Two Variants on the Plateau Problem

by

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Abstract

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The Plateau problem in $\mathbb{R}^3$ begins with a given simple, closed curve $\gamma$, and asks to find a surface $M$ with $\partial M = \gamma$ that minimizes area among all surfaces with $\gamma$ as their boundary. In 1960 Federer and Fleming generalized this idea and analyzed the currents developed by De Rham. They proved certain subclasses of currents (in particular, the integral currents) can be used as a powerful tool in area and volume minimization problems. In this thesis, we approach two generalizations—first, we prove a Homological Plateau problem in the singular setting of semi-algebraic geometry using the tools of geometric measure theory. We obtain similar results to those of Federer and Fleming even in this more singular case.

Second, an earlier solution to the Plateau problem was achieved independently by Douglas and Rado in 1931 and 1933, respectively, using mappings from the two-dimensional disk. In the second chapter we generalize this mapping to a so-called “multiple-valued” mapping of the disk. Multiple-valued maps are a cornerstone of the regularity theorems of F. Almgren and are interesting in their own right for many problems in the geometric calculus of variations. We prove existence and regularity for these Plateau solutions under fairly general conditions. We also produce a class of examples and analyze a degenerate case.
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Chapter 1

Introduction To Mass Minimization Problems

This chapter gives a broad outline to Plateau problems and the framework of geometric measure theory—where the vast majority of our results will be stated and proven.

1.1 The Classical Plateau Problem

Much time is spent in introductory calculus learning how to minimize ‘nice’ functions. We teach our students derivative tests and Lagrange multipliers and other assorted conditions for minimizers. The calculus of variations is an extension of these ideas to infinite dimensional spaces. The most well-known and simplest example of a geometric variational problem is the geodesic problem: between any two points in a Riemannian manifold, find a curve of least length connecting them. Indeed, even
this one-dimensional case has led to a huge amount of mathematics, being one of the foundational concepts of differential geometry.

Generalizing by one dimension, the Plateau problem in $\mathbb{R}^3$ begins with a given simple, closed curve $\Gamma$, and asks to find a surface $M$ that minimizes area among all surfaces with $\Gamma$ as their boundary. As in our calculus lectures, necessary conditions are much easier to come across than sufficient ones. In this case, one can find a necessary condition by considering deformations of the surface smoothly away from the boundary—any such ‘nearby’ surface must have greater area. This concept gives the celebrated Minimal Surface Equation, which leads to the deep and rich theory of minimal surfaces (for an overview, see [CM11]).

In the early 1930’s, this problem was solved independently by J. Douglas [Dou31] and T. Rado [Rad30]. This seminal work earned J. Douglas a Fields medal, and the final results, including both the existence and regularity, are stated below. In the following, $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$, and $W^{1,2}(D, \mathbb{R}^3)$ denotes the standard Sobolev space of those square integrable functions once weakly differentiable with square integrable weak derivatives.

**Theorem 1.1.1.** Given a piecewise $C^1$ simple closed curve $\Gamma \subset \mathbb{R}^3$, there exists a map $u : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ so that:

1. $u : \partial D \rightarrow \Gamma$ is a homeomorphism.

2. $u \in C^0(D) \cap W^{1,2}(\mathbb{D}, \mathbb{R}^3) \cap C^\infty(D)$. Further, $u$ is harmonic and conformal on $D$.

3. The image of $u$ minimizes area among all maps from disks with boundary $\Gamma$. 
We outline the proof here, since the ideas are standard to many arguments in the calculus of variations, and have analogues throughout the thesis. The main ideas are taken from Chapter 4 of [CM11].

First, one computes that, for a map \( f : \mathbb{D} \to \mathbb{R}^3 \), the area of the image, possibly with multiplicity, is given by:

\[
\text{Area}(f) = \int_{\mathbb{D}} (|f_x|^2|f_y|^2 - \langle f_x, f_y \rangle^2)^{\frac{1}{2}} \, dx \, dy
\]

however, simple examples of hairy (or tentacled) disks show that even minimizing sequences for this functional need not behave well—for a simple picture, imagine a map from the disk that is the identity onto \( \mathbb{D} \times \{0\} \) except for a thin, tall spire over the origin. As the spire 'thins' out, the area of this map converges to the infimum of all possible for its given boundary data, but, there is no convergence of the individual maps.

The major step in this classical work is realizing that a more useful functional to be minimized is the so-called Dirichlet Energy of the map, defined by:

\[
\text{Dir}(f) = \frac{1}{2} \int_{\mathbb{D}} |f_x|^2 + |f_y|^2 \, dx \, dy
\]

It is easy to check that:

\[
\text{Area}(f) \leq \text{Dir}(f) \quad \text{with equality if and only if} \quad f \quad \text{is conformal.}
\]

To find a Dirichlet Energy minimizing map, we define the \textit{admissible functions} as follows:
\[ \mathcal{A} := \{ f \in W^{1,2}(\mathbb{D}, \mathbb{R}^3) \mid \text{the trace } f|_{\partial\mathbb{D}} \text{ is monotone and continuous} \} \]

Next, one checks that, for a given curve \( \Gamma \), the infima of the functionals \( \text{Dir} \) and \( \text{Area} \) on all maps which monotonically map \( \partial\mathbb{D} \) onto \( \Gamma \) are equal.

Further, using the Poincaré inequality and the Banach-Alaoglu Theorem, one checks that if \( f_n \) is a sequence of functions with \( \sup \text{Dir}(f_n) < \infty \), that there exists a sequence which converges in an appropriate weak sense to a \( W^{1,2} \) function \( f \).

However, the major subtlety remaining is the boundary data. It is entirely possible that, if we choose a sequence of functions \( f_n \in \mathcal{A} \) so that:

\[ \text{Dir}(f_n) \to \inf_{\mathcal{A}} \text{Dir}(g), \]

then the ‘boundary data’, or the functions \( f_n|_{\partial\mathbb{D}} \to \Gamma \) could conceivably degenerate in the limit, and simply be a constant map.

In two dimensions, though, we are saved by a particularly technical fact (known as the Courant-Lebesgue Lemma) and a clever trick.

**Lemma 1.1.2** (Courant-Lebesgue Lemma). If \( u \in A \) has \( \text{Dir}(u) \leq \frac{K}{2} \), then, for any \( \delta < 1 \), there exists \( \rho \in [\delta, \sqrt{\delta}] \) so that, for any \( p \in \mathbb{D} \) we have:

\[ \text{Diam} \left( u(\{ q \in \mathbb{D} \mid |p - q| = \rho \}) \right)^2 < \frac{8\pi^2K}{-\log(\delta)} \]  

(1.1.1)

where, for \( A \) a subset of Euclidean space, \( \text{Diam}(A) = \sup_{x,y \in X} |x - y| \).

This Lemma, combined with the following two observations then allow us to guarantee that the boundary data does not collapse:
Remark 1.1.1. 1. If $\phi : \mathbb{D} \to \mathbb{D}$ is a conformal diffeomorphism and $u \in \mathcal{A}$, then $u \circ \phi \in \mathcal{A}$ and, further, $\text{Dir}(u \circ \phi) = \text{Dir}(u)$.

2. Given any six points $\alpha_i, \beta_i \in \partial \mathbb{D}, i = 1, 2, 3$, there exists a conformal diffeomorphism $\phi$ so that $\phi(\alpha_i) = \beta_i$.

Then, after ‘normalizing’ the boundary data of each $f_n$ by precomposing with a conformal diffeomorphism $\phi_n$ to ensure that three chosen points of $\partial \mathbb{D}$ map to three fixed points of $\Gamma$, some technical arguments involving the estimate in the Courant-Lebesgue Lemma allow us to prove that the new $f_n \circ \phi_n$’s converge on $\partial \mathbb{D}$, and thus the limit object $f$ is an element of $\mathcal{A}$. Further, standard Banach space theory gives the desired lower-semicontinuity of the Dirichlet energy and shows that $\text{Dir}(f) \leq \lim \inf \text{Dir}(f_n)$, giving our minimal element.

By computing the derivative of energy with respect to certain pertubations, one is able to show that $f$ must be conformal and harmonic and use this fact to prove that $f|_{\partial \mathbb{D}}$ is a homeomorphism, and that $f$ is also an area minimizer.

### 1.2 Some Geometric Measure Theory

To ask similar minimization questions in higher dimension (and co-dimension), one must often be willing to accept that, while minimal objects will still exist, finding a map as above is impossible. One set of machinery to approach this problem is that of Geometric Measure Theory (GMT). The main tool is the concept of a current, whose theory was pioneered in the 1960’s by H. Federer and W. Fleming [FF60]. Currents enable one to solve the existence problem for various area minimization problems easily. We begin with a brief introduction to this theory and to GMT in general.
For a finite dimensional vector space $V$, and $k \in \mathbb{N}$ with $k \leq \dim(V)$, let $\wedge^k V$ denote the alternating exterior algebra of $V$.

If $V$ has an inner product $\langle \cdot , \cdot \rangle_V$, then there is a unique inner product on $\wedge^k V$ so that, for any orthonormal basis $\{v_1, \ldots, v_{\dim(V)}\}$, the collection:

$$\{v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k} \mid i_1 < i_2 < \cdots < i_n\}$$

is an orthonormal basis for $\wedge^k V$. This inner product also defines a norm on $\wedge^k V$ which we will denote $|\cdot|$.

This allows us to define the comass of an element of $\phi \in (\wedge^k V)^* = \wedge^k V$ by:

$$||\phi|| = \sup\{\phi(\zeta) \mid \zeta \in \wedge^k V, \zeta \text{ is simple}, |\zeta| \leq 1\}$$

Using duality once more, we are able to define the mass of $\zeta \in \wedge^m V$ by:

$$||\zeta|| = \sup\{\phi(\zeta) \mid \phi \in \wedge^m V, ||\phi|| \leq 1\}$$

Denote by $D^k(U)$ the $k$-dimensional differential forms on $U$ whose coefficients have compact support.

Such a form may be written as a linear combination of simple forms of the type:

$$\phi dx_{\sigma(1)} \wedge \ldots \wedge dx_{\sigma(k)}$$

where $\sigma : \{1, \ldots, k\} \to \{1, \ldots, n\}$ is some increasing map. Note that for $\omega \in D^k(U)$ and $x \in U$, the form $\omega(x)$ is a functional on $k$-vectors. This yields a norm on $D^k(U)$, known as the comass, as follows:
\[ \|\omega\| = \sup_{x \in U} \{ \omega(x)(\psi) \mid \psi \text{ is a simple vector with } \|\psi\| \leq 1 \} \]

where \( \|\psi\| \) is the norm defined above for elements of \( \wedge_k \mathbb{R}^n \).

Let \( D_k(U) \) be the topological dual to \( D^k(U) \). We obtain a norm (dubbed ‘mass’) on \( D_k(U) \) via duality—\( T \in D_k(U) \), we set:

\[ M(T) = \sup \{ T(\phi) \mid \phi \in D^k(U), \|\phi\| \leq 1 \} \]

Observe that, for \( k \geq 1 \), the formula \( (\partial T)(\psi) = T(d\psi) \) for \( \psi \in D^{k-1} \) defines a current \( \partial T \in D_{k-1} \), and \( \partial^2 = 0 \) because \( d^2 = 0 \).

Of particular interest to us will be the subset of \( D_k(U) \) generated by integer linear combinations of currents defined via integration along \( k \)-dimensional rectifiable subsets of \( U \). To define these, we briefly recall the main definitions. The details are taken from [AFP00], [LY02] and [Fed69].

First, we recall definition of \( \mathcal{H}^k \), the \( k \)-dimensional Hausdorff measure, defined as an outer-measure on any subset \( A \subset \mathbb{R}^n \) by:

\[ \mathcal{H}^k(A) = \lim_{\delta \downarrow 0} \frac{\omega_k}{2^k} \inf \left\{ \sum_{i \in I} \text{diam}(A_i)^k : \text{diam}(E_i) < \delta, A \subset \bigcup_{i \in I} A_i \right\} . \]

where \( \omega_k \) is the volume of the unit ball in \( \mathbb{R}^k \). One readily checks that, due to the definition and a theorem of Carathéodory, given below, that \( \mathcal{H}^k \) defines a Borel measure on \( \mathbb{R}^n \) so that all open sets are measurable. For details, see [LY02].

**Theorem 1.2.1 (Carathéodory Criterion).** For a measure \( \mu \) on a metric space \( X \), all open sets are \( \mu \)-measurable if and only if:
\[ \mu(A \cup B) \geq \mu(A) + \mu(B) \]

whenever \( A \subset X, B \subset X \) have \( \text{dist}(A, B) > 0 \).

We may now define the rectifiable subsets of Euclidean space.

**Definition 1.2.1.** An \( H^k \)-measurable set \( R \subset \mathbb{R}^n \) is said to be countably \( \mathcal{H}^k \)-rectifiable if there are measurable subsets \( A_i \subset \mathbb{R}^k \) and Lipschitz maps \( f_i : A_i \to \mathbb{R}^n \) so that \( \mathcal{H}^k(R \setminus \cup f(A_i)) = 0 \). If, moreover, \( \mathcal{H}^k(R) < \infty \), we say that \( R \) is \( \mathcal{H}^k \) rectifiable.

Classical results in geometric measure theory guarantee (see, for example [AFP00]) that any \( H^k \)-rectifiable set \( R \) admits a version of a “weak tangent plane” at \( H^k \) almost every point \( x \in R \)–that is, a \( k \)-plane \( T_x R \) so that the blowups:

\[ \frac{R - x}{r} \]

tend towards \( T_x R \) as \( r \to 0 \) weakly as Radon measures. This result follows essentially from Rademacher’s theorem on the almost everywhere differentiability of the Lipschitz functions. Let \( \nu_R : R \to \wedge^k(\mathbb{R}^n) \) be an \( H^k \)-measurable function which, at each point \( x \in R \) where \( T_x R \) exists gives a unit simple \( k \)-vector spanning \( T_x R \). Thus, for an open set \( U \subset \mathbb{R}^n, R \subset U \) an \( H^k \)-rectifiable set and a measurable orientation \( \nu_R \), we may define a \( k \) current \( [R] \) which acts on \( \omega \in D^k(U) \) by:

\[ [R](w) = \int_R \omega(x)(\nu_R(x)) d\mathcal{H}^k(x). \]

Note that Stokes’s Theorem guarantees that if \( M \) is a \( C^2 \) oriented submanifold of \( \mathbb{R}^n \) with boundary \( N \), then \( N \) has an orientation induced from \( M \) so that \( [\partial M] = [N] \).
Returning to the rectifiable case and generalizing slightly, we may also define a current $T$ by including an integer weighting—that is, for an $H^k$-measurable function $\theta : R \to \mathbb{Z}$ with $\theta \in L^1(R, \mathcal{H}^k)$ we may define:

$$T = \int_R \theta(x)\omega(x)(\nu_R(x)) \, d\mathcal{H}^k(x).$$

Such a current is called rectifiable. The following is a pair of deep theorems of Federer and Fleming (originally proven in \cite{FF60}) that are tremendously helpful in analyzing the subgroup of rectifiable currents:

**Theorem 1.2.2** (Closure and Compactness Theorems). *Suppose $U \subset\subset \mathbb{R}^n$ and that $\{T_n\}_{n\in\mathbb{N}} \subset D_k(U)$ are rectifiable currents so that:

$$\sup_n \{M(T_n) + M(\partial T_n)\} < \infty$$

Then:

1. For each $n$, $\partial T_n$ is rectifiable.

2. There exists a subsequence $T_{n'}$ and a rectifiable current $T$ so that, for any differential form $\omega \in D^k(U)$:

$$T_{n'}(\omega) \to T(\omega).$$

Further, we have the lower-semicontinuity of mass with respect to this convergence:"
\[ \mathcal{M}(T) \leq \lim\inf \mathcal{M}(T_n). \]

A rectifiable current of finite mass whose boundary is of the same type is known as an integral current, and the group of all such integral currents in \( U \) is denoted by \( I_k(U) \).

Thus the integral currents (with boundary defined dually to the exterior derivative) form a chain complex. To form a homology theory, we must discuss induced homomorphisms.

The existence of \( T \) in the above theorem is a simple consequence of the Banach-Alaoglu theorem on sequential weak* compactness in a dual space–however, the deep part of the theorem is that the weakly converging subsequence converges to an integral current.

To enable us to define \( I_k(A) \) for any subset \( A \) of \( \mathbb{R}^n \), we define the notion of the support of a current:

**Definition 1.2.2.** If \( T \in D_k(U) \), then

\[
\text{spt}(T) = U \setminus \left( \bigcup_{\text{open } \phi} \{ \emptyset \mid \text{For all } \omega \in D_k(\emptyset), \text{ then } T(\omega) = 0 \} \right)
\]

Then, we may define \( I_k(A) \) to be all those integral currents supported in \( A \).

Further, one checks that, for any \( C^1 \) map \( f : U \to V \subset \mathbb{R}^m \), \( V \) open and any differential \( k \)-form \( \omega \in D^k(V) \), we may pull back \( \omega \) by evaluating, for \( x \in U \) and \( v \in \Lambda^k(\mathbb{R}^n) \):
\[ f^\sharp(\omega)(x)(v) = \omega(f(x))(\wedge^k(Df(x))(v)) \]

where we use \( \wedge^k Df(x) \) to denote the map on \( \wedge^k(\mathbb{R}^n) \) induced by the linear map \( Df(x) : \mathbb{R}^n \to \mathbb{R}^m \).

For \( T \in D_k(U) \) so that \( f|_{\text{Spt}(T)} \) is proper, we may then define the push forward \( f^\sharp(T) \) to act on a differential form \( \omega \in D^k(V) \) by:

\[ f^\sharp(T)(\omega) = T(f^*(\omega)) \]

One readily checks that if \( f \) is such that \( ||Df|| \) is bounded, then \( f^\sharp \) will send finite mass currents to finite mass currents. Further, using smooth approximation, one may define \( f^\sharp \) for any Lipschitz \( f : U \to V \), as explained in [Fed69, 4.1.7]

1.3 The Plateau Problem Viewed Through Geometric Measure Theory

The homology theory outlined in the above sections gives a way to parse ‘area’ and ‘boundary’ in a very general setting-and enables us to pose and answer the Plateau problem in higher co-dimensions without the mapping requirement easily.

In particular, we ask the following question.

**Question 1.3.1.** Let \( \Gamma \in I_k(U) \) be such that \( \partial \Gamma = 0 \). For some \( U \subset V \), does there exist a mass minimizing integral current \( S \in D_{k+1}(V) \) so that \( \partial S = \Gamma \)?
The compactness theorem above immediately answers this question affirmatively—the issue reduces to the question of whether or not there are any integral currents with boundary $\Gamma$. Fortunately, in $\mathbb{R}^n$, this sort of existence is easily established via a ‘coning’ procedure.

A more interesting question is to analyze the homology groups of sets defined by integral currents supported in those sets. Define these homology groups as follows.

$$H_k^{IC}(A, B) = \frac{\{T \in I_k(A) : \partial T \in I_{k-1}(B)\}}{\{R + \partial S : R \in I_k(B), S \in I_{k+1}(A)\}}.$$ 

Here the superscript $IC$ denotes the fact that these groups stem from arbitrary integral currents since we will later introduce other homology groups, denoted by a different subscript, defined by a subgroup of integral currents.

Recall that a subset $A \subset \mathbb{R}^n$ is said to be a Euclidean Lipschitz neighborhood retract (ELNR) if there is some open neighborhood $U$ of $A$ and a Lipschitz retraction $\varphi : U \to A$.

For pairs of compact Riemannian manifolds, or more generally ELNR’s, Federer and Fleming verified that integral currents gave homology isomorphic to the ordinary singular homology with coefficients in $\mathbb{Z}$. This isomorphism combined with an isoperimetric inequality gives (in the compact case) a mass minimizing rectifiable representative for every ordinary homology class. Further, they showed that, for a compact LNR, there is an $\epsilon > 0$ so that, if $T \in I_k(A)$ is such that $M(T) < \epsilon$ and $\partial T = 0$, then $T = \partial S$ for some $S \in I_{k+1}(A)$. For details, see [Fed69, 4.4.1, 5.1].

Obviously, their proofs are unique to their situation, but one may wonder whether
these facts hold for different classes of sets—this is the starting point for Chapter 2, where we verify these facts for the class of semi-algebraic sets.

1.4 Outline

In the first chapter, we generalize the mapping problem of Douglas and Rado to the so-called “Multiple-Valued maps” of geometric measure theory. Multiple-valued functions were first introduced by F. Almgren in [AST00] to analyze the regularity of mass-minimizing rectifiable currents in arbitrary co-dimension. Such functions have recently seen a resurgence in the works of C. De Lellis and E. N. Spadaro [DLS14] (and its continuations), [DLS11, DS13], S. Chang [Cha88], W. Zhu [Zhu08, Zhu06], C. C. Lin [Lin14], P. Bouafia [Gob06], P. Bouafia and T. De Pauw [BDPW15a], and P. Mattila [Mat83], to name just a few.

After a brief introduction to the relevant definitions, notation and previous work, we state and prove our main theorem, one analogous to the mapping theorems mentioned in the introduction, for these multiple-valued maps. We also obtain a rather strong regularity result.

