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Shake Slice and Shake Concordant Links

by

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Abstract

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The study of knots and links up to concordance has proved significant for many problems in low dimensional topology. In the 1970s, Akbulut introduced the notion of shake concordance of knots, a generalization of the study of knot concordance. Recent work of Cochran and Ray has advanced our understanding of how shake concordance relates to concordance, although fundamental questions remain, especially for the class of shake slice knots. We extend the notion of shake concordance to links, generalizing much of what is known for knots, and offer a characterization in terms of link concordance and the infection of a link by a string link. We also discuss a number of invariants and properties of link concordance which extend to shake concordance of links, as well as note several that do not. Finally, we give several obstructions to a link being shake slice.

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Ultimately, to He who "gives wisdom to the wise and knowledge to those who have understanding–revealing deep and hidden things".

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CHAPTER 1

Introduction

1. Background

An *m-component link* is a smooth embedding $f : \sqcup_{i=1}^{m} S^1 \to S^3$ of a collection of disjoint, ordered circles with assigned orientations into the three sphere. A 1-component link is called a *knot*. We say links L and L' are *isotopic* if L can be smoothly deformed into L' via embeddings in S^3. Isotopy induces an equivalence relation on the set of links; we will always consider links up to isotopy.

In the 1950s and 60s, in their study of link singularities, Fox and Milnor introduced a weaker equivalence relation on links called concordance [**FM66**]. We say two m-component links $L = L_1 \sqcup ... \sqcup L_m$ and $L' = L'_1 \sqcup ... \sqcup L'_m$ are (smoothly) *concordant* if there exists a collection of m disjointly embedded, smooth, oriented annuli $A_1, ..., A_m$ in $S^3 \times [0, 1]$ such that $\partial A_i = L_i \times \{0\} \sqcup -L'_i \times \{1\}$. Notice if L and L' are concordant, then sublinks $L_{i_1} \sqcup ... \sqcup L_{i_k}$ and $L'_{i_1} \sqcup ... \sqcup L'_{i_k}$ are also concordant, for $\{i_1, ..., i_k\} \subset \{1, ..., m\}$. A link that is concordant to the trivial link we call (smoothly) *slice*; this is equivalent to the link $L \subset S^3 = \partial D^4$ bounding m disjoint, smooth disks in D^4. Here we will limit ourselves to the smooth category, but one could consider other categories. For instance, if we relax the condition on the annuli from being smooth to merely being topologically locally flat, then we call the links *topologically concordant*.

The study of links has been essential to the study of 3- and 4-manifolds. For instance, Lickorish and Wallace showed that any closed, orientable 3-manifold can be obtained from a link in S^3 via an operation called *surgery*. Improved understanding of links up to concordance is recognized as a vital step in making progress on many open problems in low dimensional topology.

2. Shake Concordance of Knots

Given a 4-manifold, one is often interested in when interesting homology classes can be represented by some submanifold. It was in this context that Akbulut introduced the notion of a shake slice knot, defined as follows.

Let W_K^r denote the 4-manifold obtained by attaching a 2-handle with framing r to the 4-ball B^4 along a knot $K \subset S^3 = \partial B^4$. We call W_K^r the *trace* of the knot and it has homology

$$H_n(W_K^r) \cong \begin{cases} \mathbb{Z} & n = 0, 2 \\ 0 & n = 1, n \geq 3 \end{cases}.$$

A knot K is called *r-shake slice* if there exists a smoothly embedded 2-sphere Σ that represents a generator of $H_2(W_K^r) \cong \mathbb{Z}$ as in Figure 1.1. Hence, after isotopy, $-\Sigma$ intersects the added 2-handle as $2n-1$ disks, n of which have as their boundary K with its opposite orientation (since we are considering $-\Sigma$ rather than Σ) and $n-1$ of which have as boundary K with the original orientation. Deleting these disks, we obtain the following equivalent definition.

Define the *r-shaking of K* to be a collection of $2n-1$ r-framed parallel copies of K, where n are oriented in the direction of K and $n-1$ are oriented in the opposite direction. See Figure 1.2. Then we call K *r-shake slice* if some r-shaking of K bounds a smooth, properly embedded, compact, connected, genus zero surface in B^4.

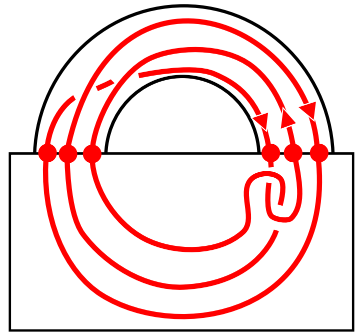


FIGURE 1.1. A schematic of an embedded sphere defining K to be shake slice in W_K.

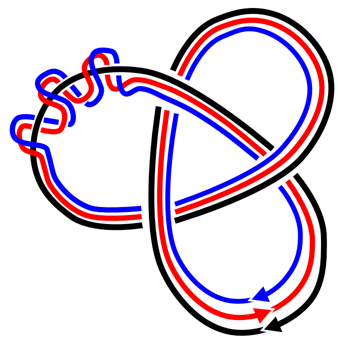


FIGURE 1.2. A 0-shaking consisting of 3 parallel copies of the trefoil.

Note that every slice knot is r-shake slice, for all r, with a representative for generator formed by the union of the slice disk for K and the core of the 2-handle attached along K. It is natural to ask if the converse is true.

In [**Akb77**], Akbulut provided examples of 1-shake slice and 2-shake slice knots that are not slice. Lickorish provided additional such examples in [**Lic79**]. More recently constructions for infinitely families of r-shake slice knots that are not slice for all nonzero r have been provided in [**Akb93**] and [**AJOT13**]. It remains an open problem to determine if 0-shake slice implies slice.

There is also a relative version. Let $W^r_{K_0,K_1}$ denote the 4-manifold obtained by attaching two 2-handles to $S^3 \times [0,1]$ along the knots $K_i \subset S^3 \times \{i\}$ with framing r. We call K_0 and K_1 *r-shake concordant* if there exists a smoothly embedded 2-sphere that represents a generator $(1,1)$ of $H_2(W^r_{K_0,K_1}) \cong \mathbb{Z}^2$. See Figure 1.3 for a schematic.

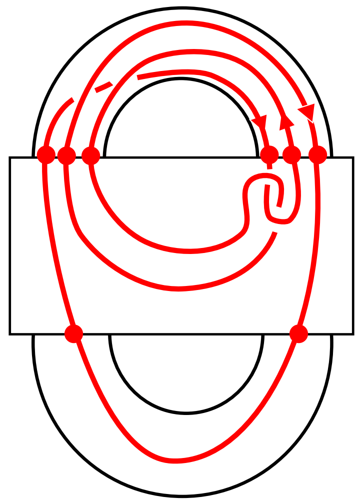


FIGURE 1.3. A schematic of the embedded sphere defining a shake concordance in W_{K_0,K_0}.

We offer an alternative definition as follows. We say K_0 is (m,n) *r-shake concordant* to K_1 if there is a smooth, properly embedded, compact, connected, genus zero surface F in $S^3 \times [0,1]$ such that $F \cap S^3 \times \{0\}$ is an r-shaking of K_0 with m components and $F \cap S^3 \times \{1\}$ is an r-shaking of K_1 with n components.

It is not hard to see that these definitions are equivalent with an argument analogous to that for the equivalence of the two definitions for shake slice knots.

Observe an r-shake slice knot is r-shake concordant to the unknot. Therefore the examples differentiating shake sliceness from sliceness also provide examples of pairs of knots that are r-shake concordant but not concordant. However, these examples only cover nonzero r. Cochran and Ray extended this result to include $r = 0$ as follows:

Theorem 1.1 (Theorem 4.1 in [**CR16**]). *For any integer r, there exist infinitely many knots which are distinct in smooth concordance but are pairwise r-shake concordant. For $r = 0$, there exist topologically slice knots with this property as well.*

Moreover, they showed that many classical knot invariants of concordance fail to be invariants of r-shake concordance. Part of this involved completely classifying knots up to shake concordance in terms of concordance and satellite operations.

3. Summary of Results

We extend the notion of r-shake concordance to links. We offer two such generalizations: an *r-shake concordance* of links and a stricter version we call *strong r-shake concordance* of links. These are both generalizations of (smooth) link concordance. They also gives rise to the notion of a link being *r-shake slice* or *strongly r-shake slice*. We largely restrict our attention to $r = 0$, for this is the setting in which the most interesting open problems remain in the setting of knots. When we omit r, it is to be understood $r = 0$.

We then offer a number of families of links that help distinguish between concordance, strong shake concordance, and shake concordance of links. In particular we prove the following two results:

Corollary 2.9. There exists an infinite family of two-component links that are pairwise shake concordant, but not pairwise strongly shake concordant.

Proposition 2.12. There exists an infinite family of 2-components links with unknotted components that are all strongly shake concordant to the Hopf link, but none of which are concordant to the Hopf link.

In fact, we show that given any two knots K and K', we may find 2-component links L and L' that are shake concordant such that $L_1 = K$ and $L'_1 = K'$. Hence, unlike concordance (or strong shake concordance), our notion of shake concordance of links completely fails to descend to sublinks. It is perhaps surprising then that we are able classify shake concordant (and strongly shake concordant links) in terms of concordance and an operation on links known as string link infection. This classification reduces to that offered by Cochran and Ray in the case of knots [**CR16**, Theorem 3.7].

Theorem 3.2. Two m-component links M and M' are shake concordant if and only if the links obtained by string link infection $I(L, J, \mathbb{E}_\varphi)$ and $I(L', J', \mathbb{E}'_{\varphi'})$ are concordant for some:

- m-component slice links L and L',
- m-component string links J, J' with closures $\widehat{L} = M$ and $\widehat{L}' = M'$, respectively,
- and embeddings of multidisks $\mathbb{E}_\varphi$ and $\mathbb{E}'_{\varphi'}$ each with m subdisks that respect L and L', respectively.

We then turn our attention to invariants of shake concordance and strong shake concordance. There are many well studied invariants of concordance. Especially significant in the study of links have been Milnor's higher order linking numbers, the

$\bar{\mu}$ invariants. Incredibly, our classification theorem allows us to recover that the first non-vanishing among these are also an invariant of shake concordance of links.

THEOREM 4.2. If two links L and L' are shake concordant, then they have equal first non-vanishing Milnor invariants.

Cochran and Ray showed that the zero surgery manifold M_K, a 3-manifold naturally obtained from a knot, is preserved up to homology cobordism under shake concordance. Since many concordance invariants are determined by the associated zero surgery manifold, this allowed them to establish these as invariants of shake concordance of knots. It follows these are also invariants of the components of strongly shake concordant links.

For shake concordance of links, we prove an analogous result:

PROPOSITION 4.4. Suppose m-component links L and L' are shake concordant. Then the zero surgery manifolds M_L and M'_L are homology cobordant.

However, since the components of shake concordant links can vary arbitrarily, we fail to recover any such invariants for the components. Nevertheless, we succeed in showing that several concordance invariants serve as obstructions to a link being shake slice. In particular, a shake slice link has all components algebraically slice and hence the signatures and Arf invariants all vanish. This follows from the following obstruction.

PROPOSITION 5.3. Suppose the m-component link L is shake slice. Then L bounds m disjoint disks in a homology 4-ball.

We also offer additional obstructions: for instance, the Milnor invariants and a generalized Arf invariant for links vanish for shake slice links. Together, these obstructions suggest that just as in the case of knots, it is a difficult problem to detect the difference between a slice and shake slice link.

CHAPTER 2

Shake Concordance of Links

1. Defining Shake Concordance for Links

In this chapter we extend the definition of shake slice to links. We'll see in section 1 that, in fact, we obtain two such generalizations: *shake concordance* of links and, stricter, *strong shake concordance* of links. Then in section 2 we'll briefly review a way to think about concordance that will prove useful for our study of shake concordance. Finally, in section 3 we offer infinite families of links that distinguish the notions of concordance, strong shake concordance, and concordance of links.

