Mathematical Properties of Variational Subdivision Schemes
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Abstract

Subdivision schemes for variational splines were introduced in the paper [WW98]. This technical report focusses on discussing the mathematical properties of these subdivision schemes in more detail. Please read the original paper before reading this analysis.

1 Introduction and Review

Variational subdivision schemes were defined by a sequence of subdivision matrices S_k which have to satisfy the fundamental relation

$$E_{k+1} S_k = U_k E_k \tag{1}$$

where E_k is the energy matrix derived from the finite element basis functions and U_k describes upsampling of coefficients from a coarse to a finer grid by inserting zeros for an new coefficients.

As the original paper demonstrated, the subdivision matrices S_k can be derived from the energy matrices E_k as a matter of linear algebra. In practice, one would like to answer several important questions concerning these subdivision schemes. Do they always converge to a solution? If so, is that solution guaranteed to be a minimizer of the associated variational problem? In this section, we analyze these questions.

2 Convergence of the Scheme

A sequence of functions $F_0(t)$, $F_1(t)$, $F_2(t)$, ... *uniformly converges* to a function $F(t)$ if

$$\lim_{k\to\infty} \|F_k(t) - F(t)\|_\infty = 0.$$

(Here, the infinity norm of a function is the maximum of its absolute value over Ω.) If $F_k(t) = p_k B_k(t)$, then the subdivision scheme defined by the matrices S_k is *uniformly convergent* if for any set of bounded initial coefficient vector p_0, the sequence of functions $F_0(t), F_1(t), F_2(t), \ldots$ uniformly converges where $p_{k+1} = S_k p_k$. Uniform convergence necessarily implies that the limit function $F(t)$ is continuous.

The key question is: Given the energy matrices E_k, does there exists a sequence of subdivision matrices S_k satisfying equation 1 that define a uniformly convergent subdivision scheme. We believe the following to be true.
Hypothesis: Given a continuous energy functional $\mathcal{E}$ of order m, there exists a sequence of energy matrices E_k and an associated sequence of subdivision matrices S_k defining a uniformly convergent subdivision scheme if and only if $2m > d$ where d is the dimension of Ω.

There are several pieces of evidence to support this belief. For example, if $2m > d$, then the space of functions for which the energy functional $\mathcal{E}$ is defined, $H_m(\Omega)$, is a subset of the space of continuous functions on Ω. (See the Sobolev embedding theorem [OR76, pp. 79-82].) Since the subdivision matrices S_k reflect the solution process for the variational problem, the subdivision should naturally converge to continuous functions. Conversely, if $2m \leq d$, then $H_m(\Omega)$ can contain discontinuous or unbounded functions. Any subdivision scheme for such a space of functions is necessarily not convergent.

For example, Laplace's equation is order one and of dimension two. Therefore, its corresponding solution space $H_1(\mathcal{R}^2)$ contains discontinuous functions. In particular, $F(t)$ may have discontinuous spikes at knots in T_0. Close analysis of the subdivision scheme given in the preceding section shows that the scheme diverges very slowly to produce spikes of infinite height at the knots of T_0. Normalizing these spikes to interpolate after each step of subdivision produces a sequence of narrower and narrower spikes that converge to the desired solution.

If the S_k define a uniformly convergent subdivision scheme, then the associated analysis is much more straightforward. If $F(t)$ is the limit function associated with the sequence of solution vectors $p_0, p_1, p_2, \ldots$, then there exists a set of basis functions $N_k(t)$ satisfying $F(t) = p_k N_k(t)$. Since $p_{k+1} = S_k p_k$, these basis functions are related to the subdivision matrices by:

$$N_k(t) = N_{k+1}(t)S_k. \tag{2}$$

If p_k is an arbitrary coefficient vector associated with the knot set T_k, then the limit function associated with p_k is

$$P_k(t) = p_k N_k(t).$$

Note that the basis functions $N_k(t)$ associated with a convergent subdivision scheme are not interpolating. Evaluating the functions $P_k(t)$ at knots T_k yields an *interpolation* matrix I_k satisfying

$$P_k(T_k) = I_k p_k.$$

For example, if p_0 is chosen such that $I_0 p_0 = P(T_0)$, then $P_0(t)$ and $P(t)$ must agree on T_0. This choice of p_0 forces the final limit function to satisfy the interpolation conditions.

