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Global Regularity and Finite-time Blow-up in Model Fluid Equations

by

Tam Do

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APPROVED, THESIS COMMITTEE:

Alexander Kiselev, Chair
Professor of Mathematics

Robert Hardt
W.L. Moody Professor of Mathematics

Maarten V. de Hoop
Simons Chair in Computational and Applied
Mathematics and Earth Science

Houston, Texas

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ABSTRACT

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Determining the long time behavior of many partial differential equations modeling fluids has been a challenge for many years. In particular, for many of these equations, the question of whether solutions exist for all time or form singularities is still open. The structure of the nonlinearity and non-locality in these equations makes their analysis difficult using classical methods. In recent years, many models have been proposed to study fluid equations. In this thesis, we will review some new result in regards to these models as well as give insight into the relation between these models and the true equations.

First, we analyze a one-dimensional model for the two-dimensional surface quasi-geostrophic equation and vortex sheets. The model gained prominence due to the work of Cordoba, Cordoba, and Fontelos and is often referred to as the CCF model. We will show that solutions are globally regular in the presence of logarithmically supercritcal dissipation and that solutions eventually gain regularity in the presence of supercritical dissipation. Finally, by analyzing a dyadic model of the equation, we will gain insight into how certain possible singularities in the CCF model can be supressed by dissipation.

For the second part of this thesis, we study some one-dimensional model equations for the Euler equations. These models are influenced by the recent numerical simulations of Tom

Hou and Guo Luo. They observed possible singularity formation for the three-dimensional Euler equation at the boundary of a cylindrical domain under certain symmetry assumptions. Under these assumptions, a singularity was observed numerically and the solution was observe to have hyperbolic structure near the singularity. Hou and Luo proposed a one-dimensional model system to study singularity formation theoritically. We will study a family of one-dimensional models generalizing their model. The results in chapter 2 are the results of joint work with A. Kiselev, V. Hoang, M. Radosz, and X. Xu.

The contents of this thesis have been published and can be found in [13, 15, 16, 14]

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To my father

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The Cordoba-Cordoba-Fontelos model

1.1 Introduction

In this chapter, we will consider solutions to the following initial value problem

$$\theta_t = (H\theta)\theta_x - (-\Delta)^\alpha \theta \tag{1.1}$$

$$\theta(x,0) = \theta_0(x)$$

for $(x,t) \in \mathbb{R} \times [0,\infty)$ and $0 \leq \alpha < 1/2$. Here, H is the Hilbert transform

$$H\theta(x) = P.V. \frac{1}{\pi} \int \frac{\theta(y)}{x-y}\, dy$$

and $(-\Delta)^\alpha$ is the fractional Laplacian. We will take the convention that if $\alpha = 0$, the fractional Laplacian term is absent and we just have a transport equation. We will refer to equation (1.1) as the Cordoba-Cordoba-Fontelos (CCF) model [10]. From scaling considerations, equation (1.1) can be thought of as a 1D model for the 2D surface quasi-geostrophic (SQG) equation

$$\theta_t = u \cdot \nabla\theta - (-\Delta)^\alpha \theta \quad \text{on } \mathbb{R}^2 \times [0,\infty) \tag{1.2}$$

$$u = \nabla^\perp (-\Delta)^{-1/2} \theta. \tag{1.3}$$

Indeed, up to a constant, equation (1.1) is the only nonlocal active scalar equation in one dimension with a velocity that is a singular integral operator. In addition, (1.1) also has similarities with the Birkhoff-Rott equations for the evolution of vortex sheets (see [10] for more references).

The CCF model has an L^∞ maximum principle which makes the analysis of the equation different depending on the value of α. For $\alpha \geq 1/2$, the problem (1.1) is globally well-posed for initial data in $H^{3/2-2\alpha}$ and locally well-posed for $0 \leq \alpha < 1/2$ [17]. For $\alpha > 1/2$ the equation can be classified as *subcritical* since the dissipation has a higher order than the nonlinearity and global well-posedness can be shown using energy estimates. For $\alpha = 1/2$, the equation can be viewed as *critical*. In this regime, the nonlinearity and dissipation are of the same order and global regularity can be proven using a delicate argument as seen in Dong [17]. For $0 \leq \alpha < 1/2$, the primary range of α considered in this thesis, the equation can be classified as *supercritical* since the dissipation is now of lower order than the nonlinearity. For $0 \leq \alpha < 1/4$, finite time blow up has been shown to be possible [32, 10]. For $1/4 \leq \alpha < 1/2$, it is an open problem as to whether finite time blow-up can occur from smooth initial data.

For the SQG equation (1.2) (1.3), it is an outstanding open problem to determine if the equation has global regularity or finite time blow-up in the supercritical regime $0 \leq \alpha < 1/2$. See [7] and [25] for recent progress and references.

To serve as contrast to the CCF model, consider Burgers' equation with fractional dissipation:

$$\theta_t = \theta\theta_x - (-\Delta)^\alpha \theta.$$

For $\alpha \geq 1/2$, there is global regularity and for $0 \leq \alpha < 1/2$, there is finite time blow-up [27].

In sections 1.2 and 1.3, we will show that two results that are true for the SQG equation also hold for this 1D model. First, we will show eventual regularization for dissipation in the supercritical regime $0 < \alpha < 1/2$ with non-negative initial data. Second, we will prove global regularity for the slightly supercritical version of this equation. For the SQG equation, the arguments rely on dissipation in the "perpendicular" direction [26] or incompressibility of the fluid velocity [12, 34], which are both absent in our 1D setup. In our results, we will need to carefully use the structure of the nonlinearity as well as the exact formula for the dissipation term.

In sections 1.4-1.7, we will propose and study a discrete model for CCF. While blow-up has been proven in the range $0 \leq \alpha < 1/4$, the exact nature of the blow-up is unclear. Numerical simulations seem to indicate that positive, even, and decaying data forms a cusp at the origin in finite time. In the setting of our discrete model, we will prove global regularity in the supercritical range $1/4 \leq \alpha < 1/2$. If such a statement were true for the full CCF model, this would be the first example of supercritical regularity in a fluid mechanics PDE.

1.2 Eventual Regularization

In this section, we will closely follow the arguments of Kiselev [26]. We will work with solutions to the dissipative regularization of (1.1):

$$\theta_t = (H\theta)\theta_x - (-\Delta)^\alpha \theta + \epsilon \Delta \theta \tag{1.4}$$

$$\theta(x, 0) = \theta_0(x).$$

The solutions of (1.4) will be smooth and we will estimate the Holder norms of these solutions uniformly in $\epsilon > 0$. The limit obtained by letting $\epsilon \to 0$ will yield a candidate for a weak solution $\theta(x,t) \in C_w([0,\infty); L^2) \cap L^2([0,\infty); H^{1/2}) \cap L^\infty([0,\infty); L^2)$. However, this regularity appears to be insufficient to conclude that θ solves (1.1) in the standard weak sense. The equation is not conservative. On the other hand, the limit will inherit our estimates on the regularization. By having control of high enough Holder norms, the following theorem allows us to conclude smoothness :

Theorem 1.2.1 *Let θ be a solution of* (1.4) *with non-negative initial data. Let $\beta > 1 - 2\alpha$ and let $0 < t_0 < t < \infty$. If $\theta \in L^\infty([t_0, t]; C^\beta(\mathbb{R}))$ then $\theta \in C^\infty((t_0, t] \times \mathbb{R})$ with bounds independent of ϵ.*

The proof of Theorem 1.2.1 is analogous to the proof of Theorem 3.1 in Constantin and Wu [8] where they showed a similar result is true for the SQG equation. Their argument for SQG uses Besov space techniques and does not rely heavily on incompressibility, the key difference between (1.1) and SQG. Since we have non-negative initial data, solutions of (1.1) are bounded in L^2 without the need of incompressibility [10], a condition necessary to show the analogous result for SQG. Also, (1.1) posess the same scaling as SQG and the Hilbert transform is bounded on the Holder and Besov spaces, like the Riesz transforms, which is needed in the proof.

The main result of this section is the following theorem.

Theorem 1.2.2 *Let $\theta(x, t)$ be the limit obtained in* (1.4) *by letting $\epsilon \to 0$ with $\theta_0 \in L^\infty \cap H^{3/2-2\alpha}$ and non-negative. Then there exist times $0 < T_1(\alpha, \theta_0) \leq T_2(\alpha, \theta_0)$ such that θ is smooth for $0 < t < T_1$ and $t > T_2$ (θ will be a classical solution of* (1.1) *for these times).*

Remark. For $T_1 \leq t \leq T_2$, it is unclear in what sense the θ above is a solution of (1.1). The theorem follows from uniform in ϵ estimates for (2) and such estimates can be regarded as the main result of this section.

To control Holder norms, we will show that a certain family of moduli of continuity is eventually preserved under the evolution.

Definition 1.2.3 *A function $\omega(\xi) : (0,\infty) \mapsto (0,\infty)$ is a modulus of continuity if ω is increasing, continuous on $(0,\infty)$, concave, and piecewise C^2 with one-sided derivatives defined at every point in $[0,\infty)$. A function $f(x)$ obeys ω if $|f(x) - f(y)| < \omega(|x-y|)$ for all $x \neq y$.*

To prove that solutions preserve a modulus of continuity we state the following lemma, which describes the scenario in which the modulus is broken.

Lemma 1.2.4 *Let $\theta(x,t)$ be a solution of (1.1). Suppose that $\omega(\xi,t)$ is continuous on $(0,\infty)\times[0,T]$, piecewise C^1 in the time variable (with one-sided derivatives defined at all points) for each fixed $\xi > 0$, and that for each fixed $t \geq 0$, $\omega(\xi,t)$ is a modulus of continuity. Assume in addition that for each $t \geq 0$, either $\omega(0+,t) > 0$, or $\partial_\xi \omega(0+,t) = \infty$, or $\partial^2_{\xi\xi}\omega(0+,t) = -\infty$, and that $\omega(0+,t)$, $\partial_\xi\omega(0+,t)$ are continuous in t with values in $\mathbb{R}\cup\infty$. Let the initial data $\theta_0(x)$ obey $\omega(\xi,0)$. Suppose that for some $t > 0$ the solution $\theta(x,t)$ no longer obeys $\omega(\xi,t)$. Then there exist $t_1 > 0$ and $x, y \in \mathbb{R}$, $x \neq y$ such that for all $t < t_1$, $\theta(x,t)$ obeys $\omega(\xi,t)$ while*

$$\theta(x,t_1) - \theta(y,t_1) = \omega(|x-y|,t_1).$$

The proof of preceding lemma can be found in [26] for the periodic case. Decay results for solutions from [17] allow the lemma to be extended to the non-periodic setting [18]. We will use the same moduli of continuity as in [26]:

$$\omega(\xi,\xi_0) = \begin{cases} \beta H\delta^{-\beta}\xi_0^{\beta-1}\xi + (1-\beta)H\delta^{-\beta}\xi_0^{\beta}, & 0 < \xi < \xi_0 \\ H(\xi/\delta)^{\beta}, & \xi_0 \leq \xi \leq \delta \\ H, & \xi > \delta \end{cases}$$

where $\beta > 1 - 2\alpha$. Observe that if $2\|\theta_0\|_{L^\infty} \leq \omega(0, \delta) = (1-\beta)H$, then θ_0 obeys $\omega(\xi, \delta)$. Thus, for every bounded initial data, we can find a modulus that is obeyed.

It is known that for $0 < \alpha < 1/2$,

$$(-\Delta)^\alpha \theta(x) = P.V. \int_{-\infty}^{\infty} \frac{\theta(x) - \theta(x+y)}{|y|^{1+2\alpha}}\, dy,$$

see [9] for a proof. We will need the following estimate of the dissipation terms:

Lemma 1.2.5 *(Dissipation Estimate) Let $\xi = |x - y|$. Then*

$$-(-\Delta)^\alpha \theta(x, t) + (-\Delta)^\alpha \theta(y, t) \leq D_\alpha(\xi, t)$$

where

$$D_\alpha(\xi, t) = c_\alpha \left(\int_0^{\frac{\xi}{2}} \frac{\omega(\xi + 2\eta, t) + \omega(\xi - 2\eta, t) - 2\omega(\xi, ft)}{\eta^{1+2\alpha}}\, d\eta \right.$$
$$\left. + \int_{\frac{\xi}{2}}^{\infty} \frac{\omega(\xi + 2\eta, t) - \omega(2\eta - \xi, t) - 2\omega(\xi, t)}{\eta^{1+2\alpha}}\, d\eta \right).$$

See [26] for the proof. Theorem 1.2.2 is an easy consequence of the following lemma, which we will prove later.

Lemma 1.2.6 *Assume that $\theta_0(x)$ of (1.1) obeys $\omega(\xi, \delta)$. Then there exist positive constants $C_{1,2} = C_{1,2}(\alpha, \beta)$ such that if $\xi_0(t)$ is a solution of*

$$\frac{d\xi_0}{dt} = -C_2 \xi_0^{1-2\alpha}, \quad \xi_0(0) = \delta, \tag{1.5}$$

and $H \leq C_1 \delta^{1-2\alpha}$, then the solution $\theta(x, t)$ obeys $\omega(\xi, \xi_0(t))$ for all t such that $\xi_0(t) \geq 0$.

Proof of Theorem 1.2.2 The solution $\xi_0(t)$ of (1.5) becomes zero and stays zero in finite time. Then eventually, the solution $\theta(x,t)$ obeys $\omega(\xi,0)$ and we can uniformly bound its C^β norm, $\beta > 1-2\alpha$. □

To prove lemma 1.2.6, we will show that the breakthrough scenario described in lemma 1.2.4 cannot happen. Suppose there exists $t_1 > 0$ such that $\theta(x,t)$ obeys $\omega(\xi,t)$ for $t < \xi_0(t_1)$ and $\theta(x,t_1) - \theta(y,t_1) = \omega(\xi,\xi_0(t_1))$ where $\xi = |x-y|$. Then it is not hard to see that $\nabla\theta(x,t_1) = \partial_\xi\omega(\xi,\xi_0(t_1))\frac{x-y}{\xi} = \nabla\theta(y,t_1)$ and $\Delta\theta(x,t_1) - \Delta\theta(y,t_1) \le 2\partial^2_{\xi\xi}\omega(\xi,t_1)$ (details are in [26]). Also, by (1.4), lemma 1.2.5,

$$\partial_t\left[\frac{\theta(x,t)-\theta(y,t)}{\omega(\xi,\xi_0(t))}\right]\Bigg|_{t=t_1} \le \frac{\Omega(x,y,t_1)\partial_\xi\omega(\xi,\xi_0(t_1)) + d_\alpha(\xi,t_1) + 2\epsilon\partial^2_{\xi\xi}\omega(\xi,t_1) - \partial_t\omega(\xi,\xi_0(t_1))}{\omega(\xi,\xi_0(t_1))} \tag{1.6}$$

where $\Omega(x,y,t_1) = H\theta(x,t_1) - H\theta(y,t_1)$ and

$$d_\alpha(\xi,t_1) = \frac{1}{2}\left(-(-\Delta)^\alpha\theta(x,t_1) + (-\Delta)^\alpha\theta(y,t_1)\right) + \frac{1}{2}D_\alpha(\xi,t_1).$$

If we can show that the numerator of the right hand side of (1.6) is negative, then the modulus of continuity must have been broken at an earlier time, a contradiction. Because of the concavity of ω, $2\epsilon\partial^2_{\xi\xi}\omega(\xi,t_1) \le 0$ and since we want our estimates to be independent of ϵ, we will ignore this term.

Lemma 1.2.7

$$\Omega(x,y) = H\theta(x) - H\theta(y) \le C\left[\xi^{2\alpha}\left((-\Delta)^\alpha\theta(x) - (-\Delta)^\alpha\theta(y)\right) + \xi\int_{\xi/2}^\infty \frac{\omega(r)}{r^2}\,dr\right]$$

For simplicity of expression, we have omitted time in our expressions.

Proof.

The first term on the right side will control the singular behavior of the Hilbert transforms near the kernel singularity and the second term will control the behavior away from the singularity. Where appropriate, integrals will be understood in the principal value sense.

Let $\widetilde{x} = \frac{x+y}{2}$. Then

$$\begin{aligned}
\left|\int_{|x-z|\geq\xi} \frac{\theta(z)}{x-z}\,dz - \int_{|y-z|\geq\xi} \frac{\theta(z)}{y-z}\,dz\right| &= \left|\int_{|x-z|\geq\xi} \frac{\theta(z)-\theta(\widetilde{x})}{x-z}\,dz - \int_{|y-z|\geq\xi} \frac{\theta(z)-\theta(\widetilde{x})}{y-z}\,dz\right| \\
&\leq \int_{|\widetilde{x}-z|\geq\xi/2} \left|\frac{1}{x-z} - \frac{1}{y-z}\right| |\theta(z)-\theta(\widetilde{x})|\,dz \\
&\leq C\xi \int_{|\widetilde{x}-z|\geq\xi/2} \frac{1}{|\widetilde{x}-z|^2} |\theta(z)-\theta(\widetilde{x})|\,dz \leq C\xi \int_{\xi/2}^{\infty} \frac{\omega(r)}{r^2}\,dr
\end{aligned}$$

Now, we will estimate the other part. Observe that

$$H\theta(x) - H\theta(y) = \int_{-\infty}^{\infty} \frac{\theta(x+z)-\theta(x)}{z}\,dz - \int_{-\infty}^{\infty} \frac{\theta(y+z)-\theta(y)}{z}\,dz$$

Then

$$\begin{aligned}
&\int_{|z|<\xi} \frac{\theta(x+z)-\theta(x)}{z}\,dz - \int_{|z|<\xi} \frac{\theta(y+z)-\theta(y)}{z}\,dz - \xi^{2\alpha}((-\Delta)^{\alpha}\theta(x) - (-\Delta)^{\alpha}\theta(y)) \\
= &\int_{|z|<\xi} \frac{\theta(x+z)-\theta(x)}{z}\,dz - \int_{|z|<\xi} \frac{\theta(y+z)-\theta(y)}{z}\,dz - \xi^{2\alpha} \int_{-\infty}^{\infty} \frac{\theta(x)-\theta(x+z)}{|z|^{1+2\alpha}}\,dz \\
&+\xi^{2\alpha} \int_{-\infty}^{\infty} \frac{\theta(y)-\theta(y+z)}{|z|^{1+2\alpha}}\,dz \\
= &\int_{|z|<\xi} (\theta(x+z)-\theta(y+z)+\theta(y)-\theta(x)) \left(\frac{1}{z} + \frac{\xi^{2\alpha}}{|z|^{1+2\alpha}}\right) \\
&+ \int_{|z|>\xi} (\theta(x+z)-\theta(y+z)+\theta(y)-\theta(x)) \frac{\xi^{2\alpha}}{|z|^{1+2\alpha}}\,dz \\
= &\int_{|z|<\xi} (\theta(x+z)-\theta(y+z)-\omega(\xi)) \left(\frac{1}{z} + \frac{\xi^{2\alpha}}{|z|^{1+2\alpha}}\right) \\
&+ \int_{|z|>\xi} (\theta(x+z)-\theta(y+z)-\omega(\xi)) \frac{\xi^{2\alpha}}{|z|^{1+2\alpha}}\,dz \leq 0
\end{aligned}$$

The last inequality follows from the facts that

$$\theta(x+z) - \theta(y+z) - \omega(\xi) \le 0$$

and that in our region of integration

$$\frac{1}{z} + \frac{\xi^{2\alpha}}{|z|^{1+2\alpha}} \ge \frac{1}{z} + \frac{1}{|z|} \ge 0$$

Thus, we have control over the Hilbert transforms near the kernel singularity. Combining our estimates, we get the result. □

Proof of Lemma 1.2.6

We want to show

$$\partial_t \omega(\xi, t_1) > (H\theta(x,t_1) - H\theta(y,t_1))\partial_\xi \omega(\xi, \xi_0(t_1)) + d_\alpha(\xi, t_1) \tag{1.7}$$

From Lemma 3.3 of [26], we can choose the constant C_2 in (1.5) small enough so that we have $\partial_t \omega(\xi, \xi_0(t)) > \frac{1}{4} D_\alpha(\xi, t)$ at $t = t_1$ ($\xi_0'(t)$ is small). By Lemma 5.3 of [26], we can replace $\xi \int_{\xi/2}^{\infty} \frac{\omega(r)}{r^2}\, dr$ in Lemma 2.7 by $\omega(\xi, \xi_0)$. Using an argument very similar to Lemma 3.3 of [26], it can be shown that for all $0 < \xi < \delta$,

$$C\omega(\xi, \xi_0(t_1))\partial_\xi \omega(\xi, \xi_0(t_1)) \le -\frac{1}{4} D_\alpha(\xi, t_1).$$

where C is the constant from Lemma 2.7. Now, for $0 < \xi < \delta$, we have

$$C\xi^{2\alpha}\partial_\xi \omega(\xi, \xi_0(t_1)) \;\; \le \;\; C\beta H \xi^{2\alpha}\delta^{-\beta}\xi^{\beta-1} = C\beta \frac{H}{\delta^{1-2\alpha}} \left(\frac{\xi}{\delta}\right)^{2\alpha+\beta-1}$$

By choosing C_1 in $H \le C_1 \delta^{1-2\alpha}$ small enough we can bound the expression above by $\frac{1}{2}$. Then

$$C\xi^{2\alpha}\partial_\xi \omega(\xi, \xi_0(t_1))\left((-\Delta)^\alpha \theta(x,t_1) - (-\Delta)^\alpha \theta(y,t_1)\right) \le \frac{1}{2}\left((-\Delta)^\alpha \theta(x,t_1) - (-\Delta)^\alpha \theta(y,t_1)\right)$$

Combining these estimates with Lemma 1.2.7, we have (1.7). □

1.3 Well-posedness for Slightly Supercritical Hilbert Model

In this section, we prove global regularity for our model for which the dissipation can be supercritical by a logarithm. Specifically, we will look at solutions of the following equation

$$\theta_t = (H\theta)\theta_x - \mathcal{L}\theta, \quad \theta(x,0) = \theta_0(x) \tag{1.8}$$

for $\theta_0 \in H^{3/2}(\mathbb{R})$, where $\mathcal{L}\theta = \frac{(-\Delta)^{1/2}}{\log(1-\Delta)}\theta$ is a Fourier multiplier operator with multiplier

$$P(\xi) = \frac{|\xi|}{\log(1+|\xi|^2)}.$$

For simplicity, we will only concern ourselves with a dissipative operator of this form. The results of this section can easily be generalized to other similar dissipative operators. The main result of this section is the following

Theorem 1.3.1 *Assume that $\theta_0 \in H^{3/2}(\mathbb{R})$. Then there exists a unique smooth solution θ of* (1.8).