1.1. Shake Slice Links. Consider an m component link $L \subset S^3 = \partial B^4$. We obtain a 4-manifold W_L by attaching a 2-handle with framing r along each component L_i of L, $i = 1, ..., m$. Note W_L^r has as boundary the 3-manifold obtained by r surgery on L, which we denote M_L^r. Moreover,

$$H_n(W_L^r) \cong \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n = 1 \\ \mathbb{Z}^m & n = 2 \\ 0 & n \geq 3 \end{cases}.$$

Extending the notion from knots, we can now define what it means for a link to be shake slice–see Figure 2.1.

Definition 2.1. We call L *r-shake slice* if there exist m disjoint spheres $\Sigma_1, ..., \Sigma_m$ embedded in W_L^r that represent the generators $(1, 0, ..., 0), ..., (0, 0, ..., 1)$ of $H_2(W_L^r) \cong \mathbb{Z}^m$.

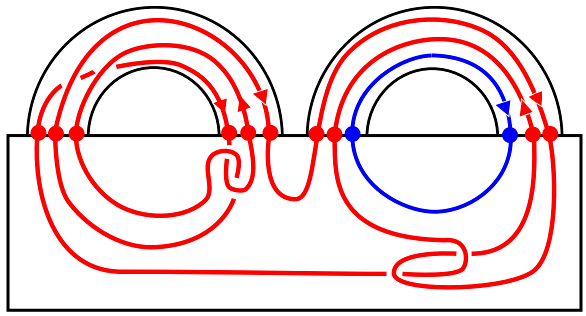


Figure 2.1. A schematic of a shake slice 2-component link.

Up to isotopy, each embedded sphere $-\Sigma_i$ intersects the i^{th} added 2-handle as $2n_{ii} - 1$ disks for some $2n_i \geq 1$ and intersects the j^{th} added 2-handle as $2n_{ij}$ disks for some $n_{ij} \geq 0$ for each $j \neq i$. In particular, after deleting these disks, the manifold $\sqcup_{k=1}^m - \Sigma_k$ has as their boundary an odd number of r-framed parallel copies of L_i for each $i = 1, ..., m$. In particular, $-\Sigma_i$ with these disks removed bounds $2n_i - 1$ r-framed parallel copies of L_i, n_i of which have the orientation of L_i and $n_{ii} - 1$ of which have the opposite orientation, and $2n_{ij}$ r-framed copies of L_j, n_{ij} of which have the same orientation of L_j and n_{ij} of which have opposite orientation.

Hence we may form an equivalent definition for a link to be shake slice as follows.

Definition 2.2. For any link L define the $(2n_1 - 1, ..., 2n_m - 1)$ *r-shaking* of L to be the link formed by taking $2n_i - 1$ r-framed parallel copies of L_i where n_i are oriented in the direction of L_i and $n_i - 1$ are oriented in the opposite direction for each $i = 1, ..., m$.

See Figure 2.2 for a $(3, 1)$ 0-shaking of a 2-component link.

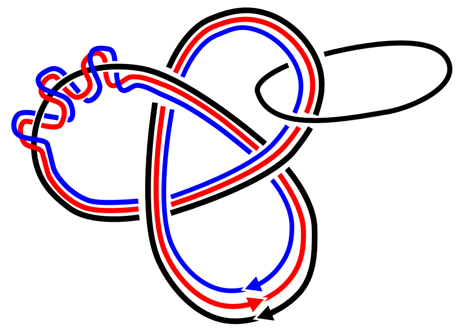


FIGURE 2.2. A (3,1) 0-shaking of the trefoil.

DEFINITION 2.3 (Alternative). We call an m-component link $L \subset S^3 = \partial D^4$ r-*shake slice* if there exists m disjoint, smooth, properly embedded, compact, connected, genus zero surfaces $\Sigma_1, ..., \Sigma_m$ in D^4 such that each Σ_i bounds $2n_{ii}-1$ r-framed parallel copies of L_i, precisely n_{ii} of which have the same orientation as L_i, and $2n_{ij}$ r-framed parallel copies of L_j, precisely n_{ij} of which have the same orientation as L_j, for all $j \neq i$, such that $\sqcup_{k=1}^{m} \Sigma_k$ bound a $(N_1, ..., N_m)$ r-shaking of L where

$$N_i = \sum_{j=1}^{m} n_{ij}.$$

One notices that a link L is r-shake slice, for all r, whenever L is slice. The converse fails for $r \neq 0$ since it is known to fail for knots. Therefore, we will limit our attention largely to 0-shake slice links. When we do not specify r, it is to be understood $r = 0$.

If we impose the restriction on each sphere Σ_i that it only intersects the i^{th} 2-handle, then we get the following, stricter, notion:

Definition 2.4. We call an m-component link L *strongly r-shake slice* if there exists m disjoint, smooth, properly embedded, compact, connected, genus zero surfaces $\Sigma_1, ..., \Sigma_m$ in D^4 such that each Σ_i bounds $2n_{ii} - 1$ r-framed parallel copies of L_i, precisely n_{ii} of which have the same orientation as L_i, such that $\sqcup_{k=1}^{m} \Sigma_k$ bound a r-shaking of L.

Then notice for a link, for all r,

$$slice \Rightarrow strong\ \ r - shake\ \ slice \Rightarrow r - shake\ \ slice.$$

1.2. Shake Concordant Links. We may now extend the notion of shake concordance defined for knots to links. Given oriented m-component links $L \hookrightarrow S^3 \times \{0\}$ and $L' \hookrightarrow S^3 \times \{1\}$ let $W^r_{L,L'}$ denote the 4-manifold obtained by adding $2m$ 2-handles with framing r to $S^3 \times [0,1]$ along the $2m$ handles of the links L and L'.

Definition 2.5. We call m-component links L and L' *shake concordant* if there exist m disjoint spheres $\Sigma_1, ...\Sigma_m$ embedded in $W^r_{L,L'}$ that represent the set of generators $\{(x_1, ..., x_m, y_1, ..., y_m) | x_i = y_i = 1, x_j = y_j = 0\ for\ j \neq i\}_{i=1,...,n}$ of $H_2(W^r_{L,L'}) \cong \mathbb{Z}^{2m}$

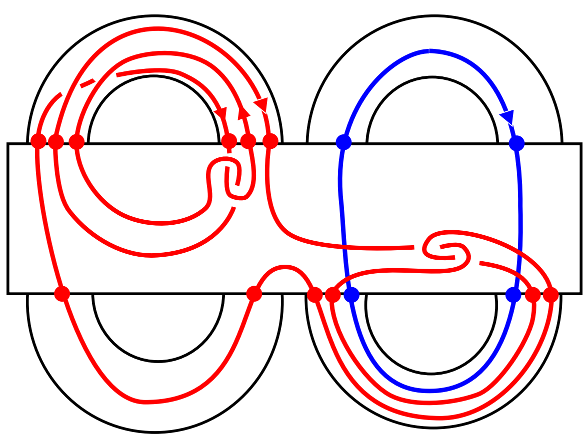


Figure 2.3. A schematic of shake concordant 2-component links.

And as before, this gives rise to an alternative definition.

Definition 2.6 (Alternative). We call the links L and L' $(2N_1-1,...,2N_m-1;2N'_1-1,...,2N'_m-1)$ *r-shake concordant* if there are disjoint smooth, properly embedded, compact, connected, genus zero surfaces $F_1,...,F_m$ in $S^3\times[0,1]$ such that $F_k\cap S^3\times\{0\}$ consists $2n_{ii}-1$ r-framed parallel copies of L_i, precisely n_{ii} of which are orientated the same as L_i, and for $j\neq i$, $2n_{ij}$ r-framed parallel copies of L_j, precisely n_{ij} of which are oriented the same as L_j, where

$$N_j=\sum_{i=1}^{m}n_{ij},\quad j=1,..,m.$$

Similarly, each $F_k\cap S^3\times\{1\}$ consists of $2n'_{ii}-1$ copies of L'_i, precisely n'_{ii} of which are orientated the same as L'_i, and for $j\neq i$, $2n'_{ij}$ copies of L'_j, precisely n'_{ij} of which are oriented the same as L'_j, where

$$N'_j=\sum_{i=1}^{m}n'_{ij},\quad j=1,..,m.$$

We say the links are *strongly r-shake concordant* if $n_{ij}=n'_{ij}=0$ whenever $i\neq j$.

See Figure 2.4 for a schematic of a $(3,1;1,3)$ shake concordance and Figure 2.5 for a schematic of a $(3,1;1,3)$ strong shake concordance.

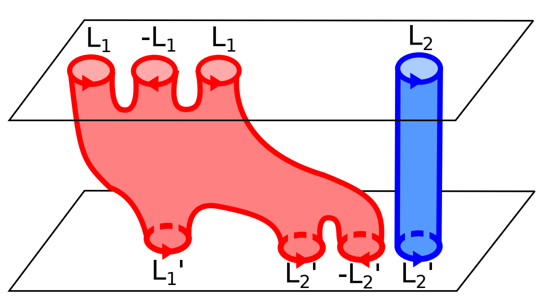


Figure 2.4. A (3,1;1,3) shake concordance.

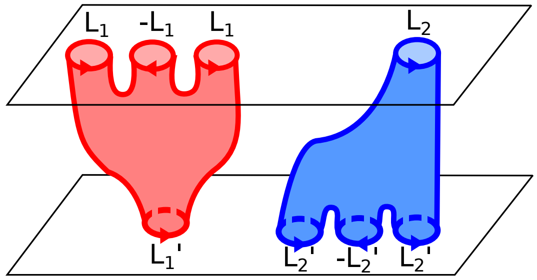


Figure 2.5. A strong shake concordance.

Notice for any two links,

$$concordance \Rightarrow strong\ \ shake\ \ concordance \Rightarrow shake\ \ concordance.$$

For knots, strong shake concordance and shake concordance are equivalent notions. Moreover, observe that if links $L = L_1 \sqcup ... \sqcup L_m$ and $L' = L'_1 \sqcup ... \sqcup L'_m$ are $(n_1, ..., n_m; n'_1, ..., n'_m)$ strongly r-shake concordant, then components L_i and L'_i are (n_i, n'_i) r-shake concordant as knots for all $i = 1, ..., m$. One would not expect from the definition, though, that something similar can be said if L and L' are r-shake concordant, but not strongly r-shake concordant. In fact, we will see that in this setting there is no such relationship between components L_i and L'_i.

Finally, note that an m-component link is (strongly) r-shake slice if and only if it is $(n_1, ..., n_m; 1, ..., 1)$ (strongly) r-shake concordant to the trivial link. We can see this in one direction by removing a ball from D^4 that hits a neighborhood of a point in each of the m genus zero surfaces and in the other direction by capping off the unlink by m disks.

1.3. Induced Equivalence Relation on Links. Notice every link is $(1, 1)$ strongly r-shake concordant to itself by the trivial concordance. Also, if L is $(n_1, ..., n_m;$

$n'_1, ..., n'_m$) (strongly) r-shake concordant to L', then L' is $(n'_1, ..., n'_m; n_1, ..., n_m)$ (resp. strongly) r-shake concordant to L. However, unlike concordance, strong shake concordance is not in general a transitive property of links. For instance, suppose L is $(1, 3)$ strongly r-shake concordant to L' and L' is $(3, 5)$ strongly r-shake concordant to L''. If we attempt to glue together the genus zero surfaces representing these shake concordances along the 3-component r-shaking of L', we obtain a surface bounding r-shakings of L and L'', but genus is introduced by the gluing, so we cannot conclude that L is $(1, 5)$ strongly r-shake concordant to L' as one may hope. However, transitivity does hold if we restrict the shake concordance as follows.

PROPOSITION 2.7. *If L is $(n_1, ..., n_m; 1, ..., 1)$ r-shake concordant to L' and L' is $(1, ..., 1; n''_1, ..., n''_m)$ r-shake concordant to L'', then L is $(n_1, ..., n_m; n''_1, ..., n''_m)$ r-shake concordant to L''.*

Moreover, if L is $(n_1, ..., n_m; n'_1, ..., n'_m)$ r-shake concordant to L' and L' is $(1, ..., 1;$ $n''_m, ..., n''_m)$ strongly r-shake concordant to L'', then L is $(n_1, ..., n_m; n'_1 n''_1, ..., n'_m n''_m)$ r-shake concordant to L''.