In the second chapter, we introduce semi-algebraic sets and semi-algebraic chains. One can ask similar questions to those answered by Federer and Fleming for ELNR’s for algebraic varieties or, more concretely, for semi-algebraic subsets of $\mathbb{R}^n$: sets defined by finitely many polynomial equalities and inequalities (see, for example, [BCR98, § 2]). There is much interest in semi-algebraic spaces since they are an integral part of the study of real algebraic geometry and an example of an o-minimal
structure, see [BCR98] and [VdD98], respectively. Simple examples of compact semi-algebraic sets have cusps and fail to be Euclidean Lipschitz neighborhood retracts, however, they are homeomorphic to finite polyhedra so singular theory gives a natural homology theory that is uniquely characterized by the Eilenberg-Steenrod axioms.

In the second chapter, after a brief outline, we verify that, for a pair $B \subseteq A$ of semi-algebraic subsets of $\mathbb{R}^n$, the ordinary singular homology groups are isomorphic with the integral current homology groups, and use this fact to generalize two classical results concerning rectifiable currents from the ELNR case to the semi-algebraic case, namely, that for any compact semi-algebraic set, there is a mass-minimizing rectifiable representative in each of its finitely many homology classes and we are able to prove (again, similarly to the ELNR case, seen in [FF60, 9.6]) that cycles of small enough mass in a compact semi-algebraic set are boundaries.

It is convenient to use other homology groups to verify these facts. Our main result is to establish an isomorphism between the rectifiable current homology groups and the more well-behaved homology given by the much smaller subclass of semi-algebraic chains [Har75a, Har75b]: those integral currents whose support and boundary support are semi-algebraic sets. Further, semi-algebraic maps (i.e. continuous maps with semi-algebraic graphs) are suitable morphisms for semi-algebraic chains—even though they may fail to be locally Lipschitz. Substituting semi-algebraic sets and maps into the definitions above, we obtain a homology theory that satisfies the Eilenberg-Steenrod Axioms (see Section 3.2). Therefore, in the compact case, the semi-algebraic homology coincides with ordinary singular theory.
Chapter 2

A Multiple-Valued Plateau Problem

In this chapter, we generalize the mapping problem of Douglas and Rado to the so-called “Multiple-Valued maps” of geometric measure theory.

A $Q$-valued function to $R^n$ is a single-valued function taking values in the set $A_Q(R^n)$ of all unordered $Q$ tuples of points in $R^n$. Equivalently $A_Q(R^n)$ may be considered as the space of all sums $\sum_{i=1}^Q [[a_i]]$ of $Q$ Dirac measures of points in $R^n$. Thus, for $\Omega \subset R^m$, a function $f : \Omega \to A_Q(R^n)$ corresponds to a $Q$-valued function from $\Omega$ to $R^n$. We will occasionally refer to such functions as *multiple-valued functions*.

One can metrize $A_Q(R^n)$ via a standard ‘translation’ invariant metric. With this metric, Almgren showed that $A_Q(R^n)$ admits a bi-Lipschitz embedding onto a Lipschitz retract of a high dimensional Euclidean space, and used this to define Sobolev multiple-valued maps. Further, he gave a well-defined notion of differentiability and Dirichlet energy, and solved the Dirichlet problem for such functions.
In this chapter we seek to formulate and prove a result analogous to the classical Plateau problem in the setting of multiple-valued functions. Our main result is the following:

**Theorem 2.0.1.** Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_K$ be disjoint, simple closed curves of finite length in $\mathbb{R}^n$ so that $\Gamma_i \cap \Gamma_K = \phi$ for $i \neq j$ and $k_1, \ldots, k_K$ positive integers so that $\sum k_i = Q$. Further, suppose that each $\Gamma_i$ is a Lipschitz Neighborhood Retract of an open neighborhood $U_i$. Denote by $\mathbb{D}$ the open unit disk in $\mathbb{R}^2$. Then, if

$$A = \{ F \in W^{1,2}(\mathbb{D}, A_Q(\mathbb{R}^n)) \cap C^0(\bar{\mathbb{D}}, A_Q(\mathbb{R}^n)) \mid F|_{S^1}(z) = \sum_i \sum_{\zeta = z^{k_i}}([f_i(\zeta)])$$

with $f_i : S^1 \to \Gamma_i$ weakly monotonic\}

then there is a map $f \in A$ so that:

$$\text{Dir}(f) = \min_{G \in A} \text{Dir}(G).$$

Recall that a map $f_i : S^1 \to \Gamma_i$ is said to be weakly monotonic if it may be written as $\varphi_i(e^{i\tau_i(\theta)})$, where $\varphi_i : S^1 \to \Gamma_i$ is a homeomorphism, and $\tau_i : [0, 2\pi] \to \mathbb{R}$ is a nondecreasing continuous, $2\pi$ periodic function.

Unfortunately, there is a subtlety in the existence of ‘wrapped’ solutions—that is, those solutions with some $k_i > 1$. For example, if the optimal $f_i$ is constant on an arc of length greater than $\frac{2\pi}{k_i}$, there will be a branch point on the boundary circle $S^1$. Simple planar examples show that this can indeed occur—and so we must introduce a condition to guarantee such branching does not occur:
Definition 2.0.1. In the setting of Theorem 3.3.3, the boundary data \((\Gamma_1, ..., \Gamma_K, k_1, ..., k_K)\) is said to admit a wrapped solution if there exists \(f\) a minimizing element of \(\mathcal{A}\) so that:

\[
f|_{S^1}(z) = \sum_i \sum_{\zeta = z^{k_i}} [f_i(\zeta)]
\]

for \(f_i : S^1 \to \Gamma_i\) weakly monotonic so that no \(f_i\) is constant on an arc of length greater than or equal to \(\frac{2\pi}{k_i}\).

One should note that, while Theorem 3.3.3 guarantees a minimizing element of \(\mathcal{A}\) under fairly general conditions, this element can fail to satisfy the condition of Definition 2.0.1 without contradicting its admissibility.

Definition 2.0.1 is an analogous condition to the Douglas condition for planar domain Plateau solutions, as in [JDK+10, 8.6] where similar degenerations must be ruled out. Interestingly, though, in contrast to the classical case, we are able to establish existence of a solution without this condition—but regularity fails.

As in the classical case, we begin with an admissible sequence \(\{f^n\}\) whose energy tends towards the infimum and we extract a subsequence which converges (\(W^{1,2}\) weakly) to some function \(f : \mathbb{D} \to A_Q(\mathbb{R}^n)\). Then the issue is showing the admissibility of the function \(f\)–a fact which ultimately comes down to proving that the boundary data of the subsequence cannot collapse. In fact, a posteriori, a selection for the optimal boundary data will consist of homeomorphisms of the boundary; a stronger result than that \(f|_{S^1}\) is a homeomorphism onto a subset of \(A_Q(\mathbb{R}^n)\).

To prevent the boundary data from collapsing, one first precomposes with a disk
automorphism to guarantee that, if \( f^n|_{S^1} \) has boundary data \( f^n_j : S^1 \to \Gamma_j \) as in the definition of \( \mathcal{A} \), then \( f^n_j \) satisfies a uniform three point condition. We will then show that (in the case that the limit \( f \) is branched) this normalization forces a normalization for each \( f^n_j \)– on a first attempt, one might try, supposing towards a contradiction, to analyze Dirichlet minimizing functions \( G : \mathbb{D} \to A_Q(\mathbb{R}^n) \) for which \( 0 \in \text{spt}(G(s)) \) for all \( s \in S^1 \). Unfortunately, such functions can exist with rather elaborate branching behavior, as exhibited in Section 2.5, so a more complicated argument is required.

Finally, using some differential geometry and classical arguments in conjunction with Theorem 3.3.3, we are able to prove the following:

**Theorem 2.0.2.** Suppose that \( \Gamma_1, ..., \Gamma_K \) and \( k_1, ..., k_K \) are given as in Theorem 3.3.3, and further suppose that the boundary data \( (\Gamma_1, ..., \Gamma_K, k_1, ..., k_K) \) admits a wrapped solution. Then there exists \( F : \mathbb{D} \to A_Q(\mathbb{R}^m) \), an area-minimizing Plateau solution for some collection of disjoint Jordan curves \( \Gamma_1, ..., \Gamma_K \), so that if \( f_1, ..., f_K \) is the associated boundary data the following holds:

1. around each regular point there exists a conformal selection for \( F \)
2. \( \text{Dir}(F) = \text{MV-Area}(F) \) and
3. Each \( f_i \) is a homeomorphism.

Here \( \text{MV-Area}(F) \) is the two-dimensional area of the set \( \{\text{spt}(F(x)) \mid x \in \mathbb{D}\} \), counting multiplicity.
This multi-valued Plateau problem may, as described in Section 2.4, lead to minimal surfaces necessarily having branch points, even in $\mathbb{R}^3$. Here the location or number of the branch points is not fixed. Near a single prescribed branch point, a minimal surface has been studied by considering corresponding solution of a “multi-valued minimal surface equation” in [SW07], [Ros10] and [Ros11].

2.1 Preliminaries

A detailed introduction to the theory of multiple-valued functions may be found in [DLS11]. Here we present only the details essential to the current thesis. We will mostly follow the notations found therein, with the exception that our definition of “affinely approximatable” is slightly weaker than the definition of “differentiable” given therein. Throughout the chapter, $m$, $n$, and $Q$ will denote natural numbers, $\Omega$ is an open subset of $\mathbb{R}^m$ with sufficiently regular boundary, and, for $x$ an element of any metric space $X$, we will use $B_r(x)$ to denote the open ball of radius $r$ centered at $x$.

We define the metric space $A_Q(\mathbb{R}^n)$ as the set of positive integer sums of Dirac measures on $\mathbb{R}^n$ with total mass $Q$. If, for $P \in \mathbb{R}^n$, $[P]$ denotes the Dirac mass at $P$, then we may define $A_Q(\mathbb{R}^n)$ in symbols as follows:

$$A_Q(\mathbb{R}^n) = \left\{ \sum_{i=1}^{Q} [P_i] \mid P_i \in \mathbb{R}^n \right\}.$$

We note that we do not require the $P_i$'s to be distinct, and that we will occasionally omit $Q$ and $n$ when they are clear from the context. Following [AST00], we metrize
$A_Q(\mathbb{R}^n)$ by a metric $\mathcal{G}$ defined as:

$$
\mathcal{G}(\sum \|P_i\|, \sum \|Q_i\|) = \min_{\sigma \in S_Q} \sqrt{\sum ||P_i - Q_{\sigma(i)}||^2}.
$$

Where $S_Q$ denotes the permutation group on $\{1, \ldots, Q\}$. It is easy to see that this gives a well-defined metric under which $A_Q(\mathbb{R}^n)$ is a complete metric space.

For an open $\Omega \subset \mathbb{R}^m$, we say a function $f : \Omega \rightarrow A_Q(\mathbb{R}^n)$ is a multiple-valued function. If $g : \Omega \rightarrow A_{Q'}(\mathbb{R}^n)$ is another multiple-valued function, we denote by $\llbracket f \rrbracket + \llbracket g \rrbracket$ the function into $A_{Q+Q'}(\mathbb{R}^n)$ whose value at the point $x \in \Omega$ is the sum of the measures $f(x)$ and $g(x)$.

If $f_i : \Omega \rightarrow \mathbb{R}^n$ are such that $f = \sum \llbracket f_i \rrbracket$, we say $\{f_i\}$ is a selection for $f$. A major problem in the theory of multiple-valued maps is attempting to derive selections with some degree of regularity from multiple-valued functions. However, we always have the following result:

**Theorem 2.1.1.** If $f : \Omega \rightarrow A_Q(\mathbb{R}^n)$ is measurable, then there exists a selection for $f$ consisting of measurable functions.

Almgren developed an extrinsic theory through the help of a bi-Lipschitz embedding of $A_Q(\mathbb{R}^n)$ onto a Lipschitz retract of higher dimensional Euclidean space. A corollary attributed to B. White (for a proof, see [BDPW15b]) strengthens this embedding to locally preserve distances, and we present this version below.

**Theorem 2.1.2.** There exists $N = N(Q,n)$, an injective map $\zeta : A_Q(\mathbb{R}^n) \rightarrow \mathbb{R}^N$ and a Lipschitz retraction $\rho : \mathbb{R}^N \rightarrow \zeta(A_Q(\mathbb{R}^n))$ so that:

1. $\text{Lip}(\zeta) \leq 1$. 
2. If \( Q = \zeta(A_Q(\mathbb{R}^n)) \), then \( \text{Lip}(\zeta^{-1}|_Q) \leq C(Q, n) \).

3. For any \( T \in A_Q(\mathbb{R}^n) \), there exists \( \delta \) so that \( \zeta|_{B_\delta(T)} \) preserves distances.

This embedding allows us to quickly define multiple-valued Sobolev functions. For
\( \Omega \subset \mathbb{R}^m \), \( k \in \mathbb{N} \) and \( 1 \leq p \leq \infty \), we define \( W^{k,p}(\Omega, A_Q(\mathbb{R}^n)) \) to be those functions
\( f : \Omega \to A_Q(\mathbb{R}^n) \) so that \( \zeta \circ f \in W^{k,p}(\Omega, \mathbb{R}^N) \). While this definition may seem
rather contrived, it has many computational benefits to the more natural equivalent
definitions discussed in [DLS11]. In particular, we note that classical results regarding
Lipschitz approximation of Sobolev functions, interpolation lemmas, traces and one-
dimensional restrictions of Sobolev functions are seen to immediately carry over to
the multiple-valued case.

We further define the Dirichlet energy of a \( W^{1,2}(\Omega, A_Q(\mathbb{R}^n)) \) function as:

\[
\text{Dir}(f, \Omega) = \int_{\Omega} |D(\zeta \circ f)|^2
\]

Again, this is simply a pragmatic definition–more natural equivalent definitions are
available in the literature.

Remark 2.1.1. From this definition it immediately follows that if \( \Omega \) is two-dimensional,
the Dirichlet energy is invariant under precomposition by conformal maps and, fur-
ther, that it is lower semi-continuous with respect to weak convergence on \( W^{1,2}(\Omega, A_Q(\mathbb{R}^n)) \).

We say that a multiple-valued function \( f \in W^{1,2}(\Omega, A_Q(\mathbb{R}^n)) \) is Dirichlet mini-
mizing if

\[
\text{Dir}(f, \Omega) \leq \text{Dir}(g, \Omega)
\]
for any $g \in W^{1,2}(\Omega, A_\mathbb{Q}(\mathbb{R}^n))$ such that $f|_{\partial \Omega} = g|_{\partial \Omega}$. Here, the last condition means that the $W^{1,2}$ traces on $\partial \Omega$ of $\zeta \circ f$ and $\zeta \circ g$ coincide. In Almgren’s original work, he proved that if $g \in W^{1,2}(\Omega, A_\mathbb{Q}(\mathbb{R}^n))$, then there exists a Dirichlet minimizing $f \in W^{1,2}(\Omega, A_\mathbb{Q}(\mathbb{R}^n))$ with $f|_{\partial \Omega} = g|_{\partial \Omega}$. He further showed that there is a constant $\alpha = \alpha(m, Q) > 0$ so that such a minimizing function is equal almost everywhere to a function which is $\alpha$ Hölder continuous on any $\Omega' \subset \subset \Omega$. In the following, we assume that this identification has been made, and in particular that Dirichlet minimizers are continuous functions “strictly defined” at every point.

In case $m = 1$, Sobolev functions automatically admit an absolutely continuous selection. We may define the space of absolutely continuous multiple-valued functions on an interval $I$, denoted by $AC(I, A_\mathbb{Q}(\mathbb{R}^n))$, to be those functions $f : I \to A_\mathbb{Q}(\mathbb{R}^n)$ so that $\zeta \circ f$ is absolutely continuous. We may now state a selection theorem for one-dimensional multiple-valued Sobolev functions—see Proposition 1.2 of [DLS11]:

**Theorem 2.1.3.** If $f \in W^{1,p}(I, A_\mathbb{Q}(\mathbb{R}^n))$, then,

1. $f \in AC(I, A_\mathbb{Q}(\mathbb{R}^n))$.

2. There exists a selection $\{f_i\} \subset W^{1,p}(I, \mathbb{R}^n)$ and so that $|D f_i| \leq |D(\zeta \circ f)|$.

Two more definitions will be useful in the following.

**Definition 2.1.1.** We say a multiple-valued function $f : \Omega \subset \mathbb{R}^m \to A_\mathbb{Q}(\mathbb{R}^n)$ is *affinely approximatable* at $x_0 \in \Omega$ if there exist affine maps $T_i : \mathbb{R}^m \to \mathbb{R}^n$ for $i = 1, \ldots, Q$ so that:

1. $\mathcal{G}(f(x_0), \sum[T_i(x)]) = o(|x - x_0|)$
2. If \( f_i(x_0) = f_j(x_0) \), then \( L_i = L_j \).

**Definition 2.1.2.** A *branch point* for a multiple-valued function \( f : \Omega \to A_Q(\mathbb{R}^n) \) is a point of discontinuity for the function \( \sigma(x) = H^0(\text{spt}(f(x))) \).

**Remark 2.1.2.** For a Dirichlet minimizer the set of branch points is at most of Hausdorff dimension \( m - 2 \), and, in case \( m = 2 \), it is locally finite. If \( x \in \Omega \) is not a branch point for a Dirichlet minimizing \( f \), then in some neighborhood around \( x \), there exists a harmonic selection for \( f \), and such a point is called a *regular point* for \( f \). Note this definition allows the possibility that at a regular point one of the selection functions for \( f \) may still have a critical point where the rank of its differential is less than \( \min\{m, n\} \).

### 2.2 A Solution to the Multiple-Valued Mapping Problem

In this section, we will prove Theorem 3.3.3. Notice that for \( K = 1, k_1 = 1 \), this theorem reduces to the classical case proven by Douglas and Rado, see [Dou31] and [Rad30].

The naive idea is the following: take a minimizing sequence and apply a version of the Courant-Lebesgue Lemma to an appropriate subsequence to get a set of maps which are weakly monotonic on the boundary and then appeal to lower semicontinuity of Dirichlet energy under weak convergence. Unfortunately, the Courant-Lebesgue Lemma requires the so called “three point condition.” In the classical case,
this condition is obtained by “normalizing” the functions by precomposing with an automorphism of the disk. In our context, the problem with this approach is that we cannot a priori simultaneously normalize all of the boundary functions—the simplest explanation for our method around this is when \( K = 2 \). We will show that, if there are certain kinds of branch points within the disk for each element of a minimizing sequence, then normalization of one of the boundary functions automatically normalizes the other. In the absence of these branch points, we may normalize separately.

To do this, we require two lemmas. The first lemma relies upon the following (slight) generalization of two Theorems of W. Zhu, found in [Zhu08].

**Theorem 2.2.1.** Let \( \epsilon > 0 \) and suppose an energy minimizing \( f \in W^{1,2}((1 + \epsilon) \mathbb{D}, A_Q(\mathbb{R}^n)) \) is strictly defined and has precisely one branch point in \( \mathbb{D} \) at the origin. Then there exists \( j \leq Q \) and \( k_1, k_2, \ldots, k_j \in \mathbb{N} \) so that \( \kappa := \sum k_i \leq Q \) and harmonic functions \( f_1, \ldots, f_{k_j}, f_{Q-\kappa}, f_{Q-\kappa+1}, \ldots, f_Q : \mathbb{D} \rightarrow \mathbb{R}^n \) so that, for all \( s \in \mathbb{D} \),

\[
f(s) = \sum_{i=1}^{Q} \left( \sum_{\zeta = s^{k_i}} \lfloor f_i(\zeta) \rfloor \right)
\]

where, to suppress notation, we set \( k_\ell = 1 \) for \( \ell \geq j \).

The next lemma is elementary, but, to the author’s knowledge, is not explicitly stated in the available literature.

**Lemma 2.2.2.** Suppose \( \Omega \subset \mathbb{R}^m \) is path connected and locally path connected and that \( f : \Omega \rightarrow A_Q(\mathbb{R}^n) \) is continuous. Additionally, suppose that \( f \) satisfies the following for some \( \alpha > 0 \):
1. \{(x,y) \in \Omega \times \mathbb{R}^n \mid y \in \text{spt}\{f(x)\}\} has K connected components \(E_1, \ldots, E_K\) so that \(d(E_i, E_j) > \alpha\) for \(i \neq j\).

Then \(f\) admits a continuous decomposition, that is, there are integers \(1', 2', \ldots, K'\) so that \(\sum i' = Q\), continuous functions \(f_j : \Omega \to A_{i'}(\mathbb{R}^n)\) so that \(f = \sum [f_i]\). Moreover, if \(\Omega\) is a region in \(\mathbb{R}^n\) and \(f \in W^{1,p}(\Omega, A_Q(\mathbb{R}^n))\) for some \(1 < p < \infty\), then each \(f_i \in W^{1,p}(\Omega, A_{i'}(\mathbb{R}^n))\).