First, we have local existence of smooth solutions that we will eventually show can be extended.

Proposition 1 *Let $0 < \alpha < 1/2$ and $\theta_0 \in H^{3/2}$. Then there exists $T > 0$ such that* (1.8) *has a unique solution θ up to time T that satisfies*

$$\sup_{0<t<T} t^{\beta/(2\alpha)} \|\theta(t,\cdot)\|_{\dot{H}^{3/2-2\alpha+\beta}} < \infty$$

for any $\beta \geq 0$ and

$$\lim_{t\to 0} t^{\beta/(2\alpha)} \|\theta(t,\cdot)\|_{\dot{H}^{3/2-2\alpha+\beta}} = 0$$

for any $\beta > 0$. Furthermore, we can extend the solution beyond T if $\|\nabla\theta\|_{L^1(0,T;L^\infty)} < \infty$.

Proof. This result is analogous to Theorem 4.1 and Proposition 6.2 of Dong [17] where it is done for the usual fractional laplacian dissipation. The argument for the dissipation we are using is very similar. We will present the modification necessary to make their proof work. The general idea is that $\mathcal{L}$ is more dissipative then $(-\Delta)^\alpha$ for $0 < \alpha < 1/2$. Let θ be a solution of (1.8) and let $\theta_j = \Delta_j\theta$ be the jth Littlewood-Paley projection. Applying Δ_j to both sides of (1.8) we get

$$\partial_t\theta_j + (H\theta)(\theta_j)_x + \mathcal{L}\theta_j = [H\theta, \Delta_j]\theta_x \tag{1.9}$$

where $[H, \Delta_j]$ is a commutator with $[H\theta, \Delta_j]\theta_x = (H\theta)(\theta_j)_x - \Delta_j((H\theta)(\theta_x))$. By applying Plancherel and using that Δ_j localizes θ in the frequency space,

$$\int_{\mathbb{R}} \theta_j \mathcal{L}\theta_j \, dx \geq 2^{2\alpha j} C \|\theta_j\|_{L^2}^2$$

for some constant C. Then by multiplying both sides of (1.9) by θ_j and integrating we get

$$\frac{1}{2}\frac{d}{dt}\|\theta_j\|_{L^2}^2 + 2^{2\alpha j} C\|\theta_j\|_{L^2}^2 \leq \int_{\mathbb{R}} \left([H\theta, \Delta_j]\theta_x + H\theta_x\theta_j/2\right)\theta_j \, dx.$$

This is the same type of inequality used in Dong [17] and one can apply the methods there to arrive at the a priori bounds needed to conclude local existence as well as higher regularity despite the absence of a divergence free property

Dong also proves a Beale-Kato-Majda type blow up criterion for (1.1) and the result still holds true for (1.8) with our logarithmic dissipation. The contribution from the dissipation term is still non-negative, which is the only fact used about dissipation in his proof. Specifically, by Plancherel, for any regular enough function f,

$$\int_{\mathbb{R}} f\mathcal{L}f \, dx \geq 0$$

□

Thus, if we can show $\|\nabla\theta\|_\infty$ is bounded uniformly in time, then Theorem 3.1 is proved. Having such a bound will allow us to extend local solutions indefinitely. To have a bound on $\|\nabla\theta\|_\infty$, we will show the evolution preserves a family of moduli of continuity. If a function $f \in C^2(\mathbb{R})$ obeys a modulus ω satisfying $\omega'(0) < \infty$ and $\omega''(0) = -\infty$, then $\|\nabla f(x)\|_\infty < \omega'(0)$(see [28]). Therefore, if θ preserves a modulus of continuity, $\|\nabla\theta\|_\infty < \omega'(0)$.

1.3.1 Writing $\mathcal{L}$ as dissipative nonlocal operator

In the proofs, it will be easier to write $\mathcal{L}$ as a nonlocal dissipative nonlocal operator, which the following version of lemmas 5.1 and 5.2 from [11] allows us to do.

Lemma 1.3.2 *The operator $\mathcal{L}$ can be written as*

$$\mathcal{L}\theta(x) = \int_{\mathbb{R}} (\theta(x) - \theta(x+y))K(y)\,dy.$$

Also, there exists a positive constant C such that

$$\frac{1}{C}\frac{1}{|y|}P(|y|^{-1}) \le K(y) \le C\frac{1}{|y|}P(|y|^{-1})$$

where the lower bound holds for $|y| < 2\sigma$ for some small constant σ.

Since are not assured positivity of the kernel K, by the previous lemma, we will not have the L^∞ maximum principle. The following result (Lemma 5.4 from [11]) allows us to circumvent this.

Lemma 1.3.3 *Let θ solve* (1.8). *Then there exists a constant $M = M(P, \theta_0)$ such that $\|\theta(\cdot,t)\|_{L^\infty} \le M$ for all $t \ge 0$.*

Using the notation from Lemma 1.3.2, let φ be a smooth radially decreasing function that is identically 1 on $|y| \leq \sigma$ and vanishes identically on $|y| \geq 2\sigma$. Let

$$\begin{aligned} K_1(y) &= K(y)\varphi(y) \\ K_2(y) &= K(y)(1-\varphi(y)) \end{aligned}$$

Now, we decompose the dissipation term $\mathcal{L}$:

$$\mathcal{L}\theta(x) = \mathcal{L}_1\theta(x)+\mathcal{L}_2\theta(x) := \int_{\mathbb{R}} (\theta(x)-\theta(x+y))K_1(y)\,dy + \int_{\mathbb{R}} (\theta(x)-\theta(x+y))K_2(y)\,dy. \quad (1.10)$$

Let

$$m(r) = \frac{1}{C}P(r^{-1})\varphi(r)$$

where C is the constant from Lemma 1.3.2. Then we have the following lower bound on $\mathcal{L}_1$ that we will use extensively:

$$\mathcal{L}_1\theta(x) \geq \int_{\mathbb{R}} (\theta(x) - \theta(x+y))\frac{m(|y|)}{|y|}\,dy$$

The operator $\mathcal{L}_1$ satisfies the following conditions satisfied by more general nonlocal dissipative operators

1. there exists a positive constant $C_0 > 0$ such that

   $$rm(r) \leq C_0 \text{ for all } r \in (0, 2\sigma)$$

   for some $r_0 > 0$.

2. there exists some $a > 0$ such that $r^a m(r)$ is non-increasing.

We also have the following dissipation estimate whose proof is analogous to Lemma 1.2.5.

Lemma 1.3.4 *Suppose θ obeys a modulus of continuity ω. Suppose there exists x, y with $|x-y| = \xi > 0$ such that $\theta(x) - \theta(y) = \omega(\xi)$. Then*

$$\mathcal{L}_1\theta(x) - \mathcal{L}_1\theta(y) \geq \mathcal{D}(\xi)$$

where

$$\begin{aligned} \mathcal{D}(\xi) = \; & A \int_0^{\xi/2} (2\omega(\xi) - \omega(\xi+2\eta) - \omega(\xi-2\eta)) \frac{m(2\eta)}{\eta}\, d\eta \\ + \; & A \int_{\xi/2}^{\infty} (2\omega(\xi) - \omega(\xi+2\eta) + \omega(2\eta-\xi)) \frac{m(2\eta)}{\eta}\, d\eta \end{aligned}$$

and A is a constant.

1.3.2 The Moduli of Continuity

The modulus from [11] will work here. Fix a small constant $\kappa > 0$. For any $B \geq 1$, define $\delta(B)$ to be the solution of

$$m(\delta(B)) = \frac{B}{\kappa}.$$

We can also assume that $\delta(B) \leq \sigma/2$ by choosing κ small enough. Let $\omega_B(\xi)$ be a continuous function with $\omega_B(0) = 0$ and

$$\omega_B'(\xi) = B - \frac{B^2}{2C_a\kappa} \int_0^{\xi} \frac{3 + \log(\delta(B)/\eta)}{\eta m(\eta)}\, d\eta, \quad \text{for } 0 < \xi < \delta(B), \tag{1.11}$$

$$\omega_B'(\xi) = \gamma m(2\xi), \quad \text{for } \xi > \delta(B), \tag{1.12}$$

where $C_a = (1+3a)/a^2$ and $\gamma > 0$ is a constant dependent on κ, A, and m. It is shown in [11] that ω_B is a indeed a modulus of continuity.

Now, we will show that solutions will initially obey some $\omega_B(\xi)$ for some B large enough. Since evolution immediately smooths out the initial data, we can assume θ_0 is a smooth as needed. By Lemma 1.3.3, it suffices to find B such that $\omega_B(\xi) \geq \min\{\xi\|\nabla\theta_0\|_{L^\infty}, 2M\}$ for all $\xi > 0$ where M is from Lemma 1.3.3. By concavity of ω, we are left to show that $\omega_B(b) \geq 2M$ where $b = 2M/\|\nabla\theta_0\|_{L^\infty}$. Choose B large so $b > \delta(B)$, so

$$\omega_B(b) = \omega_B(\delta(B)) + \int_{\delta(B)}^{b} \omega_B'(\eta)\,d\eta \geq \gamma \int_{\delta(B)}^{b} m(2\eta)\,d\eta \to \infty$$

as $\delta(B) \to 0$. By choosing B possibly even larger we can have $\omega_B(\sigma) \geq 2M \geq 2\|\theta(\cdot,t)\|_{L^\infty}$ where σ is from our decomposition of $\mathcal{L}$ earlier. Therefore, the modulus can only be broken for $0 < \xi < \sigma$ and solutions will initially obey a modulus from the family $\{\omega_B\}_{B\geq 1}$.

1.3.3 The moduli are preserved

To prove a modulus of continuity is preserved, we will rule out the breakthrough scenario described in Lemma 1.2.4. Let t_1 be the time of breakthrough. By using (1.8) and Lemma 3.5,

$$\partial_t\left(\theta(x,t) - \theta(y,t)\right)\big|_{t=t_1} \leq \left(H\theta(x,t_1) - H\theta(y,t_1)\right)\omega'(\xi) - \mathcal{D}_B(\xi) + \mathcal{L}_2\theta(y,t_1) - \mathcal{L}_2\theta(x,t_1)$$

where $\xi = |x-y|$. If the right side of the equation above is negative then the modulus was broken at an earlier time, a contradiction. In [11], they show

$$|\mathcal{L}_2\theta(x,t) - \mathcal{L}_2\theta(y,t)| \leq \frac{1}{2}\mathcal{D}_B(\xi)$$

for $0 < \xi < \sigma$ so to prove Theorem 1.3.1, it suffices to show

$$\left(H\theta(x,t_1) - H\theta(y,t_1)\right)\omega_B'(\xi) - \frac{1}{2}\mathcal{D}_B(\xi) < 0 \tag{1.13}$$

for $0 < \xi < \sigma$ where $\mathcal{D}_B$ is the expression from Lemma 1.3.4 with ω_B being the modulus. For simplicity, we will now omit t_1 from our expressions involving θ.

Case: $\xi \geq \delta(B)$

By a similar argument to the proof of Lemma 1.2.7,

$$\left| \int_{|x-z|\geq 2\xi} \frac{\theta(z)}{x-z}\, dz - \int_{|y-z|\geq 2\xi} \frac{\theta(z)}{y-z}\, dz \right| \leq C\xi \int_{\xi}^{\infty} \frac{\omega_B(\eta)}{\eta^2}\, d\eta.$$

Integrating by parts

$$\xi \int_{\xi}^{\infty} \frac{\omega_B(\eta)}{\eta^2}\, d\eta = \omega_B(\xi) + \gamma\xi \int_{\xi}^{\infty} \frac{m(2\eta)}{\eta}\, d\eta.$$

By property (2) of m,

$$\int_{\xi}^{\infty} \frac{m(2\eta)}{\eta}\, d\eta \leq \xi^a m(2\xi) \int_{\xi}^{\infty} \frac{1}{\eta^{1+a}}\, d\eta \leq \frac{m(2\xi)}{a}$$

Now, we have

$$\xi \int_{\xi}^{\infty} \frac{\omega_B(\eta)}{\eta^2}\, d\eta \leq \omega_B(\xi) + \frac{\gamma\xi m(2\xi)}{a}$$

For $\delta(B) \leq \xi \leq 2\delta(B)$, it is not hard to see that

$$\frac{\gamma\xi m(2\xi)}{a} \leq \omega_B(\xi),$$

the details are in [11]. For $\xi > 2\delta(B)$, we have $\xi - \delta(B) \geq \xi/2$ so

$$\omega_B(\xi) \geq \gamma \int_{\delta(B)}^{\xi} m(2\eta)\, d\eta \geq \gamma m(2\xi)(\xi - \delta(B)) \geq \frac{\gamma\xi m(2\xi)}{2}.$$

Thus, we have,

$$C\xi \int_{\xi}^{\infty} \frac{\omega_B(\eta)}{\eta^2}\, d\eta \leq C\left(1 + \frac{2}{a}\right)\omega_B(\xi).$$

In [11], they prove the following estimate on the dissipation term

$$-\mathcal{D}_B(\xi) \leq -\frac{2-c_a}{C}\omega_B(\xi) m(2\xi)$$

where $c_a = 1 + (3/2)^{-a}$. Then we obtain

$$\left(\int_{|x-z|\geq 2\xi} \frac{\theta(z)}{x-z}\,dz - \int_{|y-z|\geq 2\xi} \frac{\theta(z)}{y-z}\,dz\right)\omega_B'(\xi) - \frac{1}{4}\mathcal{D}_B(\xi) \leq \left(C\gamma\frac{a+2}{a} - \frac{2-c_a}{4C}\right)\omega_B(\xi)m(2\xi) < 0$$

if we set γ small enough. In other words, we have used some of the dissipation to control the modulus of the Hilbert transform away from the kernel singularity. Now, we will concern ourselves with the other part of the Hilbert transform. A novel step is that instead of using $\mathcal{D}_B$ we will use the expression for $\mathcal{L}_1\theta$ directly. We want to show

$$\left(\int_{|x-z|\leq 2\xi} \frac{\theta(z)}{x-z}\,dz - \int_{|y-z|\leq 2\xi} \frac{\theta(z)}{y-z}\,dz\right)\omega_B'(\xi) - \frac{1}{4}(\mathcal{L}_1\theta(x) - \mathcal{L}_1\theta(y)) < 0. \tag{1.14}$$

After a similar manipulation as in the proof of Lemma 1.2.7, the left side of (1.14) is precisely

$$\int_{|z|<2\xi} (\theta(y) - \theta(z+y) - \theta(x) + \theta(x+z)) \left[\frac{\omega_B'(\xi)}{z} + \frac{1}{4}\frac{m(z)}{|z|}\right] dz$$
$$+ \int_{|z|\geq 2\xi} (\theta(y) - \theta(z+y) - \theta(x) + \theta(x+z)) \frac{1}{4}\frac{m(z)}{|z|}\,dz$$

By hypothesis, we have $\theta(y) - \theta(x) = \omega_B(\xi)$, so

$$\theta(y) - \theta(z+y) - \theta(x) + \theta(x+z) = \omega_B(\xi) - \theta(z+y) + \theta(x+z) < 0.$$

If we can show

$$\frac{\omega_B'(\xi) - \frac{1}{4}m(z)}{z} > 0$$

for $-2\xi < z < 0$ then we are done. However,

$$\omega_B'(\xi) - \frac{1}{4}m(z) = \gamma m(2\xi) - \frac{1}{4}m(z) < 0$$

from the fact that m is non-increasing and choosing γ small enough. Therefore, we have (1.13) and the case when $\xi \geq \delta(B)$ is complete.

Case: $0 < \xi \leq \delta(B)$ The argument for this case is exactly the same as in [11] with no modifications. Therefore, the proof of theorem 1.3.1 is complete. □

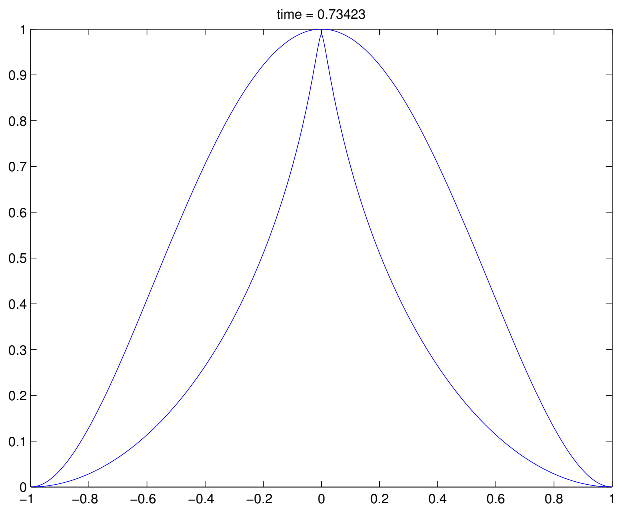


Figure 1.1 : Plot of the solution for the inviscid equation with initial data $\theta_0(x) = (1 - x^2)^2\chi_{[-1,1]}(x)$

1.4 A Dyadic Model

In this section, we will derive a dyadic model for CCF

$$\theta_t(x,t) + (H\theta)(x,t)\theta_x(x,t) + (-\Delta)^\alpha\theta(x,t) = 0 \quad (x,t) \in \mathbb{R} \times [0,\infty) \tag{1.15}$$

where H is the Hilbert transform. While existing proofs for blow-up for the range $0 < \alpha < 1/4$ do not provide a fine characterization as to how blow-up occurs, it is conjectured that solutions blow-up by forming a cusp of order $1/2$. This has important consequences as to what occurs in the regime $1/4 \leq \alpha < 1/2$.

The proof of blow-up for $0 \leq \alpha < 1/4$ provided in [32, 10] relies on the following novel inequality that for $f \in C_c^\infty(\mathbb{R}^+)$ and $0 < \delta < 1$, there exists a constant C_δ such that

$$-\int_0^\infty \frac{f_x(x)(Hf)(x)}{x^{1+\delta}}\,dx \geq C_\delta \int_0^\infty \frac{f^2(x)}{x^{2+\delta}}\,dx. \tag{1.16}$$

The proof given in [10] uses tools from complex analysis and is used to prove blow-up for even positive initial data with a maximum at 0. In [25], another more elementary proof of blow-up for $0 \leq \alpha < 1/4$ was given. The proof also goes by way of (1.16) but without appealing to complex analysis. In particular, the complicated non-locality of the Hilbert transform is handled by the following key inequality from [25]:

Proposition 2 *Suppose that the function $f(x)$ is C^1, even, $f'(x) \geq 0$ for $x > 0$ and f is bounded on $\mathbb{R}$ with $f(0) = 0$. Then for $1 < q < 2$,*

$$Hf(x) \leq \log(q-1)(f(qx) - f(q^{-1}x)).$$

Consider initial data for (1.15) that is C^1, even, and monotone decaying away from the origin. These properties are preserved by (1.15). The proposition is then applied to $f(x) = \theta(0,t) - \theta(x,t)$. To derive our model, we will look at dyadic points. Consider the following system of ODEs

$$a_k'(t) = -(a_k - a_{k-1})^2(t)2^k, \quad a_k(0) = a_k^0, \quad k \geq 1. \tag{1.17}$$

and we set $a_1'(t) = 0$, $a_0 = 0$. This system serves as a discrete model for

$$\theta_t(x,t) = -(H\theta)\theta_x(x,t). \tag{1.18}$$

One can think of $a_k(t)$ as approximating $\theta(2^{-k}, t)$. In particular, one could think that $-(a_k - a_{k-1})^2(t)2^k \approx \theta_x(2^{-k}, t)$ and that, using Proposition 2 as inspiration, we shall take $(a_k - a_{k-1})$ as our approximation of $H\theta(2^{-k}, t)$. The reason for considering a dyadic model is the nature of conjectured blow up for (1.18). A cusp appears to form in finite time at the origin (see Figure 1) and it is our hope that studying (1.17) will help us understand such phenomenon. The fact that we are restricting ourselves to even, monotone decaying data

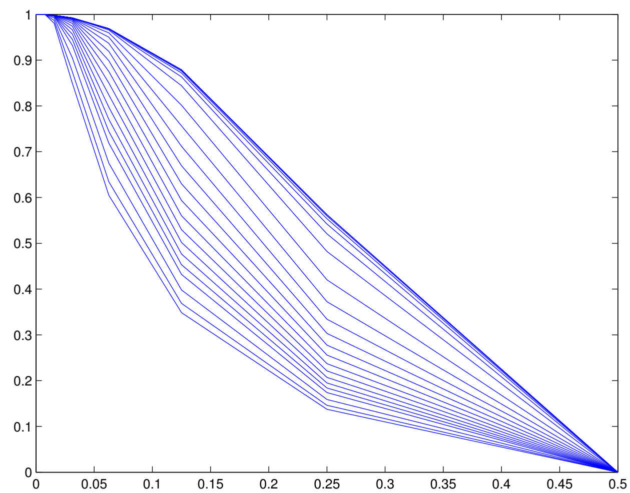


Figure 1.2 : Plot of the solution on $[0, 1/2]$ to (1.17) at various times with initial data corresponding to $\theta_0(x) = (1 - 4x^2)^2\chi_{[-1/2,1/2]}(x)$

isn't too restricting. It has been shown that blow-up occurs at any local maximum [35] and it is believed that cusps are formed at local maximums.

Dyadic models have been used to study regularity properties of Navier-Stokes and Euler equations (see [24],[31], and [2] as well as references therein). A key difference between with our model is that we use reductions in physical space to achieve our model rather than frequency space. One reason for this is that blow-up in (1.15) is believed to occur at a single point and as a cusp, which is not captured well by Fourier methods. In analyzing our model, we hope to gain insight into regularity of the continuous equation that is not apparent through standard norm estimates.