PROOF. The first case is clear as we can simply glue together the surfaces F_i and F'_i of the r-shake concordances in the obvious way for $i = 1, ..., m$. In the second case, we glue similarly, but note that this is possible since the surfaces of the r-shake concordance have trivial bundles and hence we can take parallel copies by extending the normal vector field given by the r-framing to all of the surface. Note then that taking $n'_i = 2k_i - 1$ parallel copies of each F'_i, $k_i - 1$ of which have reversed orientation, for all $i = 1, ..., m$, and glueing in the expected way, gives the desired shake concordance. □

Despite this restriction on transitivity, we can still introduce equivalence relations $\sim_r$ on the set of links where we say $L \sim_r L'$ if there exist links $L = L_1, L_2, ..., L_n = L'$ for some n such that L_i is shake concordant to L_{i+1} for all $i = 1, ..., n-1$. Similarly we may define equivalence relations for strong r-shake concordant. In the next chapter we will offer a classification theorem, Theorem 3.2, that helps us better understand these equivalence classes.

2. Visualizing Slice Disks and Concordances

We can visualize a concordance or a slice disk as a "movie", which we here illustrate with an example.

Figure 2.6 depicts a concordance between two links. Figure 2.7 shows cross sections of the surface of the concordance as you move down it. In particular, Figure 2.7 (A), (B), (C) shows the effect of the saddle, while (D), (E), (F) shows the effect of capping off a component with a disk.

More generally, there are four features of a surface as we move up it that change our diagrams beyond just isotopy. A local minimum of the surface gives birth to a new component, while a local maximum kills off a component; these correspond with capping off components with slice disks. A join saddle fuses two components of a link, as in our example, while a split saddle separates a component into two different components. Diagrammatically, this fusion of components is accomplished by attaching two components via a band while a split is accomplished by attaching both ends of a band to the same component. See Figure 2.8, although note that these bands may also be knotted or twisted.

If a slice disk has no such local maximum, we call the knot *ribbon*.

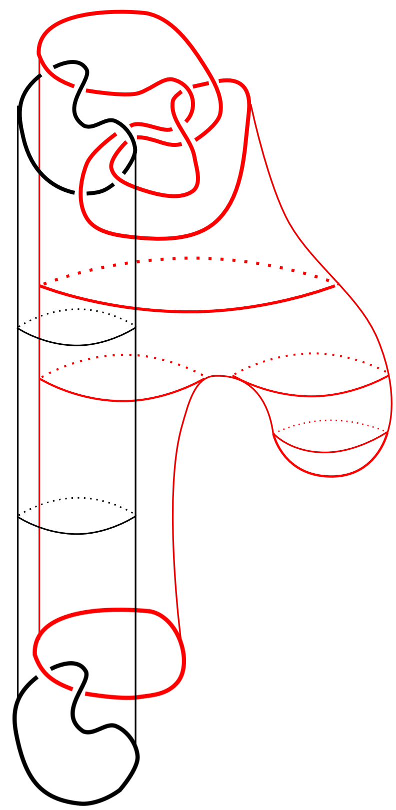
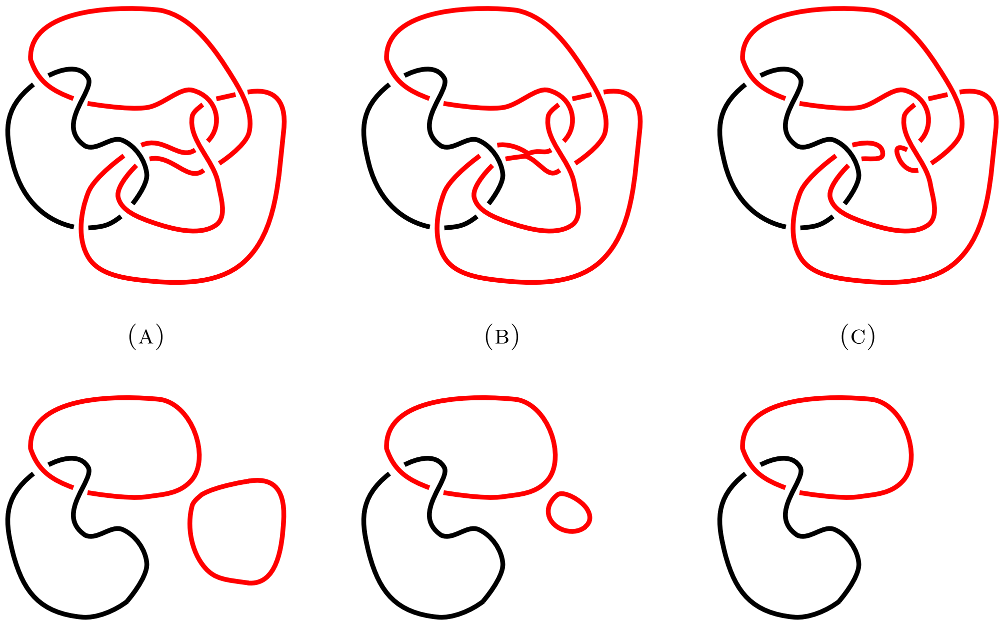


Figure 2.6. A concordance between links.

3. Differentiating Shake, Strong Shake, and Concordance

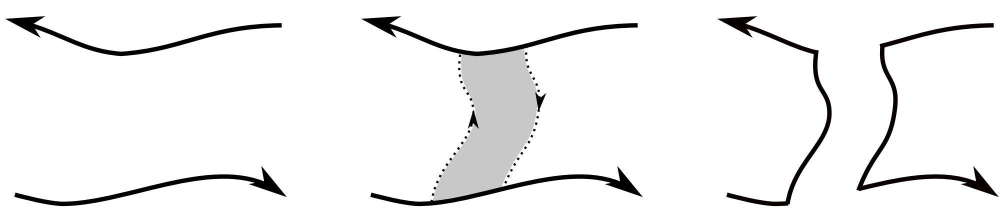
We now turn our attention to understanding how concordance, r-shake concordance, and strong r-shake concordance differ from each other. The work of Akbulut and others [**Akb77**],[**Akb93**], [**AJOT13**] demonstrating that there are r-shake slice knots that are not slice for $r \neq 0$ leaves us interested in the case $r = 0$. Here we recall the finding of Cochran and Ray [**CR16**] that there exists an infinite family of topologically slice knots that are pairwise 0-shake concordant but distinct in smooth concordance. Of remaining interest are links with at least 2-components. Note that families of links with unknotted components are especially helpful in understanding how link concordance effects links beyond its effect on individual components.



(A) (B) (C)

(D) (E) (F)

Figure 2.7. Diagrams along various steps of the concordance.



(A) (B) (C)

Figure 2.8. Effect of adding a band.

3.1. Shake Concordant But Not Strongly Concordant. For knots, shake concordance and strong shake concordance are equivalent notions. We introduce a family of 2-components links to show that strong shake concordance and shake concordance are distinct notions for links. Let $h(K)$ denote the 2-component link consisting of first component K and second component a meridian of K as in Figure 2.9.

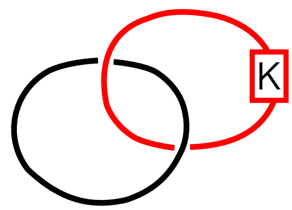


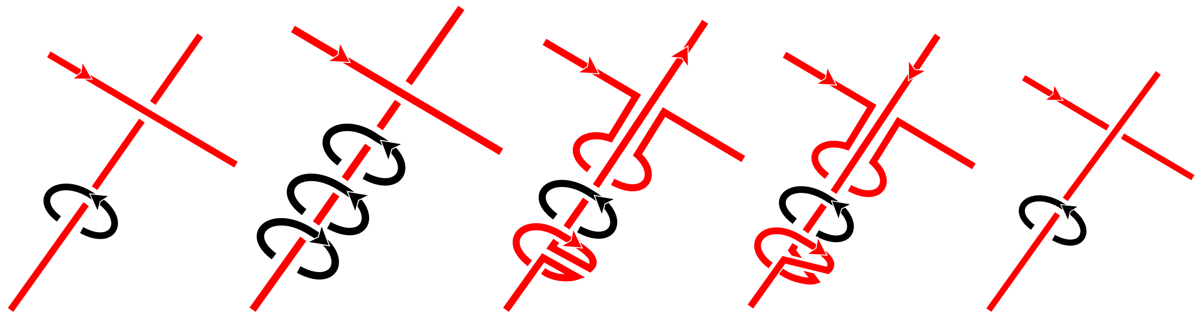
FIGURE 2.9. The two component link $h(K)$ consisting of K and its meridian.

Note for U the unknot, $h(U)$ is the hopf link.

PROPOSITION 2.8. *The links $h(K)$ and $h(J)$ are shake concordant for any knots K and J.*

PROOF. We show in Figure 2.10 how a shake concordance accomplishes a crossing change in a link of the form $h(K)$, which we now explain. Position the meridian component so that it is next to the crossing in K that we want to change as in Figure 2.10 (A). Then, consider a $(1,3)$ shaking of $h(K)$ as in Figure 2.10 (B). Take a band sum of one of the meridian components that has the appropriate orientation with K at the crossing to change it from an overcrossing to undercrossing (or vice versa). Also take a band sum of a meridian component that has opposite orientation and band it with K so as not to change K as in as in Figure 2.10 (C). Depending on orientations, we may need a half twist in the band as in Figure 2.10 (D). Attaching these bands accomplishes the desired shake concordance between $h(K)$ and $h(K)$ with a crossing change of our choice as in Figure 2.10 (E).

Notice this technique can be extended to accomplish any number of crossing changes via a shake concordance. In particular, there is a $(1, 2n+1)$ shake concordance between $h(K)$ and $h(K)$ with n crossing changes accomplished by attaching bands in a similar way at each of the n crossings. □



(A) (B) (C) (D) (E)

Figure 2.10. Steps to accomplish crossing change via shake concordance.

Not all knots are shake concordant; for instance, Cochran and Ray showed in [**CR16**] that signature is an invariant of shake concordance that distinguishing infinitely many different classes of knots up shake concordance. Therefore, we see from the above proposition that corresponding sublinks of shake concordant links are not necessarily shake concordant! Moreover, if K and J have differing signatures, $h(K)$ and $h(J)$ are not strongly shake concordant, since K and J are not shake concordant. This gives us the following desired corollary:

Corollary 2.9. *There exists an infinite family of two-component links that are pairwise shake concordant, but not pairwise strongly shake concordant.*

Proof. Consider the family $\{h(K_k)\}_{k=1,2,3,\ldots}$ where K_k is the connect sum of k trefoil knots which has signature $2k$ and therefore are distinct up to concordance. □

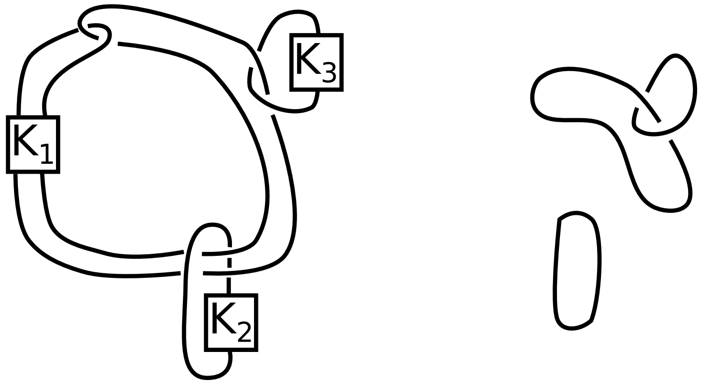
We can strengthen the above proposition using the notion of link homotopy induced by Milnor in [**Mil54**]. We say links L and L' are *link homotopic* if there is a homotopy deforming L into L' such that the components remain disjoint during the deformation.