By equation 2, an alternative way of computing the values of $N_k(t)$ at T_k is to subdivide the basis functions using S_k, compute the values of the subdivided basis functions at T_{k+1} and downsample using U_k^T. This observation yields the matrix relation:

$$U_k^T I_{k+1} S_k = I_k. \tag{3}$$

3 The energy matrix for the scheme

Given E_k and I_k, the energy matrix for a convergent subdivision scheme is particularly simple. If $P_k(t) = p_k N_k(t)$, then we claim that the energy function for $P_k(t)$ satisfies

$$\mathcal{E}[P_k] = p_k^T (E_k I_k) p_k.$$

Before proving this fact, we note the following matrix relation. Take the transpose of both sides of equation 1 and multiply by $I_{k+1} S_k$,

$$\begin{aligned} S_k^T E_{k+1} I_{k+1} S_k &= E_k U_k^T I_{k+1} S_k, \\ &= E_k I_k. \end{aligned} \tag{4}$$

Applying equation 3 to the right-hand side of the first equation yields equation 4.

Theorem 1 *Let the matrices S_k define a uniformly convergent subdivision scheme. Given a function $P_j(t)$ of the form $p_j N_j(t)$, then the energy of this function satisfies*

$$\mathcal{E}[P_j] = p_j^T(E_j I_j)p_j.$$

Proof: Given the initial vector p_j, let subsequent vectors p_k be defined by the subdivision process:

$$p_k = S_k S_{k-1} ... S_j p_j.$$

for $k \geq j$. If the $F_k(t)$ are the approximate solution produced by the finite elements, $p_k B_k(t)$, then these functions uniformly converge to $F(t) = P_j(t)$ by hypothesis. Since this convergence is uniform, their corresponding energies are also convergent.

$$\mathcal{E}[P_j] = \lim_{k\to\infty} \mathcal{E}[F_k] = \lim_{k\to\infty} p_k^T E_k p_k,$$

Subtracting $p_j^T E_j I_j p_j$ from both sides of this equation yields

$$\mathcal{E}[P_j] - p_j^T E_j I_j p_j = (\lim_{k\to\infty} p_k^T E_k p_k) - p_j^T E_j I_j p_j.$$

By equation 4,

$$p_j^T E_j I_j p_j = p_k^T E_k I_k p_k$$

for all $k \geq j$. Pushing $p_j^T E_j I_j p_j$ inside the limit yields

$$\mathcal{E}[P_j] - p_j^T E_j I_j p_j = \lim_{k\to\infty} p_k^T E_k (p_k - I_k p_k). \tag{5}$$

We conclude by showing that righthand side of this equation converges to zero.

By the construction, $I_k p_k$, are the values of the limit function sampled at the knots T_k. Due to uniform convergence of the subdivision scheme, the coefficient vectors p_k uniformly converge to the values of the limit function sampled at T_k. Therefore, $||p_k - I_k p_k||_\infty$ also uniformly converges to zero. Since $||p_k||_\infty$ is bounded, the righthand side of equation 5 converges to zero. Therefore, the theorem holds. □

This theorem can be extend to include the continuous inner product used in defining $\mathcal{E}$. In particular, the ijth entry of $E_k I_k$ is the inner product of the ith and jth basis function in $N_k(t)$. This extension allow a simple characterization of those functions that minimize $\mathcal{E}$.

4 Minimization of the energy functional

Let V_k denote the span of the basis functions $N_k(t)$ defined by the subdivision scheme. Since the energy functional $\mathcal{E}$ is defined for the basis function $N_k(t)$, $V_k \subset H_m(\Omega)$. Due to their definition through subdivision, the V_k are nested,

$$V_0 \subset ... \subset V_k \subset V_{k+1} \subset ... \subset H_m(\Omega).$$

We claim that V_0 is exactly the space of minimizers of $\mathcal{E}$ over the knots T_0.

