + \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{w^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, \quad m-1 < \alpha < m. \quad (3)$$

Such a representation can lead to algorithms for the numerical computation of fractional derivatives. Indeed, the series in eq. (3) can be expanded and the series representation of the GL derivative

$${}_0^{GL}D_t^\alpha w(t) = \lim_{\Delta t \rightarrow 0} \Delta t^{-\alpha} \sum_{k=0}^n GL_k w(t-k\Delta t), \quad (4)$$

can be derived, where GL_k are calculated recursively using the relationship

$$GL_k = \frac{k-\alpha-1}{k} GL_{k-1}; \quad GL_0 = 1. \quad (5)$$

Eq. (4) provides the G1-algorithm that is used in this paper for conducting requisite numerical integrations. Such an algorithm clearly points out the fading memory property of the fractional derivative.

2.2. Equations governing the large vibration of a continuous system

Continuous systems are described via partial differential equations of the form

$$D[v] = q, \quad (6)$$

where v is the unknown system response, $D[\cdot]$ is a nonlinear differential operator, and q is the load. Developing an exact solution to problem (6) is a daunting task. Thus, two approximate techniques are proposed herein: Boundary Element Method (BEM) based Monte Carlo simulations; and Statistical Linearization (SL). These techniques can be utilized for estimating the system response when the nonlinear operator $D[\cdot]$ includes a fractional derivative element.

The BEM is implemented in conjunction with the nonlinear problem (6) by resorting to a surrogate linear problem accommodating the numerical implementation of the BEM as commonly proposed in the literature. Specifically, an appropriate linear partial differential equation of the form

$$L[v] = b, \quad (7)$$

where $L[\cdot]$ is a linear operator and b is a fictitious time-dependent term, is selected. Such a problem admits an integral representation of the solution, which is conveniently discretized to determine the system response as

$$v = \mathbf{G} \cdot \mathbf{b}, \quad (8)$$

with \mathbf{G} being a known matrix, and \mathbf{b} being an unknown (time-dependent) vector. Finally, the unknown source term \mathbf{b} is calculated by collocating eq. (6) in each element of the continuum, so that the equation

$$\mathbf{B}[\mathbf{b}] = \mathbf{q}. \quad (9)$$

is obtained. Eq. (9) is a system of nonlinear ordinary differential equations including the fractional derivative operator that can be solved in the time domain via a numerical integration scheme. Thus, the source term \mathbf{b} is calculated by integrating eq. (9) in the time domain and the system response is readily computed via eq. (8) at each time step. Next, this procedure is repeated a large number of times to derive Monte Carlo results for the continuum response statistics.

SL involves the replacement of a nonlinear differential equation by an equivalent linear system in which the system parameters are determined by minimizing a mean square error between the nonlinear equation and the linear one. Its application in conjunction with continua is based on a prior representation of the system response via Galerkin expansion. Such an expansion is used for deriving the system of nonlinear ordinary differential equations

$$D_{nl}[w] = Q, \quad (10)$$

which is successively replaced by the equivalent linear system,

$$D_{eq}[w] = Q, \quad (11)$$

whose parameters are determined by minimizing the mean square error

$$\varepsilon = E\{(D_{nl}[w] - D_{eq}[w])^2\}. \quad (12)$$

These preceding procedures which here have been outlined succinctly, will be elucidated in the next sections, by considering specific problems pertaining to nonlinear random vibrations of a beam and of a plate.

3. Nonlinear vibration of a beam

The equation governing the large deflection $v(x,t)$ of a vibrating beam of length L is

$$EI \frac{\partial^4 v(x,t)}{\partial x^4} + \rho A \frac{\partial^2 v(x,t)}{\partial t^2} + c_0 \partial_t^\alpha v(x,t) - N \frac{\partial^2 v(x,t)}{\partial x^2} = q(x,t), \quad (13)$$

where the symbols E , I , ρ , and A denote elastic modulus, moment of inertia of the cross-section, mass density, and cross-sectional area, respectively. Further, c and α are parameters of a fractional derivative element; $q(x,t)$ is a stochastic load, and N is the axial force given by the equation

$$N = \frac{EA}{2L} \int_0^L \left(\frac{\partial v(x,t)}{\partial x} \right)^2 dx. \quad (14)$$

The load is assumed of the separable form $q(x,t) = p(x)f(t)$, where $p(x)$ is a deterministic function, and $f(t)$ is a random Gaussian process of given power spectral density function $S(\omega)$.