PROPOSITION 2.10. *If two m-component links are link homotopic, then they are sublinks of shake concordant $2m$-component links.*

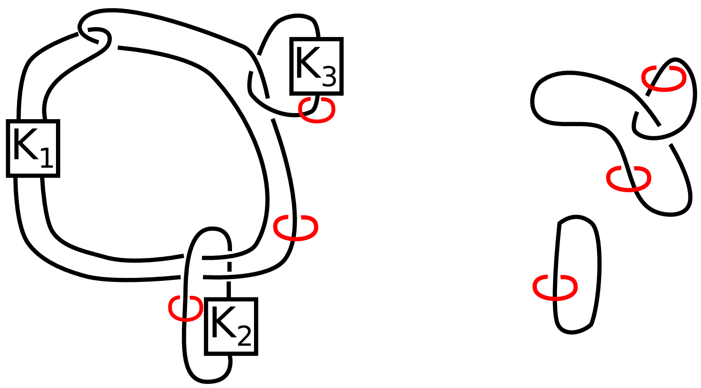
PROOF. Suppose m-component links L and L' are link homotopic, then L' can be obtained from L by ambient isotopy and crossing changes between arcs of the same component of L. However, as we saw in the proof of Proposition 2.8, these crossing changes can also be obtained via shake concordance of the components of L with added meridian components. Therefore, if we let J (resp. J') denote the $2m$-component link consisting of the m components of L (resp. L') and m meridian components for each component of L, then we have J is shake concordant to J'. See Figure 2.11. ☐

3.2. Strongly Shake Concordant But Not Concordant. We now show that strong shake concordance and concordance are distinct notions. Note that Theorem 1.1 provides a infinite family of knots that are shake concordant but not concordant. We extend this result by showing that there is an infinite family of two component links with unknotted components that are pairwise strongly shake concordant but not shake concordant.

Given a knot K, let $L(K)$ denote the two component link of Figure 2.12. Each component of $L(K)$ is unknotted and $L(U)$ is the hopf link.



(A) Homotopic 3-component links.



(B) Shake concordant 6-component links.

FIGURE 2.11

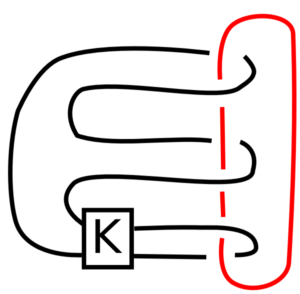


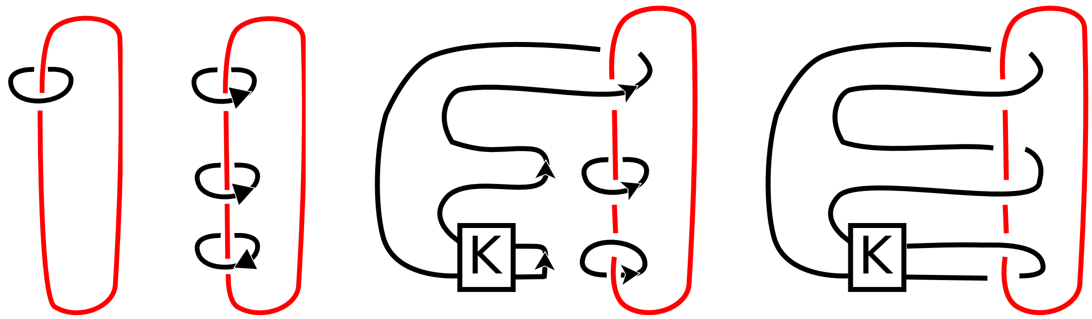
FIGURE 2.12. The link $L(K)$.

THEOREM 2.11 (Cha, Kim, Ruberman, Strle 2010). *The 2-component link* $L(K)$ *is not concordant to the Hopf link for* K *when* $\tau(K) > 0$. *Moreover, there is a infinite family of knots* K_n *such that* $L(K_n)$ *are distinct up to smooth concordance.*

Here $\tau(K)$ denotes the Ozsváth-Szabó tau invariant. It is an integer-valued invariant of knot concordance that vanishes for slice knots, is additive under connect sum, that is, $\tau(K\#K') = \tau(K) + \tau(K')$, and changes sign under changes in orientation, $\tau(-K) = -\tau(K)$. We will discuss the tau invariant in more detail when we discuss invariants of shake concordance.

Proposition 2.12. *There exists an infinite family of 2-components links with trivial components that are all strongly shake concordant to the Hopf link, but none of which are concordant to the Hopf link.*

Proof. We argue that $L(K)$ is strongly $(1,1;3,1)$ shake concordant to the Hopf link for any knot K. First take a $(3,1)$ shaking of the Hopf link as in Figure 2.13 (A) and (B). Then isotope one of the parallel copies as in Figure 2.13 (C) and band sum it with the other parallel copies to obtain Figure 2.13 (D). This gives the desired strong shake concordance; however, we have by Theorem 2.11 that $L(K)$ is not concordant to the Hopf link whenever $\tau(K) > 0$. Tau is known to be positive for many families of knots. For instance, for a (p,q)-torus knot $T_{p,q}$, where $p,q > 0$, we have $\tau(T_{p,q}) = (p-1)(q-1)/2$ [**OS03**, Corollary 1.7]. $\square$



(A) (B) (C) (D)

Figure 2.13. Steps to obtain strong shake concordance between Hopf link and $L(K)$.

CHAPTER 3

Classification Theorem

1. Classification of Shake Concordance of Knots

1.1. Satellite Operation. Cochran and Ray [**CR16**] offered a classification of shake concordance up to concordance and satellite operation. This latter notion is defined as follows. Consider a knot P essentially embedded in a solid torus $S^1 \times D^2 \subset S^3$, that is, there exists no meridinal disks that avoids P. Then given any knot $K \subset S^3$, there exists an embedding f that maps the torus to a neighborhood of K, sending the longitude of $S^1 \times \{1\}$ to the longitude of $f(S^1 \times D^2)$. Notice, the image of P under the embedding $f(P)$ is a knot. Notice, we can think of a pattern P as an operator, denoting the knot $f(P)$ obtained by the satellite operation as $P(K)$. Since P is a torus knot, the winding number of a satellite operation is well-defined.

1.2. Classification Theorem. We are now prepared to state the classification theorem for knots up to shake concordance.

Theorem 3.1 (Theorem 3.7 in [**CR16**]). *Two knots K and J are shake concordant if and only if there exist winding number one satellite operators P and Q, with $i(P)$ and $i(Q)$ ribbon where i denotes the inclusion of the standard torus into S^3, such that $P(K)$ is concordant to $Q(J)$.*

In this chapter, our goal is to generalize this theorem to a classification theorem for shake concordance of links. This first requires a generalization of satellite operation for links, which is the focus of the next section.

2. String Links and Infection

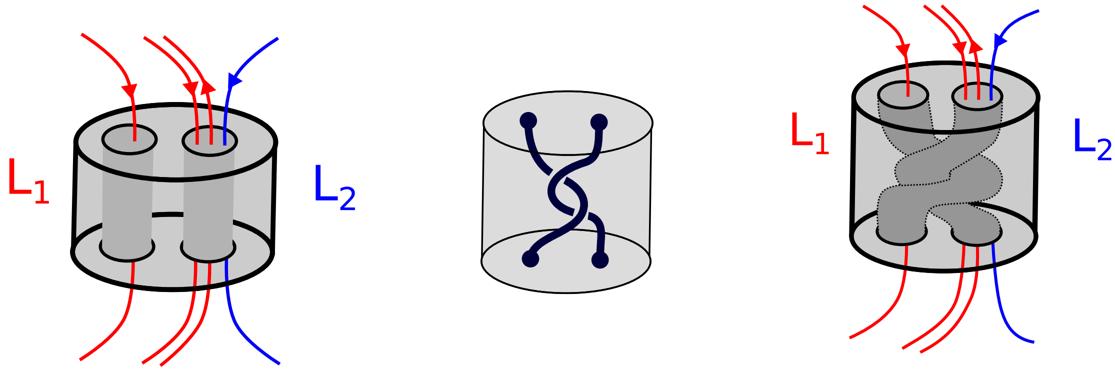
An m-component *string link* is a proper embedding

$$J : \bigsqcup_{i=1}^{m} I_i \to D^2 \times I$$

of the disjoint union of m copies of the unit interval I_i in $D^2 \times I$ where we equip D^2 with m marked points in its interior and such that the image of each I_i runs from $(x_i, 0)$ to $(x_i, 1)$. By an abuse of notation, we also refer to the image of the string link by J. We call the string link $\sqcup_{i=1}^{m}(\{x_i\} \times I)$ the *trivial* m*-component string link.* Notice, a sting link J can be closed in the obvious way to obtain an m-component link $\widehat{J}$, which we call *closure of* J. Every link is the closure of some string link. One may consider the meridians and longitudes of string links exactly analogous as those of links.

Infection by a string link [**COT04**], also called *multi-infection* [**CFT07**] and *tangle sum* [**CO94**], is a generalization of satellite construction that modifies an m-component link L by some string link J in order to obtain an infected m-component link in such a way that we here describe. See [**CFT07**, Section 2] for details.

An r-multi-disk $\mathbb{E}$ is an embedded disk D with k disjoint embedded open subdisks $D_1, ..., D_k$ contained in the interior of D. Consider an m-component link L and a map $\varphi : \mathbb{E} \to S^3$ such that the image of φ, which we'll denote $\mathbb{E}_\varphi$, intersects L transversely at points $p_1, ..., p_m$ all of which are in the images of $D_1, ..., D_r$.



(A) L with $\mathbb{E}_\varphi$ (B) J (C) $I(L, J, \mathbb{E}_\varphi)$

FIGURE 3.1. Infection of link L by string link J.

We may view $(\mathbb{E}_\varphi \setminus \sqcup_i D_i) \times [0,1]$ as the exterior of the trivial k-component string link. Note this has the same boundary as the exterior of any k-component string link J, denoted $(D^2 \times [0,1] \setminus \nu(J))$ where $\nu(J)$ is the neighborhood of J. Hence we can modify S^3 by deleting the exterior of the trivial r-component string link and glue in the exterior of the string link J in such a manner that equates the meridians and longitudes of these two string links. This gives a manifold that is homeomorphic to S^3:

$$S^3 \setminus ((\mathbb{E}_\varphi \setminus \sqcup_i D_i) \times I) \cup (D^2 \times I \setminus \nu(J)) = S^3 \setminus (\mathbb{E}_\varphi \times I) \cup (D^2 \times I \setminus \nu(J)) \cup (\overline{\sqcup_i D_i} \times I)$$

$$\cong D^3 \cup D^3 \cong S^3.$$

Effectively, this ties J into L along $\mathbb{E}_\varphi$ resulting in an infected link in S^3 we denote $I(L, J, \mathbb{E}_\varphi)$. See Figure 3.1.

We will say $\mathbb{E}_\varphi$ *respects* L if $k = m$ and each link component L_i intersects the subdisk D_j algebraically once if $i = j$ and algebraically zero times if $i \neq j$. See Figure

3.1 (A). If, moreover, the count of intersection points is geometrically zero for $i \neq j$, then we say the embedding *strongly respects* L.

3. Classification of Shake Concordance of Links

THEOREM 3.2. *The m-component links M and M' are shake concordant if and only if the links obtained by string link infection $I(L, J, \mathbb{E}_\varphi)$ and $I(L', J', \mathbb{E}'_{\varphi'})$ are concordant for some:*

- *m-component slice links L and L',*
- *m-component string links J, J' with closures $\widehat{J} = M$ and $\widehat{J'} = M'$, respectively,*
- *and embeddings of multidisks $\mathbb{E}_\varphi$ and $\mathbb{E}'_{\varphi'}$ each with m subdisks that respect L and L', respectively.*

REMARK. In the above theorem we can specify r-shake concordance for any r if we introduce r twists to each series of strands passing through each subdisk in the multiinfection, denoted $I_r(L, J, \mathbb{E}_\varphi)$. Also *shake concordant* can be strengthened to *strong shake concordant* by also strengthening the condition on multidisks from *respect* to *strongly respect.* The proof is identical to what we offer here, other than maintaining the modified conditions throughout.

4. Proof of Theorem

4.1. Proof of "if" direction. First we prove the following lemma.

LEMMA 3.3. *Given slice m-component link L, string link J, and an embedded multidisk $\mathbb{E}_\varphi$ that respects L, we have $I(L, J, \mathbb{E}_\varphi)$ is $(1, ..., 1; n_1, ..., n_m)$ shake concordant to $\widehat{J}$, the closure of J into a link.*

Proof. Suppose L intersects the subdisk D_i of $\mathbb{E}_\varphi$ geometrically n_i times for $i = 1, ..., m$. Then L can be obtained by band summing a copy of L with T, a $(n_1, ..., n_m)$ shaking of the m-component unlink, as in Figure 3.2.