1.4.1 Dyadic Fractional Laplacian

Now, we would like to formulate a discrete version of the fractional Laplacian $(-\Delta)^\alpha$. Recall for θ smooth enough,

$$(-\Delta)^\alpha\theta(x) = P.V. \int_{-\infty}^{\infty} \frac{\theta(x)-\theta(y)}{|x-y|^{1+2\alpha}}\, dy,$$

see [9] for a derivation. If we take θ to be even then

$$(-\Delta)^\alpha\theta(x) = \int_0^\infty \left[\frac{1}{|x-y|^{1+2\alpha}} + \frac{1}{|x+y|^{1+2\alpha}}\right](\theta(x)-\theta(y))\, dy$$

Let $x = 2^{-k}$ and $y = 2^{-n}$. If $k < n$ then $\frac{1}{|x-y|^{1+2\alpha}} + \frac{1}{|x+y|^{1+2\alpha}} \approx 2^{(1+2\alpha)k}$. Similarly, if $k > n$, then $\frac{1}{|x-y|^{1+2\alpha}} + \frac{1}{|x+y|^{1+2\alpha}} \approx 2^{(1+2\alpha)n}$. Also, $\theta(x)-\theta(y) \approx a_k - a_n$. Combining these observations, we can see that a reasonable model for fractional dissipation would be

$$(\mathcal{L}^\alpha a)_k = \sum_{n=0}^{k-1}(a_k-a_n)2^{2\alpha n} + \sum_{n=k+1}^{\infty}(a_k-a_n)2^{2\alpha k}2^{k-n}. \tag{1.19}$$

In our discrete formulation, we have ignored the tails of the integral defining $(-\Delta)^\alpha$. Adding the tails into (6) (i.e. letting the first sum go to $-\infty$) doesn't change the main results of this note and just adds more complications to the calculations.

Thus, our model for (1.15) is

$$a_k' = -(a_k-a_{k-1})^2 2^k - (\mathcal{L}^\alpha a)_k. \tag{1.20}$$

$$a_0' = -(\mathcal{L}^\alpha a)_0$$

For simplicity of notation, we will sometimes omit the α and simply write $\mathcal{L}$ instead. It should be noted that our model can also serve as discrete model for other non-local transport equations with fractional diffusion such as

$$\theta_t(x,t) = u(x,t)\theta_x(x,t) - (-\Delta)^\alpha(x,t)$$

where

$$u(x,t) = \begin{cases} \theta(x,t) - \theta(2x,t) & x \geq 0 \\ \theta(2x,t) - \theta(x,t) & x < 0 \end{cases}$$

It is unknown whether solutions to this type of equations with general initial data can blow-up or exist globally in time in the supercritical range $0 \leq \alpha < 1/2$. In [36], a related kind of "non-local" Burgers type equation was studied and blow-up was observed in the non-viscous case in certain situations.

In Section 1.5, we present several properties of solutions to (1.20) and show that solutions to (1.17) blow-up, strengthening the analogy between the model and (1.15). In Section 1.6, we prove an a-priori bound on solutions akin to a global in-time Hölder 1/2 bound in the continuous setting. In Section 1.7, we use this bound to prove blow-up for $0 < \alpha < 1/4$ and global regularity for $1/4 \leq \alpha < 1/2$, which can be considered the main result of this note. If this result were carried into the continuous setting, it would suggest that solutions to (1.15) in the supercritical range $1/4 \leq \alpha < 1/2$ are globally regular, contrary to natural scaling considerations.

1.5 Local existence and properties of solutions

Define the space

$$X^s = \{\{a_k\}_{k=0}^{\infty} : \|a\|_X^s := \sup_k |a_k| + \sup_{k \geq 1} |a_k - a_{k-1}| 2^{sk} < \infty\}.$$

One could think of X^s as being analogous to the Hölder spaces. However, the X^s are made to deal with behavior near the origin. Let $b_{k,s} = (a_k - a_{k-1})2^{sk}$.

Before we show local existence for the full system, we will state some facts about $\mathcal{L}$ and its associated semigroup.

Lemma 1.5.1 *Let* $0 < \alpha < 1/2$.

(a) *Suppose that for the index* k, $b_{k,s} > c_s \|a\|_{X^s}$ *where*

$$c_s = \frac{3}{4}(2^s - 1)^{-1}\left(1 - \frac{1}{2^{s+1} - 1}\right) < 1.$$

Then

$$((\mathcal{L}^\alpha a)_k - (\mathcal{L}^\alpha a)_{k-1})\, 2^{sk} \geq C(\alpha)(2^{2\alpha k} - 2^{2\alpha})b_{k,s}$$

where $C(\alpha)$ *is a positive constant only dependent on* α.

(b) *The operator* $\mathcal{L}^\alpha$ *generates a contracting semigroup* $e^{-t\mathcal{L}^\alpha}$ *on* X^s *for all* $s > 0$.

(c) *The following identity, which is analogous to* $\int(-\Delta)^\alpha\theta = 0$, *is true*

$$\sum_{k=0}^{\infty}(\mathcal{L}^\alpha a)_k 2^{-k} = 0.$$

Proof. **(a)** We will consider $k > 1$ as the case for $k = 1$ is similar. By direct computation,

$$\begin{aligned}
(\mathcal{L}a)_k - (\mathcal{L}a)_{k-1} &= \sum_{n=0}^{k-2}(a_k - a_{k-1})2^{2\alpha n} + 2(a_k - a_{k-1})2^{2\alpha(k-1)} \\
&\quad + 2^{(1+2\alpha)k}(1 - 2^{-(1+2\alpha)}) \sum_{n=k+1}^{\infty}(a_k - a_n)2^{-n} \\
&= (a_k - a_{k-1})\left(\frac{2^{2\alpha(k-1)} - 1}{2^{2\alpha} - 1}\right) + 2(a_k - a_{k-1})2^{2\alpha(k-1)} \\
&\quad + 2^{(1+2\alpha)k} \sum_{n=k+1}^{\infty}(a_k - a_n)(1 - 2^{-(1+2\alpha)})2^{-n} \\
&:= I + II + III
\end{aligned}$$

By hypothesis, we have $c_s b_{j,s} < b_{k,s}$ for $j > k$. In terms of a, this means, for $n \geq k + 1$,

$$(a_n - a_k) = \sum_{j=k+1}^{n}(a_j - a_{j-1}) \leq c_s^{-1}(a_k - a_{k-1})\sum_{j=1}^{n-k} 2^{-js} = \frac{c_s^{-1}}{2^s - 1}(a_k - a_{k-1})(1 - 2^{-(n-k)s}).$$

Then

$$III \geq -\frac{c_s^{-1}}{2^s-1}2^{(1+2\alpha)k}(a_k - a_{k-1})(1-2^{-(1+2\alpha)})\sum_{n=k+1}^{\infty}(1-2^{-(n-k)s})2^{-n}$$
$$= -\frac{4}{3}(1-2^{-(1+2\alpha)})(a_k - a_{k-1})2^{2\alpha k}.$$

From this inequality and using that $0 < \alpha < 1/2$, it is easy to see that $II + III \geq 0$. Then

$$((\mathcal{L}a)_k - (\mathcal{L}a)_{k-1})\, 2^{sk} \geq I \cdot 2^{sk} = C(\alpha)(2^{2\alpha k} - 2^{2\alpha})b_{k,s}$$

(b) Consider the system

$$a_k' = -(\mathcal{L}a)_k.$$

Using (a), it's not hard to see that $\frac{d}{dt}\|a\|_{X^s}(t) < 0$ and so $e^{-t\mathcal{L}}$ is a contracting semigroup.

(c) follows by explicit computation.

□

We will need the following version of Picard's theorem to prove local existence

Theorem 1.5.2 *(Picard Fixed Point) Let Y be a Banach space and let $\Gamma : Y \times Y \to Y$ be a bilinear operator such that for all $a, b \in Y$,*

$$\|\Gamma(a,b)\|_Y \leq \eta \|a\|_Y \|b\|_Y.$$

Then for any $a^0 \in Y$ with $4\eta\|a^0\|_Y < 1$, the equation $a = a^0 + \Gamma(a,a)$ has a unique solution $a \in Y$ such that $\|a\|_Y \leq 1/2\eta$.

Theorem 1.5.3 *(Local existence) Let $\{a_k(0)\} \in X^s$ where $s \geq 1$. Then there exists $T = T(\|a(0)\|_{X^s})$ such that there is a unique solution $\{a_k(t)\} \in C([0,T], X^s)$.*

Proof. The argument is fairly standard and we will provide a sketch. By Lemma 1.5.1, the operator $\mathcal{L}$ generates a contracting semigroup $e^{-t\mathcal{L}}$ on X^s. Observe that a solution will satisfy

$$a_k(t) = e^{-t\mathcal{L}} a_k^0 - \int_0^t e^{-(t-s)\mathcal{L}} \{2^k (a_k - a_{k-1})^2(s)\}\, ds$$

In writing $e^{-(t-s)\mathcal{L}}$ in the integral, we have slightly abused notation. The integrand is the kth element of $e^{-(t-s)\mathcal{L}}$ applied to the sequence $\{2^j(a_j - a_{j-1})^2(s)\}_{j=1}^{\infty}$.

Define a bilinear operator $\Gamma : X^s \times X^s \to X^s$ by

$$\Gamma(a,b)_j(t) = -\int_0^t e^{-(t-s)\mathcal{L}} \{2^k (a_k - a_{k-1})(s)(b_k - b_{k-1})(s)\}\, ds$$

Define $\gamma(a,b)(s)$ by

$$\gamma(a,b)_k(s) = -e^{-(t-s)\mathcal{L}} \{2^k (a_k - a_{k-1})(s)(b_k - b_{k-1})(s)\}$$

Then after doing basic estimates, one has $\|\gamma(a,b)(s)\|_{X^s} \leq C\|a(s)\|_{X^s}\|b(s)\|_{X^s}$ for $s \geq 1$. From this, we have an estimate on Γ and choosing t small enough, we can apply Picard. □

Also, we have the following preservation properties:

Lemma 1.5.4 *Let $\{a_k(t)\}$ be a solution of* (1.20) *in $C([0,T], X^s)$, $s > 1$. Suppose $\{a_k^0\}$ is non-decreasing in k and is non-negative.*

(a) *(Monotonicity and positivity) Then for all $t \leq T$, $\{a_k(t)\}$ is non-decreasing in k and is non-negative.*

(b) *(Max stays at* 0*) For $t \in [0,T]$, we have that*

$$\sup_k a_k(t) = \lim_{k\to\infty} a_k(t)$$

(c) *(ℓ^∞ maximum principle) The supremum* $\sup_k a_k(t)$ *is non-increasing in* t *and* $a_0(t)$ *is increasing in* t*. If* $\{a_k(t)\}$ *is a solution to* (1.17)*, then* $\sup_k a_k(t)$ *is constant in time*

Proof.

(a) Set $b_k(t) = (a_k - a_{k-1})(t)2^k$. For a contradiction, suppose there exists j and a time t_0 such that $b_j(t_0) < 0$. Choose a sufficiently small ϵ with $0 < \epsilon < 1$ such that $b_j(t_0) < -\epsilon/t_0$. Define a function f by $f(t) = \dfrac{-\epsilon}{2t_0 - t}$ for $0 \leq t < 2t_0$, so $b_j(t_0) < f(t_0)$.

Let $g(t) = \inf_k b_k(t)$. Let t_1 be the first time g crosses f so $\inf_k b_k(t_1) = f(t_1)$. Because $b_k(t_1) \to 0$ as $k \to \infty$, the infimum is actually achieved for some index. Let k_1 be the first index for which $b_{k_1}(t_1) = f(t_1)$. By minimality of t_1, $b'_{k_1}(t_1) \leq f'(t_1) = -\dfrac{\epsilon}{(2t_0 - t)^2}$. However, at time t_1, b_{k_1} also satisfies

$$\begin{aligned} b'_{k_1} &= -b_{k_1}^2 + 2b_{k_1-1}^2 - ((\mathcal{L}a)_{k_1} - (\mathcal{L}a)_{k_1-1})\, 2^{k_1} \\ &\geq -\frac{\epsilon^2}{(2t_0 - t)^2} - ((\mathcal{L}a)_{k_1} - (\mathcal{L}a)_{k_1-1})\, 2^{k_1} \end{aligned}$$

By an argument analogous to Lemma 1.5.1, the contribution from the dissipative terms on the right side of the inequality above is negative. From this, we arrive at a contradiction to the above inequality. Because a_0 is always non-negative, a_k stays non-negative for all time by the monotonicity just proved.

(b) follows directly from **(a)**.

(c) By **(a)**, we see that $(\mathcal{L}a)_0(t) < 0$ for all time $t \leq T$ so $a_0(t)$ is increasing in t. Now, for $t \in [0, T]$

$$\frac{d}{dt} \sup_{k>0} a_k(t) = \frac{d}{dt} \lim_{k\to\infty} a_k(t) = \lim_{k\to\infty} \frac{d}{dt} a_k(t) = \lim_{k\to\infty} \left(-(a_k - a_{k-1})^2(t)2^k - (\mathcal{L}^\alpha a)_k(t) \right)$$

Since we are on a compact time interval, the convergence above is uniform and so we are justified in interchanging limit and derivative. The limit of the first term on the far right is zero. Using that $\{a_k(t)\} \in X^s$ where $2\alpha < 1 < s$,

$$\lim_{k\to\infty} (\mathcal{L}^\alpha a)_k(t) = \lim_{k\to\infty} \left(\sum_{n=0}^{k-1} (a_k - a_n)(t) 2^{2\alpha n} + \sum_{n=k+1}^{\infty} (a_k - a_n)(t) 2^{2\alpha k} 2^{k-n} \right) > 0$$

because the limit of the first term is positive by monotonicity and the limit of the second term is zero. This completes the proof.

□

In the spirit of [25], we show that solutions of (1.17) blow up in finite time.

Theorem 1.5.5 *Let* $\{a_k\}_{k=0}^\infty$ *be a solution of* (1.17) *in* X^s*,* $s > 1$ *with initial data that is non-negative and non-decreasing in* k*. Then* $\{a_k\}$ *develops blow up in* X^s *for every* $s > 1$ *in finite time.*

Define

$$J(t) = \sum_{k=1}^{\infty} (a - a_k(t)) 2^{k\delta}$$

where $0 < \delta < 1$. Observe that, $J \leq \|a\|_{X^1} \sum_{k=1}^\infty 2^{k(\delta-1)}$. To prove we have blow up, we will show that J must become infinite in finite time. We will need the following lemma

Lemma 1.5.6 *Let* $0 < \delta < 1$*. Suppose that* $\{a_k\} \in X^s$*,* $s > 1$*, non-decreasing in* k*, and* $a := \lim_{k\to\infty} a_k < \infty$*. Then*

$$\sum_{k=1}^{\infty} (a_k(t) - a_{k-1}(t))^2 2^{k(\delta+1)} \geq C_0(\delta) \sum_{k=1}^{\infty} (a - a_k(t))^2 2^{k(\delta+1)} \tag{1.21}$$

where $C_0(\delta)$ *is constant depending only on* δ*.*

Proof. Choose $c > 0$ such that $(c+1)^{-2} 2^{\delta+1} > 1$. We call k "good" if $a_k - a_{k-1} \geq c(a - a_k)$ and bad otherwise.

Claim: If the set of all good k are finite, then both sides of (1.21) are infinite.

If the set of good k is finite, then there exists K such that for $k > K$, $a_k - a_{k-1} \leq c(a - a_k)$ or equivalently $(a - a_k) \geq (c+1)^{-1}(a - a_{k-1})$. Then

$$(a - a_k)^2 2^{k(\delta+1)} \geq (c+1)^{-2(k-K+1)}(a - a_K)^2 2^{k(\delta+1)} \to \infty$$

as $k \to \infty$ by the choice of c, so the right side of (1.21) diverges.

Define c_n by $a_k - a_{k-1} = c_k(a - a_k)$. Then $(a - a_{k-1}) = (1 + c_k)(a - a_k)$. Since $\lim_{k\to\infty}(a - a_k) = 0$, $\prod_{k=K}^{\infty}(1 - \frac{c_k}{c_k+1}) = \prod_{k=K}^{\infty} \frac{1}{1+c_k} = 0$, so $\sum_k c_k = \infty$. For the bad k,

$$(a_k - a_{k-1})^2 2^{k(\delta+1)} = c_k^2(a - a_k)^2 2^{k(\delta+1)}$$

from which it is not hard to see that the left side of (1.21) will be infinite as well. The claim is proven.

Now suppose k_{j-1}, k_j are good such that for $k_{j-1} < k < k_j$, k is bad. Then $(a - a_{k-1}) \leq (1 + c)(a - a_k)$, which implies

$$\begin{aligned}
\sum_{k=k_{j-1}+1}^{k_j} (a - a_k)^2 2^{k(\delta+1)} &\leq \sum_{k=k_{j-1}+1}^{k_j} (1+c)^{2(k_j-k)}(a - a_{k_j})^2 2^{k(\delta+1)} \\
&= (a - a_{k_j})^2 2^{k_j(\delta+1)} \sum_{k_{j-1}+1}^{k_j} (1+c)^{2(k_j-k)} 2^{(k-k_j)(\delta+1)} \\
&= (a - a_{k_j})^2 2^{k_j(\delta+1)} \sum_{n=0}^{k_j-k_{j-1}+1} (1+c)^{2n} 2^{-n(\delta+1)} \\
&\leq C(\delta)(a - a_{k_j})^2 2^{k_j(\delta+1)}
\end{aligned}$$

where the last inequality comes from the choice of c. Treating all bad k this way, the inequality (1.21) follows. □

Proof of Theorem 1.5.5 Using the Lemma 1.5.4(c) and 1.5.6 as well as Holder's inequality, we have

$$\begin{aligned}\frac{d}{dt}J(t) &= \sum_{k=1}^{\infty}(a_k(t)-a_{k-1}(t))^2 2^{k(\delta+1)} \geq C_0\sum_{k=1}^{\infty}(a-a_k(t))^2 2^{k(\delta+1)} \\ &\geq C_0\left(\sum_{k=1}^{\infty}2^{k(\delta-1)}\right)^{-1}\left(\sum_{k=1}^{\infty}(a-a_k(t))2^{k\delta}\right)^2 = C_1(\delta)J(t)^2\end{aligned}$$

The result follows from Gronwall's inequality. □

1.6 A-priori Hölder-1/2 bound

The purpose of this section is to prove an a-priori bound for solutions of (1.20) from which the results of next section will follow. If this result of this section were to be carried to the continuous setting, it would mean that solutions to (1.15) are bounded in the Hölder class $C^{1/2}$. This regularization effect has recently been conjectured in [35] for the vanishing viscosity approximation. If such a bound were to hold, it would mean that $\alpha = 1/4$ is the true critical power for regularity and would answer the open question regarding (1.15) stated in the introduction. For the SQG equation (1.2), the $C^{1-2\alpha}$ norm is critical [20]: weak solutions that are bounded in time in $C^{1-2\alpha}$ are classical solutions.

First, we will prove bounds on (1.17), the model without dissipation, as it is more elementary. Then we will generalize to the full model (1.20) with dissipation.

We rewrite (1.17) in a different form, which allows us to give a more detailed picture of how blow-up can occur. Let $b_k = (a_k - a_{k-1})2^k$. Then the b_k's satisfy

$$b_k'(t) = -b_k^2 + 2b_{k-1}^2 \tag{1.22}$$

with corresponding initial data

$$b_k(0) = -(a_k^0 - a_{k-1}^0)2^k := b_k^0$$

By convention, we set $b_1' = -b_1^2$ and $b_0 = 0$. We will take $b_k(0) > 0$ for all k so b_k will remain non-negative by Lemma 1.5.4(a). The fact that the a_k's blow up in X^s, $s > 1$, means that there exists $T > 0$ such that $\lim_{t\to T}\sup_k b_k(t)2^{k(s-1)} = \infty$. In what follows, we work with X^s, $s > 1$ solutions.

We will prove there exists an invariant region for the system of ODE's satisfied by the sequence $\{b_k(t)\}_{k=1}^{\infty}$. Define a sequence $\{\gamma_k\}$ by $\gamma_1 = \gamma_2 = 2$ and define γ_k, $k > 2$, to be the positive root of the polynomial

$$x^2 + \left(\frac{2}{\gamma_{k-1}^2} - 1\right)x - 2 = 0.$$

One can see that $\gamma_k \to \sqrt{2}$ as $k \to \infty$. Hölder 1/2 control of solutions will follow from proving that the following region is invariant for $\{b_k(t)\}_{k=1}^{\infty}$:

$$\mathcal{I} := \{\{c_k\}_{k=1}^{\infty} : c_k \geq 0 \text{ for all } k \text{ and } c_k \leq \gamma_k c_{k-1} \text{ for all } k > 1\}$$

Invariant regions have also been used to study regularity properties of dyadic models for other fluid equations [1].

Lemma 1.6.1 *Let $\{b_k\}_{k=1}^{\infty}$ be a solution of* (1.22). *Suppose that $\{b_k(0)\}_{k=1}^{\infty} \in \mathcal{I}$. Then for all $t \geq 0$, $\{b_k(t)\}_{k=1}^{\infty} \in \mathcal{I}$.*

Proof

The preservation of positivity of the b's follows from Lemma 1.5.4(a). For the other inequality, we proceed by induction. First, we will show $b_2(t) \leq 2b_1(t) = \gamma_2 b_1(t)$, for all t. It suffices to show that $(2b_1 - b_2)'(t_1) \geq 0$ at anytime t_1 such that $b_2(t) = 2b_1(t)$. This is true:

$$(2b_1 - b_2)'(t_1) = -4b_1^2(t_1) + b_2^2(t_1) = 0.$$

Now assume $b_{k-1}(t) \leq \gamma_{k-1} b_{k-2}(t)$ for all t. We want to show $b_k(t) < \gamma_k b_{k-1}(t)$ for all t. Suppose t_k is a time for which $b_k(t_k) = \gamma_k b_{k-1}(t_k)$. Then using the inductive hypothesis

$$\begin{aligned}(\gamma_k b_{k-1} - b_k)'(t_k) &= \gamma_k(-b_{k-1}^2(t) + 2b_{k-2}^2) + b_k^2(t) - 2b_{k-1}^2(t) \\ &\geq \left[\gamma_k\left(\frac{2}{\gamma_{k-1}^2} - 1\right) + \gamma_k^2 - 2\right] b_{k-1}^2(t_k) = 0.\end{aligned}$$

This completes the proof. □

Theorem 1.6.2 *(Hölder* $1/2$ *bound) Suppose we have initial data lying in* $\mathfrak{I}$*, then* $\sup_{k>0,t\geq 0} b_k(t)2^{-k/2} < \infty$

Proof. It is easy to show that the infinite product $\prod_k \frac{c_k}{\sqrt{2}}$ converges. Then by Lemma 3.1, we are done. □

The following additional lemma will give more insight into how blow-up occurs.