**Proof.** Let 

\[ \text{Gf}(z) = \sum_{y \in \text{spt}(f(z))} [(z, y)] \]

be the graph map associated to \(f\). Note that if \(f\) is continuous (or Sobolev), so too is \(Gf\). Let \(i'\) be the number of elements of \(E_i \cap \text{spt}(Gf(z))\) counted with multiplicity (i.e. the mass of the atomic measure \(Gf(z) \ll E_i\)). We will show that \(i'\) is independent of \(z \in \Omega\).

Let \(x, y \in \Omega\) be given and let \(c : [0, 1] \to \Omega\) be a curve with \(c(0) = x\) and \(c(1) = y\). Well known selection theory (see, for example the beginning chapters of [DLS11]) gives a selection for \(Gf \circ c\) as:

\[ Gf \circ c = \sum [g_i]. \]

We see that, if \(g_i(0) \in E_\ell\), then \(d(g_i(t), E_k) > \alpha\) for all \(k \neq \ell\) by assumption two. Therefore, \(g_i(1) \in E_\ell\), proving \(i'\) is well defined.

Thus for \(y \in \Omega\), we may put:
\[ f_i(y) = \sum_{(y,p) \in E_i} [p]. \]

To see that each \( f_i \) is continuous, given \( x \in A \) and \( \epsilon > 0 \), we choose \( \delta \) so that \( |x - y| < \delta \) implies \( G(Gf(x), Gf(y)) < \min(\epsilon, \frac{\alpha}{2}) \). Let \( Gf(x) = \sum [(x, p_i)] \) and \( Gf(y) \sum [(y, q_i)] \) be labeled so that:

\[ G(Gf(x), Gf(y))^2 = \sum \|x - y\|^2 + \|p_i - q_i\|^2. \]

Then, in particular, for each \( i \), \( \|x - y\|^2 + \|p_i - q_i\|^2 < \frac{\alpha^2}{2} \). So, \( (x, p_i) \in E_j \) implies that \( (y, q_i) \in E_j \), which implies that \( G(f_j(x), f_j(y)) < \epsilon \), so each \( f_j \) is continuous.

Finally we show that each \( f_j \) must be Sobolev as follows: for a.e. point \( x \) in the interior of \( \Omega \), if \( \ell(x) \) is a line parallel to the unit axis of \( \mathbb{R}^n \) passing through \( x \in \mathbb{R}^n \),\n then \( Gf|_{\ell(x)} \) is Sobolev on some neighborhood \( U \subset \ell(x) \) of \( x \). This implies, by Proposition 1.2 of [DLS11] that there exists a Sobolev selection on \( U \). However, since each \( f_i \) is continuous we must have that the Sobolev selection corresponds to a Sobolev selection for each \( f_i|U \). Hence by Theorem 2 of [EG91, §4.9.2] each \( f_j \in W^{1,2}_{\text{loc}}(\Omega, A_j'(\mathbb{R}^n)) \), and \( f \in W^{1,2}(\Omega, A_Q(\mathbb{R}^n)) \) allows us to conclude that \( f_j \in W^{1,2}(\Omega, A_{j'}(\mathbb{R}^n)) \) for all \( j \).

Our ultimate goal is to apply the following generalization of the Courant-Lebesgue Lemma to the boundary data of a minimizing sequence of multiple-valued functions. For the result in the classical setting, see [JDK+10, §4.3 Theorem 3].
Theorem 2.2.3. Suppose $\Gamma_1, \ldots, \Gamma_K$ are closed, oriented Jordan curves as in the statement of Theorem 3.3.3 and that $\{f^i_k\}_{k \in \mathbb{N}}$ are sequences of monotone continuous mappings $f^i_k : S^1 \to \Gamma^i$. Define $F_k : S^1 \to A_Q(\mathbb{R}^n)$ by:

$$F_k(s) = \sum_i \sum_{\zeta = s^{k_i}} [f^i_k(\zeta)]$$

and suppose that the Dirichlet minimizing extensions of the maps $F_k$, denoted by $\tilde{F}_k : \mathbb{D} \mapsto A_Q(\mathbb{R}^n)$ are such that

$$\text{Dir}(\tilde{F}_k; \mathbb{D}) \leq M$$

for some $M \in \mathbb{R}$ independent of $k$, then, for each $i$, the family $\{f^i_k\}_{k \in \mathbb{N}}$ is equicontinuous if they satisfy a uniform three point condition:

$$f^i_k(\alpha^i_j) = \beta^i_{k,j} \text{ for } j = 1, 2, 3,, i = 1, \ldots, Q$$

for some distinct points $\alpha^i_j \in S^1$ and $\beta^i_{k,j} \in \Gamma^i$ so that $\beta^i_{k,j} \to \beta^i_j$ holds, for $\beta^i_j$ three distinct points of $\Gamma_i$.

Once we guarantee that we can find an $A$ (independent of $k$) for which the conditions of Lemma 2.2.2 hold for each $\tilde{F}_k$, the proof of Theorem 2.2.3 follows from identical methods of [JDK+10, §4.3 Theorem 3] after applying Lemma 2.2.2 and analyzing the embeddings $\zeta \circ f_i$ for the decomposition guaranteed by the lemma. Finding such a neighborhood is the content of the next proposition.

Lemma 2.2.4. Suppose that $\{F_k\}_{k \in \mathbb{N}} \subset W^{1,2}(\mathbb{D}, A_Q(\mathbb{R}^n)) \cap C(\overline{\mathbb{D}}, A_Q(\mathbb{R}^n))$ is a sequence of Dirichlet minimizing functions with a selection for the trace of $F_k$ given, for $s \in S^1$ by
\[ F_k(s) = \sum_i \sum_{\zeta = s^{k_i}} [f_k^i(\zeta)] \]

for \( f_k^i : S^1 \rightarrow \Gamma_i \) continuous, weakly monotonic maps onto curves as in Theorem 3.3.3. Further suppose that each \( F_k \) is affinely approximatable a.e. on the interior of \( \mathbb{D} \) and that

\[ \sup_k \text{Dir}(F_k; \mathbb{D}) = M < \infty. \]

Then there exists a neighborhood \( A \subset \mathbb{D} \) of \( S^1 \) so that the conditions of Lemma 2.2.2 are satisfied for each \( F_k|_A \).

Further, if each \( \Gamma_i \) is a LNR of some open neighborhood \( U_i \) (where we assume that \( U_i \cap U_j = \emptyset \) for \( i \neq j \)), we may guarantee that: \( F_k|_A \in \sum_1^K k_i[U_i] \), where \( \sum_1^K k_i[U_i] \) is the set of all \( y \in A_Q(\mathbb{R}^n) \) so that \( H^0(\text{spt}(y) \cap U_i) = k_i \) for all \( i \).

**Remark 2.2.1.** Note that the differentiability condition is immediately satisfied by any sequence of functions which are Dirichlet minimizing for their boundary data, as in [AST00] or [DS13].

**Proof.** Let \( \epsilon < \min_{i \neq j} \text{dist}(\Gamma_i, \Gamma_j) \). We will show that for \( \delta < 1 \) close enough to one and any \( x \in S^1, \delta \leq \alpha \leq 1, \mathcal{G}(F_k(\alpha x), F_k(\gamma)) < \frac{\epsilon}{3} \), where \( \gamma \) is some arc of \( S^1 \), independent of \( k \), and \( x \in \gamma \). By our assumptions on the boundary data of each \( F_k \), this will guarantee the assumptions of Lemma 2.2.2. We will assume that \( M = 1 \), which is possible simply by dividing the functions appropriately.

Let \( 1 \geq \alpha \geq \delta \geq \frac{1}{2} \) where \( \delta \) will be chosen shortly, and \( 0 < \Delta \theta < \frac{\pi}{2} \) be given. Let \( a \in \alpha \cdot S^1 \) be given, and let \( \hat{a} = \frac{a}{\|a\|} \). Viewing \( \mathbb{R}^2 \) as \( \mathbb{C} \), we lose no generality in
assuming that \( \hat{a} = 1 \). Consider the region \( R_{\delta} \) defined by:

\[
R_{\delta} = \{ (x, y) \mid y \in [-\delta \sin(\frac{\Delta \theta}{2}), \delta \sin(\frac{\Delta \theta}{2})], \text{and } \delta^2 \leq x^2 + y^2 \leq 1 \}.
\]

For each \( y_0 \in [-\delta \sin(\frac{\Delta \theta}{2}), \delta \sin(\frac{\Delta \theta}{2})] \), let \( L(y_0) \) be the horizontal line segment \((\mathbb{R} \times \{y_0\}) \cap R_{\delta} \). Denote by \( x_{\delta}(y_0) \) and \( x_1(y_0) \) the \( x \) coordinates of where this line intersects \( \delta S^1 \) and \( S^1 \), respectively.

Since each \( F_k \) is Sobolev, for a.e. line \( \ell \) in \( \mathbb{C} \), we have that \( F_k|_{(\ell \cap \mathbb{R})} \) is absolutely continuous for all \( k \) and that \( F_k \) is affinely approximatable at \( \mathcal{H}^1 \) a.e. point on \( \ell \). This implies, by combining Proposition 1.2 of [DLS11] and Theorem 6.4 of [Gob06]<sup>1</sup> that for a.e. \( y_0 \in [-\delta \sin(\frac{\Delta \theta}{2}), \delta \sin(\frac{\Delta \theta}{2})] \)

\[
G(F_k(x_{\alpha}(y_0), y_0), F_k(x_1(y_0), y_0)) \leq \int_{L(y_0) \cap \{x \mid \delta \leq ||x|| \leq 1\}} ||\nabla F_k|| \, d\mathcal{H}^1 \leq \int_{L(y_0)} ||\nabla F_k|| \, d\mathcal{H}^1.
\]

Integrating this expression with respect to the variable \( y \) gives:

\[
\int_{-\delta \sin(\frac{\Delta \theta}{2})}^{\delta \sin(\frac{\Delta \theta}{2})} G(F_k(x_{\delta}(y), y), F_k(x_1(y), y)) \, dy \leq \int_{-\delta \sin(\frac{\Delta \theta}{2})}^{\delta \sin(\frac{\Delta \theta}{2})} \left( \int_{L(y)} ||\nabla F_k|| \, d\mathcal{H}^1 \right) \, dy.
\]

The Fubini Theorem and the Hölder Inequality implies:

\[
\int_{-\delta \sin(\frac{\Delta \theta}{2})}^{\delta \sin(\frac{\Delta \theta}{2})} \left( \int_{L(y) \cap R_{\delta}} ||\nabla F_k|| \, d\mathcal{H}^1 \right) \, dy \leq M \frac{1}{2} \left( \mathcal{H}^2(R_{\delta}) \right)^{\frac{1}{2}}.
\]

On the other hand, if \( \gamma_1 \) and \( \gamma_2 \) denote the portions of \( \alpha \cdot S^1 \) and \( S^1 \) contained in \( R_{\delta} \), we see that:

<sup>1</sup>While the statement of the theorem in [DLS11] requires that \( F_k \) be Lipschitz, the proof is identical provided that \( F_k \) admits an absolutely continuous selection, which is guaranteed by Proposition 1.2 of [DLS11].
\[ 2\delta \sin \left( \frac{\Delta \theta}{2} \right) \mathcal{G}(F_k(\gamma_1), F_k(\gamma_2)) \leq \int_{-\delta \sin \left( \frac{\Delta \theta}{2} \right)}^{\delta \sin \left( \frac{\Delta \theta}{2} \right)} \mathcal{G}(F_k(x_\alpha(y), y), F_k(x_1(y), y)) \, dy \]

which, combined with Equations (2.2.1) and (2.2.3) gives the bound:

\[ \mathcal{G}(F_k(\gamma_1), F_k(\gamma_2)) \leq \frac{\mathcal{H}^2(R_\delta)^{\frac{1}{2}}}{2 \sin \left( \frac{\Delta \theta}{2} \right) \delta} \]  

(2.2.2)

(Recall that we’ve assumed \( M = 1 \).) By rotation, this bound may be established for any arc of length equal to \( \Delta \theta \).

Next, one easily checks that \( \mathcal{H}^2(R_\delta) \rightarrow 0 \) as \( \delta \rightarrow 1 \) so we may choose (for fixed \( \Delta \theta \) ) \( \delta \) close enough to one so that:

\[ \frac{\mathcal{H}^2(R_\delta)^{\frac{1}{2}}}{2 \sin \left( \frac{\Delta \theta}{2} \right) \delta} \leq \frac{\epsilon}{6}. \]  

(2.2.3)

We now observe that, if the arc length were chosen to be small enough so that the variation of all of the functions \( F_k \) on \( \delta \gamma \) was smaller than \( \frac{\epsilon}{6} \), we could guarantee that, for each \( e^{i\psi} \in \gamma \):

\[ \mathcal{G}(F_k(\delta e^{i\psi}), F_k(\gamma)) < \frac{\epsilon}{3}. \]

Obtaining such a bound for smaller arc length is slightly more complex than it seems at first, since \( \delta \) must be chosen after the arc-length is already fixed, but it can be done via the following method of choosing \( \delta \) and the arc length \( \ell \).

Recall that, in Theorem 9 of [DLS11], the following inequality is shown for Dirichlet minimizing functions on the disk:
\[
\int_{B_r(x)} \|DF_k\|^2 d\mathcal{H}^2 \leq r^\frac{3}{2} \int_{B} \|Df\|^2 d\mathcal{H}^2 \tag{2.2.4}
\]

Whenever \( x \in \mathbb{D} \) and \( r < 1 - ||x|| \).

Following the arguments of Section 3.2 of [HL11], we see that this implies, some constant \( C \) which depends only on \( n \) and \( Q \) so that:

\[
x, y \in \delta\mathbb{D} \text{ with } ||x - y|| < \frac{1 - \delta}{2} \Rightarrow \mathcal{G}(F_k(x), F_k(y)) \leq C ||x - y||^\frac{1}{3}.
\]

Let \( \epsilon \) be as above and choose \( n \in \mathbb{N} \) so that \( \frac{2\pi^2}{\epsilon^2} \leq n \). For \( \delta \) sufficiently close to one, \( \ell = n \left( \frac{1 - \delta}{2} \right) \) defines an arc within the ranges given in the preceding arguments.

Further, we may also choose \( \delta \) close enough to one to guarantee the additional inequality:

\[
nC \left( \frac{1 - \delta}{2} \right)^\frac{1}{3} \leq \frac{\epsilon}{6}. \tag{2.2.5}
\]

Then, using the asymptotics:

\[
\mathcal{H}^2(R_\delta) \approx \ell(1 - \delta)
\]

\[
2 \sin \left( \frac{\Delta \theta}{2} \right) \approx \ell
\]

and our choices of \( n, \delta \) and \( \ell \), one checks that the above integral estimates imply (possibly upon choosing \( \delta \) even closer to one):

\[
\mathcal{G}(F_k(\gamma), F_k(\delta \gamma)) \leq \frac{\epsilon}{6}.
\]
for any arc $\gamma \subset S^1$ with length $\ell$.

Let $\gamma$ be such an arc. For any two points $x, y \in \delta \gamma$, by choice of $\ell$, we may find a chain:

$$\{x = x_1, x_2, ..., x_n = y\} \subset \delta \gamma$$

for which $||x_i - x_{i+1}|| \leq \frac{1-\delta}{2}$.

Then:

$$\mathcal{G}(F_k(x), F_k(y)) < \sum \mathcal{G}(F_k(x_i), F_k(x_{i+1})) \leq nC \left( \frac{1-\delta}{2} \right)^{\frac{1}{2}} \leq \frac{\epsilon}{6}$$

where the last inequality follows from Inequality (2.2.5).

To summarize, we have shown the following statement:

If $\gamma$ is an arc of $S^1$ of length $\ell$ and $x \in \gamma$, $k \in \mathbb{N}$, there exists $y_k \in \gamma$ so that

$$\mathcal{G}(F_k(\delta x), F_k(y_k)) \leq \frac{\epsilon}{3}.$$ 

One checks that this statement proves the initial goal, since choosing $\delta \leq \alpha$ only improves the bounds used above. By further constraining $\epsilon$ to also satisfy $\epsilon < \min_i \text{dist}(\Gamma_i, \partial U_i)$, we may obtain the final statement in the Lemma, completing the proof.

\[\square\]

Remark 2.2.2. Note that Equation (2.2.2) a posteriori gives the additional property, where $\{f_i\}$ is the selection from Lemma 2.2.2: There is an $\alpha > 0$ such that for all $0 < \alpha \leq \beta \leq 1$, $\beta S^1 \subset A$ and, and all $i$, $\mathcal{G}(f_i(\beta e^{i\theta}), f_i(\gamma)) < \epsilon$ where $\gamma$ is an arc of $S^1$ with length bounded by $C(1 - \beta)$ with $e^{i\theta} \in \gamma$.

Finally, we recall the following interpolation lemma, Lemma 2.15 of [DLS11].
Lemma 2.2.5 (Interpolation Lemma). There is a constant $C = C(m, n, Q)$ with the following property. Let $r, \epsilon > 0$, $g \in W^{1,2}(\partial B_r, A_Q(\mathbb{R}^n))$ and $f \in W^{1,2}(\partial B_{r(1-\epsilon)}, A_Q(\mathbb{R}^n))$. Then, there exists $h \in W^{1,2}(B_r \setminus B_{r(1-\epsilon)}, A_Q(\mathbb{R}^n))$ so that $h|_{\partial B_{r(1-\epsilon)}} = f$, $h|_{\partial B_r} = g$ and so that $\text{Dir}(h, B_r \setminus B_{r(1-\epsilon)})$ is less than or equal to:

$$Cer[\text{Dir}(g, \partial B_r) + \text{Dir}(f, \partial B_{r(1-\epsilon)})] + \frac{C}{er} \int_{\partial B_r} \mathcal{G}(g(x), f((1-\epsilon)x))^2 \, dx \quad (2.2.6)$$

Proof of Theorem 3.3.3. We present a proof in the case where $K = 2$, and $k_1$ and $k_2$ are given so that $k_1 + k_2 = Q$. The more general case follows by identical arguments, there are simply more cases to analyze. For more details, see Remark 2.2.4.

Let $\{F_k\}_{k=1}^{\infty} \subset W^{1,2}(\mathbb{D}, A_2(\mathbb{R}^n))$ be a minimizing sequence so that:

1. $F_k|_{S^1} = \sum_{\zeta = s_1} \mathcal{J}_f(\zeta) + \sum_{\zeta = s_2} \mathcal{J}_g(\zeta)$ for $f_k : S^1 \to \Gamma$ and $g_k : S^1 \to \Gamma$ monotone.

2. $\text{Dir}(F_k, \mathbb{D}) \to \inf_{G \in A} \text{Dir}(G, \mathbb{D})$

We may additionally assume that each $F_k$ is Dirichlet minimizing for its boundary data—since replacing each $F_k$ with a Dirichlet minimizer will only decrease the energy.

Standard compactness arguments allow us to extract a subsequence (which we won’t relabel) which weakly converges in $W^{1,2}$ to a Dirichlet minimizing function $F$. Furthermore, by Arzela-Ascoli, uniform Holder continuity on any compact $\Omega \subset \subset \mathbb{D}$ and a diagonal argument applied to a compact exhaustion of $\mathbb{D}$, we may guarantee that the $f_k$ converge uniformly on compact subsets of the disk.
Let $A$ be the neighborhood of $S^1$ provided by Lemma 2.2.4. For simplicity, we assume that $A$ is a disk of inner radius $\delta$. For reasons that will be elucidated shortly, we will mostly be interested in the branch points of $F$ within $\mathbb{D} \setminus A$, so denote by $\Sigma_k$ the branch set of $F_k$ within the compact set $\mathbb{D} \setminus A$ and $\Sigma_0$ the branch set of $F$ within $\mathbb{D} \setminus A$. Since interior branch points for two-dimensional multiple-valued functions are isolated, we know that $F$ may have only finitely many branch points in $\mathbb{D} \setminus A$. Further, since the $F_k$'s converge to $F$ uniformly on $\mathbb{D} \setminus A$, we may, by throwing out finitely many elements of the sequence $\{F_k\}$, also assume that $\mathcal{H}^0(\Sigma_k) \geq \mathcal{H}^0(\Sigma_0) = \ell$.

Put $\Sigma_0 = \{x_1, ..., x_\ell\}$.

For the remainder of the proof it will be crucial to have more control on the location and quantity of points in $\Sigma_k$—the following claim shows that we can do this without changing our sequence’s properties substantially.

**Claim 2.2.6.** We may modify the sequence $\{F_k\}$ to obtain a new sequence, $\{\tilde{F}_n\}$ still converging uniformly on compact sets to $F$ so that if $\tilde{\Sigma}_n$ is the branch set of $\tilde{F}_n$ in $\mathbb{D} \setminus A$, then $\mathcal{H}^0(\tilde{\Sigma}_n) = \ell$ and so that $\text{Dir}(\tilde{F}_n) = \text{Dir}(F_{k(n)}) + o(1)$ as $n \to \infty$ for some increasing sequence $k(n)$.

**Proof of Claim.** Using the uniform convergence of the sequence $\{F_k\}$ on $\mathbb{D} \setminus A$, we may find $r_1, ..., r_\ell$ so that the following holds for all large $k$:

1. $B_{r_i}(x_i) \cap B_{r_j}(x_j) = \phi$ for $i \neq j$.