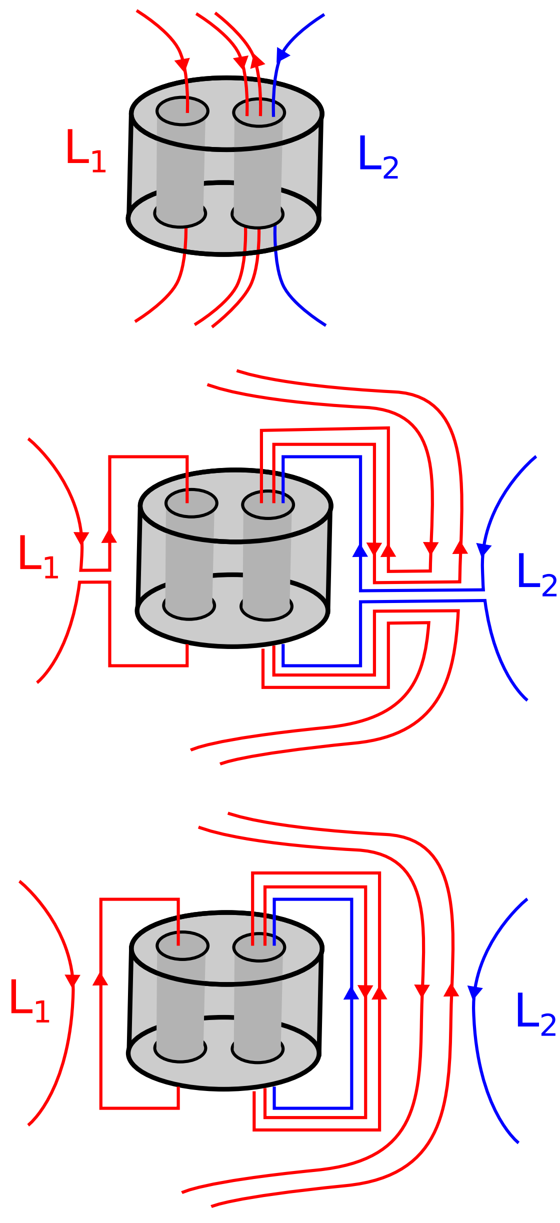


Figure 3.2. Effect of bands on fusing components.

Hence there exists a smooth, compact, connected, genus 0 surface $S \subset S^3 \times [0, 1]$ that cobounds $L \subset S^3 \times \{0\}$ and the disjoint union $L \sqcup T_m \subset S^3 \times \{1\}$.

We may construct S such that it lies entirely in the complement of

$$((\mathbb{E}_\varphi / \sqcup_i \varphi(D_i)) \times [0,1]) \times [0,1] \subset S^3 \times [0,1].$$

Replace $((\mathbb{E}_\varphi / \sqcup_i \varphi(D_i)) \times [0,1]) \times \{t\}$ at each $t \in [0,1]$ with the complement of string link $J \subset D^2 \times [0,1]$. Then S cobounds $I(L, J, \mathbb{E}_\varphi) \subset S^3 \times \{0\}$ and a disjoint union of L and a $(n_1, ..., n_m)$ shaking of $\widehat{J}$ in $S^3 \times \{1\}$. As L is slice, we can cap it off to obtain a $(1, ..., 1; n_1, ..., n_m)$ shake concordance between $I(I, J, \mathbb{E}_\varphi)$ and $\widehat{J}$. □

The proof of the "if" direction follows immediately from this lemma, for if $I(L, J, \mathbb{E}_\varphi)$ is concordant to $I(L', J', \mathbb{E}'_{\varphi'})$ then by the lemma $I(L, J, \mathbb{E}_\varphi)$ is $(1, ..., 1; n_1, ..., n_m)$ shake concordant to $\widehat{J}$ and $I(L', J', \mathbb{E}'_{\varphi'})$ is $(1, ..., 1; n'_1, ..., n'_m)$ shake concordant to $\widehat{J}'$. Hence, $\widehat{J}$ is $(n_1, ..., n_m; n'_1, ..., n'_m)$ shake concordant to $\widehat{J}'$.

4.2. Proof of "only if" direction. Similarly, we begin by proving the following lemma.

Lemma 3.4. *If L is $(1, ..., 1; n_1, ..., n_m)$ shake concordant to L', then L is concordant to $I(L'', J', \mathbb{E}_\varphi)$ for some string link J' such that $\widehat{J}' = L'$, slice link L'', and embedded multidisk $\mathbb{E}_\varphi$ that respects L''.*

Proof. Let $F_1, ..., F_m$ be the m disjoint genus zero surfaces in $S^3 \times [0,1]$ with boundary $L \subset S^3 \times \{0\}$ and $sh(L') \subset S^3 \times \{1\}$ where $sh(L')$ is a $(n_1, ..., n_m)$ shaking of L'. That is, the surfaces F_i determine a shake concordance between L and L'.

We can isotope each F_i such that the projection map $S^3 \times [0,1] \to [0,1]$ is a Morse function when restricted to each F_i and such that all local maxima occur at level $\{\frac{4}{5}\}$, split saddles at level $\{\frac{3}{5}\}$, join saddles at level $\{\frac{2}{5}\}$, and local minima at level $\{\frac{1}{5}\}$ (as in proof of Proposition 3.5 in [**CR16**]). Hence, the level of $\{\frac{1}{2}\}$ of each surface F_i is a connected component M_i of some m component link M. Notice, L is concordant to M. See Figure 3.3.

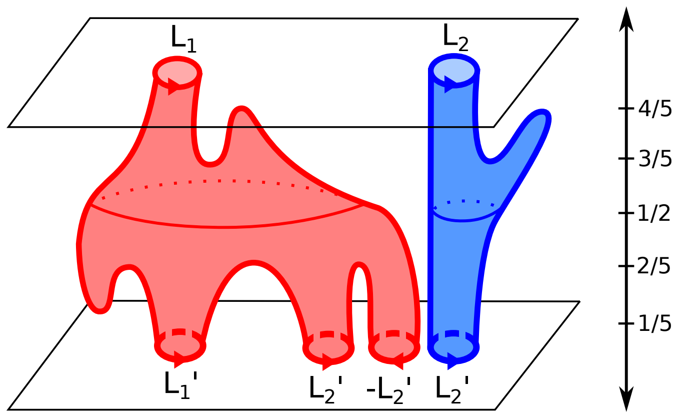


FIGURE 3.3. Morse function on the shake concordance.

Moreover, M is a fusion of $sh(L')$ and a trivial link T (corresponding to the local minima of each F_i). That is, M is obtained by attaching bands between distinct components of $sh(L')$ and T until there are only m component.

The $(n_1, ..., n_m)$ shaking of the m-component trivial link is itself a trivial link of $N = \Sigma_i n_i$ components which we denote T_N. Hence, $sh(L') = I(T_r, J', \mathbb{E}_\varphi)$ for some $\mathbb{E}_\varphi$ respecting T_N.

Suppose that we are able to isotope our fusion bands as to avoid $\mathbb{E}_\varphi \times [0,1]$. Then we have $M = I(T', J', \mathbb{E}_\varphi)$, where T' is a link obtained by fusing T_N and the trivial link T. Note T' is slice. Hence, L is concordant to $M = I(T', J', \mathbb{E}_\varphi)$, as desired.

However, suppose the fusion bands cannot be isotoped to avoid intersecting $\mathbb{E}_\varphi \times [0,1]$; such as in Figure 3.4. Then embed $\varphi' : \mathbb{E}' \hookrightarrow S^3$, where $\mathbb{E}'$ is a multidisk with subdisks $D_1, ..., D_m$, such that $\mathbb{E}'_{\varphi'} \times [0,1]$ intersects M as a trivial string link so that each $D_i \times [0,1]$ contains a trivial n_i component string link corresponding to the i^{th}

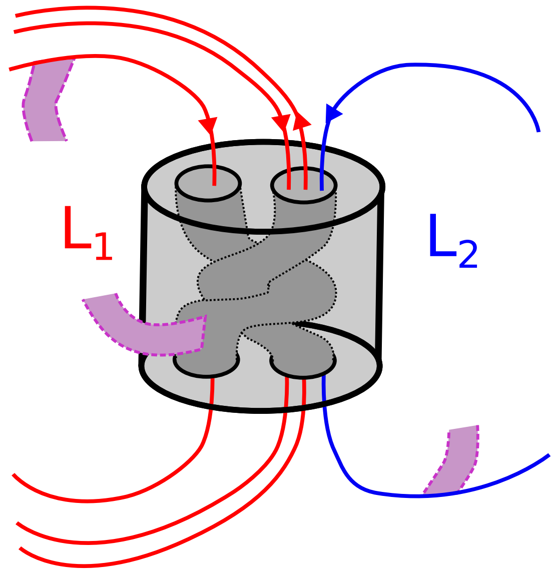


FIGURE 3.4. Infected link with fusion bands.

set of components in the $(n_1, ..., n_m)$ shaking of L'. Moreover, we may choose this embedding so that it avoids intersecting the fusion bands, T, and $\mathbb{E}_\varphi \times [0,1]$.

Now, we may infect M at $\mathbb{E}'_{\varphi'}$ with the string link $J' \# -J'$; see Figure 3.5. This is a slice string link, thus M is concordant to the $I(M, J'\# - J', \mathbb{E}'_{\varphi'})$. We can think of inflecting along $\mathbb{E}'_{\varphi'}$ by $J'\# - J'$ as infecting along $\mathbb{E}''_{\varphi''}$ by J' and along $\mathbb{E}'''_{\varphi'''}$ by $-J'$ where

$$\mathbb{E}''_{\varphi''} \times [0,1] := \mathbb{E}'_{\varphi'} \times [0, 0.5], \quad \mathbb{E}'''_{\varphi'''} \times [0,1] := \mathbb{E}'_{\varphi'} \times [0.5, 1].$$

Now define $L'' = I(M, -J', \mathbb{E}'''_{\varphi'''})$; see Figure 3.6. Then notice M is concordant to $I(L'', J', \mathbb{E}''_{\varphi''})$. Observe, L'' is slice since it's the fusion of a trivial link T and an infection of a trivial link by the slice string link $J'\# - J'$. Hence, J is concordant to $I(L'', J', E'_1)$, as desired. $\square$

Now suppose L is shake concordant to L'. Then there are some surfaces $F_1, .., F_m$ that bound a shaking of L in $S^2 \times \{0\}$ and a shaking of L' in $S^2 \times \{1\}$. As in the

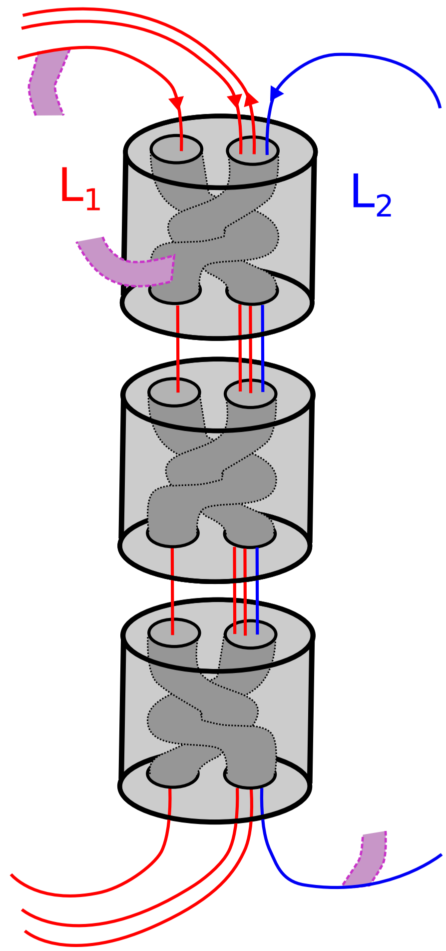


FIGURE 3.5. Infecting link to avoid fusion bands.

proof above, we may construct a Morse function $f : S^3 \times [0,1] \to [0,1]$ such that when restricted to each f_i, all maxima occur at level $\{\frac{4}{5}\}$, split saddles at level $\{\frac{3}{5}\}$, join saddles at level $\{\frac{2}{5}\}$, and local minima at level $\{\frac{1}{5}\}$. Hence, the level $\{\frac{1}{2}\}$ of each surface F_i is a connected component M_i of some m component link M. Moreover, M is $(1, ..., 1; n_1, ..., n_d)$ shake concordant to L and M is $(1, ..., 1; s_1, ..., s_d)$ shake concordant to L'. Thus, by the lemma, M is concordant to $I(L, J, \mathbb{E}_\varphi)$ and $I(L', J', \mathbb{E}'_{\varphi'})$ for L and L' slice and $\mathbb{E}_\varphi$ and $\mathbb{E}'_{\varphi'}$ meeting the conditions in the statement of the theorem.