3.1. BEM solution

The BEM solution is sought by utilizing the equations

$$\frac{\partial^4 v(x,t)}{\partial x^4} = b(x,t). \quad (15)$$

In this context, the response representation is

$$v = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \int_0^L G(x,\xi) b(\xi,t) d\xi, \quad (16)$$

where $G = |x-\xi|(x-\xi)^2/12$ and the constants c_i ($i=0,\dots,3$) are dependent on the specific boundary conditions.

By discretizing the beam into N elements, assuming that the source term $b(\xi,t)$ is constant over each element, and collocating eq. (13) in each nodal point, the set of equations

$$\rho A \underline{\underline{G}} \cdot \ddot{\underline{b}}(t) + c \underline{\underline{G}} \cdot D_t^\alpha \underline{b}(t) + EI \underline{b}(t) - \frac{EA}{2L} \underline{F}(\underline{b}(t), \underline{\underline{G}}) = \underline{q}(t), \quad (17)$$

is obtained, where $\underline{b}(t)$ is a vector containing the nodal values of the fictitious load, $\underline{\underline{G}}$ is a known matrix arising from the integration of $G(x,\xi)$ on the elements, $\underline{F}(\underline{b}(t), \underline{\underline{G}})$ is a nonlinear vector function and $\underline{q}(t)$ is a vector containing the load time histories.

Eq. (17) is integrated numerically. For this purpose, the fractional derivative element is approximated by the representation (4). In this sense, the known past values of $\underline{b}(t)$ are used for estimating an effecting excitation accounting for the influence of the past terms. This approach allows deriving the incremental equation of motion

$$\rho A \underline{\underline{G}} \cdot \Delta \ddot{\underline{b}}(t_i) + c \Delta t^{-\alpha} \underline{\underline{G}} \cdot \Delta \underline{b}(t_i) + EI \Delta \underline{b}(t_i) - \frac{EA}{2L} \Delta \underline{F}(t_i) = \Delta \underline{q}(t_i) - c \Delta t^{-\alpha} \underline{\underline{G}} \cdot \left(\sum_{k=1}^{i-1} GL_k \Delta \underline{b}(t_i - k\Delta t) + GL_i \underline{b}(0) \right), \quad (18)$$

which is implemented by a classical Newmark scheme. From a computational perspective, it is worth - mentioning that a truncation can be introduced by neglecting the small contributions provided by past terms associated with k larger than a certain threshold.

3.2. Statistical linearization solution

The vertical displacement of the beam is represented by the equation,

$$v(x, t) = \sum_{m=1}^{\infty} w_m(t) \Phi_m(x), \quad (19)$$

where $\Phi_m(x)$ are the orthogonal linear modes of beam vibration, and $w_m(t)$ are time-dependent functions. By introducing the quantities

$$K_{mn} = K_{nm} = \int_0^L \Phi'_m \Phi'_n dx, \text{ and } R_{mn} = \int_0^L \Phi_m \Phi_n'' dx, \quad (20)$$

and by algebraic manipulations reflecting error projection on the space of the beam modes, a nonlinear ordinary differential equation describing the time variation of $w_m(t)$ is derived. Specifically:

$$\ddot{w}_m + \frac{c}{\rho A} D_t^\alpha w_m + \omega_m^2 w_m - \frac{E}{2\rho L^2} \sum_n \sum_i \sum_j w_n w_i w_j K_{ij} R_{mn} = \frac{P_m}{\rho A L} f(t), \text{ for } m = 1, 2, \dots, \quad (21)$$

where ω_m is the natural frequency of vibration and

$$P_m = \int_0^L p(x) \Phi_m(x) dx. \quad (22)$$

Next, an approximate solution of eq. (21) is sought by replacing it by the system,

$$\ddot{w}_m + \frac{c}{\rho A} D_t^\alpha w_m + \omega_{eq,m}^2 w_m = \frac{P_m}{\rho A L} f(t), \text{ for } m = 1, 2, \dots, \quad (23)$$

in which $\omega_{eq,m}$ is an optimal equivalent stiffness. In this context, the procedure for determining the equivalent linear stiffness leads to the equation