2. $\Sigma_k \subset \bigcup B_{r_i}(x_i)$

3. The image of the graph map of $F_k|_{\partial B_{r_i}(x_i)}$ has the same number of connected
components for all $k$.

For each $k \in \{1, ..., \ell\}$, we consider $B_{r_i}(x_i)$ independently. For ease of notation, assume that $x_i = 0$, and, as before, let $B_s$ be the open ball of radius $s$ about the origin.

Let the image of the graph map of $F_k|_{\partial B_{r_i}}$ have $\ell$ connected components.

Then there are positive integers $\ell_{i,j}$ for $j = 1, ..., \ell$ so that $\sum_j \ell_{i,j} = Q$ and there are $\ell$ continuous functions $f_{k,i,j}$ defined on $\partial B_{r_i}$ mapping to $A_{\ell_{i,j}}(\mathbb{R}^n)$ so that:

$$F_k|_{\partial B_{r_i}} = \sum_j [f_{k,i,j}]$$  \tag{2.2.7}

and so that $\mathcal{G}(f_{k,i,j}, f_{k,i,m})$ is bounded away from zero independent of $k$ for $j \neq m$.

Since $F$ is Dirichlet minimizing, has only one branch point within $B_{r_i}(x_i)$, and has $\ell$ connected components on $\partial B_{r_i}$, we conclude via Theorem 2.2.1 that there are $\ell$ harmonic functions $h_j : B_1 \to \mathbb{R}^n$ so that:

$$F|_{B_{r_i}}(z) = \sum_j \sum_{\zeta^i,j = r_i z} [h_j(\zeta)].$$

Uniform convergence implies that (up to relabeling),

$$f_{k,i,j}(z) \to \text{Trace} \left( \sum_{\zeta^i,j = r_i z} [h_j(\zeta)] \right)$$

uniformly as $k \to \infty$. For arbitrary $0 < \epsilon < r_i$, we may apply Lemma 2.2.5 to the following sequence of functions:

1. $F_k|_{\partial B_{r_i}} = \sum_j [f_{k,i,j}]$

2. $F_\epsilon(s) = F(\frac{s}{1-\epsilon})$ defined on $\partial B_{r_i(1-\epsilon)}$. 

We obtain a sequence \( h_{k, \epsilon} \in W^{1, 2}(B_{r_i} \setminus B_{r_i(1-\epsilon)}, A_Q(\mathbb{R}^n)) \) with bounds on \( \text{Dir}(h_{k, \epsilon}, B_{r_i}) \); in particular:

\[
\text{Dir}(h_{k, \epsilon}, B_{r_i} \setminus B_{r_i(1-\epsilon)}) \leq C\epsilon r \text{Dir}(F_k, \partial B_{r_i}) + \text{Dir}(F_\epsilon, \partial B_{r_i(1-\epsilon)}) + C \epsilon r \int_{\partial B_{r_i}} g(F_k(x), F_\epsilon((1-\epsilon)x))^2 \, dx. \tag{2.2.8}
\]

Notice that in the above equation, as \( k \to \infty \), the right hand side tends to:

\[
2C\epsilon r \text{Dir}(F_\epsilon, \partial B_{r_i}) \leq M\epsilon \tag{2.2.9}
\]

where \( M \) is independent of \( \epsilon \).

Let \( \epsilon_n \) be any sequence tending to zero, and further suppose that \( 0 < \epsilon_n < \min(r_i, \epsilon) \). For each \( n \), choose an increasing sequence \( k(n) \) so that:

\[
C\epsilon_n r \text{Dir}(f_{k(n)}, \partial B_{r_i}) + \text{Dir}(h_{\epsilon_n}, \partial B_{r_i(1-\epsilon_n)}) + C \epsilon_n r \int_{\partial B_{r_i}} g(f_{k(n)}(x), h_{\epsilon_n}((1-\epsilon)x))^2 \, dx < 2M\epsilon.
\]

Now, define \( \tilde{F}_n \) by:

\[
\tilde{F}_n(x) = \begin{cases} 
F_{k(n)}(x) & : x \in \mathbb{D} \setminus B_{r_i} \\
h_{k(n), \epsilon_n}(x) & : x \in B_{r_i} \setminus B_{r_i(1-\epsilon)} \\
F(\frac{x}{1-\epsilon_n}) & : x \in B_{r_i(1-\epsilon)}
\end{cases}
\]

One may check that \( \text{Dir}(\tilde{F}_n, B_{r_i}) = \text{Dir}(F_{k(n)}, B_{r_i}) + o(1) \) holds, and further that \( \tilde{F}_n \) has only one branch point within \( B_{r_i} \) for sufficiently large \( n \)–namely 0.
By applying the above construction to each \( B_{\delta_i}(x_i) \), we obtain a new sequence \( \{ \tilde{F}_k \} \) satisfying the conclusions of the Claim.

\[ \square \]

Remark 2.2.3. Note that the new functions \( \tilde{F}_k \) are no longer Dirichlet minimizing for their given boundary data, however, they are *locally* Dirichlet minimizing in some neighborhood of their branch points which is independent of \( k \).

There is a sort of dichotomy between the two types of branch points that may occur in \( \Sigma_k \), which we now analyze.

**Definition 2.2.1.** Call \( \{ x^k_i \}_{k \in \mathbb{N}} \) connecting if there is a smoothly bounded neighborhood \( U \) so that:

1. for all large \( k \), \( U \cap \Sigma_k = \{ x^k_1 \} \) and,

2. \( U \) is smoothly contractible to \( x^k_1 \) for all \( k \) as in (1),

3. \( \partial U \) is a smooth curve whose intersection with \( A \) has a smooth arc \( \Gamma \) of positive \( \mathcal{H}^1 \) measure.

4. If \( GF_k \) denotes the graph map (as in the proof of Lemma 2.2.2), then \( \text{spt}(GF_k(\partial U)) \) has a connected component \( \Lambda \) so that \( (\Lambda \cap \text{spt}(GF_k(\Gamma))) \cap B_i \neq \emptyset \) for \( i = 1, 2 \).

The proof now reduces to two cases.

**Case I:** Suppose first that for every \( j \), \( \{ x^k_j \}_{k \in \mathbb{N}} \) are not connecting. We will show that the decomposition of \( F_k \) on \( A \) extends to a decomposition on all of \( \mathbb{D} \), which then allows us to independently normalize the boundary data to satisfy the three point condition of Theorem 2.2.3.
To see this, choose \(x_0 \in \mathbb{D} \setminus (A \cup \Sigma_0)\), and let \(\gamma : [0, 1] \to A\) be a path so that \(\gamma(0) = x_0\) and \(\gamma(1) = z \in A\) so that \(\gamma([0, 1]) \subset \mathbb{D} \setminus (A \cup \Sigma_0)\). Standard selection theory (see, for example the beginning chapters of [DLS11]) gives a unique selection

\[
F_k \circ \gamma = \sum [f_k]^i
\]

Partition \(F_k(x_0) = \sum [f_k]^i(0)\) into two multiple-valued functions by:

\[
F_k(x_0) = \sum_{(z, f_k^i(1)) \in B_1} [f_k^i(0)] + \sum_{(z, f_k^i(1)) \in B_2} [f_k^i(0)] =: [F_k^1(x_0)] + [F_k^2(x_0)]
\]

One checks that the fact that no element of \(\Sigma_0\) is connecting implies that this choice is independent of path and therefore well-defined. Since each \(F_k\) is continuous, this decomposition may be extended to the finite set \(\Sigma_0\) by continuity. It readily follows that each \(F_k^i\) is Sobolev. Further, the decomposition on \(A\) gives (up to possible relabeling)

\[
F_k^1|_{S^1}(s) = \sum_{\zeta = s^{t_1^1}} [f_k(\zeta)]
\]

\[
F_k^2|_{S^1}(s) = \sum_{\zeta = s^{t_2^1}} [g_k(\zeta)]
\]

as in (1) at the beginning of the proof. Therefore, by picking two sequences of conformal disk automorphisms, \(\{\sigma_k\}\) and \(\{\psi_k\}\) so that \(f_k \circ \sigma_k\) and \(g_k \circ \psi_k\) satisfies a three point condition, we see that \([F_k^1 \circ \sigma_k] + [F_k^2 \circ \psi_k]\) converges to an admissible map \(\tilde{F}\) as \(k \to \infty\)–but since this normalization doesn’t affect the energy, lower semicontinuity of the Dirichlet energy gives that \(\tilde{F}\) is a minimizer, as desired.

**Case II:** Suppose now that \(\{x_k\}\) is a sequence of connecting branch points. We will prove that the boundary data \(f_k\) and \(g_k\) cannot degenerate in the limit. Let
\(\alpha_1, \alpha_2, \text{ and } \alpha_3 \) be three distinct points of \( S^1 \), and let \( s_i^k = f_k(\alpha_i) \), \( t_i^k = g_k(\alpha_i) \) for \( i = 1, 2, 3 \). By compactness of \( \Gamma_1 \) and \( \Gamma_2 \), we may, upon extracting to a subsequence (which we again will not relabel) suppose that \( s_i^k \to s_i \) and \( t_i^k \to t_i \) for each \( i \) as \( k \to \infty \).

By precomposing with a disk diffeomorphism, we may assume that the set \( \{t_1, t_2, t_3\} \) consists of three distinct elements, and hence that the boundary data \( \tilde{g}_k \) converges uniformly to some \( g \) by Theorem 2.2.3. We want to show that the set \( \{s_1, s_2, s_3\} \) also consists of three distinct elements, since then we may apply Theorem 2.2.3 to obtain the result immediately.

We now assume towards a contradiction that \( \{s_1, s_2, s_3\} \) are not distinct, say that \( s_1 = s_3 \). Under these assumptions, by the monotonicity of the \( f_k \), and once again, possibly extracting a (still un-relabeled) subsequence we may select a sequence \( \beta^k \) with the following properties:

1. \( \beta^k \to \beta \notin \{\alpha_1, \alpha_2\} \) as \( k \to \infty \).

2. \( f_k(\beta^k) = \tilde{s}_3 \notin \{s_1, s_2\} \).

Note that \( g \) must be non-constant on one of the arcs \((\beta, \alpha_1)\), or \((\beta, \alpha_2)\), where \((\beta, \alpha_1)\) is the arc that doesn’t pass through \( \alpha_2 \) and vice versa. We suppose that \( g \) is non-constant on \((\beta, \alpha_1)\). Since for each \( k \) there are only finitely many branch points to avoid and their locations are constrained to balls of fixed radii, we may find a smoothly bounded, simply connected \( U \subset \mathbb{D} \) satisfying the properties in the definition of connecting branch point, with the additional property that:
• the set $\partial U$ is a smooth curve whose intersection with $A$ has an arc $L$ so that
\[ \text{dist}(L \cap (\beta, \alpha_1)) < \frac{\delta}{2}, \]
where $(\beta, \alpha_1)$ is the arc of $S^1$ connecting $\beta$ and $\alpha_1$ which does not pass through $\alpha_2$.

Using the estimate given by Inequality (2.2.2) in the proof of Lemma 2.2.4, we may further assert that if $r_f : U_f \to \Gamma_i$ is a Lipschitz retract and $\epsilon > 0$ is given, we may choose $U$ so that $(\bar{s}^3 \pm \epsilon, s_1 \pm \epsilon) \subset r_f(F_k(L) \cap U_f)$, where the $\pm$’s are independent of one another.

Let $\phi : \mathbb{D} \to U$ be the conformal map guaranteed by the Riemann Mapping Theorem. Further, by Caratheodory’s Theorem (see, for example Theorem 5.5 of [Con95] and the preceding sections), we see that $\phi$ extends to a homeomorphism $\varphi : S^1 \to \partial U$. Then since $\{x_k^i\}$ is a sequence of connecting branch points, we may put

$$F_k|_{\partial U}(z) = \sum_i \sum_{\zeta^i_i=\phi^{-1}(z)} \left[ h_{k,i}(\zeta) \right].$$

For some integers $\ell_i$ with $\sum \ell_i = Q$ and some continuous functions $h_{k,i} : S^1 \to \mathbb{R}^n$.

Without loss of generality, we may assume that $h_{k,1}$ is a parameterizing map for $\Lambda$ as in the definition of connecting branch points. Set $h_k = h_{k,i}$.

Consider now the maps $\psi_k(z) = (z, h_k(z))$ for $z \in \mathbb{D}$. Since $\partial U \subseteq \mathbb{D}$, uniform convergence on compact sets of the maps $F_k$ guarantees that the curves $\psi_k(S^1)$ converge in the sense of Fréchet (see, for example, [JDK+10, §4.2]) to a curve $\Psi$. Further, since $g_k$ converges uniformly to a non-constant function (by our choice of arc $(\beta, \alpha_1)$), by the comments in Remark 2.2.2, the traces of $\psi_k$ are monotone and satisfy the
three point condition of Theorem 2.2.3 and thus Trace($\psi_k$) converges uniformly to some $\psi : S^1 \to \mathbb{R}^n$, which is monotonic onto the curve $\Psi$. Put $\psi(z) = (z, h(z))$. Then, for each $k$, there is a segment of the image curve of $h$ contained in $((\tilde{s}_3, s_1))_\delta$ (where this is the interval inside $\Gamma_1$), which follows from Remark 2.2.2 and so, by applying the retraction $r_f : (\Gamma_1)_\delta \to \Gamma_1$, we see that this property must persist in the limit, and therefore the $f_k$ restricted to the interval $(\beta, \alpha_1)$ cannot degenerate, and hence we may choose three points and apply Theorem 2.2.3 to obtain convergence to a monotone map.

Finally, notice that we may (possibly upon extracting to another subsequence and utilizing another diagonal argument) assume that the sequence of renormalized functions weakly converges to some $\tilde{F} \in W^{1,2}(\mathbb{D}, A_Q(\mathbb{R}^n))$. Further, Remark 2.2.2 applied to smaller and smaller neighborhoods paired with uniform convergence on compact sets and uniform convergence on $S^1$, one obtains using a standard $\varepsilon/3$ argument that this subsequence converges uniformly to $\tilde{F}$ on $\mathbb{D}$, which implies that $\tilde{F}$ is admissible, and lower semi-continuity of the Dirichlet Energy then gives the result.

$\square$

*Remark 2.2.4 (Generalization to $K > 2$).* One notices that there is nothing unique for the case where $K = 2$, there are simply fewer cases of the branching behavior. Indeed, one first checks that Claim 2.2.6 holds for arbitrary $K$, and then, to guarantee the boundary convergence, one uses an identical argument as presented above, there are simply more cases to be concerned with since there are more possibilities for branching behavior.
2.3 Proof of the Regularity Theorem

In this section we prove Theorem 2.0.2 via methods gleaned from the classical situation for Douglas Minimizers. The proof involves using topological methods and complex analysis to produce a single-valued map from a planar domain with the same minimization properties and boundary data. It relies heavily on the classical results regarding the Douglas problem—see Chapter 8 of [JDK+10].

Before we begin, we need a few results from complex analysis. We start by stating a weakened version of a theorem of Bieberbach [Bie25]—see also [BK08].

**Theorem 2.3.1.** Let \( \Omega \subset \mathbb{C} \) be a bounded domain in the plane bounded by \( n \) non-intersecting Jordan curves \( \gamma_1, \gamma_2, ..., \gamma_n \), and, for each \( i \), choose a point \( b_i \in \gamma_i \). Then there exists a proper holomorphic mapping \( f : \Omega \rightarrow \mathbb{D} \) which is an \( n \)-to-one branched covering map. Further, since \( f \) is proper, \( f \) extends continuously to the boundary of \( \Omega \), and this extension (which we will also denote by \( f \)) maps each boundary curve monotonically onto the unit circle and has \( f(b_i) = 1 \) for all \( i \).

The above Theorem leads, in a fairly straightforward way, to the following result. The main idea is to translate the following Theorem into a problem on the upper half-plane, and then add functions produced via the above with clever selections of points—for full details, see [BK08, §5].

**Theorem 2.3.2.** Suppose that \( \Omega \subset \mathbb{C} \) is as in Theorem 2.3.1 and \( d_1, ..., k = d_n \) are positive integers. Then there exists a proper holomorphic map \( f : \Omega \rightarrow \mathbb{D} \) such that:

1. \( f \) is a \( d_1 + d_2 + ... + d_n \)-to-one branched cover of \( \mathbb{D} \)
2. \( f \) admits a continuous extension to \( \Omega \) (which we will also denote by \( f \)) so that, for each \( i \), \( f|_{\gamma_i} \) has degree \( k_i \).

**Proof of Theorem 2.0.2.** Let \( F, \Gamma_1, \ldots, \Gamma_K \) and \( k_1, \ldots, k_K \) be as in Definition 2.0.1. Consider the set \( \Gamma(F) = \{(x, y) \mid x \in \mathbb{D}, y \in \text{spt}(F(x))\} \), the graph of \( F \) (note that \( \Gamma(F) \) is also the multiple-valued image of the map \( GF \) from the above sections).

Notice that, since the boundary data admits a wrapped solution, \( F \) can have no branch points on the boundary \( S^1 \), and thus by Remark 2.2.2 \( F \) has no branch points on some annulus \( A \), and hence only finitely many branch points in \( \mathbb{D} \).

Next let \( \pi_1 \) denotes the projection of \( \mathbb{R}^2 \times \mathbb{R}^n \) onto the \( \mathbb{R}^2 \) coordinates, and \( \pi_2 \) denote the projection onto \( \mathbb{R}^n \) coordinates. Comments in the above paragraph show that \( \pi_1 : \Gamma(F) \to \mathbb{D} \) is a branched cover.

Further, one readily checks that \( \Gamma(F) \) is a Riemann surface with boundary for which any loop is either null-homotopic or homotopic to a concatenation of boundary curves—and so \( \Gamma(F) \) is therefore topologically a punctured Riemann Sphere. By Koebe’s General Uniformization Theorem (see, for example, [Sim89]) that there is a planar domain \( \Omega \) with \( K \) boundary components and a conformal diffeomorphism \( \varphi : \Omega \to \Gamma(F) \).

By conformal invariance of energy and construction, \( \kappa := \pi_2 \circ \varphi : \Omega \to \mathbb{R}^n \) has the same energy as (the multiple-valued function) \( F \).

Our goal will be to show that that \( \kappa \) is conformal, as this will imply (2) of Theorem 2.0.2. Then, (2) paired with the energy/area inequality will then imply (1).

Supposing towards a contradiction, suppose that \( \kappa \) is not conformal. Following
[JDK+10, §8.2 and §4.5], we find a domain $\tilde{\Omega}$ and a diffeomorphism $\sigma : \tilde{\Omega} \to \Omega$ so that $\text{Dir}(\kappa \circ \sigma) < \text{Dir}(\kappa)$. Further, we may assume that $\sigma$ is arbitrarily close to the identity in the uniform topology, and thus that $\tilde{\Omega}$ has the same number of boundary components as $\Omega$.

Our next goal is to use $\tilde{\Omega}$ and $\sigma$ to produce a competitor for $F$ with less energy, which will contradict $F$'s minimizing property.

Let $\gamma_1, \ldots, \gamma_k$ be the boundary curves of $\tilde{\Omega}$ ordered so that $\kappa \circ \sigma(\gamma_i) \subset \Gamma_i$. Then, apply Theorem 2.3.2 to $\tilde{\Omega}$ with $d_i = k_i$ to obtain a $Q$-to-one branched cover $\Phi : \tilde{\Omega} \to \mathbb{D}$ which extends to a map on the closure of $\tilde{\Omega}$ for which the maps $\Phi|_{\gamma_i}$ are monotonic and of degree $k_i$ onto the circle $S^1$.

We then define a multiple-valued map $G : \mathbb{D} \to A_Q$ as follows: for $z$ a regular point of the branched cover $\Phi$,

$$G(z) = \sum_{y \in \Phi^{-1}(z)} \left[ \kappa \circ \sigma(y) \right]$$

and extended by continuity to the finitely many branch points of $\Phi$. One readily checks that the energy of $G$ is (since $\Phi$ is holomorphic and extends to a monotonic map on the boundary, and hence conformal except at finitely many interior points) equal to the energy of $\kappa \circ \sigma$, a contradiction, since $G$ is also admissible. Therefore $\kappa$ is conformal, proving (2) and hence (1).

To prove (3), we follow an argument similar to the one presented in [JDK+10, Theorem 3, §4.5]. If one of the $f_i$'s were not a homeomorphism, since it is monotonic it must map some arc $\gamma$ along $S^1$ to a constant $\alpha \in \mathbb{R}^n$. Let $x_0^{k_i}$ be some point on the
interior of this arc. We then find a harmonic selection on some small ball $B_r(x_0) \cap \mathbb{D}$ by Lemma 2.2.4. By (1), and since this selection is unique (up to reordering), it must be conformal. Let $F_i$ be the function on $B_r(x_0) \cap \mathbb{D}$ which maps $B_r(x_0) \cap \gamma$ to a constant. By Schwarz reflection and conformality, we obtain that $F_i \equiv \alpha$ and hence, since $\mathbb{D} \setminus \sigma(F)$ is path connected (where $\sigma(F)$ is the set of branch points of $F$), that $\alpha \in \text{spt}(F(z))$ for all $z \in \mathbb{D}$, an obvious impossibility given the separation assumptions on the curves $\Gamma_i$.

\[ \square \]

2.4 A Class of Examples

In this section we produce an example of two curves $\Gamma_1$ and $\Gamma_2$ for which the Plateau solution produced in the above section does not allow for a global selection so that there is necessarily a branch point and the image is connected. Further, we give a method for producing a large number of such examples.