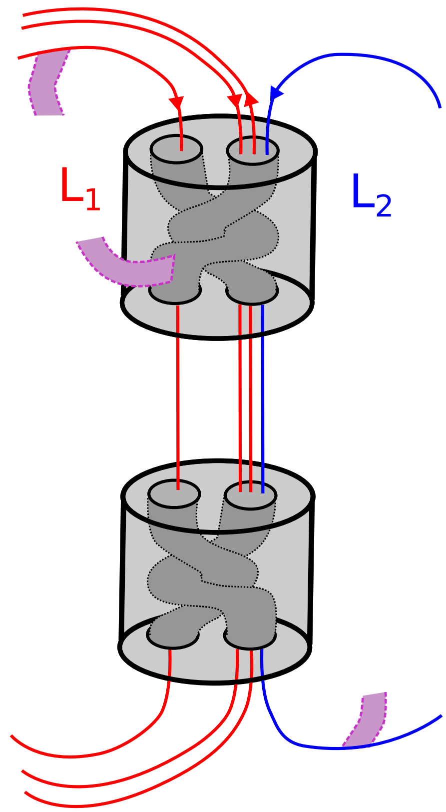


FIGURE 3.6. The link that is infected to obtain M.

5. Classification of Shake Slice Links

We are also able to classify shake slice links.

COROLLARY 3.5. *The m-component link M is (strongly) shake slice if and only if the link obtained by string link infection $I(L, J, \mathbb{E}_\varphi)$ is slice for some:*

- *m-component slice link L,*
- *m-component string link J with closures $\widehat{J} = M$,*
- *and embedding of multidisk $\mathbb{E}_\varphi$ with m subdisks that (strongly) respect L.*

PROOF. A link M is (strongly) shake slice if and only if it is $(n_1, ..., n_m; 1, ..., 1)$ (strongly) shake concordant to the trivial link. By Theorem 3.2 this is equivalent to the links obtained by string link infection $I(L, J, \mathbb{E}_\varphi)$ and $I(L', J, \mathbb{E}'_{\varphi'})$ being concordant. However, we note that $I(L', J, \mathbb{E}'_{\varphi'})$ is slice since L' is slice and $\widehat{J'} = T$. $\square$

Therefore, to find a shake slice link that is not slice, one need just find a link M that is not slice, a slice link L, and a proper infection $I(L, J, \mathbb{E}_\varphi)$ that is slice. There are a number of results of this type. For instance, Cochran, Friedl, and Teichner proved in [**CFT09**, Theorem 1.5] that slice links can be generated by infecting a slice link by any link meeting some simple conditions, but their work places restrictions on the multidisk that prevent it from being a proper embedding. In fact, as we'll see in chapter 5 section 6, it appears to be a hard problem to find shake slice links that are not slice.

CHAPTER 4

Invariants of Shake Concordance

Our goal in this chapter is to see what invariants of concordance are also preserved up to strong shake concordance and shake concordance. Our classification theorem from the last chapter will be useful, as it implies that invariants that are preserved under concordance and a particular class of string link infection will be invariants also of shake concordance. This will let us show in section 1 that the first non-vanishing Milnor invariant is preserved under shake concordance. In section 2 we'll show that the zero surgery manifolds obtained from shake concordant links are homology cobordant. We will then consider a number of other classical invariants in section 3.

1. Milnor Invariants

1.1. Definition. In [**Mil54**] and [**Mil57**] Milnor defined a family of invariants for links, the Milnor $\bar{\mu}$ invariants. For an m component link L, the Milnor invariants $\bar{\mu}_L(I)$ are defined for each multi-index $I = i_1 i_2 ... i_k$ where $1 \leq i_j \leq m$ and can be thought of as the higher order linking numbers of L. We say that the Milnor invariant has *length* $|I| = k$. Indeed, $\bar{\mu}_L(ij) = lk(L_i, L_j)$, the linking number between components L_i and L_j.

The invariants are defined algebraically from the link group by measuring how deep in the lower central series of the group longitudes lie. This gives rise to an indeterminacy in the higher order invariants if the lower order invariants do not vanish.

In particular, $\bar{\mu}_L(I)$ is defined modulo the greatest common divisor of all $\bar{\mu}_L(J)$ where J is obtained by removing at least one index from I and possibly permuting the remaining elements cyclicly. For this reason, it is often of particular interest to study the first non-vanishing Milnor invariant, that is multiindex I such that $\bar{\mu}_L(I) \neq 0$ and $\bar{\mu}_L(J) = 0$ for all $|J| < |I|$.

If L' is a sublink of L and I contains only indices that correspond with components contained in the sublink L', then $\bar{\mu}_{L'}(I)$ is well-defined and in fact $\bar{\mu}_{L'}(I) = \bar{\mu}_L(I)$. All $\bar{\mu}_T(I)$ vanish for a trivial link T. The Milnor invariants are invariants of concordance [**Cas75**], and hence they all vanish for L slice.

The Milnor invariant $\bar{\mu}_L(ijk)$ has a geometric interpretation [**Coc90**]. Let Σ_i, Σ_j, and Σ_k be Seifert surfaces for components L_i, L_j, and L_k of L. Then a count of the points which constitute $\Sigma_i \cap \Sigma_j \cap \Sigma_k$, with signs determined from the orientation induced by the Seifert surfaces, gives $\bar{\mu}_L(ijk)$. In particular, if $L = L_1 \sqcup L_2 \sqcup L_3$ is the Borromean rings depicted in Figure 4.1, then $\bar{\mu}_L(123) = \pm 1$, the sign depending on which orientation we assign the components, and $\bar{\mu}_L(ij) = lk(L_i, L_j) = 0$ for $1 \leq i, j \leq 3$.

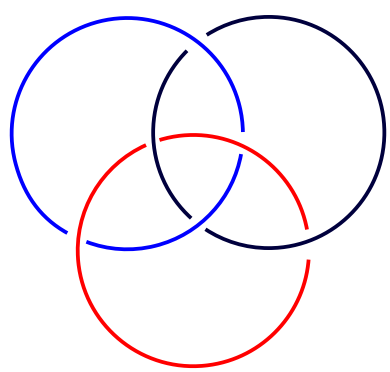


FIGURE 4.1. Borromean Rings

1.2. A useful lemma about Milnor invariants. We'll need the following lemma in the next subsection where we prove that the first non-vanish Milnor invariant is preserved under shake concordance. To do this, we'll apply [**Coc90**, Theorem 8.13] that shows that the first nonvanishing Milnor invariants are additive under exterior band sums.

LEMMA 4.1. *Let I be a multi-index which contains the indices $\{1, ..., m\}$ and let k_i be the number of occurrences of the index i in I ($k_i \geq 1$). Let $L = L_1 \sqcup ... \sqcup L_m$ be an m-component link with $\bar{\mu}(I') = 0$ whenever $|I'| < |I|$ and let J be an m-component string link whose closure $\widehat{J}$ has $\bar{\mu}_{\widehat{J}}(I') = 0$ whenever $|I'| < |I|$. Let $\varphi : \mathbb{E} \to S^3$ be a proper m-multi-disc in (S^3, L) that respects L. Then $I(L, J, \mathbb{E}_\varphi)$ is also a link with $\bar{\mu}_{I(L,J,\mathbb{E}_\varphi)}(I') = 0$ whenever $|I'| < |I|$ and*

$$\bar{\mu}_{I(L,J,\mathbb{E}_\varphi)}(I) = \bar{\mu}_L(I) + \mu_{\widehat{J}}(I).$$

PROOF. If the embedded multidisk $\mathbb{E}_\varphi$ strongly respects L, then the desired result follows immediately from [**JKP⁺14**, Lemma 4.1]. Otherwise, we do not meet the conditions of the lemma, however the proof generalizes to accommodate this case, which we offer here for completeness.

Suppose $D_i \cap L_j$ contains a_{ij} positive b_{ij} negative intersection points. Note $a_{ii} - b_{ii} = 1$ for all $1 \leq i \leq m$ and $a_{ij} - b_{ij} = 0$ for $i \neq j$. Denote $a_i = \sum_j a_{ij}$ and $b_i = \sum_j b_{ij}$. Let J' be the oriented string link generated by taking a_i parallel copies of the i-th component J_i of J and b_i parallel copies of J_i with opposite orientation, for $i = 1, ..., m$. Notice, $I(L, J, \mathbb{E}_\varphi)$ is the outcome of performing band sums on the split union of L and $\widehat{J'}$ as in Figure 3.2.

In $\widehat{J}'$ label each parallel copy of $\widehat{J}_i$ with an index $j \in \{1, ..., a_i + b_i\}$ for $i = 1, ..., m$. Define the function $g : \{1, ..., \sum_i (a_i + b_i)\} \to \{1, ..., m\}$ which sends the index of a parallel copy of $\widehat{J}_i$ to i and the function $h : \{1, ..., \sum_i (a_i + b_i)\} \to \{1, ..., m\}$ which sends the index of a parallel copy of $\widehat{J}_i$ to j where L_j is the component of L that the parallel copy is adjoined to by band sum. Choose a parallel copy of $\widehat{J}_i$ for each occurrence of i in I, for all $i = 1, ..., m$, and form the multi-index I' by replacing each occurrence of i in I by the index of the parallel copy of $\widehat{J}_i$ chosen. We need to sum over all such multi-indices:

$$\sum_{\{I' | h(I') = I\}} \bar{\mu}_{\widehat{J}'}(I') = \sum_{\{I' | h(I') = I,\, g(I') = I\}} \bar{\mu}_{\widehat{J}'}(I') + \sum_{\{I' | h(I') = I,\, g(I') \neq I\}} \bar{\mu}_{\widehat{J}'}(I').$$

Recall that reversing the orientation on a single component L_i of a link L changes the sign of the Milnor invariant $\bar{\mu}_L(I)$ by $(-1)^{k_i}$ where k_i is the number of times i appears in I. Therefore $\bar{\mu}_{\widehat{J}'}(I')$ vanishes when $g(I') \neq I$ since $a_{ij} = b_{ij}$ for $i \neq j$ and therefore the multi-indices I' satisfying $h(I') = I$ and $g(I') \neq I$ occur in pairs with $\bar{\mu}_{\widehat{J}'}(I')$ of opposite sign by replacing for some i the choice of the parallel copy of $\widehat{J}_i$ that is adjoined to L_j by band sum for $i \neq j$ with a parallel copy of $\widehat{J}_i$ of opposite orientation that is also adjoined to L_i band sum. We are left with,

$$\sum_{\{I' | h(I') = I,\, g(I') = I\}} \bar{\mu}_{\widehat{J}'}(I') = \sum_{\{I' | h(I') = I,\, g(I') = I\}} \bar{\mu}_{\widehat{J}}(I) \cdot \prod_{j \in I'} r_j^{\lambda_j}$$

where $r_j \in \{\pm 1\}$ is -1 if the parallel copy of a component of $\widehat{J}$ with index j chosen uses the reverse orientation, and is $+1$ otherwise, and λ_j is defined to be the number of times that j appears in I'. Therefore,

$$\sum_{\{I' | h(I') = I,\, g(I') = I\}} \bar{\mu}_{\widehat{J}'}(I') = \bar{\mu}_{\widehat{J}}(I) \cdot \sum_{\{I' | h(I') = I,\, g(I') = I\}} \prod_{j \in I'} (a_i - b_i) = \bar{\mu}_{\widehat{J}}(I)$$

and we have by [**Coc90**, Theorem 8.13]

$$\bar{\mu}_{I(L,J,\mathbb{E}_\varphi)}(I) = \bar{\mu}_L(I) + \bar{\mu}_{\widehat{J}}(I).$$