Lemma 1.6.3 *Let* $\{b_k\}$ *be a positive solution to* (1.22)*. Suppose that for* $T_1 \leq t \leq T_2$ *and some* $k \geq 2$*,* $b_{k-1}(t) > b_{k-2}(t)$ *(or* $<$*) and* $b_k(T_1) > b_{k-1}(T_1)$ *(or* $<$*). Then for* $T_1 \leq t < T_2$*, we have* $b_k(t) > b_{k-1}(t)$ *(or* $<$*).*

Proof

We first treat the case $k = 2$ and prove the "$>$" case as the other direction will be analogous. For a contradiction, suppose the set $A_2 := \{T_2 \geq t > T_1 : b_2(t) > b_1(t)\}$ is non-empty. By continuity, A_2 has a minimum which we denote by t_2. Then $b_2(t_2) = b_1(t_2)$, which implies

$$(b_2 - b_1)'(t_2) = -b_2^2(t_2) + 3b_1^2(t_2) = 2b_1^2(t_2) > 0.$$

This leads to contradiction to the minimality of t_2 so this is case is settled.

Now, let $k > 2$. The argument is similar to the previous case. It suffices to show that $(b_k - b_{k-1})'(t) > 0$ at a time $t = t_k$ such that $b_k(t_k) = b_{k-1}(t_k)$, given that $b_{k-1}(t_k) > b_{k-2}(t_k)$. By a quick calculation, we can check that this is true. $\square$.

Consider initial data of the following form

$$\begin{aligned} b_k^0 &< b_{k-1}^0, \quad \text{for } k > K_0 \\ b_k^0 &> b_{k-1}^0 > 0, \quad \text{for } k \leq K_0. \end{aligned}$$

for some index $K_0 > 1$. Then by the previous lemma, we know that for all t, there exists $K_t \geq K_0$ such that

$$\begin{aligned} b_k(t) &< b_{k-1}(t), \quad \text{for } k > K_t \\ b_k(t) &> b_{k-1}(t) > 0, \quad \text{for } k \leq K_t. \end{aligned}$$

The scenario we describe is analogous to describing the concavity of solutions in the continuous setting. We view $b_k > b_{k-1}$ as saying that the solution is concave up on the interval $(2^{-k}, 2^{-k+2})$ and vice versa. The index K_t can be thought as encoding the inflection point. In order for the solution to blow-up, $K_t \to \infty$ as $t \to T$.

1.6.1 With Dissipation

In this section, we prove the Hölder 1/2 bound for the system with fractional dissipation. We still consider initial data $\{a_k^0\}_{k=0}^{\infty} \in X^s$, $s > 1$, that is positive and non-decreasing in k.

As before we consider the system satisfied by $b_k = (a_k - a_{k-1})2^k$:

$$b'_k(t) = -b_k^2 + 2b_{k-1}^2 + 2^k((\mathcal{L}a)_{k-1} - (\mathcal{L}a)_k) \tag{1.23}$$

The bound will follow quickly from the following lemma about dissipation.

Lemma 1.6.4 *Let $0 < \alpha \leq 1/2$. Fix a time t. Suppose $b_j(t) = \gamma_j b_{j-1}(t)$ and that $b_k(t) < \gamma_k b_{k-1}(t)$, for $k > j$. Then for $j \geq 2$,*

$$(\mathcal{L}a)_j(t) - (\mathcal{L}a)_{j-1}(t) > \frac{\gamma_j}{2}\left[(\mathcal{L}a)_{j-1}(t) - (\mathcal{L}a)_{j-2}(t)\right].$$

Proof.

For simplicity of expression, all expressions are evaluated at t. By direction calculation,

$$\begin{aligned}(\mathcal{L}a)_j - (\mathcal{L}a)_{j-1} &= \sum_{n=0}^{j-2}(a_j - a_{j-1})2^{2\alpha n} + 2(a_j - a_{j-1})2^{2\alpha(j-1)} \\ &\quad + 2^{(1+2\alpha)j}(1 - 2^{-(1+2\alpha)})\sum_{n=j+1}^{\infty}(a_j - a_n)2^{-n}.\end{aligned} \tag{1.24}$$

Also, we have

$$\begin{aligned}\frac{\gamma_j}{2}\left[(\mathcal{L}a)_{j-1} - (\mathcal{L}a)_{j-2}\right] &= \sum_{n=0}^{j-3}\frac{\gamma_j}{2}(a_{j-1} - a_{j-2})2^{2\alpha n} + \gamma_j(a_{j-1} - a_{j-2})2^{2\alpha(j-2)} \\ &\quad + \gamma_j 2^{(1+2\alpha)(j-1)-1}(1 - 2^{-(1+2\alpha)})\sum_{n=j}^{\infty}(a_{j-1} - a_n)2^{-n} \\ &= \sum_{n=0}^{j-3}(a_j - a_{j-1})2^{2\alpha n} + 2(a_j - a_{j-1})2^{2\alpha(j-2)} \\ &\quad + \gamma_j 2^{(1+2\alpha)(j-1)-1}(1 - 2^{-(1+2\alpha)})\sum_{n=j}^{\infty}(a_{j-1} - a_n)2^{-n}\end{aligned}$$

where we have used $b_k = \gamma_k b_{k-1}$ in the last equality. Then

$$\begin{aligned}(\mathcal{L}a)_j \quad - \quad & (\mathcal{L}a)_{j-1} - \frac{c_j}{2}\left[(\mathcal{L}a)_{j-1} - (\mathcal{L}a)_{j-2}\right] = (2-2^{-2\alpha})(a_j - a_{j-1})2^{2\alpha(j-1)} \\ & + \frac{\gamma_j}{4}(1-2^{-(1+2\alpha)})(a_j - a_{j-1})2^{2\alpha(j-1)} + 2^{(1+2\alpha)j}(1-2^{-(1+2\alpha)})\sum_{n=j+1}^{\infty}(a_j - a_n)2^{-n} \\ & -\gamma_j 2^{(1+2\alpha)(j-1)-1}(1-2^{-(1+2\alpha)})\sum_{n=j+1}^{\infty}(a_{j-1} - a_n)2^{-n} \\ & = (2 + \frac{\gamma_j}{2} - 2^{-2\alpha} - \gamma_j 2^{-(2+2\alpha)})(a_j - a_{j-1})2^{2\alpha(j-1)} \\ & \quad + \gamma_j(1-2^{-(1+2\alpha)})2^{(1+2\alpha)(j-1)-1}\sum_{n=j+1}^{\infty}(a_j - a_{j-1})2^{-n} \\ & \quad + (1-2^{-(1+2\alpha)})(1-\gamma_j 2^{-(2+2\alpha)})2^{(1+2\alpha)j}\sum_{n=j+1}^{\infty}(a_j - a_n)2^{-n}.\end{aligned}$$

Using that $b_{k+1} < \gamma_k b_k$ for $k \geq j$ and that γ_k is decreasing, $a_j - a_n \geq \dfrac{\left(\frac{\gamma_j}{2}\right)^{n-j+1} - \frac{\gamma_j}{2}}{1 - \frac{\gamma_j}{2}}(a_j - a_{j-1})$ for $n > j$. From this we can get $\displaystyle\sum_{n=j+1}^{\infty}(a_j - a_n)2^{-n} \geq -\frac{2\gamma_j}{4-\gamma_j}(a_j - a_{j-1})2^{-j}$. Inserting this into the above estimate we get that the above expression is bounded below by

$$\begin{aligned}& \left(2 + \frac{\gamma_j}{2} - \frac{\gamma_j 2^{1+2\alpha}}{4-\gamma_j}(1-\gamma_j 2^{-(2+2\alpha)})\right)(1-2^{-(1+2\alpha)})(a_j - a_{j-1})2^{2\alpha(j-1)} \\ & = \frac{2(4-2^{2\alpha}\gamma_j)}{4-\gamma_j}(1-2^{-(1+2\alpha)})(a_j - a_{j-1})2^{2\alpha(j-1)}\end{aligned}$$

Since $\gamma_j \leq 2$, we see that this expression is positive for $0 < \alpha \leq 1/2$. The case for when $j = 2$ is done analogously. This completes the proof $\square$.

Combining this lemma with the proof from Lemma 1.6.1, we have the main result of this subsection.

Theorem 1.6.5 *Let* $\{b_k\}_{k=1}^{\infty}$ *be a solution of* (1.23). *Suppose that* $\{b_k(0)\}_{k=1}^{\infty} \in \mathcal{I}$. *Then for all* $t \geq 0$, $\{b_k(t)\}_{k=1}^{\infty} \in \mathcal{I}$. *In particular,* $\sup_{k>0,t\geq 0} b_k(t)2^{-k/2} < \infty$.

Proof. First, we show $b_2(t) \leq 2b_1(t)$, for all t. It suffices to show that $(2b_1 - b_2)'(t_1) \geq 0$ at anytime t_1 such that $b_2(t_1) = 2b_1(t_1)$ while $b_k(t_1) < \gamma_k b_{k-1}(t_1)$ for $k > 2$. This is true:

$$(2b_1 - b_2)'(t_1) = -4b_1^2(t) + b_2^2(t) - 4\left[(\mathcal{L}a)_1 - (\mathcal{L}a)_0\right] + 4\left[(\mathcal{L}a)_2 - (\mathcal{L}a)_1\right] > 0.$$

by the previous Lemma. Now we want to show $b_k(t) < \gamma_k b_{k-1}(t)$ for all t. Suppose t_k is the first time for which $b_k(t_k) = \gamma_k b_{k-1}(t_k)$. Then $b_j(t_k) < \gamma_j b_{j-1}(t_k)$ for $j \neq k$. By direct computation we get

$$\begin{aligned}(\gamma_k b_{k-1} - b_k)'(t_k) &= \gamma_k(-b_{k-1}^2(t) + 2b_{k-2}^2) + b_k^2(t) - 2b_{k-1}^2(t) \\ &\quad - \gamma_k 2^{k-1}\left[(\mathcal{L}a)_{k-1} - (\mathcal{L}a)_{k-2}\right] + 2^k\left[(\mathcal{L}a)_k - (\mathcal{L}a)_{k-1}\right] \\ &\geq \left[\gamma_k\left(\frac{2}{\gamma_{k-1}^2} - 1\right) + \gamma_k^2 - 2\right] b_{k-1}^2(t_k) = 0.\end{aligned}$$

where we have used the previous lemma in the inequality. □

1.7 Regularity for $1/4 < \alpha \leq 1/2$ and Blow-up for $\alpha < 1/4$

The previous theorem will allow us to adapt the argument of Theorem 1.5.5 and prove blow-up for (1.20) for $0 < \alpha < 1/4$.

Theorem 1.7.1 *Let $0 < \alpha < 1/4$ Suppose $\{a_k^0\}$ is increasing in k and is non-negative. Then there exists initial datum in X^s, $s > 1$, such that solutions blow-up in finite time.*

Proof.

As before we let

$$J(t) = \sum_{k=0}^{\infty}(a_\infty(t) - a_k(t))2^{k\delta}.$$

for $0 < \delta < 1 - 4\alpha$ where $a_\infty(t) = \lim_{k\to\infty} a_k(t)$. Then computing and using Lemma 1.5.6 we get

$$\begin{aligned}\frac{d}{dt}J(t) &= \sum_{k=1}^{\infty}(a_k(t) - a_{k-1}(t))^2 2^{k(\delta+1)} - \sum_{k=1}^{\infty}\left((\mathcal{L}a)_\infty - (\mathcal{L}a)_k\right) 2^{k\delta} \\ &\geq C_0\sum_{k=1}^{\infty}(a_\infty(t) - a_k(t))^2 2^{k(\delta+1)} - \sum_{k=1}^{\infty}\left((\mathcal{L}a)_\infty - (\mathcal{L}a)_k\right) 2^{k\delta}\end{aligned}$$

where

$$(\mathcal{L}a)_\infty = \lim_{k\to\infty}(\mathcal{L}a)_k = \sum_{n=0}^{\infty}(a_\infty - a_n)2^{2\alpha n}$$

Now,

$$(\mathcal{L}a)_\infty - (\mathcal{L}a)_k = \sum_{n=0}^{k-1}(a_\infty - a_k)2^{2\alpha n} + \sum_{n=k}^{\infty}(a_\infty - a_n)2^{2\alpha n} - \sum_{n=k+1}^{\infty}(a_k - a_n)2^{(1+2\alpha)k}2^{-n}$$

We analyze each of the above three terms separately.

$$\sum_{n=0}^{k-1}(a_\infty - a_k)2^{2\alpha n} \leq C(a_\infty - a_k)2^{2\alpha k}$$

By Theorem 1.6.5,

$$\sum_{n=k}^{\infty}(a_\infty - a_n)2^{2\alpha n} \leq C\sum_{n=k}^{\infty}2^{(2\alpha-1/2)n} \leq C2^{(2\alpha-1/2)(k-1)}$$

For the last term we have

$$\sum_{n=k+1}^{\infty}(a_n - a_k)2^{(1+2\alpha)k}2^{-n} \leq (a_\infty - a_k)2^{(1+2\alpha)k}\sum_{n=k+1}^{\infty}2^{-n} \leq C(a_\infty - a_k)2^{2\alpha k}.$$

Then using $\alpha < 1/4$ and $0 < \delta < 1 - 4\alpha$,

$$\sum_{k=1}^{\infty}\left((\mathcal{L}a)_\infty - (\mathcal{L}a)_k\right) 2^{k\delta} \leq C_1 + C_2\sum_{k=1}^{\infty}(a_\infty - a_k)2^{(2\alpha+\delta)k}$$

For every $\epsilon > 0$, by an application of Holder's inequality and using that $\alpha < 1/4$,

$$\sum_{k=1}^{\infty}(a_\infty - a_k)2^{(2\alpha+\delta)k} \leq \epsilon\sum_{k=1}^{\infty}(a_\infty - a_k)^2 2^{(\delta+1)k} + C_\epsilon\|a\|_{\ell^\infty}$$

Given these bounds and following the proof of Theorem 1.5.5 we can show

$$J'(t) \geq C(\delta)J(t)^2 - C(1 + \|a\|_{\ell^\infty}).$$

By choosing initial data appropriately, the inequality above leads to blow-up. □

Now, we will move towards proving regularity for $1/4 < \alpha < 1/2$.

Theorem 1.7.2 *Let $1/4 < \alpha < 1/2$. Suppose we have the same hypotheses as in Theorem 1.6.5. Then solutions exist for all time, in particular, for $s > 1$*

$$\sup_{t>0} \|a(t)\|_{X^s} < \infty.$$

Proof.

After a computation,

$$(b_{k,s})'(t) = -b_k \cdot b_{k,s}(t) + 2^s b_{k-1} \cdot b_{k-1,s}(t) - ((\mathcal{L}a)_k - (\mathcal{L}a)_{k-1})(t) 2^{sk} \tag{1.25}$$

From Theorem 1.6.5, $b_{k-1}(t) \leq C_1 2^{k/2}$ for all $t \in [t_0, t_1]$ where the constant C_1 is independent of the time interval. Then there exists K' such that if $k \geq K'$ and $b_{k,s}(t) > c_s \|a\|_{X^s}(t)$ where c_s is the constant from Lemma 1.5.1,

$$(b_{k,s})'(t) \leq b_{k,s} \left(C_1 2^{k/2+s} - C(\alpha)(2^{2\alpha k} - 2^{2\alpha}) \right) < 0.$$

In the above inequality, we use $0 < \alpha < 1/4$. This implies that for any $T > 0$, $\lim_{t \to T, k \to \infty} b_{k,s}(t) \neq \infty$. Therefore, the solution exists globally in time. □

Remark: It is unclear whether the hypothesis that $\{b_k^0\}_{k=1}^\infty \in \mathfrak{I}$ can be weakened as it is crucial to the proof of the a-priori bound. We conjecture that any reasonable initial data will eventually satisfy such a condition at least for large enough indexes.

Summary: We have shown that when $\alpha > 1/4$, under mild assumptions on the initial data, it is possible to have global solutions: the dissipation terms win over the nonlinearity. In order to have blow-up, energy initially present in the lower modes must reach the higher modes. By making use of a-priori Holder-type control and an estimate on the dissipative terms, we have shown that such an energy transfer must stop at some time.

It is important to note that our conditions on the initial data are not a condition of "smallness" with regard to some norm. Our results of global regularity can hold for large initial data. Also, one can easily find initial data for which our model blows up for $0 < \alpha < 1/4$ and is regular for $\alpha > 1/4$.

Chapter 2

On Model Equations for the Euler Equations

2.1 On some models for the 2D Euler Equations and related equations

The following transport equation

$$\omega_t + u \cdot \nabla \omega = 0. \tag{2.1}$$

is a basic mathematical model in fluid dynamics. If u depends on ω, (2.1) is called an active scalar equation. The problem of deciding whether blowup can occur for smooth initial data becomes very hard if the dependence of ω is nonlocal in space.

The relationship expressing u in terms of ω is commonly called Biot-Savart law. We have the following examples in 2D:

$$u = \nabla^{\perp}(-\triangle)^{-1}\omega, \tag{2.2}$$

where $\nabla^{\perp} = (-\partial_y, \partial_x)$ is the perpendicular gradient. Equations (2.1) and (2.2) are the vorticity form of 2D Euler equation. When we take

$$u = \nabla^{\perp}(-\Delta)^{-\frac{1}{2}}\omega,$$

(2.1) becomes the surface quasi-geostrophic (SQG) equation, which has important applications in geophysics, or can be regarded as a toy model for the 3D-Euler equations. For more details we refer to [6].

A question of great importance is whether solutions for these equations form singularities in finite time. A promising new approach for the construction of singular solutions is to use the *hyperbolic flow scenario.* In [23], [22], such a scenario was proposed to obtain singular solutions for the 3D Euler equations, and in [30], the long-standing question of existence of solutions to the 2D Euler with double-exponential gradient growth was settled using hyperbolic flow.

The hyperbolic flow scenario in two dimensions can be explained in the following way. Consider e.g. a flow in the upper half-plane $\{x_2 > 0\}$. The essential properties required are (see Figure 2.1 for an illustration):

- There is a stagnant point of the flow at one boundary point (e.g. the origin) for all times.
- Along the boundary, the flow is essentially directed towards that point for all times.

Such flows can be created by imposing symmetry and other conditions on the initial data. For incompressible flows the stagnant point is a hyperbolic point of the velocity field, hence the name.

The scenario is a natural candidate for creating flows with strong gradient growth or finite-time blowup, since the fluid is compressed along the boundary. Due to non-linear and non-local interactions however, the flow remains hard to control, so a rigorous proof of blowup for the 3D Euler equations using hyperbolic flow remains a challenge. The crucial issue is to stabilize the scenario up to the singular time.

One way to make progress in understanding and to gain insight into the hyperbolic blowup scenario is to study it in the context of one-dimensional model equations. This was begun in [4, 3], where one-dimensional models for the 2D-Boussinesq and 3D axisymmetric

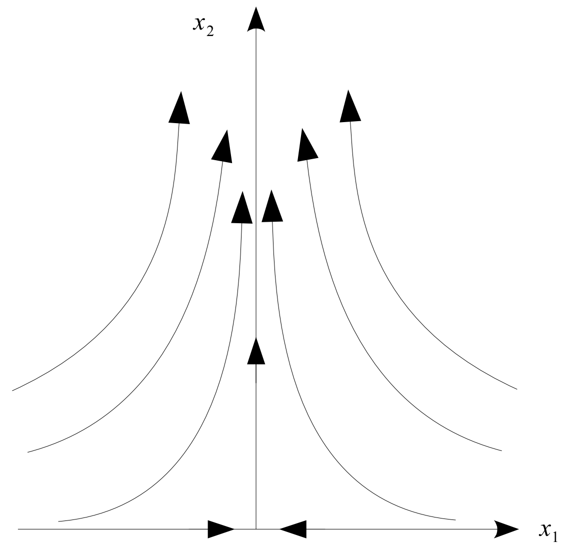


Figure 2.1 : Illustration of hyperbolic flow scenario in two dimensions.

Euler equations were introduced and blowup was proven.

One-dimensional models capturing other aspects of fluid dynamical equations have a long-standing tradition, one of the earliest being the celebrated Constantin-Lax-Majda model [5]. We refer to the introduction of [35] for a more thorough review of known one-dimensional model equations, and to [3] for discussion of the aspects relating to the hyperbolic flow scenario.

In this paper, we will study 1D models of (2.1) on $\mathbb{R}$ with the following two choices of u:

$$u_x = H\omega, \tag{2.3}$$

$$u = (-\Delta)^{-\frac{\alpha}{2}}\omega = -c_\alpha \int_{\mathbb{R}} |y-x|^{-(1-\alpha)}\omega(y,t)\, dy. \tag{2.4}$$

The choice (2.3) leads to a 1D analogue of the 2D Euler equation. This model is derived simply by restricting the dynamics to the boundary. In section 2.2 we give a brief heuristic argument which works by assuming that ω is concentrated in a small boundary layer.

We note that the model (2.3) was mentioned in [3], where it was stated that (2.3) has properties analogous to the 2D Euler equation, without giving details. In particular, in [3] a 1D model of the 2D Boussinesq equations (an extended version of (2.3)) was introduced and studied. One of our goals here is to validate the 1D model introduced in [3] in a setting where comparison with 2D results are available. The fact shown below, that the solutions to the model problem (2.3) behave similarly to the full 2D Euler case, provides support to the usefulness of the extended version of this model in [3] for getting insight into behavior of solutions to 2D Boussinesq system and 3D Euler equation.

The model defined by (2.4) is called α-patch model and appears in [19], where also a viscosity term is present. From the regularity standpoint, the α-patch model is between 1D Euler $u_x = H\omega$ and the Córdoba-Córdoba-Fontelos model $u = H\omega$ (see [10, 35]), which is an analogue of the SQG equation. These two models differ however from a geometric perspective, since the symmetry properties of the Biot-Savart laws are different. For the CCF model, the velocity field is odd for even ω, whereas (2.4) is odd for odd ω. It is important to choose data with the right symmetry to make u odd, and thus to create a stagnant point of the flow at the origin for all times.