Our last proposition is the following:

**Proposition 2.4.1.** Suppose that $V \subset \mathbb{C} \times \mathbb{C}^n$ is a holomorphic variety for which the projection $\pi(z_1, \ldots, z_{n+1}) = (z_1)$ is a $Q$ to 1 cover of the disk $\mathbb{D}$ except for possibly finitely many points on the interior of $\mathbb{D}$, and for which $S^1 \times \mathbb{C}^n$ intersects $V$ in $Q$ disjoint curves. Then the multiple-valued function $z \mapsto \pi^{-1}(z) \cap V$ is the Plateau solution for the given curves.

The proof of the proposition relies on the following elementary lemma.
Lemma 2.4.2. Suppose $\Omega \subset \mathbb{R}^2$ is compact, and that $F : \Omega \rightarrow \mathbb{R}^n$ is injective and Lipschitz. Then, if $\tilde{F}(x,y) = (x,y,F(x,y)) \in \mathbb{R}^2 \times \mathbb{R}^n$, and $[\Omega]$ is the canonically-oriented two-dimensional current in $\mathbb{R}^2$, we have:

$$M(F_\sharp([\Omega])) + |\Omega| \leq M(\tilde{F}_\sharp([\Omega]))$$

Proof. One readily checks that, if $a$ and $b$ are non-negative so that:

$$DF^T \circ DF = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then,

$$\tilde{D}\tilde{F}^T \circ \tilde{D}\tilde{F} = \begin{pmatrix} 1 + a & b \\ c & 1 + d \end{pmatrix}.$$ 

Therefore,

$$1 + \sqrt{\det(DF^T \circ DF)} \leq \sqrt{\det(\tilde{D}\tilde{F}^T \circ \tilde{D}\tilde{F})}$$

which combined with the area formula implies the result.

To prove the proposition, we recall two results from [Spa10], see Lemma 1.8 and Proposition 2.2 and the definitions preceeding them for more details:

Theorem 2.4.3. Let $f \in W^{1,2}(\Omega,A_Q(\mathbb{R}^n)$ be so that $\mathcal{M}(T_{f,\Omega}) < \infty$. Then:

$$\mathcal{M}(T_{\lambda f,\Omega}) = Q|\Omega| + \frac{\lambda^2}{2}\text{Dir}(f,\Omega) + o(\lambda^2) \text{ as } \lambda \rightarrow 0$$
Further, if $F$ is the associated $Q$-valued function of a holomorphic variety over $\Omega \subset \mathbb{C}$, then:

$$M([V] \land \Omega \times \mathbb{C}^n) = Q|\Omega| + \frac{\text{Dir}(F, \omega)}{2}$$

Proof of Proposition 2.4.1. Note that $V_\lambda = \{(z, \lambda w) \mid (z, w) \in V\}$ is also a holomorphic variety, and, if $F_\lambda$ is the corresponding $Q$-valued function, Theorem 2.4.3 yields:

$$M([V_\lambda] \land D \times \mathbb{C}^n) = Q|D| + \frac{\text{Dir}(F_\lambda, D)}{2} = Q|D| + \frac{\lambda^2}{2} \text{Dir}(F, D)$$

Let $G : D \to \mathbb{C} \times \mathbb{C}^n$ be any admissible map for $\Gamma_1, ..., \Gamma_Q$. Then, note that if $P : \mathbb{C} \times \mathbb{C} \times \mathbb{C}^n$ is the projection onto the last components, $\pi(z, w, v) = (w, v)$ for $z, w \in \mathbb{C}$ and $v \in \mathbb{C}^n$ and $\lambda > 0$ is given,

$$M([V_\lambda] \land D \times \mathbb{C}^n) \leq M(P_\sharp(T_{\lambda G,D}))$$

since complex varieties are absolutely mass minimizing and $G$ is admissible.

However, using the above inequalities, this amounts to:

$$Q|D| + \frac{\lambda^2}{2} \text{Dir}(F, D) \leq M(P_\sharp(T_{\lambda G,D}))$$

However, by the definition of the current $T_{\lambda G,D}$ and Lemma 2.4.2, we see that the above gives:

$$2Q|D| + \frac{\lambda^2 \text{Dir}(F, D)}{2} \leq M(P_\sharp(T_{\lambda G,D})) + Q|D| \leq M(T_{\lambda G,D}) = Q|D| + \frac{\lambda^2}{2} \text{Dir}(f, D) + o(\lambda^2)$$
and therefore,
\[
Q|\mathbb{D}| + \frac{\lambda^2}{2} \text{Dir}(F, \mathbb{D}) \leq \frac{\lambda^2}{2} \text{Dir}(G, \mathbb{D}) + o(\lambda^2)
\]

Finally, notice that \(\lambda F\) allows for a local conformal selection, and, if \(\tilde{F}\) denotes the graph map as in Lemma 2.4.2, that \(1 + \frac{\lambda^2}{2} |DF|^2 = \sqrt{\det(D\tilde{F}^T \circ \tilde{F})}\), so the area-energy equality for conformal maps, the above gives:
\[
\frac{\lambda^2}{2} \text{Dir}(\tilde{F}, \mathbb{D}) \leq \frac{\lambda^2}{2} \text{Dir}(G, \mathbb{D}) + o(\lambda^2)
\]
which implies \(\text{Dir}(\tilde{F}, \mathbb{D}) \leq \text{Dir}(G, \mathbb{D})\), proving the proposition.

\(\square\)

For a specific example, consider the multiple-valued map \(F\) defined by \(z \mapsto (z, \sqrt{z^2 - \frac{1}{4}})\) for \(z \in \mathbb{C}\). It is clear that \(F\) has two connecting branch points located at the points \(z = \pm \frac{1}{2}\).

Note that \(F|_{S^1}\) has a selection given by:

\[
\begin{align*}
\bullet \ f_1(\theta) &= (\exp(i\theta), \sqrt{(1 + \frac{1}{16} - \frac{\cos(2\theta)}{2}) \exp(i \cdot \frac{\text{atan}2(\sin(2\theta), \cos(2\theta) - \frac{1}{4})}{2})}) \\
\bullet \ f_2(\theta) &= (\exp(i\theta), \sqrt{(1 + \frac{1}{16} - \frac{\cos(2\theta)}{2}) \exp(i \cdot \frac{\text{atan2}(\sin(2\theta), \cos(2\theta) - \frac{1}{4})}{2} + \pi)})
\end{align*}
\]

Let \(f_i = (f_i^1, f_i^2, f_i^3)\). Then, even though the above selection is discontinuous it may be used to show that the boundary has two connected components, as is suggested by the following images of the real and imaginary parts of the functions \(f_i^3\).

One may check that \(\Re(f_i^3(\theta)) = \Re(f_i^3(\theta))\) if and only if \(\theta \in \{\pm \frac{\pi}{2}\}\) and \(\Im(f_i^3(\theta)) = \Im(f_i^3(\theta))\) if and only if \(\theta \in \{0, \pi\}\).
Further, these images suggest that the following will give a continuous selection:

\[
g_1(\theta) = \begin{cases} 
  f_1(\theta) : & \theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \\
  f_2(\theta) : & \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]
\end{cases}
\]

\[
g_2(\theta) = \begin{cases} 
  f_2(\theta) : & \theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \\
  f_1(\theta) : & \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]
\end{cases}
\]

Which one checks provides a continuous selection of the boundary data. Further, by (locally) holomorphically defining \( z \mapsto \sqrt{z \pm \frac{1}{2}} \) we see that, for sufficiently small \( \rho > 0 \), \( F|_{\partial B_\rho(\pm \frac{1}{2})} \) is a connecting branch point, since then it has structure \( \varphi \circ \sqrt{z \mp \frac{1}{2}} \), for \( \varphi \) holomorphic. Therefore, Proposition 2.4.1 guarantees that \( F \) is the Plateau solution, and thus that the Plateau solution is branched.

Branch points could also occur in \( \mathbb{R}^3 \) for essentially topological reasons. For \( 0 < \epsilon < 1 \), consider on the circle \( S^1 \), a single branch of the 2-valued function \( \phi(z) = (\sqrt{z - \epsilon}, \text{Re}\sqrt{z - \epsilon}) \). Note that the image, under this 2-valued map is a single
smoothly embedded circle in $\mathbb{R}^2 \times \mathbb{R}$ whose projection onto $\mathbb{R}^2$ circulates the origin twice. Then $\phi$ does not admit a continuous 2-valued extension $F : \mathbb{D} \to A_2(\mathbb{R}^2 \times \mathbb{R})$ which is a locally smooth embedding without branch points. In fact otherwise one could start at the origin with two values and smoothly extend to obtain a selection on the whole disk, the image of whose boundary values would, unlike $\phi$, be two embedded curves. In particular, any 2-valued Plateau solution in $\mathbb{R}^3$ starting with this “doubly-wrapped” 2-valued boundary $\phi$ must have branch point. By the same argument we see that:

*Any solution of the Plateau problem in Theorem 1.2 starting with boundary parameterization as in Theorem 1.1 with multiple wrapping (i.e. some $k_i > 1$), whose boundary curves satisfy condition 2.0.1 must have a branch point. In case $N = 3$, the image then must curves of self-intersection.*

It is the subject of current work to determine whether or not one can verify the conditions of Definition 2.0.1 in $\mathbb{R}^3$.

However, with $Q = 2$ and $k_1 = k_2 = 1$, for two closed curves lying in different planes in $\mathbb{R}^3$, one can, using similar arguments to those in [JDK+10, §8.8] to guarantee that our Plateau solutions do in fact admit branch points in three-dimensional space. Further, constancy along a boundary curve can have elaborate branching behavior, as illustrated in the example below—although this example does differ slightly from the type of degeneracy that would occur if given boundary data did not admit a wrapped solution.

In particular, we may apply the following general argument to produce a minimizing mapping with branch points. For a curve $\Gamma \subset \mathbb{R}^3$, we choose $Q = k_1 = 2$, and
find \( F : \mathbb{D} \to A_2(\mathbb{R}^3) \) as in Theorem 3.3.3. We put:

\[
F|_{S^1}(z) = \sum_{\zeta^2 = z} \llbracket f(\zeta) \rrbracket
\]

for some monotone map \( f : S^1 \to \Gamma \). The following dichotomy then applies:

1. If \( f \) is not constant on any interval of length \( \pi \). Then \((\Gamma, 2)\) admits a wrapped solution, and thus \( F \) admits branch points, and the multiple-valued images has curves of self intersection.

2. If \( f \) is constant (say identically equal to \( c \in \Gamma \)) on an interval of length \( \pi \), then on \( S^1 \), \( F \) admits a selection:

\[
F|_{S^1}(z) = \llbracket c \rrbracket + \llbracket g(z) \rrbracket.
\]

If \( F \) has no branch points, then \( F \) admits and global selection and \( c \in \text{spt}(F(z)) \) for all \( z \in \mathbb{D} \). Therefore, by harmonically extending \( g \) to \( \mathbb{D} \) (still denoted by \( g \)), we produce a new 2-valued function:

\[
G(z) = \sum_{\zeta^2 = z} \llbracket g(\zeta) \rrbracket
\]

which has the same energy as \( F \) and a branch point at 0.

On the other hand, if \( F \) has branch points, then we have produced a map as desired.
The next section will prove that the second case can actually occur—it should be noted, however, that Theorem 2.0.2 fails in this case. Indeed, the proof of Theorem 2.0.2 required that the set
\[ \Gamma(F) = \{(z, w) \mid w \in \text{spt}(F(z))\} \]
be a smooth, Riemannian manifold. In this degenerate case, \( \Gamma(F) \)'s boundary is a wedge of two circles.

### 2.5 An Example of a Degenerate Behavior

In this section we provide a negative answer to the following question for values of \( Q > 1 \):

**Question 2.5.1.** Let \( Q \geq 2 \) and suppose that \( F : \mathbb{D} \to A_Q(\mathbb{R}^n) \) is Dirichlet minimizing, and \( F|_{S^1} = \sum_{i=1}^{Q-1} [f_i] + [0] \), for monotone, continuous maps \( f_i : S^1 \to \Gamma_i \), with \( \Gamma_i \) closed Jordan curves with \( 0 \not\in \Gamma_i \). Then, is \( 0 \in \text{spt}(F(x)) \) for all \( x \in \mathbb{D} \)?

Note that the answer is *yes* for \( Q = 1 \), by the maximum principle for harmonic functions. Similarly, for \( n = 1 \), the answer to Question 2.5.1 is again *yes*, since one may find a continuous (and therefore Sobolev, since \( F \) will be affinely approximatable almost everywhere, with \( |DF| \) square integrable) selection for \( F \), one of whose traces is zero on the boundary.

These facts leads to the question for higher \( Q \) – and is an *a priori* type of degeneracy that can occur for certain types of sequences that could have occurred in our existence
proof. The answer to this question also illustrates just how weak the maximum principal (as in Section 3.2 of [DLS11]) for multiple-valued functions actually is.

Our example is in the case $Q = 2$ and $n = 2$, and our proof is unique to this case. However, simple examples show that this example may be extended to all $Q \geq 2$, provided $n > 1$.

The key is to look instead for zero average multiple-valued functions on $D$ that admit a selection on $S^1$. The following notation will be useful in proving an equivalence between the two concepts.

**Notation 2.5.2.** For $f : \mathbb{D} \to \mathbb{R}^n$ and $H \in W^{1,2}(\mathbb{D}, A_2(\mathbb{R}^n))$, we denote by $H + f$ the function $[h_1 + f] + [h_2 + f]$, where $h_1$ and $h_2$ is a measurable selection for $H$. Note that $H + f$ is independent of the selection chosen.

Next we recall a result of [Spa10]:

**Theorem 2.5.3.** Let $V \subset \mathbb{C}^\mu \times \mathbb{C}^\nu$ be an irreducible holomorphic variety which is a $Q$-to-one cover of the ball $B_2 \subset \mathbb{C}^\mu$ under the orthogonal projection. Then there is a Dirichlet minimizing $f \in W^{1,2}(B_1, A_Q(\mathbb{R}^{2\nu}))$ so that $\text{graph}(f) = V \cap (B_1 \times C^\mu)$.

Consider the variety $V = \{(z, w) \mid z^2 - \frac{1}{4} = w^2\}$. It is easy to see that $p(z, w) = w^2 - z^2 - \frac{1}{4}$ is irreducible. Notice that for $z \neq \pm \frac{1}{2}$, $V$ is $2 : 1$ cover under the orthogonal projection. Therefore, Theorem 2.5.3\(^2\) gives that the multiple-valued function $z \mapsto \sqrt{z^2 - \frac{1}{4}}$ is Dirichlet Minimizing over the disk $\mathbb{D}$.

\(^2\)In the proof of Theorem 2.5.3, [Spa10] actually only requires (using standard notation for currents in Euclidean space) that $\pi_2([V]) = 2[B_2]$
Further, as shown in Section 6, $F$ admits a continuous selection on the boundary $S^1$ of $\mathbb{D}$, since this result follows from the fact that the graph map $z \mapsto (z, \sqrt{z^2 - 1/4})$ maps $S^1$ to two components. We also notice that $F$ is symmetric—i.e. that if $f_1$ and $f_2$ represent a measurable selection for $F$, then, since $f_1 + f_2$ is harmonic and zero on the boundary, we must have $f_1 + f_2 = 0$.

Therefore, let $F|_{S^1} = [f] + [-f]$. With a slight abuse of notation, let $f$ also denote the harmonic extension of $f$ to the disk $\mathbb{D}$. We finally check that, since $F$ is Dirichlet minimizing, $\tilde{F} = F + f$ defined as in Notation 2.5.2 is also Dirichlet minimizing for its boundary data.

This follows immediately from the following proposition.

**Proposition 2.5.4.** Let $H \in W^{1,2}(\mathbb{D}, A_2(\mathbb{R}^n))$ be Dirichlet minimizing and suppose that $H|_{S^1} = [H_1] + [H_2]$ for $H_1, H_2 : S^1 \to \mathbb{R}^n$ continuous. Let $h$ denote the harmonic extension of $H_1$ to $\mathbb{D}$. Then:

1. if $H_1 = -H_2$, then $\text{Dir}(H + h, \mathbb{D}) = \text{Dir}(H, D) + 2\text{Dir}(h, \mathbb{D})$

2. if $H_2 = 0$, then $\text{Dir}(H - \frac{h}{2}, \mathbb{D}) = \text{Dir}(H, D) - 2\text{Dir}(\frac{h}{2}, \mathbb{D})$

**Proof of Proposition.** Denote for two $m \times n$ matrices $(a_{ij})$ and $(b_{ij})$ the matrix inner product:

$$(a_{ij} : b_{ij}) = \sum_{i,j} a_{ij} \overline{b_{ij}}.$$  

Note that the norm induced by this inner product is the Hilbert-Schmidt norm.
Let $h_1$ and $h_2$ denote a measurable selection for $G$. In case (1), properties of the inner product above imply that at any point where $G$ is affinely approximatable (which constitutes a full measure set of $\mathbb{D}$) we have:

\[ ||Dh_1 + Dh||^2 + ||Dh_2 + Dh||^2 = ||DH||^2 + 2||Dh||^2 + 2(Dh_1 + Dh_2 : Dh). \]

But, $2(Dh_1 + Dh_2 : Df) = 0$ since $h_1 + h_2 = 0$, and this gives the result.

For case (2):

\[ ||Dh_1 - \frac{Dh}{2}||^2 + ||Dh_2 - \frac{Dh}{2}||^2 = ||DH||^2 + 2\left(\frac{Dh}{2}\right)^2 - 2(Dh_1 + Dh_2 : \frac{Dh}{2}). \]

In this case, though, since $h_1 + h_2$ is harmonic and has the same boundary data as $h$ it must be the case that $h_1 + h_2 = h$, so:

\[ 2(Dh_1 + Dh_2 : \frac{Dh}{2}) = 2(Dh, \frac{Dh}{2}) = 4\left|\frac{Dh}{2}\right|^2. \]

Which implies (2).

Now we prove $\tilde{F}$ is Dirichlet minimizing. Statement (1) of the above proposition guarantees:

\[
\text{Dir}(\tilde{F}, \mathbb{D}) = \text{Dir}(F + f, \mathbb{D}) = \text{Dir}(F, \mathbb{D}) + 2\text{Dir}(f, \mathbb{D})
\]

Let $G \in W^{1,2}(\mathbb{D}, A_2(\mathbb{R}^n))$ be Dirichlet minimizing and suppose that $G|_{S^1} = \tilde{F}|_{S^1} = [2f] + [0]$. Then case (2) of the proposition implies:
\[ \text{Dir}(G - f, \mathbb{D}) = \text{Dir}(G, \mathbb{D}) - 2\text{Dir}(f, \mathbb{D}). \]

So, if

\[ \text{Dir}(G, \mathbb{D}) < \text{Dir}(\tilde{F}, \mathbb{D}) \]

then:

\[ \text{Dir}(G, \mathbb{D}) < \text{Dir}(F, \mathbb{D}) + 2\text{Dir}(f, \mathbb{D}) \]

which implies:

\[ \text{Dir}(G - f, \mathbb{D}) = \text{Dir}(G, \mathbb{D}) - 2\text{Dir}(f, \mathbb{D}) < \text{Dir}(F, \mathbb{D}). \]

However, this last inequality is impossible since \( F \) is minimizing, so \( \tilde{F} \) is Dirichlet minimizing.

However, since \( F \) admits no global selection (due to its connecting branch points), \( \tilde{F} \) also admits no such selection, and therefore \( 0 \not\in \text{spt}(\tilde{F}(x)) \) for some \( x \in \mathbb{D} \), so \( \tilde{F} \) is a counterexample to Question 2.5.1.
Chapter 3

Semi-Algebraic Homology

In this chapter we give a brief introduction to semi-algebraic sets, define semi-algebraic chains (a particular subclass of currents) and prove results on the integral current homology groups on semi-algebraic sets analogous to those obtained by Federer and Fleming in the ELNR case.

The main idea for this chapter comes from an attempt to mimic the proof of the mass minimization result for compact Riemannian manifolds, proven by H. Federer in [Fed69]. The main point of that proof was that any such manifold could be isometrically embedded in Euclidean space as a Lipschitz retract of some of its open neighborhoods on which tools such as the deformation theorem could be applied. Unfortunately, a semi-algebraic set need not be a Lipschitz retract of any of its open neighborhoods—as can be seen by simple examples such as \( \{(t^2, t^3) | t \in \mathbb{R} \} \subset \mathbb{R}^2 \).

However, in loose analogy with the ELNR case, given a semi-algebraic \( X \subset \mathbb{R}^n \), there are arbitrarily small neighborhoods which are semi-algebraically contractible—and thus the inclusion of \( X \) into these neighborhoods induces a surjective chain map...
between the groups of semi-algebraic chains, and it is this surjection that enables us to
generalize the other results to this context once the above isomorphism is established.