$$\omega_{eq,m}^2 = \omega_m^2 - \frac{E}{2\rho^3 A^2 L^4} \frac{1}{P_m S_{mm}} \sum_n \sum_i \sum_j K_{ij} R_{mn} P_n P_i P_j (S_{mn} S_{ij} + S_{mi} S_{nj} + S_{mj} S_{ni}), \text{ for } m=1, 2, \dots \quad (24)$$

where

$$S_{mn} = \int_{-\infty}^{+\infty} H_m(\omega) S(\omega) H_n(-\omega) d\omega, \quad (25)$$

and $H_m(\omega)$ is the system transfer function

$$H_m(\omega) = [-\omega^2 + c(i\omega)^\alpha / (\rho A) + \omega_{eq,m}^2]^{-1}. \quad (26)$$

collocated on boundary points. By doing so, the linear system of equations

$$\begin{bmatrix} [A_{11}] & [A_{12}] & [0] & [A_{14}] \\ [A_{21}] & [A_{22}] & [A_{23}] & [0] \\ [A_{31}] & [A_{32}] & [A_{33}] & [A_{34}] \\ [0] & [0] & [A_{43}] & [A_{44}] \end{bmatrix} \begin{Bmatrix} \{u\} \\ \{u_n\} \\ \{\nabla^2 u\} \\ \{\nabla^2 u\}_n \end{Bmatrix} = \begin{Bmatrix} \{B_1\} \\ \{B_2\} \\ \{0\} \\ \{0\} \end{Bmatrix} + \begin{bmatrix} 0 \\ 0 \\ [C_3] \\ [C_4] \end{bmatrix} [b_1]. \quad (33)$$

is derived. The sub-matrices composing the first two rows are determined by the boundary conditions. System (33) allows determining the boundary quantities in terms of the fictitious load b_1 . Thus, the response of the plate is calculated by the equation

$$\underline{u} = \underline{G}_1 \underline{b}_1, \quad (34)$$

where \underline{G}_1 is a known matrix, \underline{b}_1 is a vector containing the values of the fictitious load at each point of the domain and \underline{u} is a vector containing the response at that points. A similar procedure can be used for representing the stress function:

$$\underline{\phi} = \underline{G}_2 \underline{b}_2, \quad (35)$$

where it is observed that the only difference with the determination of \underline{u} relates to the different boundary conditions. The representation obtained in this manner is used for collocating the displacements and stress values into the original equations (28) - (29) in order to derive a set of fractional nonlinear ordinary differential equations for the fictitious loads \underline{b}_1 and \underline{b}_2 . That is,

$$\rho h \underline{G}_{=1} \ddot{\underline{b}}_1 + c \underline{G}_{=1} D_t^\alpha \underline{b}_1 + D \underline{b}_1 - \underline{F}_1(G_{=1,xx}, G_{=2,xx}, G_{=1,yy}, G_{=2,yy}, G_{=1,xy}, G_{=2,xy}) = \underline{q}, \text{ and } \underline{b}_2 = \underline{E} \underline{F}_2(G_{=1,xx}, G_{=1,yy}, G_{=1,xy}), \quad (36)$$

where \underline{F}_1 and \underline{F}_2 are nonlinear functions encapsulating the nonlinear elements of the original system.

The numerical solution of this fractional differential equation is obtained by a Newmark based algorithm by utilizing the same approach described for the beam vibration problem. The resulting incremental equation of motion is

$$\rho h \underline{G}_{=1} \cdot \Delta \ddot{\underline{b}}_1(t_i) + c \Delta t^{-\alpha} \underline{G}_{=1} \cdot \Delta \underline{b}_1(t_i) + D \Delta \underline{b}_1(t_i) - \Delta \underline{F}_1(t_i) = \Delta \underline{q}(t_i) - c \Delta t^{-\alpha} \underline{G}_{=1} \cdot \left(\sum_{k=1}^{i-1} GL_k \Delta \underline{b}_1(t_i - k \Delta t) + GL_i \underline{b}_1(0) \right). \quad (37)$$