We note that local existence and blowup results for (2.4) were given in [19], where also dissipation is allowed. There the authors rely on a suitable Lyapunov function to show blowup, whereas we emphasize the more geometric aspects in this paper. That is, we will be studying the analogue of the hyperbolic flow scenario for the above 1D models and show that this leads to natural and intuitive constructions of solutions with strong gradient growth and finite-time blowup.

Another blowup result related to hyperbolic flow was recently proven by A. Kiselev, L.

Ryzhik, Y. Yao and A. Zlatoš [29] and concerns a α-patch model in 2D for small $\alpha > 0$.

2.2 Euler 1D model

2.2.1 Heuristic derivation.

Recall the 2D Euler equations in vorticity form

$$\omega_t + u \cdot \nabla\omega = 0$$

where $u = \nabla^{\perp}(-\Delta)^{-1}\omega$.

We first indicate a simple heuristic motivation for the choice (2.3) (see also [3]). Consider the 2D Euler equation in a half-space $\{x_2 \geq 0\}$ and denote $\bar{x} = (x_1, -x_2)$. The x_1-component of the velocity (up to a normalization constant) for compactly supported vorticity ω is given by

$$u_1(x,t) = -\int_{\mathbb{R}^2} \frac{(y_2 - x_2)}{|y-x|^2}\omega(y,t)\ dy \tag{2.5}$$

where ω has been extended to $\{x_2 \leq 0\}$ by odd reflection ($\omega(\bar{x}, t) = -\omega(x,t)$).

Suppose now that ω is concentrated in a boundary layer of width $a > 0$ and that $\omega(x_1, x_2, t) = \omega(x_1, t)$ in this boundary layer. Then a calculation gives

$$u_1(x_1, 0, t) = -2\int_{\mathbb{R}} \log\left(\frac{(y_1 - x_2)^2 + a^2}{(y_1 - x_1)^2}\right)\omega(y_1, t)\ dy_1. \tag{2.6}$$

If we now retain only the singular part of the kernel $\log\left(\frac{z^2+a^2}{z^2}\right) \sim -2\log|z|$ and identify u with u_1, we get (dropping the constants)

$$u(x,t) = \int_{\mathbb{R}} \log|y-x|\omega(y,t)\ dy.$$

So a reasonable 1D model is

$$\omega_t + u\omega_x = 0, \quad u_x = H\omega, \quad \omega(x,0) = \omega_0(x) \quad (x,t) \in \mathbb{R}\times[0,\infty). \tag{2.7}$$

where H is the Hilbert transform, using the convention

$$H\omega(x,t) = P.V.\int \frac{\omega(y,t)}{x-y}\, dy.$$

For this model, we have the following local well-posedness property:

Proposition 3 *Given initial data $\omega_0 \in H_0^m((0,1))$ with $m \geq 2$, there exists $T = T(\|\omega_0\|_{H_0^m}) > 0$ such that the system has a unique classical solution $\omega \in C([0,T]; H_0^m)$.*

The proof is standard so we skip it here.

An alternative argument to motivate (2.3) is to observe that the gradient of the 2D Euler velocity is given by a zero-th order operator acting on ω. In one dimension, this leaves only the choice $u_x = cH\omega$ or $u_x = c\omega$, c being a nonzero constant. So we could also consider the model

$$\omega_t + u\omega_x = 0, \quad u_x = -\omega, \quad \omega(x,0) = \omega_0(x) \quad (x,t) \in \mathbb{R} \times [0,\infty). \tag{2.8}$$

(2.8) is however not a close analogue of 2D Euler (see Remark 2.2.4).

2.2.2 Sharp a-priori bounds for gradient growth.

We will first prove the global regularity of the solution to equation (2.7) by showing that ω_x can grow at most with double exponential rate in time. Then we will give an example of a smooth solution to (2.7) where such growth of the gradient of ω is achieved, meaning the bound is sharp.

Due to the Biot-Savart law relating u and ω, the proof of an upper bound for $\|\omega_x(\cdot,t)\|_\infty$ is very similar to the proof for the full 2D Euler equations. For the reader's convenience, we give the proof. Recall first the definition of the Hölder norm

$$||\omega||_{C^\alpha} = \sup_{|x-y|\leq 1, x\neq y} \frac{|\omega(x)-\omega(y)|}{|x-y|^\alpha}$$

for compactly supported ω.

We will need an estimate on the Hilbert transform:

Lemma 2.2.1 *Let $0 < \alpha < 1$. Suppose supp$(\omega) \subset [-D(t), D(t)]$ and assume without loss of generality that $||\omega_0||_{L^\infty} = 1$. Then*

$$\|u_x\|_\infty \leq C(\alpha)\left(1 + |\log(D(t))| + \log(1 + \|\omega\|_{C^\alpha}\right))$$

Proof. For any $\delta > 0$, we have

$$\left|\int_{[-D(t),D(t)]\setminus(x-\delta,x+\delta)} \frac{\omega(y)}{x-y}\, dy\right| \leq C\int_\delta^{D(t)} \frac{1}{y}\, dy \leq C(|\log\delta| + |\log(D(t))|).$$

Using the oddness of $\frac{1}{x}$, we have

$$\left|P.V.\int_{x-\delta}^{x+\delta} \frac{\omega(y)}{x-y}\, dy\right| = \left|\int_{x-\delta}^{x+\delta} \frac{1}{x-y}(\omega(y) - \omega(x))\, dy\right| \leq C(\alpha)\|\omega(x,t)\|_{C^\alpha}\delta^\alpha.$$

Choosing $\delta = \min\left\{1, (\frac{1}{\|\omega\|_{C^\alpha}})^{\frac{1}{\alpha}}\right\}$, we get the desired estimate of $\|u_x\|_\infty$. $\square$

The following Lemma gives an estimate on $D(t)$.

Lemma 2.2.2 *Suppose the support of ω_0 is in $[-1, 1]$ and $||\omega_0||_{L^\infty} = 1$. Then the support of $\omega(x,t)$ will be inside $[-C\exp(Ce^{Ct}), C\exp(Ce^{Ct})]$, for some universal constant $C > 0$.*

Proof. Suppose $\operatorname{supp}\omega = [-D(t), D(t)]$. Then for any point x inside of this interval, we have

$$|u(x)| \leq \int_{-D(t)}^{D(t)} |\log|x-y||\, dy \leq C\int_0^{2D(t)} |\log|s||\, ds \leq CD(t)(|\log(D(t))| + 1).$$

By following the trajectory of the particle at $D(t)$,

$$D'(t) \leq CD(t)(|\log(D(t))| + 1).$$

A simple argument using differential inequalities shows that $D(t)$ is always less than $z(t)$, where $z(t)$ is the solution of

$$z'(t) = Cz(t)(\log z(t) + 1), \quad z(0) = \min\{D(0), 2\}.$$

This yields the double-exponential upper bound on $D(t)$. □

The following Theorem gives the double exponential upper bound for ω_x.

Theorem 2.2.3 *There is universal constant C such that if ω_0 is smooth, compactly supported with supp $\omega_0 \subset [-1, 1]$ and $\|\omega\|_{L^\infty} = 1$,*

$$\log(1 + \|\omega_x\|_{L^\infty}) \leq C \log(1 + \|(\omega_0)_x\|_{L^\infty}) e^{Ct} \quad (t \geq 0). \tag{2.9}$$

Proof. We follow the proof in [30]. Let us denote the flow map corresponding to the evolution by $\Phi_t(x)$. Then

$$\frac{\partial}{\partial t}\Phi_t(x) = u(\Phi_t(x), t), \quad \Phi_0(x) = x,$$

and

$$\left| \frac{\partial_t |\Phi_t(x) - \Phi_t(y)|}{|\Phi_t(x) - \Phi_t(y)|} \right| \leq ||u_x||_{L^\infty}.$$

After integration, and by Lemma 2.2.1 and Lemma 2.2.2, this gives

$$f(t)^{-1} \leq \frac{|\Phi_t(x) - \Phi_t(y)|}{|x - y|} \leq f(t),$$

where

$$f(t) = \exp\left(C \int_0^t (1 + \exp(Cs) + \log(1 + ||\omega_x||_{L^\infty}))\, ds \right).$$

This bound also holds for Φ_t^{-1}. On the other hand,

$$\|\omega_x\|_{L^\infty} = \sup_{x \neq y} \frac{|\omega_0(\Phi_t^{-1}(x)) - \omega_0(\Phi_t^{-1}(y))|}{|x - y|} \leq \|(\omega_0)_x\| \sup_{x \neq y} \frac{|\Phi_t^{-1}(x) - \Phi_t^{-1}(y)|}{|x - y|}.$$

Which means we have

$$(1+\|\omega_x\|_{L^\infty}) \le (1+\|(\omega_0)_x\|_{L^\infty})\exp\left(C\int_0^t 1+\exp(Cs)+\log(1+\|\omega_x\|_{L^\infty})\ ds\right),$$

or

$$\log(1+\|\omega_x\|_{L^\infty}) \le \log(1+\|(\omega_0)_x\|_{L^\infty}) + C\exp(Ct) + C\int_0^t (1+\log(1+\|\omega_x\|_{L^\infty}))\ ds.$$

So $y(t) := \log(1+\|\omega_x\|_{L^\infty})$ satisfies the integral inequality

$$y'(t) \le y(0) + Ce^{Ct} + \int_0^t (1+y(s))\ ds$$

and by the integral form Gronwall's inequality and some elementary manipulations, we arrive at the bound $y(t) \le C_1 y(0) e^{C_2 t}$. This yields the desired bound on $\|\omega_x\|_\infty$. □

Remark 2.2.4 *If we choose our Biot-Savart law to be $u_x = -\omega$, then from a modification of the above proof we get an exponential upper bound for $\|\omega_x\|_{L^\infty}$. This is different from the 2D Euler equation, which suggests that* (2.7) *is a better analogue of the 2D Euler equation than* (2.8)*. Moreover the equation* (2.8) *also has different symmetry properties.*

Next we construct initial data ω_0 such that $\|\omega_x(\cdot,t)\|_{L^\infty}$ grows with double-exponential rate, proving the sharpness of the a-priori bound (2.9). The hyperbolic flow scenario is created in the following way: First, we require that the initial data ω_0 is odd with respect to the origin, and has compact support. By Proposition 3, the oddness is easily seen to be preserved by the evolution. Consequently, the velocity field (which is also an odd function) can be written as

$$u(x,t) = -x\int_0^\infty K\left(\frac{x}{y}\right)\frac{\omega(y,t)}{y}\ dy \quad (x>0), \tag{2.10}$$

where

$$K(s) := \frac{1}{s}\log\left|\frac{s+1}{s-1}\right|. \tag{2.11}$$

Note that the origin is a stagnant point of the flow for all times. By taking ω_0 to be positive on the right, the direction of the flow is towards the origin. More precisely, ω_0 is defined as follows (see Figure 2.2):

- Let ω_0 be supported on $[-1, 1]$, smooth and odd. Choose numbers $0 < x_1(0) < 2x_2(0) < 1$ such that $Mx_1(0) \leq x_2(0)$ where M will be determined later. Require that ω_0 is increasing on $[0, x_1(0)]$, decreasing on $[x_2(0), 1]$ and identically 1 on $[x_1(0), x_2(0)]$.

Using the earlier notation Φ_t for the flow map associated to (2.7), let

$$x_1(t) := \Phi_t(x_1(0))$$

$$x_2(t) := \Phi_t(x_2(0))$$

It is easy to see that the general structure of ω_0 will be preserved by the flow: For fixed t, $\omega(x, t)$ will be increasing on $[0, x_1(t)]$, decreasing on $[x_2(t), 1]$ and identically 1 on $[x_1(t), x_2(t)]$. In fact, since $u(x, t) \leq 0$ for $x \geq 0$, $x_1(t)$ and $x_2(t)$ will be moving towards the origin in time. We will show that the quantity $\dfrac{x_2(t)}{x_1(t)}$ increases double exponentially in time. This is sufficient to conclude the desired growth of $\|\omega_x(\cdot, t)\|_{L^\infty}$.

Theorem 2.2.5 *Assume our initial data is defined as above, then*

$$\log \frac{x_2(t)}{x_1(t)} \geq \log \frac{x_2(0)}{x_1(0)} \exp(Ct) \quad (t > 0),$$

for some positive constant C. As a consequence,

$$\log \|\omega_x(\cdot, t)\|_{L^\infty} \geq C_1 \exp(C_2 t) \quad (t > 0)$$

for some $C_1, C_2 > 0$.

Theorem 2.2.5 quickly follows from the following Lemma:

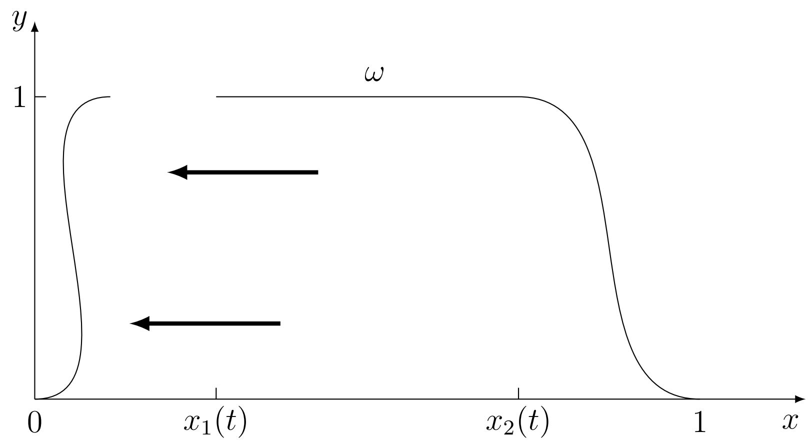


Figure 2.2 : Structure of $\omega(x,t)$.

Lemma 2.2.6 *Suppose $1 \geq x_2 \geq 8x_1$. There are universal constants C_0 and C_1 so that*

$$\frac{d}{dt}\left(\frac{x_2}{x_1}\right) \geq C_1 \frac{x_2}{x_1}\left(\log\left(\frac{x_2}{x_1}\right) - C_0\right).$$

Proof. First observe

$$\begin{aligned}\frac{d}{dt}\left(\frac{x_2}{x_1}\right) &= \frac{x_2' x_1 - x_1' x_2}{x_1^2} = \frac{u(x_2)x_1 - u(x_1)x_2}{x_1^2} = \frac{x_2}{x_1}\left(\frac{u(x_2)}{x_2} - \frac{u(x_1)}{x_1}\right)\\ &= \frac{x_2}{x_1}\int_0^1 \left[K\left(\frac{x_1}{y}\right) - K\left(\frac{x_2}{y}\right)\right]\frac{\omega(y)}{y}\,dy.\end{aligned}$$

We decompose the integral into 4 pieces which we will estimate separately:

$$\begin{aligned}&\int_0^1 \left[K\left(\frac{x_1}{y}\right) - K\left(\frac{x_2}{y}\right)\right]\frac{\omega(y)}{y}\,dy\\ =&\int_0^{2x_1} + \int_{2x_1}^{\frac{1}{2}x_2} + \int_{\frac{1}{2}x_2}^{2x_2} + \int_{2x_2}^1 \left[K\left(\frac{x_1}{y}\right) - K\left(\frac{x_2}{y}\right)\right]\frac{\omega(y)}{y}\,dy\\ =&I + II + III + IV.\end{aligned}$$

For I, we use $0 \leq \omega(y) \leq 1$ and $2x_1 \leq x_2 \leq 1$:

$$0 \leq I \leq \int_0^{2x_1} \frac{1}{x_1} \log \frac{(x_1+y)}{|x_1-y|}\, dy + \int_0^{2x_1} \frac{1}{x_2} \log \frac{(x_2+y)}{|x_2-y|}\, dy$$
$$= \frac{1}{x_1} 3x_1 \log 3 + \frac{1}{x_2}\left[2x_1 \log \frac{1+\frac{2x_1}{x_2}}{1-\frac{2x_1}{x_2}} + x_2 \log\left(1-\frac{2x_1}{x_2}\right) + x_2 \log\left(1+\frac{2x_1}{x_2}\right)\right]$$
$$\leq 3\log 3 + 2\log 2.$$

Using the fact that $K(s)$ is increasing in $[0,1)$ and decreasing in $(1,\infty]$ and that $\omega(y)=1$ for $y \in (2x_1, \frac{1}{2}x_2)$ we get

$$II = \int_{2x_1}^{\frac{1}{2}x_2} \left[K\left(\frac{x_1}{y}\right) - K\left(\frac{x_2}{y}\right)\right] \frac{\omega(y)}{y}\, dy \geq \int_{2x_1}^{\frac{1}{2}x_2} (2-\frac{1}{2}\log(3))\frac{1}{y}\, dy$$
$$= (2-\frac{1}{2}\log(3))\log\left(\frac{x_2}{x_1}\right) - C.$$

Using the positivity of K,

$$III = \int_{\frac{1}{2}x_2}^{2x_2} \left[K\left(\frac{x_1}{y}\right) - K\left(\frac{x_2}{y}\right)\right] \frac{\omega(y)}{y}\, dy \geq -\int_{\frac{1}{2}x_2}^{2x_2} K\left(\frac{x_2}{y}\right)\omega(y)\frac{1}{y}\, dy$$
$$\geq -\int_{\frac{1}{2}}^{2} \frac{1}{s^2} \log\frac{|s+1|}{|s-1|}\, ds \geq -C.$$

We estimate IV in the following way, using that $\omega(y) \leq 1$ and $\frac{x_1}{y} \leq \frac{x_2}{y} \leq 1$ for $2x_2 \leq y \leq 1$:

$$|IV| = \left|\int_{2x_2}^{1} \left[K\left(\frac{x_1}{y}\right) - K\left(\frac{x_2}{y}\right)\right] \frac{\omega(y)}{y}\, dy\right| \leq \int_{2x_2}^{1} \left[K\left(\frac{x_2}{y}\right) - K\left(\frac{x_1}{y}\right)\right] \frac{1}{y}\, dy$$
$$\leq \int_{2x_2}^{1} \frac{1}{x_2} \log\frac{y+x_2}{y-x_2}\, dy - \int_{2x_2}^{1} \frac{1}{x_1} \log\frac{y+x_1}{y-x_1}$$
$$= (i) - (ii).$$

We can compute (i) directly and get

$$(i) = \frac{1}{x_2}\log\frac{1+x_2}{1-x_2} + \log(1+x_2)(1-x_2) - 2\log(x_2) - 3\log(3).$$

Similarly, for (ii), we have

$$(ii) = \frac{1}{x_1}\log\frac{1+x_1}{1-x_1} + \log(x_1+1)(1-x_1) - 2\frac{x_2}{x_1}\log\frac{2x_2+x_1}{2x_2-x_1} - \log(2x_2+x_1)(2x_2-x_1).$$

Note that in the expressions for (i) and (ii), all terms can be bounded by universal constants except for $-2\log(x_2)$ and $\log(2x_2+x_1)(2x_2-x_1)$. However, using $x_1 < x_2$, we get

$$|IV| \leq C - 2\log(x_2) + \log(2x_2+x_1)(2x_2-x_1) = C + \log\left(4 - \left(\frac{x_1}{x_2}\right)^2\right) \leq C.$$

$\square$ The proof of Theorem 2.2.5 is now completed as follows: choose $M > 8$ so large such that $\frac{1}{2}\log(M) - C_0 \geq 0$. We have thus $\frac{1}{2}\log\left(\frac{x_2^0}{x_1^0}\right) - C_0 \geq 0$. From Lemma 2.2.6 it follows that $\frac{x_2(t)}{x_1(t)}$ is growing in time and that we have

$$\frac{d}{dt}\left(\frac{x_2}{x_1}\right) \geq \frac{C_1}{2}\frac{x_2}{x_1}\log\left(\frac{x_2}{x_1}\right),$$

or $\frac{d}{dt}\log\left(\frac{x_2}{x_1}\right) \geq \frac{C_1}{2}\log\left(\frac{x_2}{x_1}\right)$ for all times. This clearly implies that $\frac{x_2}{x_1}$ grows double-exponentially.

Remark 2.2.7 *In [30], the Biot-Savart law is decomposed into a main contribution and an error term. In our case* (2.10)*, the main contribution would be*

$$-x\int_x^\infty \frac{\omega(y)}{y}\,dy. \tag{2.12}$$

If we replace (2.10) *by* (2.12)*, then double-exponential growth of* $\frac{x_2}{x_1}$ *can be proven by a straightforward argument. In this case, the computation for the estimate in Lemma 2.2.6 becomes much easier.*

2.3 α-patch 1D model

In this section, we consider the 1D model equation

$$\omega_t + u\omega_x = 0 \tag{2.13}$$

with a different Biot-Savart law

$$u(x,t) = (-\Delta)^{-\alpha/2}\omega(x,t) = -c_\alpha \int_{\mathbb{R}} |y-x|^{-(1-\alpha)}\omega(y,t)\ dy, \qquad \alpha \in (0,1) \tag{2.14}$$

For convenience, we will assume the constant c_α associated with the fractional Laplacian is 1, and we write $\gamma = 1-\alpha$.

This problem has been studied in [19], where local existence and uniqueness results for smooth initial data are proven. From these, we can show that this equation preserves oddness and $u(0,t) = 0$ holds with odd initial datum. For odd data, we can write

$$u(x,t) = -\int_0^\infty k(x,y)\omega(y,t)\ dy \tag{2.15}$$

where $k(x,y) = |y-x|^{-\gamma} - |y+x|^{-\gamma}$. Note that $k(x,y) \geq 0$ for $x \neq y \in (0,\infty)$.

Following similar ideas as for 1D Euler, we specifiy our initial data ω_0 as follows:

- Pick $0 < x_1(0), x_2(0)$ with $Mx_1(0) < x_2(0)$. Let ω_0 be smooth, odd, $\omega_0(x) \geq 0$ for $x > 0$ and have its support in $[-2x_2(0), 2x_2(0)]$. $M > 1$ is to be chosen below. Let ω_0 moreover be bounded by 1, smoothly increasing in the interval $[0, x_1(0)]$ and $\omega_0 = 1$ between $x_1(0)$ and $x_2(0)$.

As long as the solution remains smooth, the general structure of the solution does not change. Let $x_1(t), x_2(t)$ be again the position of the particles starting at $x_1(0), x_2(0)$.

Theorem 2.3.1 *There exists a choice of $x_1(0), x_2(0), M$ and a time $T > 0$ such that the smooth solution of* (2.13) *for the above initial data cannot be continued beyond T. Provided the solution remains smooth on the time interval $[0,T)$, the particle starting at $x_1(0)$ reaches the origin at time $t = T$, i.e.*

$$\lim_{t\to T} x_1(t) = 0. \tag{2.16}$$

In this sense, the solution forms a "shock".