Since each semi-algebraic chain is an integral current, proving the isomorphism
reduces to showing the following is true in (pairs of) compact semi-algebraic sets:

1. Every semi-algebraic cycle that bounds an integral chain also bounds a semi-
   algebraic chain.

2. Every rectifiable cycle is homologous to a semi-algebraic cycle.

We verify these facts in Section 3.3, see Propositions 3.3.4 and 3.3.7, respectively.
Statement (1) follows from retracting certain deformations of semi-algebraic cycles
supported on $X$ within the open neighborhoods described above. Statement (2) is
more difficult and relies on the local Lipschitz contractibility of semi-algebraic sets,
as shown by L. Shartser and G. Valette in [SV10] (see also the independent work of
L. Shartser, [Sha11]).

Related topics are also addressed in the interesting paper of T. De Pauw [DP07]
which treats relations among integral currents, various homology theories, and Plateau
problems. Similarly, integral current homology for chains in metric spaces is discussed
in [DPHP].

The main reference for the geometric measure theory contained in the following
is H. Federer’s treatise [Fed69], but much of the theory can be found in the books
of F. H. Lin and X. Yang, and the work of L. Simon, [LY02], [Sim83], respectively.
Similarly, we repeatedly use well-known properties of semi-algebraic sets, such as
triangulability and the existence of stratifications. To the author’s knowledge, the
most comprehensive reference for this material is the book of J. Bochnak, M. Coste and M. Roy, [BCR98]. Similar results for the more general case of $\sigma$-minimality may be found in the book of L. van den Dries, [VdD98]. See also the paper of M. Edmundo and A. Woerheide [EW08], where classical homology theories were analyzed in this more general setting.

3.1 Semi-Algebraic Chains: Definitions and Elementary Properties

We denote by $\mathcal{D}^k(\mathbb{R}^n)$ the set of compactly supported differential $k$-forms on $\mathbb{R}^n$, and by $\mathcal{H}^\alpha$ the $\alpha$-dimensional Hausdorff measure on $\mathbb{R}^n$. Finally, for a semi-algebraic subset $X$ of $\mathbb{R}^n$, we denote $\text{Fron}(X) = \overline{X} \setminus X$ the frontier of $X$.

If $f : M \subset \mathbb{R}^m \to \mathbb{R}^n$ is a semi-algebraic map with $M$ a semi-algebraic subset of $\mathbb{R}^m$, then by [HLTV11, 2.5] we may find a finite collection $\{V_i\}_{i=1}^q$ of subsets of $M$ so that $M = \bigcup V_i$ and so that the following holds:

1. each $V_i$ is a connected, real analytic, embedded submanifold of $\mathbb{R}^n$ with $\text{Fron}(V_i)$ being a union of lower dimensional elements of $\{V_i\}_{i=1}^q$.

2. The set $f(V_i)$ is a real analytic, semi-algebraic manifold and either $f|_{V_i}$ is a real analytic diffeomorphism onto $f(V_i)$, or $f|_{V_i}$ is real analytic and of constant rank less than $\dim(V_i)$.

Such a partition of $M$ is known as a stratification of $M$ and we call the elements of $\{V_i\}_{i=1}^q$ the strata. We may also require that $\{f(V_i)\}_{i=1}^q$ is a stratification for $f(M)$ and
further, if $A \subset M$ is a semi-algebraic subset, that $A$ is a union of some subcollection of $\{V_i\}_{i=1}^q$. A stratification satisfying this last property is said to be *compatible* with $A$.

Now suppose that $M$ is an oriented, compact, $k$-dimensional smooth semi-algebraic submanifold of $\mathbb{R}^n$. Relabeling if necessary, let $\{V_i\}_{i=1}^\ell$ denote the strata such that $f|_{V_j}$ is of rank $k$. Set $N = \{f(V_i)\}_{i=1}^\ell$, and choose the orientation on each $f(V_i)$ induced by pushing forward the orientation of $V_i$ inherited from the orientation of $M$ by $f|_{V_i}$. Denote the inclusion map $f(V_i) \hookrightarrow \mathbb{R}^n$ by $\phi_{V_i}$ and define a current $f_\sharp(\llbracket M \rrbracket)$ such that:

$$f_\sharp(\llbracket M \rrbracket)(\omega) = \sum_{i=1}^\ell \int_{f(V_i)} \phi_{V_i}^*(\omega)$$

for all $\omega \in \mathcal{D}^k(\mathbb{R}^n)$.

Even though $f$ is not Lipschitz, $f(V_i)$ has finite $\mathcal{H}^k$ measure by [Fed69, 3.4.8 (3)] and by Proposition 3.1.1, we see that any such current is an integral current.

By combining terms where the sets $f(V_i)$ and $f(V_j)$ are equal, we may simplify the expression to:

$$f_\sharp(\llbracket M \rrbracket)(\omega) = \sum_{N \in N} n_N \int_N \phi_N^*(\omega)$$

where each integer $n_N$ depends on a choice of orientation of $N$ and whether each $f|_{V_i}$ preserves or reverses the orientation. If $\nu_N$ is an orientation form for $N \in \mathcal{N}$ consistent with the orientation chosen (i.e. a unit simple $k$-vector field on $N$ whose vectors span $T_xN$ at every point $x \in N$), then we have the alternate representation:
\[
 f_\sharp(\mathbb{M})(\omega) = \sum_{N \in \mathbb{N}} n_N \int_{V_i} \phi_N^*(\omega)(x)(\nu_N(x)) \, d\mathcal{H}^k(x).
\]

Our next goal is to show that the integral above may be re-written by the area formula as an integral over \( V_i \) of the pullback form \((f|_{V_i})^*(\phi_V^*(\omega))\).

Note that since \( f|_{V_i} \) is smooth, for any compact exhaustion \( \{V_{i,j}\}_{j \in \mathbb{N}} \) of \( V_i \), we have, for all \( j \) that \( f|_{V_{i,j}} \) has bounded derivative and is therefore Lipschitz. Therefore, denoting the inclusion \( f(V_{i,j}) \hookrightarrow \mathbb{R}^n \) by \( \phi_{V_{i,j}} \), the area formula implies that:

\[
 \int_{V_{i,j}} J_k df|_{V_{i,j}}(x) \, d\mathcal{H}^k(x) = \mathcal{H}^k(f(V_{i,j})) \leq \mathcal{H}^k(f(V_i)) < \infty. \tag{3.1.3}
\]

Therefore by applying the Monotone Convergence Theorem as \( j \to \infty \), we obtain that the Jacobian \( J_k df|_{V_i} \) is integrable over \( V_i \). Next notice that

\[
 \int_{f(V_{i,j})} \phi_{V_{i,j}}^*(\omega)(x)(\nu_{f(V_{i,j})}) \, d\mathcal{H}^k(x) = \int_{V_{i,j}} (f|_{V_{i,j}})^*(\phi_{V_{i,j}}^*(\omega))(\nu_{V_{i,j}}) \, d\mathcal{H}^k(x). \tag{3.1.4}
\]

Since \( \phi \) is smooth on \( \mathbb{R}^n \) and thus bounded in some neighborhood of \( X \), the integrability of \( J_k df|_{V_i} \) allows us to apply the Dominated Convergence Theorem (again, as \( j \to \infty \)) to the above equality. Summing over \( i \) then gives:

\[
 f_\sharp(\mathbb{M})(\omega) = \sum_i \int_{V_i} (f|_{V_i})^*(\phi_{V_i}^*(\omega)). \tag{3.1.5}
\]

The above equality implies that our definition is invariant under refinement of the stratification.

Therefore, if we are given a different stratification of \( M \), say \( \{W_i\}_{i=1}^\ell \), by finding
a common refinement for the two stratifications we see that the rule $f_\sharp([M])$ is well defined.

For the remainder of this chapter, we fix $X$, a semi-algebraic subset of $\mathbb{R}^n$.

**Definition 3.1.1.** A *semi-algebraic k chain in $\mathbb{R}^n$* is a current constructed as above. Denote by $S_k(\mathbb{R}^n)$ the set of all such currents. We define

$$S_k(X) = \{ S \in S_k(\mathbb{R}^n) \mid \text{spt}(S) \subset X \}$$

to be the set of semi-algebraic k chains of $X$.

It is easy to see that $S_k(X)$ is an Abelian group.

The following notation will be used frequently throughout the remainder of the chapter: for any smooth, compact oriented $k$-dimensional submanifold $N \subset \mathbb{R}^n$ we denote by $[N]$ the $k$-current given by $[N](\omega) = \int_N \phi^*(\omega)$, where $\phi$ denotes the inclusion $N \hookrightarrow \mathbb{R}^n$.

If $\nu_N$ is an orientation form for $N$ the area formula gives:

$$\int_N \phi^*(\omega) = \int_N \omega(x)(\nu_N(x)) d\mathcal{H}^k(x) \quad (3.1.6)$$

Proposition 3.1.3 will show that this causes no ambiguity with the notation $f_\sharp([M])$ defined above.

Arguing with the constancy theorem of [Fed69, 4.1.31] one can show the following result.

**Proposition 3.1.1 ([HLTV11, 3.5]).** For $M$ and $f$ as above $[M] \in S_k(\mathbb{R}^m)$, $[\partial M] \in S_{k-1}(\mathbb{R}^m)$ and, further, $\partial(f_\sharp([M])) = f_\sharp([\partial M])$. In particular, since the boundary
of a semi-algebraic submanifold is a finite, disjoint union of lower dimensional semi-
algebraic submanifolds, the chain $\partial(f_{\sharp}(\([M]\])) \in S_{k-1}(\mathbb{R}^n)$.

We can use this to classify semi-algebraic chains without the need for the maps $f$:

**Theorem 3.1.2** (Alternative description of semi-algebraic chains [HLTV11]). If $V_1, \ldots, V_r$ are disjoint real-analytic semi-algebraic oriented $k$-dimensional submanifolds of $\mathbb{R}^n$ such that each $\overline{V}_i$ is a compact subset of $X$, and $n_1, \ldots, n_r$ are integers, then:

$$\sum_{i=1}^{r} n_i [V_i] \in S_k(X).$$

Conversely, every element of $S_k(X)$ admits such a representation.

Thus the set of real-analytic semi-algebraic oriented $k$-dimensional submanifolds generate the group $S_k(X)$ of semi-algebraic $k$-chains in $X$ just as the set of oriented $k$-simplices in $\mathbb{R}^n$ generate the group $\mathcal{P}_k(\mathbb{R}^n)$ of $k$-dimensional polyhedral chains in $\mathbb{R}^n$. Clearly $\mathcal{P}_k(\mathbb{R}^n)$ is a subgroup of $S_k(\mathbb{R}^n)$.

We now turn to functoriality of the semi-algebraic homology theory. To any semi-algebraic map $g : X \to Y$, we associate a map $g_{\sharp} : S_k(X) \to S_k(Y)$ by:

$$g_{\sharp}(f_{\sharp}(\([M]\))) = (g \circ f)_{\sharp}(\([M]\)).$$

To see that the rule $g_{\sharp}$ is well-defined suppose $h_{\sharp}(\([N]\)) = f_{\sharp}(\([M]\))$. Choose a stratification $\{M_i\}_{i=1}^{m}$ for $M$ and $\{N_j\}_{j=1}^{\ell}$ for $N$ compatible with the maps $f$ and $g \circ f$ (respectively, $h$ and $g \circ h$). Such stratifications may be obtained, for example, by refining stratifications compatible with $f$ (or $h$). For any $\omega \in \mathcal{D}^k(\mathbb{R}^n)$:
\[
\sum_{i=1}^{m} \int_{f(M_i)} \omega(x)(\nu_{f(M_i)}(x))d\mathcal{H}^k(x) = \sum_{j=1}^{\ell} \int_{h(N_j)} \omega(x)(\nu_{h(N_j)}(x))d\mathcal{H}^k(x).
\]

However, since each \(h(n_j)\) and \(f(M_i)\) have finite \(\mathcal{H}^k\) measure this equality holds for any \(\mathcal{H}^k\) summable \(k\)-form on \(f(M) \cup h(N)\) by approximation. In particular, for \(\omega \in \mathcal{D}^k(Y)\), \(g^*(\omega)\) is such a summable \(k\)-form, which gives:

\[
\sum_{i=1}^{m} \int_{f(M_i)} g^*(\omega)(x)(\nu_{f(M_i)}(x))d\mathcal{H}^k(x) = \sum_{j=1}^{\ell} \int_{h(N_j)} g^*(\omega)(x)(\nu_{h(N_j)}(x))d\mathcal{H}^k(x)
\]

However, by Equation 3.1.1 the left side of this equation is \((g \circ f)_\sharp(\llbracket M \rrbracket)\) and the right side is \((g \circ h)_\sharp(\llbracket N \rrbracket)\), so the rule \(g_\sharp\) given above is well-defined.

The following proposition allows us to relate classical constructions using currents to semi-algebraic chains as defined above. We follow the notations for currents found in [Fed69, §4].

In particular, for a Lipschitz map \(f : X \to Y\), and a \(k\)-current of finite mass \(T\), such that \(f|_{\text{spt}(T)}\) is proper, we denote the classical *push forward* of the current \(T\) (defined by smooth approximation in [Fed69, 4.1.7]) by \(f^I_{\sharp}(T)\).

**Proposition 3.1.3.** For \(S \in S_k(X)\), \(A \subseteq X\) a semi-algebraic subset, if \(Y \subseteq \mathbb{R}^m\) semi-algebraic and \(f : X \to Y \subseteq \mathbb{R}^m\) a smooth semi-algebraic map. Then

1. The current \(S \ll A\) is a semi-algebraic chain.

2. The relation \(f_\sharp(S) = f^I_{\sharp}(S)\) holds whenever both sides are defined. That is, whenever \(f\) is a Lipschitz semi-algebraic map such that \(f|_{\text{spt}(S)}\) is proper.
3. The current \([0, 1] \times S\) is a semi-algebraic chain.

Proof. By linearity and Theorem 3.1.2, we lose no generality in assuming that \(S\) is of the form \([V]\), where \(V\) is a smooth semi-algebraic \(k\) manifold in \(\mathbb{R}^n\) with compact closure. Let \(\nu_V\) be the orientation form for \(V\).

Applying the stratification theorem [BCR98, 9.1.8] to the set \(A\) and the singleton family \(\{V \cap A\}\), we get a stratification \(\{A_i\}_{i=1}^p\) for \(A\), where, for some \(\ell \leq q\), \(V \cap A = \bigcup_{i=1}^\ell A_i\) a union of smooth, semi-algebraic strata. Relabeling if necessary, we remove from this union any stratum which is of dimension less than \(k\). The remaining strata are smooth semi-algebraic manifolds of dimension \(k\), and hence \(k\)-dimensional rectifiable sets. This implies that, for \(\mathscr{H}^k\) a.e. \(x \in A_i \cap V\), the vector field \(\nu_V\) is an orientation form for \(A_i\) (see, for example, [AFP00, 2.85]). Thus, for any \(k\)-form \(\omega\):

\[
S \llcorner A(\omega) = \int_{V \cap A} \omega(x)(\nu_V(x))d\mathcal{H}^k(x) = \sum_{i=1}^\ell \int_{A_i} \omega(x)(\nu_V(x))d\mathcal{H}^k(x) = \sum_{i=1}^\ell [A_i]\omega
\]

and hence \(S \llcorner A\) is a semi-algebraic chain, proving (1).

Since \(V = \text{spt}(S)\), we may stratify the map \(f|_{\text{spt}(S)} : \text{spt}(S) \to Y\) as in [HLTV11, 2.5] to get a stratification of \(V\) with each strata \(A_i\) a smooth submanifold with \(f|_{A_i}\) a diffeomorphism. By [Sim83, 26.21], for any \(\omega \in \mathcal{D}^k(\mathbb{R}^m)\),

\[
f^IC(S)(\omega) = \int_{V} \omega(f(x))(Df_\sharp\nu_V(x))d\mathcal{H}^k(x) = \sum_{i=1}^n \int_{A_i} \omega(f(x))(Df_\sharp\nu_V(x))d\mathcal{H}^k(x)
\]

Since \(f\) is a diffeomorphism on each \(A_i\), each integrand in the last sum is
\[ \epsilon_i \int_{f(A_i)} \omega(x)(\nu(x))d\mathcal{H}^k(x) \]

where \( \epsilon_i = \pm 1 \), depending on orientations. So, \( f^IC(S) = \sum_{i=1}^n \epsilon_i \|f(A_i)\| \).

On the other hand, this semi-algebraic chain coincides with \( f^\bullet(S) \) as defined in the previous section, giving (2).

Finally, (3) follows immediately by applying (1) to the set \( A = [0, 1] \times \mathbb{R}^n \) and the semi-algebraic chain \( \llbracket(-1, 2) \times V\rrbracket \).

Remark 3.1.1. The preceding Proposition allows us to generalize the well-known homotopy formula for currents to the semi-algebraic context. To this end, suppose that \( M \subset \mathbb{R}^m \) is a smooth, oriented compact semi-algebraic manifold of dimension \( k \) and \( h : M \times [0, 1] \to \mathbb{R}^n \) is a semi-algebraic map. Proposition 3.1.1 guarantees that \( \partial h_\sharp(\llbracket M \times [0, 1]\rrbracket) = h_\sharp(\llbracket \partial (M \times [0, 1])\rrbracket) \). Consider the three sets \( M \times \{0\} \), \( M \times \{1\} \) and \( \partial M \times [0, 1] \). Since any two of these three sets intersect in a semi-algebraic set of dimension at most \( k - 2 \), by stratifying the set \( \partial (M \times [0, 1]) \) compatibly with the three sets listed above and their pairwise intersections, we see that Equation 3.1.2 guarantees that the current \( h_\sharp(\llbracket \partial (M \times [0, 1])\rrbracket) \) decomposes (up to sets of \( \mathcal{H}^k \)-measure zero) into \( h_\sharp(\llbracket M \times \{1\}\rrbracket) - h_\sharp(\llbracket M \times \{0\}\rrbracket) + h_\sharp(\llbracket \partial M \times [0, 1]\rrbracket) \) as in the classical case.

### 3.2 The Homology of Semi-Algebraic Chains

By Proposition 3.1.1 the semi-algebraic chains give a chain complex, \( S_*(X) \). Similarly, for any semi-algebraic subset \( A \subseteq X \), we can define a chain complex for the semi-
algebraic pair \((X, A)\) by defining the Abelian group of relative cycles:

\[
Z_k(X, A) = \{ S \mid S \in S_k(X), \partial S \in S_{k-1}(A) \}
\]

(with the convention that if \(k = 0\) the condition \(\partial S \in S_{k-1}(A)\) is trivially satisfied)

and the subgroup of relative boundaries:

\[
B_k(X, A) = \{ S + \partial L \mid S \in S_k(A), L \in S_{k+1}(X) \}.
\]

we then define the \(k\)-dimensional semi-algebraic homology group of the pair \((X, A)\) as the quotient group:

\[
H^\text{SA}_k(X, A) = Z_k(X, A) / B_k(X, A).
\]

As usual, in the case where \(A = \phi\), we simply denote these sets as \(Z_k(X)\), \(B_k(X)\) and \(H^\text{SA}_k(X)\).

Working in the category of semi-algebraic pairs of sets and (continuous) semi-algebraic maps, we next show the following:

**Theorem 3.2.1.** The homology groups associated to the chain complex of semi-algebraic chains on a semi-algebraic set \(X\) satisfy the Eilenberg-Steenrod Axioms (1)-(7) listed below.

Moreover, by the compactness requirements of Theorem 3.1.2, they also satisfy the axiom of compact support (see, for example, [Mun84, § 26]).

This theorem is similar in spirit and method to the result proved in [Fed69, 4.4.1], where it is shown that the homology groups associated to the chain complex of integral
currents on a Lipschitz neighborhood retract satisfies the same axioms.

To this end, let \((X, A)\) and \((Y, B)\) be such pairs and \(f : (X, A) \to (Y, B)\) be a semi-algebraic map of pairs. By Proposition 3.1.1 the map \(f_\sharp\) induces to a map on the homology groups defined above; for convenience, also denote the induced map by \(f_\sharp\). Similarly, for \(A \subset X_0 \subset X\) we also let \(\partial\) denote the induced map on homology \(\partial : H_k(X, X_0) \to H_k(X_0, A)\). The Eilenberg-Steenrod axioms are as follows:

1. \((id_X)_\sharp\) is the identity map on \(H_k(X, A)\).

2. If \(g : (Y, B) \to (Z, C)\) is another admissible map, \((g \circ f)_\sharp = g_\sharp \circ f_\sharp\).

3. If \(C \subset A \subset X, C' \subset B \subset Y\) and \(f|_B : (A, C) \to (B, C')\) is admissible, then

\[
(f|_A)_\sharp \circ \partial = \partial \circ f_\sharp
\]

4. If \((X_0, A) \xrightarrow{i} (X, A) \xrightarrow{j} (X, X_0)\) is a series of admissible inclusions, then:

\[
\begin{array}{rcl}
H_k(X_0, A) & \xrightarrow{i_\sharp} & H_k(X, A) \\
& \xrightarrow{\partial} & H_k(X, X_0) \\
H_{k-1}(X_0, A) & \xrightarrow{i_\sharp} & H_{k-1}(X, A) \\
& \xrightarrow{j_\sharp} & H_{k-1}(X, A_0)
\end{array}
\]

is exact for \(k > 0\), and, for \(k = 0\), \(i_\sharp(H_0(X_0, A_0)) = H_0(X, A)\).