To show this fact, we construct a multi-resolution expansion in terms of the V_k for a function $P(t)$ in $H_m(\Omega)$. Define the complimentary spaces W_k satisfying

$$W_k = span\{R_k(t) \in V_{k+1} | R_k(T_k) = 0\}.$$

W_k consists of those functions in V_{k+1} that vanish at the knots T_k of the coarser space V_k. A function $P(t) \in V_{k+1}$ can be written as a combination of a function P_k in V_k and a residual function R_k in W_k such that

$$\begin{aligned} P_k(T_k) &= P(T_k), \\ R_k(T_k) &= 0. \end{aligned}$$

Therefore, the space V_{k+1} can be written as the sum of the spaces V_k and W_k,

$$V_{k+1} = V_k + W_k.$$

The beauty of this particular multi-resolution expansion is that the spaces V_i and W_i are orthogonal with respect to inner product associated with $\mathcal{E}$,

$$\langle F, G \rangle = \int_{\Omega} \sum_i c_i (\mathcal{D}_i F(t))(\mathcal{D}_i G(t)) dt.$$

Theorem 2 *If $P_k \in V_k$ and $R_k \in W_k$, then*

$$\langle P_k, R_k \rangle = 0.$$

Proof: Since $P_k(t)$ is in V_k, $P_k(t)$ can be written as $p_k N_k(t)$. Subdividing once, $P_k(t)$ can also be expressed as $(S_k p_k) N_{k+1}(t)$. Since $R_k(t) \in W_k \in V_{k+1}$, $R_k(t)$ can be written as $r_k N_{k+1}(t)$. As noted in the previous section, the continuous inner product satisfies:

$$\langle P_k, R_k \rangle = p_k^T S_k^T E_{k+1} I_{k+1} r_k.$$

Taking the transpose of equation 1 and multiplying both sides by I_{k+1} yields

$$S_k^T E_{k+1} I_{k+1} = E_k U_k^T I_{k+1}.$$

Substitution in the previous equation yields that

$$\langle P_k, R_k \rangle = p_k^T E_k U_k^T I_{k+1} r_k.$$

Since $R_k(t)$ is in W_k, $R_k(t)$ vanishes on T_k. An equivalent interpretation is that sampling $R_k(t)$ on T_{k+1} and downsampling to T_k also yields zero. In matrix terms, this condition is $U_k^T I_{k+1} r_k = 0$. Therefore, the inner product above is zero. □

Consider the function $P(t)$ written as the infinite expansion

$$P(t) = P_0(t) + \sum_{i=0}^{\infty} R_i(t), \tag{6}$$

where $P_0(t) \in V_0$ and $R_i(t) \in W_i$. Due to the bi-linearity of the inner product, the energy of $P(t)$ satisfies

$$\begin{aligned} \mathcal{E}[P] &= \langle P, P \rangle, \\ &= \langle P_0 + \sum_{i=0}^{\infty} R_i, P_0 + \sum_{i=0}^{\infty} R_i \rangle, \\ &= \langle P_0, P_0 \rangle + 2 \sum_i \langle P_0, R_i \rangle + \sum_i \sum_j \langle R_i, R_j \rangle. \end{aligned}$$

By theorem 2, the inner product of P_0 and R_i is zero. Likewise, the inner product of R_i and R_j is zero when $i \neq j$. Therefore,

$$\begin{aligned} \mathcal{E}[P] &= \langle P_0, P_0 \rangle + \sum_i \langle R_i, R_i \rangle, \\ &= \mathcal{E}[P_0] + \sum_i \mathcal{E}[R_i]. \end{aligned}$$

Based on this equation, it is clear that the minimum energy function that agrees with $P(t)$ on T_0 is simply $P_0(t)$, i.e. the function defined by the subdivision scheme.

In fact, this observation allows fast reconstruction of the minimum energy function that agrees with $P(t)$ on T_k. This function $P_k(t)$ is simply the truncation of the infinite expansions of $P(t)$ at level k,

$$P_k(t) = P_0(t) + \sum_{i=0}^{k-1} R_i(t).$$

Since $R_i(T_k) = 0$ for $i \geq k$, $P_k(t)$ agrees with $P(t)$ on T_k. We intend to explore the multi-resolution aspect of this idea more completely in a future paper.

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References

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