Remark 2.3.2 *In [19], the existence of blowup solutions to* (2.13) *is shown using energy methods. The advantage is that they are able to include a dissipation term. However, it is difficult to see the geometric blowup mechanism clearly using energy methods. Our proof for the inviscid case uses the dynamics of the solution and gives a more intuitive picture of the blowup, and is easily generalized to other even kernels having the same singular behavior.*

In the rest of this section, we will prove Theorem 2.3.1. So assume that for arbitrary choice of $x_1(0), x_2(0), M$, we have a smooth solution ω defined for all times $t \in [0, \infty)$.

First of all, we track the movement of the particle starting at $x_1(0)$, which is the following Lemma.

Lemma 2.3.3 *There exists a universal constant $M > 2$ so that if $Mx_1(t) \leq x_2(t)$, the velocity at $x_1(t)$ will satisfy*

$$u(x_1(t), t) \leq -Cx_1(t)^{1-\gamma}, \tag{2.17}$$

for some universal constant C.

Proof. Let $u_1 = u(x_1(t), t)$. Since $k, \omega \geq 0$ on $(0, \infty)$

$$\begin{aligned} -u_1 &\geq \int_{2x_1}^{x_2} k(x_1, y)\ dy \\ &= c_\gamma \left[-(x_2 + x_1)^{1-\gamma} + (x_2 - x_1)^{1-\gamma} + (3x_1)^{1-\gamma} - x_1^{1-\gamma} \right] \\ &= c_\gamma \left[(3^{1-\gamma} - 1)x_1^{1-\gamma} + (x_2 - x_1)^{1-\gamma} - (x_2 + x_1)^{1-\gamma} \right] \\ &= c_\gamma x_1^{1-\gamma} \left[(3^{1-\gamma} - 1) + \frac{1}{x_1^{1-\gamma}} \left((x_2 - x_1)^{1-\gamma} - (x_2 + x_1)^{1-\gamma} \right) \right] \end{aligned}$$

for some constant $c_\gamma > 0$. Note that $(3^{1-\gamma} - 1) > 0$. We can write

$$\frac{1}{x_1^{1-\gamma}}(x_2 - x_1)^{1-\gamma} - (x_2 + x_1)^{1-\gamma} = \frac{x_2^{1-\gamma}}{x_1^{1-\gamma}}\left[\left(1 - \frac{x_1}{x_2}\right)^{1-\gamma} - \left(1 + \frac{x_1}{x_2}\right)^{1-\gamma}\right]$$
$$=: \frac{x_2^{1-\gamma}}{x_1^{1-\gamma}} f(x_1/x_2).$$

There exists a constant $C > 0$ with $|f(x_1/x_2)| \leq C|x_1/x_2|$ for $|x_1/x_2| \leq 1/2$, and so

$$-u_1 \geq c_\gamma x_1^{1-\gamma}\left[(3^{1-\gamma} - 1) - CM^{-\gamma}\right]$$

if $Mx_1(t) \leq x_2(t)$. Now choose M large enough so that $CM^{-\gamma}$ is smaller than the number $\frac{1}{2}(3^{1-\gamma} - 1)$. □ This estimate of velocity field will lead to a blowup in finite time, provided we can show $Mx_1(t) \leq x_2(t)$. More precisely,

$$\frac{d}{dt}x_1(t) = u(x_1) \leq -Cx_1^{1-\gamma},$$

implying

$$x_1 \leq C(x_1(0)^\gamma - Ct)^{\frac{1}{\gamma}}.$$

This shows that no later than $T_0 := C^{-1}x_1(0)^\gamma$, the particle $x_1(t)$ will reach the origin, and the solution cannot be continued smoothly. Note that T_0 does not depend on $x_2(0)$.

It remains therefore to control the motion of $x_2(t)$, concluding the proof.

Lemma 2.3.4 *For $x_2(0)$ large enough, $Mx_1(t) < x_2(t)$ for $t \in [0, T_0)$.*

Proof. We write $u(x_2(t), t) = u_2$. Observe that the support of $\omega(\cdot, t)$ is always contained in $[-2x_2(0), 2x_2(0)]$ because of $u(x,t) \leq 0$ for $x > 0$.

Next we find an upper bound on u_2:

$$|u_2(t)| \leq \int_{-2x_2(0)}^{2x_2(0)} |y - x|^{-\gamma} \leq Cx_2(0)^{1-\gamma}. \tag{2.18}$$

Hence,

$$x_2(t) \geq x_2(0) - \int_0^{T_0} |u_2(s)|\, ds \geq x_2(0)(1 - Cx_2(0)^{-\gamma}T_0). \tag{2.19}$$

Now choose $x_2(0)$ so large that $Mx_1(0) < x_2(0)(1 - Cx_2(0)^{-\gamma}T_0)$. But then

$$Mx_1(t) \leq Mx_1(0) < x_2(0)(1 - Cx_2(0)^{-\gamma}T_0) \leq x_2(t),$$

giving the statement of the Lemma. □

2.4 Blow-up for a 1D model system for the 3D Euler equations

In this section, we contribute to the analysis of a recently discovered hyperbolic flow scenario mentioned earlier for singularity formation in solutions of 3D Euler equation. The 3D axisymmetric Euler equation with swirl is given by

$$\partial_t \left(\frac{\omega^\theta}{r}\right) + u^r \left(\frac{\omega^\theta}{r}\right)_r + u^z \left(\frac{\omega^\theta}{r}\right)_z = \left(\frac{(ru^\theta)^2}{r^4}\right)_z, \tag{2.20}$$

$$\partial_t(ru^\theta) + u^r(ru^\theta)_r + u^z(ru^\theta)_z = 0, \tag{2.21}$$

where u^r and u^z can be calculated via

$$u^r = \frac{\psi_z}{r},\ \ u^z = -\frac{\psi_r}{r}, \tag{2.22}$$

and the stream function ψ satisfies the elliptic equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial z^2} = \omega. \tag{2.23}$$

One can write u^r and u^z in terms of ω by computing the Green's function of the above elliptic PDE; more details can be found on [33].

The numerical simulations performed in [23] consider fluid contained in an infinite cylinder with periodic boundary conditions in z and no flux condition at the boundary of the cylinder. The initial data is given by non-zero swirl u^θ, which is odd in z, while the angular vorticity is originally zero. For a particular example of such initial data, very fast growth of angular vorticity is observed at a ring of hyperbolic points defined by the boundary of the cylinder and $z = 0$. As the first step towards rigorous analysis of this scenario, a 1D model inspired by the numerics has been proposed in [23, 22]. We will refer to this 1D model as Hou-Luo (HL) model. The HL model is given by

$$\omega_t + u\omega_x = \theta_x \tag{2.24}$$

$$\theta_t + u\theta_x = 0 \tag{2.25}$$

$$u_x = H\omega, \tag{2.26}$$

where H is the Hilbert transform and the space domain is taken to be periodic, $\mathbb{S}^1$ (the $\mathbb{R}^1$ setting can also be considered). One should think of the x coordinate as corresponding to the z direction in the original equation. Equivalently, if ω is mean zero over the period, we can write the Biot-Savart law for u as

$$u(x,t) = k * \omega(x,t) \quad \text{where} \quad k(x) = \frac{1}{\pi}\log|x|. \tag{2.27}$$

In the periodic case, ω in the formula above is extended to all real line where the convolution is applied. The convergence of the integral is understood in the appropriate principal value sense. In [3], finite time blow up is shown for (2.24)-(2.26) for a large class of smooth initial data.

There has been other work motivated by Hou-Luo computations and relevant to understanding the hyperbolic boundary blow up scenario. Kiselev and Sverak [30] show very fast (in fact, optimal) growth of $\nabla\omega$ in solutions of 2D Euler equation in a geometry related to the Hou-Luo scenario. Choi, Kiselev and Yao [4] analyzed a 1D model related to the HL model, but with a simplified Biot-Savart law inspired by [30]. They established finite time blow up for a broad class of initial data. Hou and Liu [21] have described the blow up solutions in the CKY model in more detail, and showed that these solutions possess self-similar structure.

In this section, our first theorem is the generalization of the results of [3] to the model with the following adjusted choice of Biot-Savart law:

$$u(x,t) = k * \omega(x,t) \quad \text{where} \quad k(x) = \frac{1}{\pi} \log \frac{|x|}{\sqrt{x^2 + a^2}}. \tag{2.28}$$

It has been observed already in [23, 3] that the kernel (2.28) appears naturally in the reduction of the 3D Euler equation to the 1D model of hyperbolic blow up scenario. Nevertheless, the simpler kernel (2.27) has been considered as the first step. The difference between (2.28) and the original choice (2.27) is smooth, so one can expect that the properties of the equations should be similar. However, the actual proof of finite time blow up in [3] relies on fairly fine properties of the Biot-Savart kernel, so the extension to (2.28) is far from immediate. In Section 2.6, we prove finite time blow up of solutions to the system (2.24) and (2.25) with law (2.28). While we will be able to follow the framework of the blow up proof developed in [3], many new estimates will be needed. Similarly to [3], the proof shows finite time blow up for a rather wide class of the initial data.

For our second main result, we prove that the solutions to (2.24), (2.25) with even more general kernels in the Biot-Savart law exhibit finite time blow up as well. We will modify (2.27) by adding a smooth function which preserves the symmetries of (2.24), (2.25) (and of the initial data). The details will appear in Section 2.7. To prove blow up, roughly speaking, we isolate the "leading term" of dynamics that leads to blow up and persists even with a more general Biot-Savart law. The proof is quite different from the first result: the proof of finite time blow up for the Biot-Savart law (2.28) relies, in the spirit of [3], on algebraic estimates which show that certain key quantities are positive definite. On the other hand, the more general blow up stability result is proved in a perturbative fashion, utilizing a global bound on the L^1 norm of vorticity. It may appear that our second result includes the first one, but it is not literally true as in the second case we have to work with a much more restrictive class of initial data.

One can think of our results as strengthening the case for studying the hyperbolic blow up scenario for the 3D Euler equation. By proving singularity formation for more general Biot-Savart laws, one can view the blow up of (2.24)-(2.26) as a robust phenomenon not dependent on the fine structure of the model. This may help to build a base for the next step - rigorous analysis of the higher dimensional problems.

2.5 Derivation of the Model Equations

To obtain a simplified model of (2.20),(2.21) the first step is to consider reduction to the 2D inviscid Boussinesq equations. This system on a half plane $\mathbb{R} \times [0, \infty)$ is given by

$$\omega_t + u^x \omega_x + u^y \omega_y = \theta_x \tag{2.29}$$

$$\theta_t + u^x \theta_x + u^y \theta_y = 0$$

where $u = (u^x, u^y)$ and is derived from ω by the usual 2D Euler Biot-Savart law $u = \nabla^\perp (-\Delta_D)^{-1} \omega$, with $\nabla^\perp = (\partial_2, -\partial_1)$ and Δ_D Dirichlet Lapalcian. The system is classical and describes motion of 2D ideal buoyant fluid in the field of gravity. The global regularity of solutions to 2D inviscid Boussinesq system is also open. This problem is featured in the Yudovich's list of "eleven great problems of mathematical hydrodynamics" [37].

The fact that 2D inviscid Boussinesq equation is a close proxy for 3D axisymmetric Euler equation, at least away from the axis $r = 0$, is well known (see e.g. [**?**]). Indeed, if in (2.20), (2.21), (2.22), (2.23) we re-label $\omega^\theta/r \equiv \omega$, $ru^\theta \equiv \theta$, $r = y$, $z = x$, and set $r = 1$ in the coefficients, we obtain (2.29). Since in the Hou-Luo scenario, the fastest growth of vorticity is observed at the boundary of the cylinder $r = 1$, and in particular away from the axis, the analogy should apply. In [3], to derive the HL model, the authors consider the system (2.29) in the half-plane and restrict the system to the boundary $\{(x, y) : y = 0\}$ so we have $u^y = 0$. To derive a closed form Biot-Savart law for the 1D system, ω is assumed to be constant in y in a boundary layer close to the boundary of width $a > 0$, and zero elsewhere. Such assumption leads to a law defined by convolution with the following kernel:

$$k(x_1) = \int_0^a \frac{\partial}{\partial x_2}\bigg|_{x_2=0} G_D((x_1, x_2), (0, y_2))\, dy_2$$

where G_D is the Green's function of Laplacian in the upper half-plane with Dirichlet boundary conditions. We know that

$$G_D(z, w) = \frac{1}{2\pi} \log |z - w| - \frac{1}{2\pi} \log |z - w^*|, \quad w^* = (w_1, -w_2),$$

and by a simple calculation one gets

$$u(x) = \tilde{k} * \omega(x), \tag{2.30}$$

where

$$\tilde{k}(x) = \frac{1}{\pi} \log \frac{|x|}{\sqrt{x^2 + a^2}}. \tag{2.31}$$

In [3], the authors discard the smooth part of $\tilde{k}$ (namely, $\frac{1}{\pi}\log(\sqrt{x^2+a^2})$). In this paper we will consider $\tilde{k}$ directly or even more general perturbed kernels.

While the boundary layer assumption is strong and clearly does not hold for the higher dimensional case precisely, it is noted in [23] that the numerical simulations of the full 3D Euler equation and of the reduced 1D model exhibit striking similarity. Based on the numerical results about potential singularity profile for 3D axisymmetric Euler equation ([23]), we are particularly interested in the case when ω is periodic in x (formerly z) variable and will treat this case in the next section. The periodic assumption is not crucial; in the appendix we will outline the arguments which adjust the proof to the real line case.

We complete this section by stating a local well-posedness and a conditional regularity result that we will need later. The system (2.24), (2.25), (2.28) is locally well posed and possesses a Beale-Kato-Majda type criterion. We formalize this below.

Proposition 4 *(Local Existence and Blow Up Criteria) Suppose* $(\omega_0,\theta_0)\in H^m(\mathbb{S}^1)\times H^{m+1}(\mathbb{S}^1)$ *where* $m\geq 2$*. Then there exists* $T=T(\omega_0,\theta_0)>0$ *such that there exists a unique classical solution* (ω,θ) *of* (2.24), (2.25), (2.28) *and* $(\omega,\theta)\in C([0,T];H^m\times H^{m+1})$*. In particular, if* T^* *is the maximal time of existence of such solution then*

$$\lim_{t\nearrow T^*}\int_0^t \|u_x(\cdot,\tau)\|_{L^\infty}\,d\tau=\infty. \tag{2.32}$$

The proof of the proposition is relatively standard. A short discussion of this topic can be found in [3]. A similar statement is also proved in detail in [4]. An analogous result will apply to the systems with more general Biot-Savart law that we will introduce later.

2.6 The Modified Hou-Luo Kernel: Periodic Case

In this section, we prove finite time blow up of the system with the kernel given by (2.28) and periodic initial data. From now on, we will refer to the kernel given by (2.27) as the Hou-Lou kernel, and to the kernel (2.28) as the modified Hou-Luo kernel. We will denote the velocity corresponding to the Hou-Luo kernel as u_{HL}. In addition, we will consider solutions with mean zero vorticity. A straightforward calculation shows that the mean zero property is conserved for all times for regular solutions.

Let us start by deriving a simpler expression for the Biot-Savart law in the case when the solution is periodic with period L. Our computations will be formal, ignoring the lack of absolute convergence of the integrals involved; they can be made fully rigorous using standard regularization and approximation procedures at infinity. We periodize the kernel associated with our velocity

$$
\begin{aligned}
u(x,t) &= \frac{1}{\pi}\int_{-\infty}^{\infty}\omega(y)\log\frac{|x-y|}{\sqrt{(x-y)^2+a^2}}\,dy = \frac{1}{\pi}\sum_{n\in\mathbb{Z}}\int_0^L \omega(y)\log\frac{|x-y+nL|}{\sqrt{(x-y+nL)^2+a^2}}\,dy\\
&= \frac{1}{\pi}\sum_{n\in\mathbb{Z}}\int_0^L \omega(y)\log|x-y+nL|\,dy\\
&- \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_0^L \omega(y)\left(\log((x+ia-y)+nL)+\log((x-ia-y)+nL)\right)\,dy\\
&= \frac{1}{\pi}\int_0^L \omega(y)\log\left|(x-y)\prod_{n=1}^{\infty}\left(1-\frac{(\mu(x-y))^2}{\pi^2n^2}\right)\right|\,dy\\
&\quad - \frac{1}{2\pi}\int_0^L \omega(y)\log\left|(x+ia-y)\prod_{n=1}^{\infty}\left(1-\frac{(\mu(x+ia-y))^2}{\pi^2n^2}\right)\right|\,dy\\
&\quad - \frac{1}{2\pi}\int_0^L \omega(y)\log\left|(x-ia-y)\prod_{n=1}^{\infty}\left(1-\frac{(\mu(x-ia-y))^2}{\pi^2n^2}\right)\right|\,dy\\
&= \frac{1}{\pi}\int_0^L \omega(y)\log|\sin[\mu(x-y)]|\,dy - \frac{1}{2\pi}\int_0^L \omega(y)\log|\sin(\mu(x-ia-y))\sin(\mu(x+ia-y))|\,dy
\end{aligned}
$$

where we set $\mu = \pi/L$. In the last step we used the fact that

$$f(z) = z\prod_{n=1}^{\infty}\left(1 - \left(\frac{\mu z}{\pi n}\right)^2\right)$$

is an entire function, its zeroes coincide with those of $\sin(\mu z)$, and $f'(z)|_{z=0} = 1$. A quick computation leads to

$$\begin{aligned}\sin\mu(x-ia)\sin\mu(x+ia) &= \frac{e^{i\mu(x-ia)} - e^{-i\mu(x-ia)}}{2i}\,\frac{e^{i\mu(x+ia)} - e^{-i\mu(x+ia)}}{2i}\\ &= \frac{e^{2\mu a} + e^{-2\mu a}}{4} - \frac{e^{2i\mu x} + e^{-2i\mu x}}{4}\\ &= \frac{1}{2}(\cosh(2\mu a) - \cos(2\mu x)) = \frac{1}{2}(\cosh(2\mu a) - 1) + \sin^2(\mu x)\end{aligned}$$

By a slight abuse of notation let us rename the quantity $(1/2)(\cosh(2\mu a) - 1)$ to be our new $a > 0$. We generally think of a as being small, though our estimates later will be true for arbitrary positive a. Note that the new a has dimension of *length*^{2}. Combining the above calculations, our velocity u can be now written as

$$u(x) = \frac{1}{2\pi}\int_0^L \omega(y)\left(\log|\sin^2[\mu(x-y)]| - \log|\sin^2[\mu(x-y)] + a|\right)\,dy. \qquad (2.33)$$

The main result of this section is the following

Theorem 2.6.1 *There exist initial data with mean zero vorticity such that solutions to* (2.24) *and* (2.25)*, with velocity given by* (2.33) *blow up in finite time. That is, there exists a time T^* such that we have* (2.32)*.*

We will consider the following type of initial data:

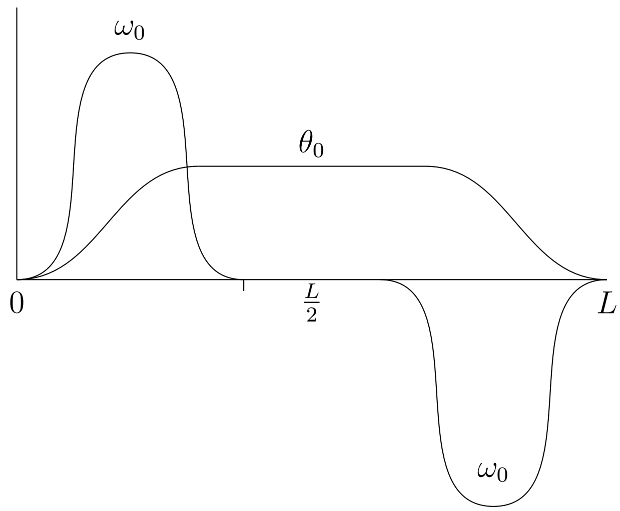
1. θ_{0x}, ω_0 smooth, odd, periodic with period L

2. $\theta_{0x}, \omega_0 \geq 0$ on $[0, \frac{1}{2}L]$.

3. $\theta_0(0) = 0$

4. $\|\theta_0\|_\infty \leq M$

This can be visualized as follows:



Here we will need the following lemma to show that the solution will have a similar structure as the above graph.

Lemma 2.6.2 *Suppose* (θ, ω) *is the solution to the system* (2.24)(2.25)(2.33) *described in Proposition 4. Then all the properties* (a)(b)(c)(d) *for our choice of initial data will be propagated in time up until possible blow up time.*

Proof. We provide a sketch the proof. From **Proposition 4** one has the local well-posedness for our system((2.24)(2.25)(2.33)), specifically the solution is unique. We can directly verify that $\theta(-x,t), -\omega(-x,t)$ or $\theta(x+L,t), \omega(x+L,t)$ are also solutions of our system. By assumed properties of the initial data and the uniqueness of solutions, we obtain that these solutions coincide with $\theta(x,t), \omega(x,t)$. This implies that ω and θ_x are odd and periodic with period L as long as ω_0 and θ_{0x} are odd and periodic.

Meanwhile, by the transport structure (2.25) and the non-positivity of $u(x)$ for $0 < x < \frac{L}{2}$, we get that $\theta_x \geq 0$ as long as the solution is smooth. As a consequence, $\omega \geq 0$ from the equation (2.24). Similarly, the properties $(c)(d)$ are also consequences of the transport structure. □

The proof of singularity formation will follow by contradiction. The overall plan of the proof is based on finding appropriate functional of the solutions that blows up in finite time. The motivation for the choice of initial data above is the following possible blow up scenario: we will have $u \leq 0$ on $[0, L/2]$ and so θ will be pushed towards the origin by the flow increasing its derivative. This also causes ω to be pushed towards the origin while increasing its L^∞ norm until there is velocity gradient blow up at the origin. The argument is similar in spirit to [10] where the authors consider the quantity

$$\int_0^{x_0} \frac{\omega(x,t)}{x} dx.$$

Due to the periodic structure, the more natural quantity to monitor is, similarly to [3],

$$\int_0^{\frac{L}{2}} \theta(x,t) \cot(\mu x) dx.$$

Since $x = 0$ is the stagnant point of the flow for all times while solution remains smooth, and since $\theta_0(0) = 0$, blow up of the above integral implies loss of regularity of the solution.