5. If \(h : I \times X \to Y\) is an admissible homotopy between \(f\) and \(f' : X \to Y\), then

\(f_\sharp = f'_\sharp\).

6. If \(g : (X, A) \to (X', A')\) is the inclusion map and \((X' \setminus A') \cap (X' \setminus X) = \phi\), then

\(g_\sharp : H_k(X, A) \to H_k(X', A')\) is an isomorphism.
7. For any $a \in \mathbb{R}^n$, $H_0(\{a\}) = \mathbb{Z}$ and, for $k > 0$, $H_k(\{a\}) = 0$.

The proofs follow as in [Fed69, 4.4.1], from various properties of currents and semi-algebraic chains:

Proof. Statement (1) follows from well-definedness of the map $id_x : S_k(X) \to S_k(X)$. Indeed, $f_z(\llbracket M \rrbracket) = (id_X \circ f)_z(\llbracket M \rrbracket)$.

Statement (2) is functoriality of the map $f \mapsto f_z$ and follows from the discussion in the previous section.

By Proposition 3.1.1 $\partial(f_z(g_z(\llbracket M \rrbracket))) = f_z(\partial g_z(\llbracket M \rrbracket))$. If $f(A) \subseteq B$ and spt($\partial g_z(\llbracket M \rrbracket)) \subset A$, then spt($f_z(\partial g_z(\llbracket M \rrbracket))) \subset B$. Equivalently,

$$(f|_A)_z \circ \partial = \partial \circ f_z$$

which shows (3).

Basic properties of currents give (4) upon noticing that the inclusion maps satisfy (2) of Proposition 3.1.3.

The homotopy formula for currents (extended to the semi-algebraic context as in Remark 3.1.1) and part (3) of Proposition 3.1.3 give (5).

Preceding the proof of (6), we make the following claim. For a separate proof, see Proposition 2.2.8 of [BCR98].

Claim 3.2.2. For any semi-algebraic $A \subset \mathbb{R}^n$, the function $\text{dist}(x, A)$ is semi-algebraic.

Proof of Claim. For simplicity, we may as well assume that $A$ is closed, since passing to the closure will not change the function in question. Let $B = \{(x, r) \mid x \in \ldots$
\( \mathbb{R}^n \) and \( \exists \ y \in A \) with \( ||x - y|| = r \). Note that \( B \) is the projection of \( \{(x, y, r) \mid x \in \mathbb{R}^n \ y \in A \) and \( ||x - y|| = r \} \), which is clearly semi-algebraic. Next let \( L \) be the connected component of the compliment of \( B \) which contains \( 0 \in \mathbb{R}^{n+1} \). This set is semi-algebraic, and, further, the frontier of \( L \) is the graph of \( x \mapsto \text{dist}(x, A) \).

The proof of (6) then follows as in [Fed69, 4.4.1] since the set:

\[
E = X' \cap \{x \mid \text{dist}(x, X' \setminus A') \leq \text{dist}(x, X' \setminus X)\}
\]

is semi-algebraic. By repeatedly applying Proposition 3.1.3, all arguments follow exactly as stated.

For statement (7), note that the only non-empty semi-algebraic subset of the singleton set \( \{a\} \) is itself—and hence the only allowable semi-algebraic chains are \( n[\{a\}] \) for \( n \in \mathbb{Z} \), which implies the result.

\[ \square \]

3.3 Coincidence of Semi-Algebraic Homology and Integral Current Homology

Our next goal is to show that the homology groups for semi-algebraic chains coincide with the homology groups of integral currents on a given semi-algebraic set \( X \).

As in the introduction, let \( I_k(X) \) denote the Abelian group of \( k \)-dimensional integral currents, and \( H_k^I(X) \) denote the respective homology groups (see [Fed69,
4.1.24]). By comments in Section 3.1 every semi-algebraic chain is necessarily an integral current which gives an inclusion $S_k(X) \rightarrow i I_k(X)$. We will show that $i$ induces an isomorphism on the respective homology groups.

We first state a simplified version of the Deformation Theorem for integral currents:

**Theorem 3.3.1** (Deformation Theorem, [Fed69, 4.2.9]). For any integral current $T \in I_k(\mathbb{R}^n)$ and $\epsilon > 0$, there exists a polyhedral chain $P \in \mathcal{P}_k(\mathbb{R}^n)$, and integral currents $Q \in I_k(\mathbb{R}^n)$ and $L \in I_{k+1}(\mathbb{R}^n)$ such that $T = P + Q + \partial L$, with $\text{spt}(P) \cup \text{spt}(L) \cup \text{spt}(Q) \subset \{x | \text{dist}(x, \text{spt}(T)) \leq 2n\epsilon\}$. Further, there exists a constant $\kappa$ so that:

1. $\mathcal{M}(P) \leq \kappa (\mathcal{M}(T) + \epsilon \mathcal{M}(\partial T))$

2. $\mathcal{M}(\partial P) \leq \kappa \mathcal{M}(\partial T)$

3. $\mathcal{M}(Q) \leq \epsilon \kappa \mathcal{M}(\partial T)$

4. $\mathcal{M}(L) \leq \epsilon \kappa \mathcal{M}(T)$

Finally, given a countable collection $\{T_j\}_{j \in \mathbb{N}}$ of integral currents, we may set $T_j = P_j + Q_j + \partial L_j$ so that each $P_j$ is a finite sum of integer multiples of $k$-dimensional cubes from the same standard cubical decomposition of $\mathbb{R}^n$ with edge-length $\epsilon$.

That we may choose the same cubical decomposition as the support for a countable collection of integral currents is crucial for generalizing mass minimization. While not explicitly stated in the theorem statement in [Fed69], it follows easily from the
proof, where it is shown that for any integral current $T$, almost any translation of the standard cubical decomposition may be taken as the support for $P$ in the above decomposition.

As stated, this version of Theorem 3.3.1 is insufficient for our purposes. We need the additional proposition:

**Proposition 3.3.2.** In Theorem 3.3.1, if $\partial T$ is a semi-algebraic chain, then $Q$ may be chosen to be semi-algebraic as well.

The above proposition follows from analyzing the definition of $Q$ in the proof of Theorem 3.3.1 in [Fed69, 4.2.9]; we will follow the definitions and notations given there. There are many known proofs of the Deformation Theorem (see, for example, the work of B. White [Whi99]), however, the proof presented in [Fed69] is very constructive and allows for a fairly simple proof of the above proposition.

The proof of Theorem 3.3.1 in [Fed69] produces $Q$ as a deformation of $\partial T$, and so $Q$ will be semi-algebraic inasmuch as $\partial T$ and the deformations are semi-algebraic.

**Proof.** $Q$ is defined to be a sum of deformations of $\partial T$ so it suffices to show that any element of this sum is semi-algebraic.

Elements of the sum are of the form:

$$\lim_{r \to 0^+} h^i_p(\llbracket I \rrbracket \times \partial T \sqcap U_r).$$

By transversality arguments, we may assume that the chosen $a$ has the additional property that $\text{spt}(\partial T) \cap (W_{n-m-1}'' + a)$ is empty. Therefore, by shifting the skeletons
by \(a\) we may drop the \(\tau_a\) terms from our analysis and, since \(\text{spt}(\partial T)\) is compact disregard the limit in the definition and consider only the chains \(h^i_I(\square I \times \partial T)\).

By Proposition 3.1.3, it suffices to show that \(h^i\) is a semi-algebraic map. This would follow immediately from showing that the retractions \(\sigma_i\) are semi-algebraic–however, a weaker condition is sufficient: since any compact semi-algebraic set (in particular, \(\text{spt}(\partial T)\)) may intersect only finitely many cubes of the decomposition, all we must show is that the retraction maps are semi-algebraic on a given cube.

Given such a cube \(C\) with center \(q\), take \(A = \{\Delta \mid \Delta \text{ is an } i\text{-face of } C\}\) and define for any \(i\)-face \(\Delta\):

\[
S_{\Delta} = \{(x, y, t) \in (C \setminus W''_{n-i-1}) \times \Delta \times \mathbb{R} \mid x = q + t(y - q), t > 0\}
\]

Note that \(S_{\Delta}\) is semi-algebraic.

Using the geometric description of \(\sigma_i\) given in [Fed69, 4.2.6], the graph of \(\sigma_i\) over the cube \(C\) is given by the finite union

\[
\bigcup_{\Delta \in A} \Pi_{x,y}(S_{\Delta})
\]

where \(\Pi_{x,y}\) denotes the projection onto the first components. Since this union is finite, the Tarski-Seidenberg principle [BCR98, 1.4] gives that the graph is semi-algebraic. Therefore \(\sigma_i|_K\) is semi-algebraic on any such cube, completing the proof.

The next remark will allow us to reduce to the compact case.

**Remark 3.3.1.** Given any compact set \(A \subset X\), there exists a compact semi-algebraic set \(B\) such that \(A \subset B \subset X\). To see this, note that since \(A\) is compact \(A \subset B_r(0)\)
for some $r \geq 0$. Further, $A \cap \text{Fron}(X) = \phi$ and moreover, there is an $\epsilon > 0$ so that $\text{dist}(x, \text{Fron}(X)) > \epsilon$ on $A$. Since, by Claim 3.2.2, $f(x) = \text{dist}(x, \text{Fron}(X))$ is semi-algebraic (see the comment in the proof of (6) of Theorem 3.2.1), so too is the closed set

$$f^{-1}([\frac{\epsilon}{2}, \infty]) \cap X.$$ 

So, the semi-algebraic set

$$B = (f^{-1}([\frac{\epsilon}{2}, \infty]) \cap X) \cap B_r(0)$$

is a compact subset of $X$ with the desired properties.

Thus, for any finite collection of currents $T_1, T_2, \ldots, T_n$, by applying the above to $A = \bigcup \text{spt}(T_i)$ and replacing $X$ by $B$, we may do all of our computations in $B$.

Our main result is to show the following:

**Theorem 3.3.3.** The inclusion map $i : S_k(X, A) \to I_k(X, A)$ induces an isomorphism on the respective homology groups $i_* : H^S_k(X, A) \to H^I_k(X, A)$.

We will prove this result first in the special case where $A = \phi$ and then derive the general result from this special case. We first prove injectivity of the induced map. Surjectivity relies on this fact, and will be proven shortly.

**Proposition 3.3.4.** The inclusion map $i : S_k(X) \to I_k(X)$ induces an injection on the respective homology groups $i_* : H^S_k(X) \to H^I_k(X)$.

The proof has two steps—the first is to show that bounded subsets of $X$ are semi-algebraic retractions of certain arbitrarily small open neighborhoods—this is simply
due to the triangulability of any compact semi-algebraic subset of \( \mathbb{R}^n \). The second is to notice that in any open set, the Deformation Theorem and Proposition 3.3.2 give that any integral current with semi-algebraic boundary is homologous to a semi-algebraic chain, and that semi-algebraic chains may be pushed forward through the above retraction.

**Proof of Proposition 3.3.4.** It is clear that \( i \) commutes with the boundary operator, and hence extends to a map \( i_* \) on the associated homology groups.

To prove the proposition, it suffices to show that if \( S \) is a semi-algebraic chain supported on \( X \) which bounds an integral current supported on \( X \), then \( S \) also bounds a semi-algebraic chain on \( X \). To this end, let \( S = \partial T \), where \( T \in I_{k+1}(X) \). Applying Remark 3.3.1 to the currents \( S \) and \( T \), we may assume that \( X \) is compact. Enclose \( X \) within the interior of a cube \( C \). Triangulate the semi-algebraic set \( C \) compatibly with the closed semi-algebraic subset \( X \) (as in [BCR98]) to obtain a simplicial complex \( K \) and a semi-algebraic homeomorphism \( h : |K| \to C \). Since \( X \) is closed, compatibility implies that \( X \) is a sub-complex of the triangulation—that is, that there exists a subcomplex \( L \) of \( K \) such that \( h|_L : |L| \to X \) is a semi-algebraic homeomorphism. Via (repeated) barycentric subdivision (up to relabeling) we may assume that \( L \) is a full subcomplex of \( K \), and hence that there exists an open neighborhood (the so-called regular neighborhood) \( N(L) \subset |K| \) of \( |L| \) on which when any \( \alpha \in N(L) \) is written in barycentric coordinates as a (non-negative) linear combination of vertices from \( K \):

\[
\alpha = \sum_{a \text{ is a vertex of } K} \alpha(a) a
\]

we have that
\[ \sum_{a \text{ is a vertex of } L} \alpha(a) > 0. \]

This enables us to define a retraction map \( r : N(L) \to X \) given as follows:

\[ r(\alpha) = \frac{\sum_{a \text{ is a vertex of } L} \alpha(a)a}{\sum_{a \text{ is a vertex of } L} \alpha(a)}. \]

For more details, see [ES52, II, § 9]. Since \(|L|\) is semi-algebraic and \( r \) is piecewise linear, we obtain that \( r \) is semi-algebraic on \(|L|\), and hence by composing with the semi-algebraic triangulating homeomorphism, there is a semi-algebraic retraction \( \tilde{r} : h(N(L)) \to X \). Since \( h(N(L)) \) is open in \( C \), by restricting to \( E = h(N(L)) \cap \hat{C} \)

we may assume that \( X \) is a Euclidean neighborhood retract in \( \mathbb{R}^n \). Further, since \( X \) is compactly contained in \( E \), \( E \) contains some \( \delta \) neighborhood of \( X \), and so for \( \epsilon > 0 \) small enough, we may apply Theorem 3.3.1 and Proposition 3.3.2 to get a polyhedral \( P \), and integral currents \( Q \) and \( R \) in \( S_{k+1}(E) \) and \( I_{k+2}(E) \), respectively, such that \( T = P + Q + \partial R \). Applying the semi-algebraic retraction to both sides of \( \partial T = \partial P + \partial Q = \partial(P + Q) \) gives \( S = \partial T = \tilde{r}_*^{\partial}(\partial T) = \tilde{r}_*(\partial(P + Q)) = \partial\tilde{r}_*(P + Q) \), where the last equality is justified by Proposition 3.1.1 since all terms involved are semi-algebraic chains. Thus \( S \) bounds the semi-algebraic chain \( \tilde{r}_*(P + Q) \), which proves the injectivity of \( i_* \).

\( \square \)

Surjectivity requires another property of semi-algebraic sets which is a corollary of Theorem 5.1 of [SV10]. A simplified version is given below, although it should be noted that the statement in [SV10] is more general and stronger.
**Theorem 3.3.5.** [SV10, Theorem 5.1] Let $X_1, \ldots, X_k$ be a stratification for $X = \bigcup X_i \subset \mathbb{R}^n$, and let $p \in X$.

1. There exists a neighborhood $U$ of $p$ in $X$ and a stratification $U_1, \ldots, U_\ell$ of $U$ so that each $X_i \cap U$ is a union of some subcollection of $U_1, \ldots, U_\ell$.

2. There exists a semi-algebraic $N \subset U$ with $p \in N$ and $\dim(N) < \dim(U)$ and a Lipschitz strong deformation retraction $r : U \times [0,1] \to U$ to $N$ such that:
   
   (a) $r(x,0) \in N$ and $r(x,1) = x$ for $x \in X$.

   (b) For any $j$, $r(U_j \times (0,1]) \subset U_j$.

We can then prove the following corollary by inducting on the dimension of the set $N$ obtained in the previous theorem. For a more elementary proof, see Theorem 4.1.5 of [Sha11].

**Corollary 3.3.6.** Let $X_1, \ldots, X_m$ be semi-algebraic subsets of $\mathbb{R}^m$ such that $X = \bigcup_{j=1}^m X_j \subset \mathbb{R}^n$ is closed. Further, suppose that $0 \in \overline{X}_j \cap X$ for all $j$. Then there exists a neighborhood $U$ of $0$ in $\mathbb{R}^n$ and a semi-algebraic Lipschitz deformation retraction to $0$, $r : U \times I \to U$ that preserves the $X_j$’s.

To assist in the proof, we make the following remark.

**Remark 3.3.2.** If $h : X \to Y$ is semi-algebraic and a homeomorphism, then its inverse is also semi-algebraic, since the graph of $h$, $\{(x,y) \in X \times Y \mid y = h(x)\} = \{(x,y) \in X \times Y \mid h^{-1}(y) = x\}$ is a coordinate permutation of the graph of $h^{-1}$, and thus the graph of $h$ is semi-algebraic if and only if the graph of $h^{-1}$ is. For brevity, we call such
a map a semi-algebraic homeomorphism. If we further suppose that $h$ is bilipschitz, then the following diagram commutes:

$$
\begin{array}{c}
H^*_{SA}(X) \xrightarrow{i_*} H^*_{IC}(X) \\
\downarrow{h_*} \quad \downarrow{h^*_IC} \\
H^*_{SA}(Y) \xrightarrow{i'_*} H^*_{IC}(Y)
\end{array}
$$

The vertical arrows are isomorphisms since $f$ is invertible in both the Lipschitz and semi-algebraic categories, so the bijectivity of the inclusion induced map $i_* : H^*_k(X) \to H^*_k(X)$ is invariant under such transformations.

**Proposition 3.3.7.** If $X \subset \mathbb{R}^n$ is semi-algebraic then the inclusion map $i : S_k(X) \to I_k(X)$ induces a surjection on the respective homology groups $i_* : H^*_k(X) \to H^*_k(X)$.

The proof will require basic knowledge of slicing of currents; the aspects that we require are documented below. The main reference for the theory of slicing is [Fed65]. In particular, the following Proposition comprises a part of [Fed65], Corollary 3.6.

**Proposition 3.3.8.** If $f : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz and $T \in I_k(\mathbb{R}^n)$ has $\partial T = 0$. Then, for almost every $y \in \mathbb{R}$, there is a current $< T, f, y > \in I_{k-1}(\mathbb{R}^n)$ so that:

1. $\text{spt}(< T, f, y >) \subseteq \text{spt}(T) \cap f^{-1}(y)$

2. $< T, f, y > = \partial(T \cap \{ f > y \})$

The current $< T, f, y >$ is referred to as the **slice of $T$ in $f^{-1}(y)$**.

We will also require the following consequence of Hardt’s trivialization theorem—for details, see [BCR98, §9.3].
Theorem 3.3.9 (Local Conic Structure, [BCR98, Theorem 9.3.6]). Let $A \subset \mathbb{R}^n$ be semi-algebraic and $x$ a nonisolated point of $E$. There exists $\epsilon > 0$ and a semi-algebraic homeomorphism $\varphi : \overline{B_{\epsilon}(x)} \to \overline{B_{\epsilon}(x)}$ so that:

1. $||\varphi(y) - x|| = ||y - x||$ for all $y \in \overline{B_{\epsilon}(x)}$.

2. $\varphi|_{\partial B_{\epsilon}(x)}$ is the identity mapping.

3. $\varphi^{-1}(A \cap \overline{B_{\epsilon}(x)})$ is the cone with vertex $x$ and basis $A \cap \partial B_{\epsilon}(x)$.

The following notation will be used heavily in the proof:

- For $y > 0$, denote by $C(x_0, y)$ the cube with faces perpendicular to the coordinate axes, side length $2y$ and centered at $x_0 \in \mathbb{R}^n$.

- For $x, z \in \mathbb{R}^n$ with components $x_i$ and $z_i$, respectively, let $D(x, z) = \max_{i=1,\ldots,n} |x_i - z_i|$.

- For $x_0 \in \mathbb{R}^n$, denote by $D_{x_0} : \mathbb{R}^n \to \mathbb{R}$ the function $D_{x_0}(y) = D(y, x_0)$.

- For $x_0 \in \mathbb{R}^n$ and $y > 0$ and $T \in I_k(X)$, denote by $< T, D_{x_0}, y >$ the slice of $T$ at $y$ by the Lipschitz function $D_{x_0}$, as defined in Proposition 3.3.8.

The main idea of the proof is induction on the ambient dimension. For the inductive step, we choose some integral current $T \in I_k(X)$ with $\partial T = 0$ and we consider the intersection of $X$ with the elements of a (finite) cover of $X$ by cubes whose boundaries slice the current $T$ well, intersect $X$ with strictly smaller dimension, and inside of which $X$ is contractible. Roughly speaking, for a fixed cube $C$ of the decomposition, we then apply the inductive assumption to the slice of $T$ in the boundary of $C$ and
use the contraction to construct a semi-algebraic current within $C$ whose boundary is contained in the boundary of $C$ and is homologous in the boundary of $C$ to the boundary of $T \mathbb{L}(X \setminus C)$, then we use injectivity of $i_*$ (Proposition 3.3.4) to restart the process in $X \setminus C$, which is covered by fewer cubes. Iterating this process and keeping track of the changes we make in each cube then gives the result.