4.2. Statistical linearization solution

The response of the system is represented by Galerkin expansions of the vertical displacement and of the stress function having time-dependent coefficients. Specifically,

$$u = \sum_{m,n} w_{mn}(t) U_{mn}(x, y), \text{ and } \phi = \frac{P_x y^2}{2bh} + \frac{P_y x^2}{2ah} + \sum_{m,n} w_{mn}^{(2)}(t) \varphi_{mn}(x, y), \quad (38)$$

with P_x and P_y being the total tension loads applied on the sides $x=0, a$ and $y=0, b$ of the plate, respectively, the eigen-functions U_{mn} and φ_{mn} depend upon the specific boundary conditions, and are orthogonal to each other. Further, the quantities

$$R_{xx}(M, N, m, n) = \iint_A \frac{\partial^2 U_{mn}}{\partial x^2} U_{MN} dA \text{ and } R_{yy}(M, N, m, n) = \iint_A \frac{\partial^2 U_{mn}}{\partial y^2} U_{MN} dA. \tag{39}$$

are introduced. By substituting eq. (38) into eq. (28) and (29), doing algebraic manipulations reflecting error projection in the space of eigen-functions, and observing that the stress function amplitudes $w^{(2)}_{mn}$ can be expressed in terms of w_{mn} , a nonlinear fractional ordinary differential equation for the time-dependent amplitudes w_{mn} is found. Specifically,

$$\begin{aligned} \ddot{w}_{MN} + \frac{c}{\rho h} D_t^\alpha w_{MN} + \omega_{MN}^2 w_{MN} - \frac{4}{ab\rho h} \left(\frac{P_x}{b} \sum_{m,n} w_{mn} R_{xx}(M, N, m, n) + \frac{P_y}{a} \sum_{m,n} w_{mn} R_{yy}(M, N, m, n) \right) \\ - \frac{4}{ab} \frac{E}{\rho} \sum_{m,n} \sum_{k,l} \sum_{p,q} w_{mn} w_{kl} w_{pq} I(M, N, m, n, k, l, p, q) = \frac{4}{ab\rho h} P_{MN} f(t), \text{ for } M, N = 1, 2, \dots \end{aligned} \tag{40}$$

where

$$\begin{aligned} I(M, N, m, n, k, l, p, q) = \\ = \sum_{i,j} \iint_A \frac{\partial^2 U_{mn}}{\partial x^2} \frac{\partial^2 \varphi_{ij}}{\partial y^2} U_{MN} + \frac{\partial^2 U_{mn}}{\partial y^2} \frac{\partial^2 \varphi_{ij}}{\partial x^2} U_{MN} - 2 \frac{\partial^2 U_{mn}}{\partial x \partial y} \frac{\partial^2 \varphi_{ij}}{\partial x \partial y} U_{MN} dA \\ \times \frac{\iint_A \frac{\partial^2 U_{kl}}{\partial x \partial y} \frac{\partial^2 U_{pq}}{\partial x \partial y} \varphi_{ij} - \frac{\partial^2 U_{kl}}{\partial x^2} \frac{\partial^2 U_{pq}}{\partial y^2} \varphi_{ij} dA}{\iint_A \frac{\partial^4 \varphi_{ij}}{\partial x^2 \partial y^2} \varphi_{ij} + \frac{\partial^2 \varphi_{ij}}{\partial x^2 \partial y^2} \varphi_{ij} + 2 \frac{\partial^2 \varphi_{ij}}{\partial x \partial y} \varphi_{ij} dA}, \end{aligned} \tag{41}$$

$$P_{MN} = \iint_A p(x, y) U_{MN} dA, \tag{42}$$

and ω_{MN} denotes the natural frequency of the linear plate. An approximate solution of eq. (40) is sought by replacing this nonlinear system by the equivalent linear system

$$\ddot{w}_{MN} + c/(\rho h) D_t^\alpha w_{MN} + \omega_{eq,MN}^2 w_{MN} = 4P_{MN} f(t)/(\rho h a b), \text{ for } M, N = 1, 2, \dots \tag{43}$$