We begin with derivation of some useful estimates for $u(x)$. Using that, due to our symmetry assumptions, our initial data is also odd with respect to $x = \frac{L}{2}$, we can write u as

$$u(x) = \frac{1}{\pi}\left[\int_0^{L/2} + \int_{L/2}^{L}\right]\omega(y)\left(\log|\sin^2[\mu(x-y)]| - \log|\sin^2[\mu(x-y)] + a|\right)\,dy$$
$$= \frac{1}{\pi}\int_0^{L/2}\left(\log\left|\frac{\sin^2\mu(x-y)}{\sin^2\mu(x+y)}\right| + \log\left|\frac{\sin^2\mu(x+y)+a}{\sin^2\mu(x-y)+a}\right|\right)\omega(y)\,dy.$$

Define

$$F(x,y,a) = \frac{\tan\mu y}{\tan\mu x}\left(\log\left|\frac{\sin^2\mu(x-y)}{\sin^2\mu(x+y)}\right| + \log\left|\frac{\sin^2\mu(x+y)+a}{\sin^2\mu(x-y)+a}\right|\right). \tag{2.34}$$

Then the Biot-Savart law (2.33) can be written in the following form, which will be handy in the proof:

$$u(x)\cot(\mu x) = \frac{1}{\pi}\int_0^{L/2} F(x,y,a)\omega(y)\cot(\mu y)\,dy \tag{2.35}$$

The majority of this section will be devoted to establishing properties of F that will allow for a proof of finite time blow up analogous to the one for HL model in [3]. These properties are contained in the following lemma.

Lemma 2.6.3 **(a)** *There exists a positive constant C depending on a such that $F(x,y,a) \leq -C < 0$ for $0 < x < y < L/2$.*

(b) *For any $0 < y < x < \frac{L}{2}$, $F(x,y,a)$ is increasing in x.*

(c) *For any $0 < x, y < \frac{L}{2}$, $\cot(\mu y)(\partial_x F)(x,y,a) + \cot(\mu x)(\partial_x F)(y,x,a)$ is positive.*

Note that F is not symmetric in x and y. Define

$$K(x,y) = \frac{\tan\mu y}{\tan\mu x}\log\left|\frac{\sin\mu(x+y)}{\sin\mu(x-y)}\right|,$$

then

$$F(x,y,a) = -2K(x,y) + \frac{\tan\mu y}{\tan\mu x}\log\left|\frac{\sin^2\mu(x+y)+a}{\sin^2\mu(x-y)+a}\right|. \tag{2.36}$$

The term $K(x, y)$ arises from the original HL model and one can view it as the main contributor from F towards the blow up. In order to show Lemma 2.6.3, we first need the following technical lemma for $K(x, y)$.

Lemma 2.6.4 *For simplicity, let us write $K(x, y)$ in the following form:*

$$K(x,y) = s \log \left| \frac{s+1}{s-1} \right|, \quad \textit{with} \quad s = \frac{\tan(\mu y)}{\tan(\mu x)}. \tag{2.37}$$

Then it has the following properties:

(a) $K(x, y) \geq 0$ *for all* $x, y \in (0, \frac{1}{2}L)$ *with* $x \neq y$

(b) $K(x, y) \geq 2$ *and* $K_x(x, y) \geq 0$ *for all* $0 < x < y < \frac{1}{2}L$

(c) $K(x, y) \geq 2s^2$ *and* $K_x(x, y) \geq 0$ *for all* $0 < y < x < \frac{1}{2}L$

The detailed proof of Lemma 2.6.4 can be found in [3], Lemma 4.1.

Proof of Lemma (2.6.3)**(a)**

First, it is easy to see that F is non-positive. Indeed

$$\left| \frac{\sin^2 \mu(x-y)}{\sin^2 \mu(x+y)} \right| \left| \frac{\sin^2 \mu(x+y) + a}{\sin^2 \mu(x-y) + a} \right| = \left| \frac{1 + \frac{a}{\sin^2 \mu(x+y)}}{1 + \frac{a}{\sin^2 \mu(x-y)}} \right| \leq 1 \tag{2.38}$$

because $\sin^2 \mu(x-y) \leq \sin^2 \mu(x+y)$ if $x, y \in [0, L/2]$.

For the better upper bound, we first consider the region $0 < x < y < L/4$. For the region $L/4 < x < y < L/2$, if we take $x^* = \frac{L}{2} - x$, $y^* = \frac{L}{2} - y$, then $0 < y^* < x^* < L/4$, and relabelling of the variables brings the kernel to the original form. This means the argument for this region would follow from that for the region $0 < x < y < L/4$. We divide our estimate of this region into four separate cases. Let $a^* = \min\{a, \frac{1}{16}\}$.

Case 1: $\frac{\sqrt{a^*}}{\pi} L = \frac{\sqrt{a^*}}{\mu} < x < y < L/4$

In this domain we have $\sin\mu y > \sin\mu x > \frac{\sin(\frac{\pi}{4})}{\frac{\pi}{4}}\mu x > \frac{1}{\sqrt{2}}\mu x > \frac{1}{\sqrt{2}}\sqrt{a^*}$, $\cos\mu x > \cos\mu y > \frac{1}{\sqrt{2}}$, hence

$$\sin^2\mu(x-y) = \sin^2\mu(x+y) - 4\sin\mu x\sin\mu y\cos\mu x\cos\mu y < \sin^2\mu(x+y) - a^*,$$

so

$$F(x,y,a) \le \log\left|\frac{\sin^2\mu(x-y)}{\sin^2\mu(x+y)}\right| + \log\left|\frac{\sin^2\mu(x+y)+a^*}{\sin^2\mu(x-y)+a^*}\right| = \log\left|\frac{1+\frac{a^*}{\sin^2\mu(x+y)}}{1+\frac{a^*}{\sin^2\mu(x-y)}}\right| \tag{2.39}$$

$$\le \log\left|\frac{1+\frac{a^*}{\sin^2\mu(x+y)}}{1+\frac{a^*}{\sin^2\mu(x+y)-a^*}}\right| \le -C_0(a) < 0 \tag{2.40}$$

where $C_0(a)$ is a positive constant independent of x, y. In the last step we use the fact that the function $\left(1+\dfrac{a^*}{z}\right)\left(1+\dfrac{a^*}{z-a^*}\right)^{-1} = 1 - \dfrac{(a^*)^2}{z^2}$ is increasing in z for $a^* < z < 1$ and fixed a^*.

Case 2: $0 < x < y < \frac{\sqrt{a^*}}{\mu} < L/4$

From Lemma 2.6.4 (b), we know

$$-4 \ge -2K(x,y) = \frac{\tan\mu y}{\tan\mu x}\log\left|\frac{\sin^2\mu(x-y)}{\sin^2\mu(x+y)}\right|. \tag{2.41}$$

so if we can show the contribution from the other part of $F(x,y,a)$ is bounded above by some constant less than 4, we are done. Expanding, we have that second term in (2.36) is equal to

$$\frac{\tan\mu y}{\tan\mu x}\log\left|\frac{\sin^2\mu x\cos^2\mu y + 2\sin\mu x\cos\mu y\sin\mu y\cos\mu x + \sin^2\mu y\cos^2\mu x + a}{\sin^2\mu x\cos^2\mu y - 2\sin\mu x\cos\mu y\sin\mu y\cos\mu x + \sin^2\mu y\cos^2\mu x + a}\right|. \tag{2.42}$$

Since $0 < y < \frac{\sqrt{a^*}}{\mu} \le \frac{\sqrt{a}}{\mu}$, we know $\sin^2\mu y\cos^2\mu x < \sin^2\sqrt{a}\cdot 1 < a$. Then we have that (2.42) is bounded above by

$$\frac{\tan\mu y}{\tan\mu x}\log\left|\frac{\sin^2\mu x\cos^2\mu y + 2\sin\mu x\cos\mu y\sin\mu y\cos\mu x + 2\sin^2\mu y\cos^2\mu x}{\sin^2\mu x\cos^2\mu y - 2\sin\mu x\cos\mu y\sin\mu y\cos\mu x + 2\sin^2\mu y\cos^2\mu x}\right| = s\log\left|\frac{2s+\frac{1}{s}+2}{2s+\frac{1}{s}-2}\right| \tag{2.43}$$

where $s = \dfrac{\tan \mu y}{\tan \mu x}$. As a function of s, by direct calculation we find the derivative of the right hand side of (2.43) is

$$\frac{4s - 8s^3}{1 + 4s^4} + \log\left|\frac{2s + \frac{1}{s} + 2}{2s + \frac{1}{s} - 2}\right|. \tag{2.44}$$

By taking the derivative of (2.44), we find the second derivative of (2.43) is

$$-\frac{8(4s^4 + 4s^2 - 1)}{(4s^4 + 1)^2},$$

which is negative for $s \geq 1$. And we know that

$$\lim_{s\to\infty}\left(\frac{4s - 8s^3}{1 + 4s^4} + \log\left|\frac{2s + \frac{1}{s} + 2}{2s + \frac{1}{s} - 2}\right|\right) = 0$$

which means the right hand side of (2.43) is increasing in s for $s > 1$ and

$$\lim_{s\to\infty} s\log\left|\frac{2s + \frac{1}{s} + 2}{2s + \frac{1}{s} - 2}\right| = 2.$$

Case 3: $\frac{\sqrt{a^*}}{2\mu} < x < \frac{\sqrt{a^*}}{\mu} < y < L/4$

In this case, because we know that x is bounded away from zero, we have $s = \dfrac{\tan \mu y}{\tan \mu x} \leq C_1(a)$ for some constant depending on a. Also, $\cos^2 \mu y \sin^2 \mu x \leq 1 \cdot \sin^2 \sqrt{a} \leq a$. Then (2.42) is bounded above by

$$s\log\left|\frac{s + 2 + \frac{2}{s}}{s - 2 + \frac{2}{s}}\right|. \tag{2.45}$$

Similarly to the previous case, the second derivative of (2.45) is negative for $s > 1$ and the limit of the first derivative of (2.45) as s goes to infinity is zero, which means (2.45) monotonically (while $s \geq 1$) increases to 4 as $s \to \infty$. However, since s is bounded above, the expression (2.45) can be bounded by some constant $C_2(a)$ which is strictly less than 4. On the other hand, note that (2.41) still applies.

Case 4: $0 < x < \frac{\sqrt{a^*}}{2\mu} < \frac{\sqrt{a^*}}{\mu} < y < L/4$

On the set $A = \{(x,y) : 0 \le x \le \frac{\sqrt{a^*}}{2\mu}, \frac{\sqrt{a^*}}{\mu} \le y \le L/4\}$, $F(x,y,a)$ is a continuous negative function (since $|x-y|$ has a positive lower bound and points where $x = 0$ are removable singularities). Since $F \neq 0$ on A and A is compact, F achieves a maximum $C_3(a)$ which is strictly less than 0.

This completes the analysis for the region $0 < x < y < L/4$, and therefore for the region $L/4 < x < y < L/2$ by symmetry considerations. Now, we are left the domain $0 < x < L/4 < y < L/2$.

This case is simpler and the analysis is divided in the following two cases. First, suppose $0 < L/8 < x < L/4 < y < 3L/8 < L/4$ Then $\frac{3\pi}{8} < \mu(x+y) < \frac{5\pi}{8}$ and $0 < \mu(y-x) < \frac{\pi}{4}$ so there exists $\epsilon > 0$ such that $\sin^2 \mu(x+y) \ge \frac{1}{2} + \epsilon$. However, $\sin^2 \mu(x-y) < \frac{1}{2}$. From this, we get $\sin^2 \mu(x+y) - \sin^2 \mu(x-y) \ge \epsilon^*$ for some constant ϵ^*, which means (2.39) follows if we replace the a^* by ϵ^*. Then we get the desired estimate. If x and y are not in this region, there exists a constant $c > 0$ such that $y - x > c > 0$, then again by the same argument as in the **Case 4** and we get the desired inequality.

This completes the proof of (a). □

Proof of 2.6.3(b) We compute directly and get

$$\begin{aligned}
\cot(\mu y)(\partial_x F)(x,y,a) &= -\mu\csc^2(\mu x)\left(\log\left(\frac{\sin^2\mu(x-y)}{\sin^2\mu(x+y)}\right)+\log\left(\frac{\sin^2\mu(x+y)+a}{\sin^2\mu(x-y)+a}\right)\right)\\
&+\mu\cot(\mu x)\left[\frac{2\sin\mu(x-y)\cos\mu(x-y)}{\sin^2\mu(x-y)}-\frac{2\sin\mu(x-y)\cos\mu(x-y)}{\sin^2\mu(x-y)+a}\right]\\
&-\mu\cot(\mu x)\left[\frac{2\sin\mu(x+y)\cos\mu(x+y)}{\sin^2\mu(x+y)}-\frac{2\sin\mu(x+y)\cos\mu(x+y)}{\sin^2\mu(x+y)+a}\right]\\
&= -\mu\csc^2(\mu x)\left(\log\left(\frac{\sin^2\mu(x-y)}{\sin^2\mu(x+y)}\right)+\log\left(\frac{\sin^2\mu(x+y)+a}{\sin^2\mu(x-y)+a}\right)\right)\\
&+\mu\cot(\mu x)\left[\frac{2a\sin\mu(x-y)\cos\mu(x-y)}{\sin^2\mu(x-y)(\sin^2\mu(x-y)+a)}-\frac{2a\sin\mu(x+y)\cos\mu(x+y)}{\sin^2\mu(x+y)(\sin^2\mu(x+y)+a)}\right]\\
&= I+II.
\end{aligned}$$

The term I, by the same calculation as (2.38), is positive. The term II, when $x>y$, can be expressed as

$$\cot(\mu x)(g(x-y)-g(x+y)),$$

where $g(t)=\frac{\cos(\mu t)}{\sin(\mu t)(\sin^2(\mu t)+a)}$. It is easy to see that whenever $0<y<x<\frac{L}{2}$, $\cos\mu(x-y)\geq\cos\mu(x+y)$, $0\leq\sin\mu(x-y)\leq\sin\mu(x+y)$. This means that $g(x-y)\geq g(x+y)$, which implies $II\geq 0$. This completes the proof of (b). □

Proof of 2.6.3(c)

Now, for the final part of the lemma. First of all, we set

$$
\begin{aligned}
G(x,y,a) &= \cot(\mu y)(\partial_x F)(x,y,a) + \cot(\mu x)(\partial_x F)(y,x,a) \\
&= -\mu(\csc^2(\mu x) + \csc^2(\mu y)) \left[\log\left(\frac{\sin^2\mu(x-y)}{\sin^2\mu(x+y)}\right) + \log\left(\frac{\sin^2\mu(x+y)+a}{\sin^2\mu(x-y)+a}\right)\right] \\
&+ \mu(\cot(\mu x) - \cot(\mu y)) \frac{2a\sin\mu(x-y)\cos\mu(x-y)}{\sin^2\mu(x-y)(\sin^2\mu(x-y)+a)} \\
&- \mu(\cot(\mu x) + \cot(\mu y)) \frac{2a\sin\mu(x+y)\cos\mu(x+y)}{\sin^2\mu(x+y)(\sin^2\mu(x+y)+a)}. \\
&= -\mu(\cot^2(\mu x) + \cot^2(\mu y) + 2) \left[\log\left(\frac{\sin^2\mu(x-y)}{\sin^2\mu(x+y)}\right) + \log\left(\frac{\sin^2\mu(x+y)+a}{\sin^2\mu(x-y)+a}\right)\right] \\
&- \mu\frac{2a\cos\mu(x-y)}{(\sin^2\mu(x-y)+a)\sin(\mu x)\sin(\mu y)} - \mu\frac{2a\cos\mu(x+y)}{(\sin^2\mu(x+y)+a)\sin(\mu x)\sin(\mu y)} \\
&= -\mu(\cot^2(\mu x) + \cot^2(\mu y) + 2) \left[\log\left(\frac{\sin^2\mu(x-y)}{\sin^2\mu(x+y)}\right) + \log\left(\frac{\sin^2\mu(x+y)+a}{\sin^2\mu(x-y)+a}\right)\right] \\
&- 2\mu\cot(\mu x)\cot(\mu y)\left[\frac{a}{\sin^2\mu(x-y)+a} + \frac{a}{\sin^2\mu(x+y)+a}\right] \\
&- 2\mu\left[\frac{a}{\sin^2\mu(x-y)+a} - \frac{a}{\sin^2\mu(x+y)+a}\right]
\end{aligned}
$$

Now our aim is to prove the positivity of $G(x,y,a)$. Notice that when $a=0$, $G(x,y,a)=0$, as a consequence, to prove the positivity of $G(x,y,a)$, the only thing we need to show is that this function is increasing in a for any x, y in the domain. On the other hand,

$$\begin{aligned}
\frac{1}{\mu}\partial_a G(x,y,a) &= (\cot^2(\mu x) + \cot^2(\mu y) + 2)\left[\frac{1}{\sin^2\mu(x-y)+a} - \frac{1}{\sin^2\mu(x+y)+a}\right] \\
&- 2\cot(\mu x)\cot(\mu y)\left[\frac{\sin^2\mu(x-y)}{(\sin^2\mu(x-y)+a)^2} + \frac{\sin^2\mu(x+y)}{(\sin^2\mu(x-y)+a)^2}\right] \\
&- 2\left[\frac{\sin^2\mu(x-y)}{(\sin^2\mu(x-y)+a)^2} - \frac{\sin^2\mu(x+y)}{(\sin^2\mu(x+y)+a)^2}\right] \\
&= (\cot^2(\mu x) + \cot^2(\mu y) + 2)\frac{\sin^2\mu(x+y) - \sin^2\mu(x-y)}{(\sin^2\mu(x-y)+a)(\sin^2\mu(x+y)+a)} \\
&- 2\cot(\mu x)\cot(\mu y)\left[\frac{\sin^2\mu(x-y)}{(\sin^2\mu(x-y)+a)^2} + \frac{\sin^2\mu(x+y)}{(\sin^2\mu(x-y)+a)^2}\right] \\
&- 2\left[\frac{\sin^2\mu(x-y)}{(\sin^2\mu(x-y)+a)^2} - \frac{\sin^2\mu(x+y)}{(\sin^2\mu(x+y)+a)^2}\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{\mu}((\sin^2\mu(x-y)+a)(\sin^2\mu(x+y)+a))^2\partial_a G(x,y,a) \\
&= (\cot^2(\mu x) + \cot^2(\mu y) + 2)(\sin^2\mu(x+y) - \sin^2\mu(x-y))(\sin^2\mu(x-y)+a)(\sin^2\mu(x+y)+a) \\
&- 2\cot(\mu x)\cot(\mu y)\left[\sin^2\mu(x-y)(\sin^2\mu(x+y)+a)^2 + \sin^2\mu(x+y)(\sin^2\mu(x-y)+a)^2\right] \\
&- 2\left[\sin^2\mu(x-y)(\sin^2\mu(x+y)+a)^2 - \sin^2\mu(x+y)(\sin^2\mu(x-y)+a)^2\right].
\end{aligned}$$

It is easy to see that this is a quadratic polynomial in a of the form $A_2a^2 + A_1a + A_0$. We will explicitly compute A_2, A_1, and A_0 and show each term is non-negative. For the second order term we get

$$\begin{aligned}
A_2 = &(\cot^2(\mu x) + \cot^2(\mu y) + 2)(\sin^2 \mu(x-y) - \sin^2 \mu(x+y)) \\
&- 2\cot(\mu x)\cot(\mu y)[\sin^2 \mu(x-y) + \sin^2 \mu(x+y)] \\
&- 2[\sin^2 \mu(x-y) - \sin^2 \mu(x+y)]. \\
&= (\cot^2(\mu x) + \cot^2(\mu y))(\sin^2 \mu(x+y) - \sin^2 \mu(x-y)) \\
&- 2\cot(\mu x)\cot(\mu y)[\sin^2 \mu(x-y) + \sin^2 \mu(x+y)].
\end{aligned}$$

This means

$$\begin{aligned}
\tan(\mu x)\tan(\mu y)A_2 =&(\frac{\tan(\mu x)}{\tan(\mu y)} + \frac{\tan(\mu y)}{\tan(\mu x)})(\sin^2 \mu(x+y) - \sin^2 \mu(x-y)) \\
&- 2[\sin^2 \mu(x-y) + \sin^2 \mu(x+y)].
\end{aligned}$$

If we set $\frac{\tan(\mu x)}{\tan(\mu y)} = s$, we get

$$\frac{\tan(\mu x)\tan(\mu y)}{\cos(\mu y)\cos(\mu x)\sin(\mu y)\sin(\mu x)}A_2 = (s+\frac{1}{s})\cdot 4 - 2[2\cdot(s+\frac{1}{s})] = 0.$$

This means as long as $0 < x, y < \frac{L}{2}$, $A_2 = 0$. Similarly, for coefficient of the first order term A_1, we have

$$\begin{aligned}
A_1 = &(\cot^2(\mu x) + \cot^2(\mu y) + 2)(\sin^2 \mu(x+y) - \sin^2 \mu(x-y))(\sin^2 \mu(x+y) + \sin^2 \mu(x-y)) \\
&- 2\cot(\mu x)\cot(\mu y)[2\sin^2 \mu(x-y)\sin^2 \mu(x+y) + 2\sin^2 \mu(x+y)\sin^2 \mu(x-y)] \\
&- 2[2\sin^2 \mu(x-y)\sin^2 \mu(x+y) - 2\sin^2 \mu(x+y)\sin^2 \mu(x-y)] \\
&\geq (\cot^2(\mu x) + \cot^2(\mu y) + 2)[\sin^4 \mu(x+y) - \sin^4 \mu(x-y)] \\
&- 8\cot(\mu x)\cot(\mu y)[\sin^2 \mu(x-y)\sin^2 \mu(x+y)].
\end{aligned}$$

Again, by setting $\frac{\tan(\mu x)}{\tan(\mu y)} = s$, we get

$$\frac{\tan(\mu x)\tan(\mu y)}{\cos(\mu x)\cos(\mu y)\sin(\mu x)\sin(\mu y)}A_1 \geq (s+\frac{1}{s})\cdot 4\cdot 2(s+\frac{1}{s}) - 8(s+\frac{1}{s}-2)(s+\frac{1}{s}+2) \geq 32.$$

Lastly, for the coefficient of the constant term A_0, we have

$$\begin{aligned}
A_0 &= (\cot^2(\mu x) + \cot^2(\mu y))(\sin^2\mu(x+y) - \sin^2\mu(x-y))\sin^2\mu(x+y)\sin^2\mu(x-y) \\
&- 2\cot(\mu x)\cot(\mu y)[\sin^2\mu(x-y)\sin^2\mu(x+y)(\sin^2\mu(x+y) + \sin^2\mu(x-y))] \\
&- 2\sin^2\mu(x-y)\sin^2\mu(x+y)[\sin^2\mu(x+y) - \sin^2\mu(x-y)] \\
&= (\cot^2(\mu x) + \cot^2(\mu y))(\sin^2\mu(x+y) - \sin^2\mu(x-y))\sin^2\mu(x+y)\sin^2\mu(x-y) \\
&- 2\cot(\mu x)\cot(\mu y)\sin^2\mu(x-y)\sin^2\mu(x+y)[\sin^2\mu(x+y) + \sin^2\mu(x-y)].
\end{aligned}$$

Setting again $s = \frac{\tan(\mu x)}{\tan(\mu y)}$, after computation we have

$$\frac{\tan(\mu x)\tan(\mu y)}{\sin^2\mu(x-y)\sin^2\mu(x+y)\cos(\mu x)\cos(\mu y)\sin(\mu x)\sin(\mu y)}A_0 = (s + \frac{1}{s})\cdot 4 - 2\cdot(2s + \frac{2}{s}) = 0.$$

In all, we have $\partial_a G(x,y,a) \geq 0$ for $0 < x, y < \frac{L}{2}$. This completes the proof. □

Remark 2.6.5 *One may notice that when $a \to \infty$, $\frac{1}{\mu}G(x,y,a)$ tends to*

$$-(\cot^2(\mu x) + \cot^2(\mu y) + 2)\left[\log\left(\frac{\sin^2\mu(x-y)}{\sin^2\mu(x+y)}\right)\right] - 4\cot(\mu x)\cot(\mu y). \tag{2.46}$$

The positivity of this quantity is also proved by Lemma 4.2 in [3], in which the authors use technical trigonometric inequalities. Our proof of the above lemma provides another approach to estimating this quantity.