**Proof of Proposition 3.3.7.** We first prove surjectivity in the case that $\dim(X) \leq n - 1$, and we prove the statement by induction on the ambient dimension. If $X \subset \mathbb{R}$, $X$ is a finite union of points, and the proof of the proposition is obvious.

Now suppose $X \subset \mathbb{R}^n$, and choose $T \in I_k(X)$ with $\partial T = 0$. By Remark 3.3.1 applied to $A = \text{spt}(T)$ we may assume $X$ is compact. The following claim helps us to decompose $X$.

**Claim 3.3.10.** If $x_0 \in X$, then there exists an $\epsilon > 0$ so that, for $0 < \delta < \epsilon$, $\dim(\partial C(x_0, \delta) \cap X) \leq n - 2$.

**Proof of Claim.** First note that if $x_0$ is an isolated point of $X$, there is nothing to prove. If $x_0$ is nonisolated, by the coarea formula, if $C_1 d^X(D_{x_0})_y$ denotes the one-dimensional coarea factor of the tangential differential of the Lipschitz function $D_{x_0}$ on the $H^{n-1}$ rectifiable set $X$ at the point $y$, then:

$$
\int_0^1 \mathcal{H}^{n-2}(X \cap \partial C(x_0, s)) \, ds = \int_{X \cap C(x_0, 1)} C_1 d^X(D_{x_0})_y \, d\mathcal{H}^{n-1}(y)
$$

(see, for example, Theorem 2.93 of [AFP00]). However, we may directly compute that $C_1 d^X(D_{x_0})_y < M$ for some constant $M$, which gives:
\[
\int_0^1 \mathcal{H}^{n-2}(X \cap \partial C(x_0, s)) \, ds \leq M\mathcal{H}^{n-1}(X \cap C(x_0, 1)) < \infty
\]

Therefore, for a.e. \( s \in [0, 1] \), \( \mathcal{H}^{n-2}(X \cap \partial C(x_0, s)) < \infty \). In particular, there are arbitrarily small \( \delta \) so that \( \mathcal{H}^{n-2}(X \cap \partial C(x_0, \delta)) < \infty \). However, for any such \( \delta \), \( X \cap \partial C(x_0, \delta) \) is semi-algebraic and therefore the Hausdorff measure bound implies \( \dim(\partial C(x_0, \delta) \cap X) \leq n - 2 \). Finally, Theorem 3.3.9 guarantees that, for \( \epsilon > 0 \) small enough and \( 0 < \delta < \epsilon \) the dimension of the sets, \( X \cap \partial C(x_0, \delta) \) is constant, and this gives the claim. \( \square \)

Invoking Corollary 3.3.6, the compactness of \( X \) and the above claim, we find a finite set of cubes \( \{(C(x_i, y_i))_{i=1}^\ell \) such that:

1. For all \( i \), \( x_i \in X \) and \( y_i > 0 \).

2. For every \( i \), there exists a semi-algebraic, Lipschitz deformation retraction to the point \( x_i \), \( f_i : C(x_i, y_i) \times I \to U(x_i, y_i) \) which preserves \( X \), where \( U(x_i, y_i) \) is some open neighborhood of \( C(x_i, y_i) \).

3. For every \( i \), \( y_i \) is such that the slice \( <T, D_{x_i}, y_i > \in I_{k-1}(X) \) exists.

4. For the semi-algebraic set \( C(x_i, y_i) \cap X \) we have that \( \dim(\partial C(x_i, y_i) \cap X) \leq n - 2 \).

We now “push off” of each cube individually. If \( X \) can be contained in just one cube then it is (semi-algebraically) Lipschitz contractible by Corollary 3.3.6, and there is nothing to prove, so we assume in the following that \( \ell > 1 \) and that
\[ X \subset \bigcup_{i=1}^{\ell} C(x_i, y_i). \]

Consider the slice \(<T, D_{x_1}, y_1>\). By Proposition 3.3.8 (1), we have:

\[ \text{spt}(<T, D_{x_1}, y_1>) \subset \{ x \in \mathbb{R}^n \mid D(x, x_1) = y_1 \} \cap X \]

and so \(<T, D_{x_1}, y_1>\) is supported in \( \partial C(x_1, y_1) \cap X \).

Further, Proposition 3.3.8 (2) guarantees that

\[ <T, D_{x_1}, y_1> = \partial(T \sqcap \{D_{x_1} > y_1\}); \]

so we have that \( \partial <T, D_{x_1}, y_1> = 0. \)

However, \( \partial C(x_1, y_1) \cap X \) is, by assumption, a semi-algebraic set of dimension at most \( n - 2 \). By stratifying the cube compatibly with this intersection we may find a point \( a_1 \in \partial C(x_1, y_1) \) and an \( \epsilon > 0 \) such that \( B(a_1, \epsilon) \cap \partial(C(x_1, y_1)) \cap X = \phi. \)

Since \( \partial C(x_1, y_1) \setminus B(a_1, \epsilon) \) is bilipschitz semi-algebraically homeomorphic to a subset of \( \mathbb{R}^{n-1} \). Noting Remark 3.3.2 and our inductive hypothesis, we get:

\[ <T, D_{x_1}, y_1> = L_1 + \partial F_1 \]

for \( L_1 \in S_{k-1}(X \cap \partial C(x_1, y_1)) \) and \( F_1 \in I_k(X \cap \partial C(x_1, y_1)) \).

Applying the homotopy formula we find:

\[ \partial (T \sqcap C(x_1, y_1) - (f_1)_2(L_1 \times [0, 1]) - F_1) = 0 \]

since \( C(x_1, y_1) \) is contractible in a way that preserves \( X \), we have that

\[ T \sqcap C(x_1, y_1) - (f_1)_2(L_1 \times [0, 1]) - F_1 = \partial K_1 \]
for some $K_1 \in I_{k+1}(X \cap C(x_1, y_1))$.

Denote by $X_j = X \cap \left( \bigcup_{k=j+1}^\ell C(x_k, y_k) \right)$. Adding $T$ to both sides of the above and rearranging gives:

$$T \mathbin{\mathcal{L}} (X \setminus C(x_1, y_1)) = T - \partial K_1 - (f_1)_\sharp(L_1 \times [0, 1]) - F_1 \quad (3.3.1)$$

We now shift our focus to

$$T \mathbin{\mathcal{L}} (X \setminus C(x_1, y_1)) + F_1 \in I_{k+1}(X_1).$$

Equation 3.3.1 yields that:

$$-\partial((T \mathbin{\mathcal{L}} X \setminus C(x_1, y_1)) + F_1) = \partial((f_1)_\sharp(L_1 \times [0, 1])).$$

Notice that $\partial((f_1)_\sharp(L_1 \times [0, 1])) \in S_{k-1}(X_1)$. So, by Proposition 3.3.4, since $\partial((f_1)_\sharp(L_1 \times [0, 1]))$ is the boundary of a rectifiable current in $I_k(X_1)$ there is a semi-algebraic chain $S_1 \in S_k(X_1)$ such that:

$$\partial S_1 = -\partial((T \mathbin{\mathcal{L}} X \setminus C(x_1, y_1)) + F_1)$$

Then, $S_1 + (T \mathbin{\mathcal{L}} X \setminus C(x_1, y_1)) + F_1$ is a rectifiable cycle supported in $X_1$. Using Equation 3.3.1, we get:

$$S_1 + (T - \partial K_1 - (f_1)_\sharp(L_1 \times [0, 1]) - F_1) + F_1$$

is a cycle of $I_k(X_1)$. Applying the above discussion to this current and the semi-algebraic set $X_1$ gives:

$$S_2 + (S_1 + T - \partial K_1 - (f_1)_\sharp(L_1 \times [0, 1])) \mathbin{\mathcal{L}} (X_1 \setminus C(x_2, y_2)) + F_2 \quad (3.3.2)$$
is an element of $I_k(X_2)$, for $S_2 \in S_k(X_2)$ and $F_2 \in I_k(X \cap \partial C(x_2, y_2))$.

On the other hand, in analogy with Equation 3.3.1 applied to:

\[
(S_1 + T - \partial K_1 - (f_1)_z(L_1 \times [0, 1])) \sqsubset (X_1 \setminus C(x_2, y_2))
\]

we may find $L_2 \in S_{k-1}(X_1 \cap \partial C(x_2, y_2))$ and a $K_2 \in I_{k+1}(X_1 \cap \partial C(x_2, y_2))$ so that:

\[
(S_1 + T - \partial K_1 - (f_1)_z(L_1 \times [0, 1])) \sqsubset (X_1 \setminus C(x_2, y_2)) = (S_1 + T - \partial K_1 - (f_1)_z(L_1 \times [0, 1])) - \partial K_2 - (f_2)_z(L_2 \times [0, 1]) - F_2.
\]

(3.3.3)

Plugging this into Equation 3.3.2, we get that:

\[
S_2 + S_1 + T - \partial K_1 - (f_1)_z(L_1 \times [0, 1])
\]

\[
- \partial K_2 - (f_2)_z(L_2 \times [0, 1])
\]

is a cycle of $I_k(X_2)$. Continuing in this way, we get:

\[
T + \sum_{i=1}^{\ell-1} (S_1 - \partial K_1 - (f_i)_z(L_i \times [0, 1]))
\]

(3.3.4)

is a cycle of $I_k(X_\ell)$, where $F_i \in I_k(X \cap \partial C(x_i, y_i))$, $S_i \in S_k(X_i)$, $L_i \in S_{k-1}(X_i \cap \partial C(x_i, y_i))$. However, $X_\ell = X \cap C(x_\ell, y_\ell)$ is Lipschitz semi-algebraically contractible in a way which preserves $X$, so we get:

\[
T + \sum_{i=1}^{\ell-1} (S_1 - \partial K_1 - (f_i)_z(L_i \times [0, 1])) = \partial K_\ell
\]

for some $K_\ell \in I_{k+1}(X_\ell)$. Rearranging terms gives:

\[
T - \sum_{i=1}^{\ell} \partial K_i = - \sum_{i=1}^{\ell-1} (S_1 - (f_i)_z(L_i \times [0, 1])).
\]
However, all the terms on the right hand side are semi-algebraic, so we have proved the proposition when $X \subset \mathbb{R}^n$ and $\dim(X) \leq n - 1$.

The general case follows immediately from this special case–since any semi-algebraic set may be embedded in a higher dimensional space by simply adding on dimensions. More precisely, $X$ is semi-algebraically bilipschitz equivalent to $X \times \{0\} \subset \mathbb{R}^{n+1}$.

We now show that $i_*$ inducing an isomorphism between $H_*^{SA}(X)$ and $H_*^{IC}(X)$ implies the seemingly stronger result that $i_*$ induces an isomorphism on the relative homology groups, i.e. $H_*^{SA}(X, A) \cong H_*^{IC}(X, A)$, and this is Theorem 3.3.3.

The proof has essentially the same two parts–for each $[T] \in H_*^{IC}(X, A)$, we need to find a semi-algebraic chain $S \in \mathcal{Z}_* (X, A) \cap [T]$, and, if $S \in [0] \in H_*^{IC}(X, A)$, we need to find $Q \in S_{k+1}(X)$ and $L \in S_k(X)$ so that $S = L + \partial Q$.

**Proof of Theorem 3.3.3.** First, let $[T] \in H_*^{IC}(X, A)$ be given. Then spt($\partial T$) $\subset A$ and so $\partial T \in I_{k-1}(A)$. Applying Proposition 3.3.7 to the homology class $[\partial T] \in H_{k-1}^{IC}(A)$, we know that $\partial T = S_1 + \partial K_1$ for $S_1 \in S_{k-1}(A)$ and $K_1 \in I_k(A)$.

As an element of $S_{k-1}(X)$, $S_1$ is the boundary of the rectifiable current $T - K_1$. So, by Proposition 3.3.4 we know $S_1 = \partial S_2$ for $S_2 \in S_k(X)$. But then $\partial(T - K_1 - S_2) = 0$ so again by Proposition 3.3.7 we find $S_3 \in S_k(X)$ and $K_2 \in I_{k+1}(X)$ so that $(T - K_1 - S_2) = S_3 + \partial K_2$. Rearranging this gives $T - (S_2 + S_3) = K_1 + \partial K_2$, which is to say that $(S_2 + S_3) \in [T]$.

Next suppose that $S \in [0] \in H_*^{IC}(X, A)$. This is equivalent to saying spt($\partial(S)$) $\subset A$ and the existence of $T \in I_k(A)$ and $F \in I_{k+1}(X)$ so that $S = T + \partial F$. Then,
as an element of $I_k(A)$, $\partial S = \partial T$, and hence by Proposition 3.3.4 we may find a semi-algebraic $S_1 \in S_k(A)$ so that $\partial S = \partial S_1$ (note that in general we could not have chosen $S_1 = S$ since $S$ need not be supported in $A$). Then, $\partial(T - S_1) = 0$ so by Proposition 3.3.7 applied within $A$, we may find $S_2 \in S_{k+1}(A)$ and $K \in I_{k+1}(A)$ so that $T - S_1 = S_2 + \partial K$. Then:

$$S - S_1 = (T - S_1) + \partial F = (S_2 + \partial K) + \partial F = S_2 + \partial(K + F)$$

rearranging this equation gives:

$$S - S_1 - S_2 = \partial(K + F)$$

and so the semi-algebraic chain $S - S_1 - S_2$ bounds a rectifiable current in $X$, so applying Proposition 3.3.4 gives $S - S_1 - S_2 = \partial S_3$ for some $S_3 \in S_{k+1}(X)$. Therefore, $S = (S_1 + S_2) + \partial(S_3)$, completing the proof.

While the above isomorphism holds in all cases, if $X$ is compact and $A \subset X$ is closed, then by semi-algebraicity, the pair $(X, A)$ is triangulable as a finite simplicial complex. Further, one may check that the proof of the uniqueness theorem, Theorem 10.1 of [ES52, III] holds provided the admissible category includes all simplicial complexes embedded in Euclidean space and all simplicial maps from them to other elements of the category—the point is that the main isomorphism for the uniqueness result is defined by axiomatically computing homology groups for very basic simplicial complexes which may be assumed to be embedded in high dimensional space and
pushing forward generators of these groups. This fact, combined with the results of Section 3.2 proves the following:

**Corollary 3.3.11.** If \((X, A)\) is a compact pair of semi-algebraic sets, then the homology groups \(H^I_C(X, A)\) are isomorphic to the ordinary singular homology groups of the pair \((X, A)\).

### 3.4 Mass Minimization in Homology Classes

As mentioned in the introduction, our main result gives mass minimization in the homology classes of semi-algebraic sets. More precisely, we have the following:

**Theorem 3.4.1.** For a compact semi-algebraic set \(X \subset \mathbb{R}^m\) and \([S] \in H_k(X)\) a semi-algebraic homology class there exists a rectifiable representative \(T \in [S]\) with

\[
\mathcal{M}(T) = \inf_{L \in [S]} \mathcal{M}(L).
\]

**Proof.** Let \(\tilde{r} : X_\delta \to X\) denote the retraction from some \(\delta\) neighborhood of \(X\) contained in \(h(N(L))\) constructed in the proof of Proposition 3.3.4, and choose a minimizing sequence \(\{T_\ell\}_{\ell=1}^\infty \subset I_k(X)\) such that:

1. \(T_\ell \in [S]\), i.e. there is a sequence \(\{R_\ell\}_{\ell=1}^\infty \subset I_{k+1}\) so that \(S - T_\ell = \partial R_\ell\)

2. \(\mathcal{M}(T_\ell) \to \inf_{L \in [S]} \mathcal{M}(L)\).

Let \(\epsilon < \frac{\delta}{2n}\). Since \(\{R_\ell\}_{\ell=1}^\infty\) is countable, we may find a cubical decomposition of \(\mathbb{R}^m\) which is in good position for every \(R_\ell\) of side length \(\epsilon\) so that all cubes intersecting \(X\) are contained in \(X_\delta\). By compactness, there are only finitely many cubes of the
decomposition which intersect $X_\delta$; denote by $\{A_i\}^q_{i=1}$ and $\{B_j\}^p_{j=1}$ their $k+1$ and $k$ faces, respectively.

Applying the Deformation Theorem (Theorem 3.3.1) to each $R_\ell$ gives: $R_\ell = P_\ell + Q_\ell + \partial L_\ell$ so that:

1. $P_\ell$, $Q_\ell$ and $L_\ell$ are supported in $X_\delta$ for every $\ell$

2. $\mathcal{M}(Q_\ell) \leq \epsilon \kappa \mathcal{M}(\partial R_\ell) = \epsilon \kappa \mathcal{M}(S - T_\ell) \leq 2 \epsilon \kappa \mathcal{M}(S)$

3. $\mathcal{M}(\partial P_\ell) < \kappa \mathcal{M}(\partial R_\ell) = \kappa \mathcal{M}(S - T_\ell)$

4. $P_\ell$ is a sum of the form $\sum^q_{i=1} \alpha(i, \ell)[A_i]$.

This gives $S - T_\ell = \partial P_\ell + \partial Q_\ell$. So, since $\{\mathcal{M}(\partial P_\ell)\}^\infty_{\ell=1}$ is bounded, the collection $\{\mathcal{M}(\partial Q_\ell)\}^\infty_{\ell=1}$ is also bounded, and therefore we may apply the well known compactness theorem for integral currents to obtain (up to relabeling) weakly convergent subsequences and currents $T \in I_k(X)$, $Q \in I_{k+1}(X_\delta)$ so that $T_\ell \rightharpoonup T$, $Q_\ell \rightharpoonup Q$.

To apply a similar compactness result to the collection $\{P_\ell\}^\infty_{\ell=1}$, we need to uniformly bound the mass of the collection. Since each $P_\ell$ is supported on the same finite polyhedral skeleton, this amounts to proving bounds on the coefficients $\alpha(i, \ell)$. Let $V$ be the vector space generated by the $A_i$’s and $W$ be the vector space generated by the $B_i$’s. Then the boundary operator may be represented by a linear map $D : V \to W$ whose entries are all $\pm 1$ or $0$ and, since $V$ and $W$ are finite dimensional is a bounded operator.

Put $\partial P_\ell = \sum^p_{j=1} \beta(j, \ell)[B_j]$ and $\vec{\beta}_\ell = (\beta(1, \ell), \ldots, \beta(p, \ell))$. Then, for each $\ell \in \mathbb{N}$, we have a linear system $D\vec{x} = \vec{\beta}_\ell$. However, since the mass of $\partial P_\ell$ is uniformly
bounded, $|\tilde{\beta}_\ell|$ is uniformly bounded. Since each $\tilde{\beta}_\ell$ is integer valued, this gives only finitely many such vectors. Finally, by assumption, for each $\ell$ we have a solution to the linear system $D\tilde{\ell} = \tilde{\beta}_\ell$, but since there are only finitely many such systems, by choosing a particular solution for each one we immediately get density bounds for some polyhedral sequence $F_\ell \in I_{k+1}(X_\delta)$ with $\partial F_\ell = \partial P_\ell$. Compactness gives $F \in I_{k+1}(X_\delta)$ so that $F_\ell \rightarrow F$. This gives:

$$S - T = \partial F + \partial Q.$$ 

However, this implies that $S - T$ is also null-homologous in $I_k(X)$ since, applying Proposition 3.3.7 in $X_\delta$ we get $[S - T] = [L] = [0]$ for $L \in S_k(X_\delta)$. Proposition 3.3.4 then guarantees $L = \partial B$ for some $B \in S_{k+1}(X_\delta)$ Applying the retraction (in $I_k(X)$) gives $[S - T] = [\tilde{\partial} B] = \partial (\tilde{\partial} B) = 0$. The proof is completed upon referencing the lower semicontinuity of mass with regards to weak convergence.

3.5 Homological Triviality of Small Cycles

Finally, we use our main theorem to show that, in the compact semi-algebraic case, we can guarantee that all rectifiable cycles of sufficiently small mass bound. This follows from an analogous result for Lipschitz neighborhood retracts, shown in [FF60].

**Proposition 3.5.1.** For any compact semi-algebraic set $X \subset \mathbb{R}^n$, there is an $\epsilon > 0$ so that if $T \in I_k(X)$ has $\partial T = 0$ and $M(T) \leq \epsilon$, then there is a $F \in I_{k+1}(X)$ so that $T = \partial F$. 
Proof. Let \( \tilde{r} : X_\delta \to X \) denote the neighborhood of \( X \) and the retraction constructed in the proof of Proposition 3.3.4. By [FF60, 9.6 (3)] there is an \( \epsilon > 0 \) so that the proposition holds for \( X_\delta \). Let \( T \in I_k(X) \) be as above with this choice of \( \epsilon \).

By Proposition 3.3.7, there is a semi-algebraic \( S \in [T] \). However, in \( X_\delta \), \( T = \partial F \), so, if \( S - T = \partial K \), then \( S = \partial (K + F) \). Applying Proposition 3.3.4 to \( S \) in \( X_\delta \), we find \( S = \partial L \) for \( L \in S_{k+1}(X_\delta) \). However, then \( [T] = [S] = [\partial \tilde{r}_\sharp (L)] = [0] \), completing the proof. \( \square \)
Bibliography


[BDPW15b] Philippe Bouafia, Thierry De Pauw, and Changyou Wang. Multiple valued maps into a separable hilbert space that almost minimize their