In this context, the equivalent stiffness is determined using the equation

$$\begin{aligned} \omega_{eq,MN}^2 = \omega_{MN}^2 - \frac{4}{ab\rho h} \frac{1}{P_{MN} S_{MN,MN}} \sum_{m,n} P_{mn} S_{MN,mn} \left[\frac{P_x}{b} R_{xx}(M, N, m, n) + \frac{P_y}{a} R_{yy}(M, N, m, n) \right] \\ - \left(\frac{4}{ab} \right)^3 \frac{E}{\rho^3 h^2} \frac{1}{P_{MN} S_{MN,MN}} \sum_{m,n} \sum_{k,l} \sum_{p,q} P_{mn} P_{kl} P_{pq} (S_{MN,mn} S_{kl,pq} + S_{MN,kl} S_{mn,pq} + S_{MN,pq} S_{mn,kl}) \\ \times I(M, N, m, n, k, l, p, q); \text{ for } M, N = 1, 2, \dots \end{aligned} \tag{44}$$

where

$$S_{MN,mn} = \int_{-\infty}^{+\infty} H_{MN}(-\omega) S(\omega) H_{mn}(\omega) d\omega, \tag{45}$$

and $H_{MN}(\omega)$ is the system transfer function

$$H_{MN}(\omega) = \frac{1}{-\omega^2 + c(i\omega)^\alpha / (\rho h) + \omega_{eq,MN}^2} \tag{46}$$

Next, the standard deviation of the transverse displacement can be determined by the equation

$$\sigma^2(x, y) \approx u^2(x, y) \approx \left(\frac{4}{ab\rho h} \right)^2 \sum_{m,n} \sum_{k,l} P_{mn} P_{kl} U_{mn} U_{kl} S_{mn,kl} \tag{47}$$

5. Concluding remarks

The random response of nonlinear continua endowed with fractional elements has been determined approximately by Boundary Element Method (BEM) based Monte Carlo simulations, and by Statistical Linearization (SL). The techniques have been described in conjunction with two structural dynamic problems: a nonlinear beam vibration and a nonlinear plate vibration. Figure 1 and 2 show typical representations of the response statistics obtained for a simply supported and for a rectangular plate. It is seen that SL is not only an efficient but also a reliable technique for calculating the response statistics.

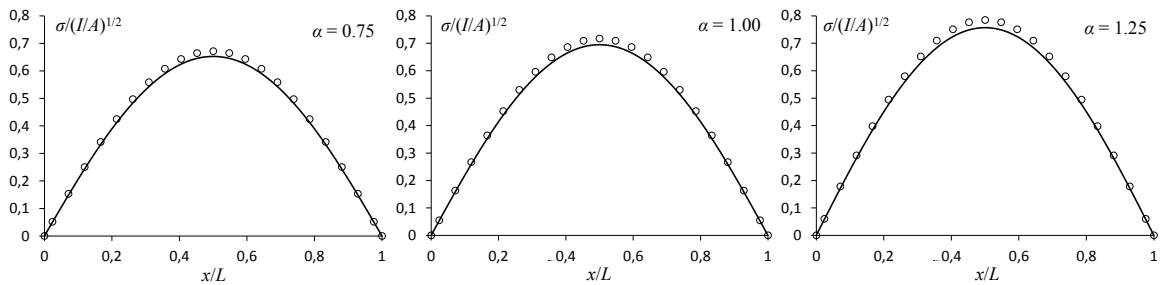


Fig. 1. Standard deviation of the vertical displacement of a simply supported beam for various fractional derivative orders. Continuous lines: SL; dotted lines: BEM.

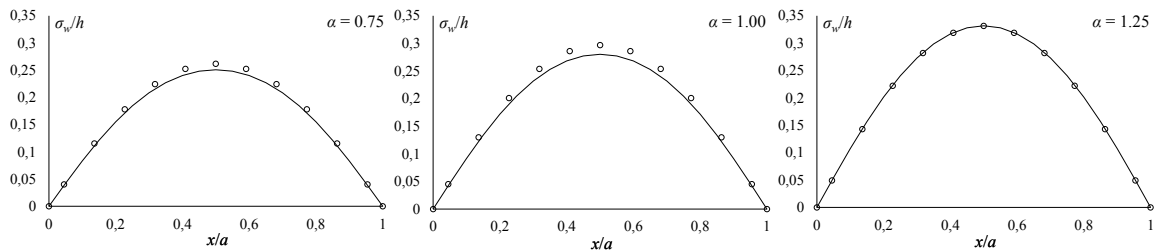


Fig. 2. Standard deviation of the vertical displacement of a rectangular plate along the section $y = b/2$ for various fractional derivative orders. Continuous lines: SL; dotted lines: BEM.

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