□

1.3. Invariance of First Non-vanishing Milnor Invariants.

Theorem 4.2. *If two links L and L' are shake concordant, then they have equal first non-vanishing Milnor invariants. That is, if for some multi-index I, $\bar{\mu}_L(I) \neq 0$ and $\bar{\mu}_L(J) = 0$ for all $|J| < |I|$, then $\bar{\mu}_L(I) = \bar{\mu}_{L'}(I)$ and $\bar{\mu}_{L'}(J) = 0$ for all $|J| < |I|$.*

Proof. Suppose J is shake concordant to J'. Then we can find slice links L and L', string links s and s' such that $\widehat{s} = J$ and $\widehat{s'} = J'$, and multidisks $\mathbb{E}_\varphi$ and $\mathbb{E}_{\varphi'}$ respecting L and L', respectively, such that $I(L, s, \mathbb{E}_\varphi)$ is concordant to $I(L', s', \mathbb{E}_{\varphi'})$. Hence for any multi-index I,

$$\bar{\mu}_{I(L,s,\mathbb{E}_\varphi)}(I) = \bar{\mu}_{I(L',s',\mathbb{E}_{\varphi'})}(I).$$

Moreover, by Lemma 4.1 we have

$$\bar{\mu}_{I(L,s,\mathbb{E}_\varphi)}(I) = \bar{\mu}_J(I) + \bar{\mu}_J(I), \quad \bar{\mu}_{I(L',s',\mathbb{E}_{\varphi'})}(I) = \bar{\mu}_{L'}(I) + \bar{\mu}_{J'}(I).$$

Since L and L' are slice, all of their Milnor invariants vanish, hence we have,

$$\bar{\mu}_J(I) = \bar{\mu}_{I(L,s,\mathbb{E}_\varphi)}(I) = \bar{\mu}_{I(L',s',\mathbb{E}_{\varphi'})}(I) = \bar{\mu}_{J'}(I).$$

□

Corollary 4.3. *Linking number is an invariant of shake concordance. That is, if L and L' are shake concordant, then $lk(L_i, L_j) = lk(L'_i, L'_j)$ where $i \neq j$.*

In general, though, not all Milnor invariants are preserved by shake concordance. For instance, consider the links L in Figure 4.2 (A) and L' in Figure 4.2 (D). L is shake concordant to L'. We can see this by taking a 3-component shaking of the component L_4 and adjoining two of them via band sum to L_2 as shown in Figure 4.2 (B) and (C).

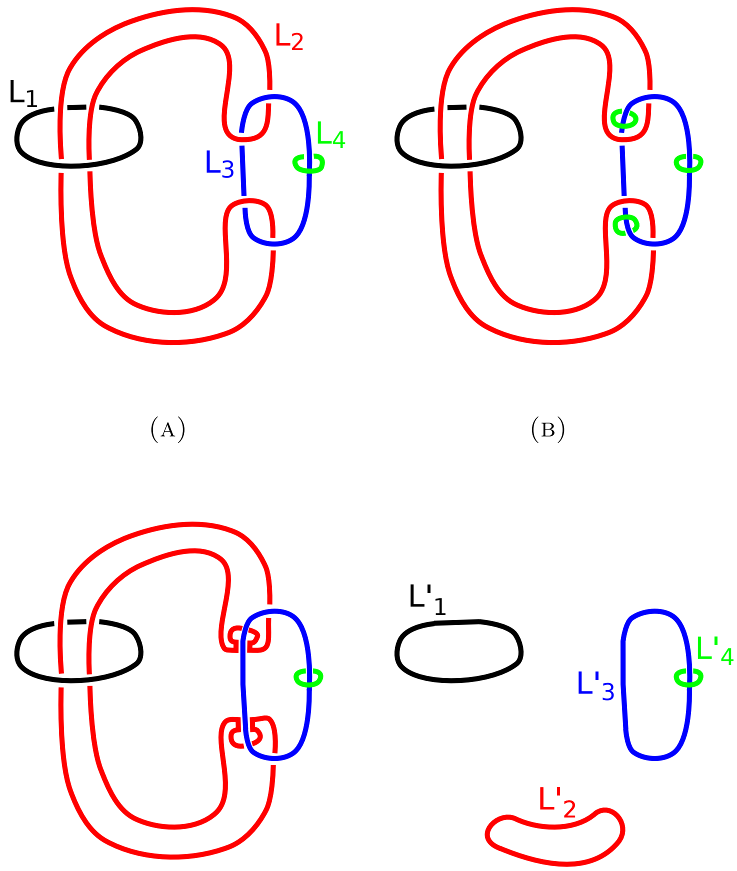
Notice, however, that the sublink $S = L_1 \sqcup L_2 \sqcup L_3$ of L is the Borromean rings and hence we have $\bar{\mu}_L(123) = \bar{\mu}_S(123) = 1$. Whereas the sublink $S' = L'_1 \sqcup L'_2 \sqcup L'_3$ of L' is a trivial link and hence we have $\bar{\mu}_{L'}(123) = \bar{\mu}_{S'}(123) = 0$. This does not violate the above theorem since $\bar{\mu}_L(34) = 1 = \bar{\mu}_{L'}(34)$.

2. Homology Cobordism

We call two closed, oriented 3-manifold M_1 and M_2 *homology cobordant* if there exists a compact, oriented 4-manifold W such that $\partial W = M_1 \sqcup -M_2$ and the maps induced by inclusion $H_n(M_i;\mathbb{Z}) \to H_n(W;\mathbb{Z})$, $i = 1, 2$, are isomorphisms for all n.

It is well known that if two links L and L' are concordant, then their zero surgery manifolds M_L and $M_{L'}$ are homology cobordant.

However, it is not necessary that the links be concordant for their zero surgery manifolds to be homology cobordant. For instance, Cochran, Franklin, Hedden, and Horn [**CFHH13**] exhibit non-concordant, topologically slice knots with homology cobordant zero surgery manifolds. Moreover, Cha and Powell [**CP14**] provide an infinite family of links with unknotted components that all have identical Milnor invariants and homeomorphic zero surgery manifolds with homotopy class of meridians preserved, but none of which are pairwise concordant.



(A) (B) (C) (D)

FIGURE 4.2. Shake concordant links with differing $\bar{\mu}(123)$.

We show that shake concordance of links is a sufficient condition for homology cobordism of the associated zero surgery manifolds. Hence, families of links that are shake concordant but not concordant offer further examples of non-concordant links with homology cobordant zero surgery manifolds.

PROPOSITION 4.4. *Suppose m-component links L and L' are shake concordant. Then the zero surgery manifolds M_L and M'_L are homology cobordant.*

Proof. This has already been shown when $m = 1$ in ([**CR16**], Proposition 5.1) which generalizes as follows. Recall $W_{L,L'}$, the 4-manifold obtained by attaching 2-handles along L and L', has boundary components M_L and $-M_{L'}$. We have

$$H_n(W_{L,L'}) \cong \begin{cases} \mathbb{Z} & n = 0, 3 \\ 0 & n = 1 \\ \mathbb{Z}^{2m} & n = 2 \\ 0 & n \geq 4 \end{cases}, \qquad H_n(M_L) \cong H_n(M_{L'}) \cong \begin{cases} \mathbb{Z} & n = 0, 3 \\ \mathbb{Z}^m & n = 1 \\ \mathbb{Z}^m & n = 2 \\ 0 & n \geq 4 \end{cases}.$$

We will modify $W_{L,L'}$ such that the inclusion maps from M_L and $M_{L'}$ into the modified 4-manifold induce isomorphisms on homology. Let $\Sigma_1, ..., \Sigma_m \hookrightarrow W_{L,L'}$ be the embedded spheres guaranteed by the definition of shake concordance of links. We can perform surgery on each Σ_i by removing a neighborhood of Σ_i, which is diffeomorphic to $S^2 \times D^2$, and gluing in a copy of $D^3 \times S^1$, which we can do since $\partial(S^2 \times D^2) = S^2 \times S^1 = \partial(D^3 \times S^1)$. Denote the resulting 4-manifold $\overline{W}$. Notice, this the effect of killing half the generators of the second homology group by killing $(\bar{e}_i, \bar{e}_i)$, for $i = 1, ..., m$. Also, this introduces m generators for the first homology group. A Mayer-Vietoris argument verifies

$$H_n(\overline{W}) \cong \begin{cases} \mathbb{Z} & n = 0, 3 \\ \mathbb{Z}^m & n = 1 \\ \mathbb{Z}^m & n = 2 \\ 0 & n \geq 4 \end{cases}$$

and that the induced maps from inclusion give the desired isomorphisms. □

In [**Har08**] Harvey introduced the real-valued homology cobordism invariants ρ_n for closed 3-manifolds. It follows from the above proposition that ρ_n, and other homology cobordism invariants, can be treated be invariants of shake concordance of links.

3. Classical Invariants

We begin be recalling the definitions of some classical invariants of concordance. Then we discuss if these remain invariant under shake concordance or strong shake concordance of links.

3.1. Algebraic Concordance Class. Every knot $K \subset S^3$ bounds an oriented Seifert surface F in S^3. There is a Seifert form V_K defined on $H_1(F;\mathbb{Z})$, which can be represented be a $2g \times 2g$ Seifert matrix where $2g$ is the rank of $H_1(F;\mathbb{Z})$. We call K *algebraically slice* if V_K is metabolic, that is, if V_K vanishes on a half-dimensional summand of $H_1(F;\mathbb{Z})$. We call knots K_1 and K_2 *algebraically concordant* if $V_{K_1} \oplus -V_{K_2}$ is metabolic. Concordance of knots implies algebraic concordance. And if a knot is slice, then it is algebraically slice; although the converse does not hold [**CG86**], [**CG78**].

3.2. Signature of a knot. Let M be the Seifert matrix for a knot K. Then the *Tristram-Levine* signature σ_ω is defined [**Tri69**] to be the signature of the hermitian form

$$(1-\omega)M + (1-\bar{\omega})M^T$$

where $\omega \in \mathbb{C}$ such that $|\omega| = 1$ and $\omega \neq 1$. Note this form is nonsingular when ω is not a root of the Alexander polynomial of M, $\Delta_M(t) = \det(M - tM^T)$. In this setting, signature is an invariant of algebraic concordance and the signatures of an algebraically slice knot vanish [**Mur65**], [**Liv05**].

3.3. Arf Invariant. For F the Seifert surface of a knot K, $H_1(F;\mathbb{Z}/2\mathbb{Z})$ has a quadratic form which counts the number of full twists in the neighborhood of an element of the homology group modulo 2. The Arf invariant of K is the Arf invariant of this quadratic form, taking values 0 or 1. It is determined by the algebraic concordance class of a knot and vanishes for algebraically slice knots.

3.4. Invariance Under Strong Shake Concordance. It follows from Proposition 4.4 that invariants of homology cobordism are invariants of shake concordance. In particular, Levine's algebraic knot concordance class [**Lev69b**], [**Lev69a**] is determined by the zero surgery manifold of a knot via the Blanchfield form and preserved under homology cobordism [**Tro73**]. This give rise to the following corollary of Proposition 4.4:

Corollary 4.5 (Corollary 5.2 in [**CR16**]). *If knots K and K' are shake concordant, then the algebraic concordance class of K and K' agree and hence K and K' have equal signatures and Arf invariants.*

What can be said in the case of links? If L and L' are strongly shake concordant, then each corresponding pair of components L_i and L_i' are shake concordant. Therefore we conclude:

Corollary 4.6. *If $L = L_1 \sqcup ... \sqcup L_m$ and $L_1' \sqcup ... \sqcup L_m'$ are strongly shake concordant, then L_i and L_i' have the same algebraic concordance class and hence equal signatures and Arf invariant for all $i = 1, ..., m$.*

3.5. Noninvariance Under Shake Concordance. If $L = L_1 \sqcup ... \sqcup L_m$ and $L' = L'_1 \sqcup ... \sqcup L'_m$ are shake concordant, but not strongly shake concordant, it doesn't necessarily follow that the components L_i and L'_i are shake concordant as knots. In fact, as the proof of Proposition 2.8 shows, we can have *any* two knots K and J that are corresponding components of shake concordant links. Therefore, no knot invariant of concordance is preserved in the components of a link under shake concordance. Nevertheless, we can still find numerous obstructions to a knot being shake slice, as we'll see in the next chapter.

CHAPTER 5

Obstructions to Shake Sliceness

In the previous chapter, we showed that the first non-vanishing Milnor invariant is an invariant of shake concordance; this gives us a significant obstruction to shake concordance. Moreover, we have seen that no concordance invariant is preserved for a component of a link under shake-concordance. However, we here show that they may still serve as obstructions to a link being shake slice: such as the Milnor invariants in section 1 and the Arf invariant in section 2. In fact, we are able to show that shake slice links are slice in a homology 4-ball in section 3. This let's us consider the tau invariant in section 5. We also show shake slice links are link homotopic to the trivial link in section 6. We close by discussing in section 7 the difficulty of finding a link that is shake slice but not slice.

1. Milnor Invariants

Proposition 5.1. *Suppose $L = L_1 \sqcup ... \sqcup L_m$ is shake slice. Then all of Milnor's $\overline{\mu}$ invariants for L vanish. In particular, $lk(L_i, L_j) = 0$ for all $i \neq j$.*

Proof. Since L is slice, it is shake concordance to a trivial link T. All Milnor invariants of T vanish. The result then follows immediately from Theorem 4.2 □

2. Arf Invariant

We argue that the the Arf invariant vanishes for all components of a shake slice link. Moreover, recall that the definition of the Arf invariant can be extended to proper links, that is, links L such that

$$\sum_{i \neq j} lk(L_i.L_j) \equiv 0 \mod 2.$$

Note that shake slice links and sublinks of shake slice links are proper as $lk(L_i, L_j) = 0$ for $i \neq j$. Suppose a planar surface $S^3 \times [0,1]$ bounds $L \times \{0\} \cup K \times \{1\}$ for a proper link L and some knot K. Then $Arf(K)$ depends only on L so we may define $Arf(L) := Arf(K)$ for some such K [**Hil12**, p. 45].

THEOREM 5.2. *If a link $L = L_1 \sqcup L_2 \sqcup ... \sqcup L_m$ is shake slice, then:*

- $Arf(L) = 0$*,*
- $Arf(L_i) = 0$ *for* $i = 1, ..., m$*, and*
- $Arf(L_i \sqcup L_j) = 0$ *for* $i \neq j$*.*

PROOF. Since L is shake slice, there exists L' a shaking of L and a planar surface with boundary $L' \times \{0\} \cup T \times \{1\}$ where T is a trivial link. Hence, by capping off all but one unknotted component of T, we have a planar surface cobounding L' and an unknot U, hence $Arf(L') = Arf(U)$. By fusing components of L we obtain a knot K and hence there exists a planar surface bounding $L \times \{0\} \cup K \times \{1\}$, therefore $Arf(L) = Arf(K)$. Note there also exists a planar surface that bounding $L \times \{0\} \cup K \times \{1\}$ obtained by fusing pairs of parallel copies of each component of L_i with opposite orientation to obtain L then fusing the components of L as before to obtain K. Hence,

$$Arf(L) = Arf(K) = Arf(L') = Arf(U) = 0.$$

Moreover, since L is shake slice, there exists a sublink L'_i of L' consisting of an odd number of parallel copies of L_i and even number of parallel copies of each L_j for $j \neq i$ such that a planar surface has boundary $L'_i \times \{0\} \sqcup S \times \{1\}$ for some trivial link S. Again, capping off all but one component of S, we have $Arf(L'_i) = Arf(U)$. But also notice that we can fuse pairs of parallel copies constituting L'_i to obtain a planar surface cobounding L'_i and L_i. Hence,

$$Arf(L_i) = Arf(L'_i) = Arf(U) = 0$$

for all $i = 1, ..., m$.

Finally, Beiss [**SB90**] has shown that for a two component link $L_{12} = L_1 \sqcup L_2$ we have

$$Arf(L_{12}) = Arf(L_1) + Arf(L_2) + \bar{\mu}_{L_{12}}(1122) \mod 2.$$

Hence, since the Milnor invariants of L all vanish, we have for any two-component sublink $L_i \sqcup L_j$ of L,

$$Arf(L_i \sqcup L_j) = 0,$$

where $1 \leq i < j \leq m$.

□

3. Homologically Slice

Given a shake slice link L, we are interested in if it is slice, that is, if L bounds m disjoint disks in a 4-ball. We can show something slightly more general is true.

PROPOSITION 5.3. *Suppose the m-component link L is shake slice. Then L bounds m disjoint disks in a homology 4-ball. That is, L is homologically slice.*

PROOF. Consider an m-component shake slice link L. Then L is shake concordant to the m-component trivial link T_m. Note the zero surgery manifold M_{T_m} is diffeomorphic to $\#_{i=1}^m S^1 \times S^2$. Hence by Proposition 4.4 the zero surgery manifold M_L is homology cobordant to $\#_{i=1}^m S^1 \times S^2$. Let W denote the 4-manifold of the homology cobordism. We modify W to obtain a homology 4-ball. First, cap off $\#_{i=1}^m S^1 \times S^2$ with $\natural m\, S^1 \times D^3$ to obtain a 4-manifold which we denote W'. Notice $\partial W' = M_L$ and

$$H_n(W') \cong \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}^m & n = 1 \\ 0 & n \geq 2 \end{cases} .$$

Attach a 0-framed 2-handle to W' along each of the m meridians of L, denote the resulting 4-manifold W''. Note $\partial W'' = 0$. This kills the first homology group of W', so that W'' homology ball. To see this consider the Mayer-Vietoris exact sequence:

$$... \to H_1(\#_{i=1}^m S^1 \times D^2) \xrightarrow{(i_*, j_*)} H_1(W') \oplus H_1(D^2 \times D^2) \xrightarrow{k_* - l_*} H_1(W'') \xrightarrow{\partial_*} H_0(\#_{i=1}^m S^1 \times D^2) \to ...$$

We observe $H_1(D^2 \times D^2) = 0$ and i_* is an isomorphism since $H_1(\#_{i=1}^m S^1 \times D^2)$ is generated by the meridians of L. Hence, $k_* - l_* = 0$. Moreover, ∂_* is the zero map since W'' is connected and hence $H_1(W'') = 0$. The co-core of each 2-handle is a disk bounded by a component of L_i in W'', $i = 1, .., m$. Note these disks are disjoint. □

In particular, each component L_i of L is slice in a homology 4-ball. Cha, Livingston, and Ruberman have shown in [**CLR08**, Theorem 3] that it then follows that L_i is algebraically slice. Hence we obtain:

Corollary 5.4. *If $L = L_1 \sqcup ... \sqcup L_m$ is shake slice, then each L_i is algebraically slice. In particular, the signatures and Arf invariant vanish for each L_i, $i = 1, ..., m$.*

4. Tau

Ozsváth and Szabó have defined an integer invariant τ for knots using the knot filtration on the Heegaard Floer complex $\widehat{CF}$. Like the knot signature, it is additive under connect sum and preserved under concordance, vanishing for slice knots. The tau invariant serves as a lower bound for the slice genus $g_*(K)$ of a knot, the minimal genus of a oriented surface in D^4 that has boundary $K \subset S^3 = \partial D^4$. In fact, something stronger is true:

Theorem 5.5 (Theorem 1.1 in [**OS03**]). *Let W be a smooth, oriented four-manifold with $b_2^+(W) = 0 = b_1(W)$ and $\partial W = S^3$. If Σ is any smoothly embedded surface-with-boundary in W whose boundary lies on S^3, where it is embedded as the knot K, then we have the following inequality:*

$$2\tau(K) + |[K]| + [\Sigma] \cdot [\Sigma] \leq 2g(\Sigma)$$

where $|[K]|$ denotes the L_1 norm of $[K] \in H_2(W)$ which evaluates $[K] = s_1 \cdot e_1 + ... + s_b \cdot e_b$, for some orthonormal basis e_i and $s_i \in \mathbb{Z}$, to be

$$|[\Sigma]| = |s_1| + ... + |s_b|.$$

Cochran and Ray have shown that τ is not invariant under shake concordance for knots, in fact, they offer an infinite family of knots that are pairwise shake concordant, but which take values for τ that increase without bound [**CR16**, Proposition 4.10]. It follows that τ cannot be an invariant of (strong) shake concordance of links. However, they also show that τ vanishes for shake slice knots [**CR16**, Corollary 5.3] from which it immediately follows that τ vanishes for each component of a strongly slice link. We note that this is also true more generally for each component of a shake slice link.

Corollary 5.6. *If L is shake slice then $\tau(L_i) = 0$ for each component L_i of L, $i = 1, ..., m$.*

Proof. Follows immediately from Proposition 5.3 and Theorem 5.5 ([**OS03**, Theorem 1.1]). □

5. Link Homotopy and Band Pass Equivalence

Any link can be transformed into the unlink via a series of crossing changes. If we restrict the crossing changes so that both strands belong to the same link component, then we recover the notion of link homotopy. It is well known that slice links are link homotopic to the trivial link [**Gol79**], [**Gif80**]. We will see the same is true of shake slice links. In fact, we can say something stronger.

A *band pass move* on a link is accomplished by keeping a diagram fixed outside a local change as in Figure 5.1 where both strands of each band belong to the same link component. We call links L and L' *band pass equivalent* if L can be deformed into L' via band-pass moves and isotopy.

Building off of work of Taniyama and Yasuhara in [**TY02**], Martin showed the following:

Theorem 5.7 (Corollary 5.2 in [**Mar13**]). *For links $L = L_1 \sqcup ... \sqcup L_m$ and $L' = L'_1 \sqcup ... \sqcup L'_m$ with vanishing pairwise linking numbers, L and L' are band-pass equivalent*

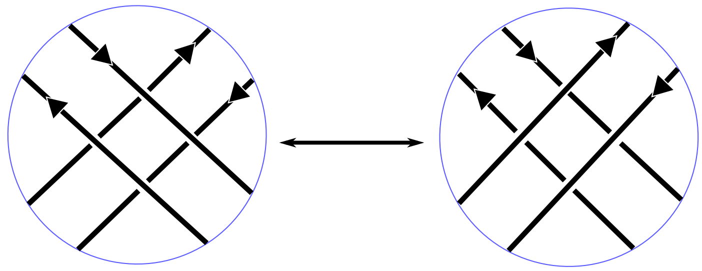


Figure 5.1. A band pass move.

if and only if:

$$Arf(L_i) = Arf(L'_i)$$

$$\bar{\mu}_L(ijk) = \bar{\mu}_{L'}(ijk)$$

$$\bar{\mu}_L(iijj) \equiv \bar{\mu}_{L'}(iijj) \mod 2$$

for all $i, j, k \in \{1, ..., m\}$.

This gives another obstruction to shake concordance.

Corollary 5.8. *If L is shake slice, then L is band-pass equivalent to the trivial link.*

Proof. Since $L = L_1 \sqcup ... \sqcup L_m$ is shake slice, it has vanishing pairwise linking numbers. Moreover, we showed $Arf(L_i) = 0$ for all $i = 1, ..., m$ and that the Milnor invariants of L all vanish. □

Since the strands of each band of a band pass move belong to the same link, band pass equivalence between links implies that the links are link homotopic. Hence we also recover the following obstruction:

Corollary 5.9. *If L is shake slice, then L is link homotopic to the trivial link.*

6. In Pursuit of a Non-Slice, Shake Slice Link

The problem of determining if every shake slice knot is slice has been open for 40 years; it is listed in the Kirby's problem list [**Kir95**]. The essential difficulty in answering it is found in the fact that shake slice knots are slice in a homology 4-ball and there is no invariant known to distinguish slice knots from homologically slice knots.

While one may hope for the situation to be better for links for an arbitrary number of components, one sees we run into a similar difficulty. We have from Proposition 5.3 that shake slice links are homologically slice and thus each component is homologically slice. Hence, we cannot expect to distinguish a shake slice link from a slice link by examining its components. It is then natural to consider the linking of the components, but Proposition 5.1 tells us that all of Milnor's higher order linking numbers vanish for shake slice links, as they do for slice links. Similarly, we saw that the generalized Arf invariant vanishes for shake slice links. Of course, there are numerous other invariants one could study. For instance, it is believed that Rasmussen's s-invariant may be able to distinguish between a knot being slice in D^4 and it being slice in a homology D^4.

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