With these lemmas at our disposal, we are ready to prove finite-time blow up.

Proof of Theorem 2.6.1.

Suppose we have a global smooth solution. We will show blow up of the following quantity:

$$I(t) := \int_0^{L/2} \theta(x,t)\cot(\mu x)\,dx.$$

thereby arriving at a contradiction since

$$|I(t)| \leq C\|\theta_x(\cdot,t)\|_{L^\infty} \leq C\|\theta_{0x}\|_{L^\infty} \exp\left(\int_0^t \|u_x(\cdot,s)\|_{L^\infty}\, ds\right).$$

If I were to become infinite in finite time, we would be able to use Beale-Kato-Majda type condition for the system as stated in equation (2.32) from which we can conclude finite time blow up.

We first compute the derivative of $I(t)$:

$$\frac{d}{dt}I(t) = -\frac{1}{\pi}\int_0^{L/2} \theta_x(x,t) \int_0^{L/2} \omega(y,t)\cot(\mu y) F(x,y,a)\, dy\, dx.$$

By the negativity of F and part (a) of the lemma, the expression above is bounded below by

$$\frac{C}{\pi}\int_0^{L/2} \theta_x(x,t)\int_x^{L/2} \omega(y,t)\cot(\mu y)\, dy\, dx = \frac{C}{\pi}\int_0^{L/2} \theta(y,t)\omega(y,t)\cot(\mu y)\, dy := CJ(t)$$

(where $J(t) = \frac{2}{\pi}\int_0^{L/2} \theta(x,t)\omega(x,t)\cot(\mu x)\, dx$). Then

$$\frac{d}{dt}(J(t)) = \frac{C}{\pi}\int_0^{L/2} \theta(x,t)\omega(x,t)\left(u(x,t)\cot(\mu x)\right)_x\, dx + \frac{C\mu}{2\pi}\int_0^{L/2} \theta^2(x,t)\csc^2(\mu x)\, dx. \tag{2.47}$$

By Cauchy-Schwarz inequality, the second integral is bounded below by $\frac{C}{L^2}I(t)^2$ for some constant C. The first integral is given by

$$\frac{C}{\pi}\int_0^{L/2} \theta_y(y)\left[\int_y^{L/2} \omega(x)\left(u(x)\cot(\mu x)\right)_x\, dx\right] dy \tag{2.48}$$

Observe that since θ is non-decreasing on $[0, L/2]$, the expression (2.48) is positive if we can show the integral in the brackets is positive as well. This is our next task. For $x, y \in [0, \frac{1}{2}L]$, $\omega(x)$ can be decomposed as

$$\omega(x) = \omega(x)\chi_{[0,y]}(x) + \omega(x)\chi_{[y,\frac{1}{2}L]}(x) =: \omega_\ell(x) + \omega_r(x).$$

Then we can decompose the integral:

$$\int_y^{L/2} \omega(x)[u(x)\cot(\mu x)]_x\,dx = \frac{1}{\pi}\int_0^{L/2} \omega_r(x)\int_0^{L/2} \omega_\ell(y)\cot(\mu y)(\partial_x F)(x,y,a)\,dy\,dx$$
$$+\frac{1}{\pi}\int_0^{L/2} \omega_r(x)\int_0^{L/2} \omega_r(y)\cot(\mu y)(\partial_x F)(x,y,a)\,dy\,dx$$

By positivity of ω on $[0,\frac{1}{2}L]$ and part (b) of the key lemma, the first integral is non-negative. By using symmetry, the second integral is equal to

$$\frac{1}{2\pi}\int_0^{L/2}\int_0^{L/2} \omega_r(x)\omega_r(y)G(x,y,a)\,dy\,dx$$

where as before $G(x,y,a) = \cot(\mu y)(\partial_x F)(x,y,a) + \cot(\mu x)(\partial_x F)(y,x,a)$. However, by part (c) of the lemma, this is positive. Together with (2.47) and (2.48) we have:

$$\frac{d^2}{dt^2}I \geq CI^2, \tag{2.49}$$

for some constant C. To close the proof, we only need the following lemma:

Lemma 2.6.6 *Suppose $I(t)$ solves the following initial value problem:*

$$\frac{d}{dt}I(t) \geq C\int_0^t I^2(s)ds, \quad I(0) = I_0. \tag{2.50}$$

Then there exists $T = T(C, I_0)$ so that $\lim_{t\to T} I(t) = \infty$.

Moreover, for fixed C and any $\epsilon > 0$, there is an $A > 0$ (depending on C, ϵ), so that for any $I_0 \geq A$, the blow up time $T < \epsilon$.

The proof of this lemma is straightforward, and one can also find a sketch of the proof in [3].

2.7 Stability of Blow Up with Respect to Perturbations

In this section, we consider our system (2.24) and (2.25) but with a Biot-Savart law which is a perturbation of the Hou-Lou kernel. As before, we will work with periodic solutions with period L, and assume that the vorticity is odd (this property will be conserved in time for the perturbations we consider). The velocity u is given by the following choice of Biot-Savart law

$$u(x) = \frac{1}{\pi}\int_0^L \left(\log|\sin[\mu(x-y)]| + f(x,y)\right)\omega(y)\,dy, \quad \mu := \pi/L \tag{2.51}$$

$$:= u_{HL}(x) + u_f(x) \tag{2.52}$$

where f is a smooth function whose precise properties we will specify later. We view f as a perturbation and we will show solutions to the system (2.24) and (2.25) with (2.51) can still blow up in finite time. As with the previous system (2.24), (2.25), (2.28), it is not hard to show that we will still have a local well-posedness result akin to Proposition (4). In particular, if T^* is a maximal time of existence of a solution then we must have

$$\lim_{t\nearrow T^*}\int_0^t \|u_x(\cdot,\tau)\|_{L^\infty}\,d\tau = \infty \tag{2.53}$$

We show below that such a time will exist for some initial data.

Theorem 2.7.1 *Let $f \in C^2(\mathbb{R}^2)$, periodic with period L with respect to both variables and such that $f(x,y) = f(-x,-y)$ for all x, y. Then there exists initial data ω_0, θ_0 such that solutions of* (2.24) *and* (2.25)*, with velocity given by (2.51), blow up in finite time. Again, that means there exists a time T^* such that we have (2.53).*

We will consider the following class of initial data:

- θ_{0x}, ω_0 smooth odd periodic with period L

- $\theta_{0x}, \omega_0 \geq 0$ on $[0, \frac{1}{2}L]$.
- $\theta_0(0) = 0$
- $(\operatorname{supp}\theta_{0x} \cup \operatorname{supp}\omega_0) \cap [0, \frac{1}{2}L] \subset [0, \epsilon]$
- $\|\theta_0\|_\infty \leq M$

We will make the choice of specific ϵ below. Observe that by the assumptions, ω_0 and θ_{0x} are also odd with respect to $\frac{1}{2}L$. By the following Lemma 2.7.3, we can choose ϵ sufficiently small so that the mass of ω near the origin gets closer to the origin leading to a scenario where blow up can be achieved.

Here similar to **Lemma 2.6.2**, we can get the above properties will propagate as long as the solution keep smooth.

Remark 2.7.2 *With the choice of $f(x, y) = \log\sqrt{\sin^2\mu(x-y)+a}$, we have the kernel from the previous section. However, in the previous section, we proved blow up for a larger class of initial data.*

Lemma 2.7.3 *With the initial data ω_0 and θ_0 as given above, we can choose ϵ_1 sufficiently small so that for $\epsilon < \epsilon_1$, $u(x) < 0$ for $x \leq \epsilon$ where u is defined as* (2.51)*.*

Proof: By periodicity and support property of ω,

$$
\begin{aligned}
u(x) &= \frac{1}{\pi}\int_0^{L/2}\left(\log\left|\frac{\tan(\mu x)-\tan(\mu y)}{\tan(\mu x)+\tan(\mu y)}\right| + f(x,y) - f(x,-y)\right)\omega(y)\,dy \\
&= \frac{1}{\pi}\int_0^{\epsilon}\left(\log\left|\frac{\tan(\mu x)-\tan(\mu y)}{\tan(\mu x)+\tan(\mu y)}\right| + f(x,y) - f(x,-y)\right)\omega(y)\,dy.
\end{aligned}
$$

By the mean value theorem, for $0 \le y \le \epsilon$, $|f(x,y)-f(x,-y)| \le 2\epsilon\|f\|_{C^1}$. By the singularity of the HL kernel when $x = y = 0$, we can choose ϵ_1 such that the expression in the parentheses is negative for $0 < x, y \le \epsilon$. □

It follows that under our assumptions on the initial data, $\omega(x,t)$ and $\theta_x(x,t)$ are supported on $[0,\epsilon]$ for all times while regular solution exists. We will also need the following lemma controlling the integral of ω over half the period.

Lemma 2.7.4 *There exists $\epsilon_2 > 0$ such that for $\epsilon < \epsilon_2$, with ω_0 and θ_0 as chosen above, solutions of* (2.24)*,* (2.25)*,* (2.51) *satisfy*

$$\int_0^{L/2} \omega(y,t)\,dy \le Mt.$$

Proof: Integrating both sides of (2.24) and integrating by parts we get

$$\int_0^{L/2} \omega_t(y,t)\,dy = \int_0^{L/2} u_x(y)\omega(y,t)\,dy + \int_0^{L/2} \theta_x(y,t)\,dy \le M + \int_0^{L/2} u_x(y)\omega(y,t)\,dy$$

If we can show the remaining integral on the right is negative, we are done. Due to our symmetry assumptions, the integral can be written as

$$\frac{1}{\pi}\int_0^{L/2} P.V. \int_0^{L/2} \left(\mu\cot[\mu(x-y)] - \mu\cot[\mu(x+y)] + f_x(x,y) - f_x(x,-y)\right)\omega(x,t)\omega(y,t)\,dy\,dx.$$

By symmetry, the integral with $\cot[\mu(x-y)]$ is 0 and using the support property of ω, the above expression is equal to

$$\frac{1}{\pi}\int_0^{\epsilon}\int_0^{\epsilon} \left(-\cot[\mu(x+y)] + f_x(x,y) - f_x(x,-y)\right)\omega(x,t)\omega(y,t)\,dy\,dx$$

Since f is smooth and ω is positive, we can make ϵ_2 small enough so that the kernel in the parentheses above in the integrand is negative. □

Now, so we can take advantage of our lemmas, we choose $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ for the support of our initial data.

Proof: [Proof of Theorem 2.7.1] Throughout, $C(f)$ will be a positive constant that only depends on f and not ω_0. We will show that

$$I(t) := \int_0^{L/2} \theta(x,t)\cot(\mu x)\,dx \tag{2.54}$$

must blow up. Taking time derivative of I and using Lemma 2.6.4, we get

$$\begin{aligned}\frac{d}{dt}I(t) &= -\int_0^{L/2} u(x)\theta_x(x)\cot(\mu x)\,dx \\ &= \frac{1}{\pi}\int_0^{L/2} \theta_x(x)\int_0^{L/2} \omega(y)\cot(\mu y)K(x,y)\,dy\,dx \\ &\quad + \int_0^{L/2} \theta_x(x)\left(u_f(x)\cot(\mu x)\right)\,dx \geq J(t) + \int_0^{L/2} \theta_x(x)\left(u_f(x)\cot(\mu x)\right)\,dx\end{aligned}$$

where, using the same notation as before,

$$J(t) = \frac{2}{\pi}\int_0^{L/2} \theta(x)\omega(x)\cot(\mu x)\,dx$$

Now, we would like to bound the extra term arising because of f. Since f is smooth and ω is supported near the origin,

$$|u_f(x)\cot(\mu x)| = \left|\int_0^{\epsilon} \left[\cot(\mu x)(f(x,y) - f(x,-y))\right]\omega(y)\,dy\right| \leq C(f)\cdot\left(\int_0^{L/2}\omega(y)dy\right).$$

Therefore, we have

$$\frac{d}{dt}I(t) \geq J(t) - C(f)M\left(\int_0^{L/2}\omega(y)dy\right) \geq J(t) - C(f)M^2 t \tag{2.55}$$

Now, we derive a differential inequality for $J(t)$.

$$\begin{aligned}\frac{d}{dt}J(t) &= \frac{2}{\pi}\int_0^{L/2} -(\theta(x)\omega(x))_x u(x)\cot(\mu x) + \theta_x(x)\theta(x)\cot(\mu x)\,dx \\ &= \frac{2}{\pi}\int_0^{L/2} \theta(x)\omega(x)(u(x)\cot(\mu x))_x\,dx + \frac{\mu}{\pi}\int_0^{L/2} \theta^2(x)\csc^2(\mu x)\,dx\end{aligned}$$

As before, by Cauchy-Schwarz inequality the second integral is bounded below by $\frac{2}{L^2}I(t)^2$. We split the first integral into two parts:

$$\frac{2}{\pi}\int_0^{L/2}\theta(x)\omega(x)(u_{HL}(x)\cot(\mu x))_x\,dx+\frac{2}{\pi}\int_0^{L/2}\theta(x)\omega(x)(u_f(x)\cot(\mu x))_x\,dx.$$

By the arguments in the proof of theorem 2.6.1, the first integral is positive. The second integral is equal to

$$\frac{2}{\pi}\int_0^{L/2}\theta_y(y)\left[\int_y^{L/2}\omega(x)(u_f(x)\cot(\mu x))_x\,dx\right]dy \tag{2.56}$$

Using the smoothness, boundedness, and symmetries of f, we have

$$|\partial_x(u_f(x)\cot(\mu x))|=\left|\int_0^{\epsilon}\partial_x\left[\cot(\mu x)(f(x,y)-f(x,-y))\right]\omega(y)\,dy\right| \tag{2.57}$$

Now let $h(x,y)=\cot(\mu x)(f(x,y)-f(x,-y))$. Then it is easy to see that $h\in C^1$ when $f\in C^2$, which means that $|\partial_x h(x,y)|$ is bounded above. This implies that the right hand side of (2.57) can be bounded above by

$$C(f)\cdot\left(\int_0^{L/2}\omega(y)dy\right).$$

Inserting this estimate into (2.56), and using monotonicity of θ, we get that (2.56) is bounded below by

$$-C(f)M\left(\int_0^{L/2}\omega(y)dy\right)^2.$$

Putting things together, we get

$$\frac{d}{dt}J(t)\geq\frac{2}{L^2}I(t)^2-C(f)M\left(\int_0^{L/2}\omega(y)dy\right)^2\geq\frac{2}{L^2}I(t)^2-C(f)M^3t^2 \tag{2.58}$$

Now, we will show that the differential inequalities we have established will lead to finite time blow up. By (2.55) and (2.58) we obtain

$$\begin{aligned}\frac{d}{dt}I(t) &\geq \frac{2}{L^2}\int_0^t I^2(s)\,ds + J(0) - c(f)M^2t - C(f)M^3\frac{t^3}{3} \\ &\geq \frac{2}{L^2}\int_0^t I^2(s)\,ds - c(f)M^2t - C(f)M^3\frac{t^3}{3}.\end{aligned} \tag{2.59}$$

We claim that one can choose $I(0)$ large enough so that the effect of the negative terms is controlled. By a rather crude estimate we have

$$\frac{d}{dt}I(t) \geq -c(f)M^2t - C(f)M^3\frac{t^3}{3}.$$

After integration, this implies

$$I(t) \geq I(0) - C(f)M^2\left(\frac{t^2}{2} + M\frac{t^4}{12}\right). \tag{2.60}$$

Now fix a time, say 1. We will show that $I(0)$ can be chosen large enough so that $I(t)$ blows up before time 1. Note that assuming $I(0) \geq C(f)M^2\left(\frac{1}{2} + M\frac{1}{12}\right)$, we have for $t \leq 1$,

$$\frac{1}{L^2}\int_0^t I^2(s)\,ds \geq \frac{t}{L^2}\left[I(0) - C(f)M^2\left(\frac{1}{2} + \frac{M}{12}\right)\right]^2$$

Choose $I(0)$ so that

$$I(0) \geq C(f)M^2\left(\frac{1}{2} + \frac{M}{12}\right) + L\sqrt{c(f)M^2 + C(f)\frac{M^3}{3}} \tag{2.61}$$

Then, for $0 \leq t \leq 1$, with this choice of $I(0)$ and using (2.59) and (2.61), we get

$$\begin{aligned}\frac{d}{dt}I(t) &\geq \frac{1}{L^2}\int_0^t I(s)^2\,ds + t\left(c(f)M^2 + C(f)\frac{M^3}{3}\right) - c(f)M^2t - C(f)M^3\frac{t^3}{3} \\ &\geq \frac{1}{L^2}\int_0^t I(s)^2\,ds\end{aligned}$$

By perhaps making $I(0)$ a little larger, if needed, we can show $I(t)$ becomes infinite before time 1 by Lemma 2.6.6. □

2.8 The Real Line Case

One can also consider the model equation (2.24) and (2.25) with the law (2.28) for compactly supported data on $\mathbb{R}$. we only outline main ideas and changes involved, leaving all details to the interested reader. Without loss of generality we assume the domain of the initial data is $[-1, 1]$. In this case, similar argument like in Section 2.5 can show that the corresponding modified Hou-Luo kernel will be

$$F(x, y, a) = \frac{y}{x}\left[\log\left(\frac{(x-y)^2}{(x+y)^2}\right) + \log\left(\frac{(x+y)^2+a}{(x-y)^2+a}\right)\right], \tag{2.62}$$

for $a > 0$.

The analogue of Lemma 2.6.3 will be the following:

Lemma 2.8.1 **(a)** *For any $a \neq 0$, there is a constant $C(a) > 0$ such that for any $0 < x < y < 1$, $F(x, y, a) \leq -C(a)$.*

(b) *For any $0 < y < x < \infty$, $F(x, y, a)$ is increasing in x.*

(c) *For any $0 < x, y < \infty$, $\frac{1}{y}(\partial_x F)(x, y, a) + \frac{1}{x}(\partial_x F)(y, x, a)$ is positive.*

Proof: First it is easy to see that $F(x, y, a)$ is non-positive. For part (a), one can follow the similar but easier argument as in the proof of part (a) of Lemma 2.6.3. Now let us prove part (b) and (c).

Proof of (b)

By direct computation

$$
\begin{aligned}
\frac{1}{y}\partial_x F(x,y,a) &= -\frac{1}{x^2}\left[\log\left(\frac{(x-y)^2}{(x+y)^2}\right)+\log\left(\frac{(x+y)^2+a}{(x-y)^2+a}\right)\right] \\
&+\frac{1}{x}\left[\frac{2(x-y)}{(x-y)^2}-\frac{2(x-y)}{(x-y)^2+a}-\frac{2(x+y)}{(x+y)^2}+\frac{2(x+y)}{(x+y)^2+a}\right] \\
&= -\frac{1}{x^2}\left[\log\left(\frac{(x-y)^2}{(x+y)^2}\right)+\log\left(\frac{(x+y)^2+a}{(x-y)^2+a}\right)\right] \\
&+\frac{1}{x}\left[\frac{2a(x-y)}{(x-y)^2((x-y)^2+a)}-\frac{2a(x+y)}{(x+y)^2((x+y)^2+a)}\right] \\
&= I + II.
\end{aligned}
$$

The term I, by the same argument as in the proof of the periodic analog, is positive. For the term II, we have

$$
II = \frac{1}{x}(g(x-y)-g(x+y)),
$$

where $g(t) = \frac{2a}{t(t^2+a)}$. It is easy to see that for $t > 0$, $g(t)$ is decreasing in t, which means $II \geq 0$ whenever $0 < y < x$.

Proof of (c)

First of all, let us call our target function $G(x,y,a)$, which means

$$
\begin{aligned}
G(x,y,a) &= \frac{1}{y}(\partial_x F)(x,y,a)+\frac{1}{x}(\partial_x F)(y,x,a) \\
&= -\left(\frac{1}{x^2}+\frac{1}{y^2}\right)\left[\log\left(\frac{(x-y)^2}{(x+y)^2}\right)+\log\left(\frac{(x+y)^2+a}{(x-y)^2+a}\right)\right] \\
&+\left(\frac{1}{x}-\frac{1}{y}\right)\left(\frac{2a(x-y)}{(x-y)^2((x-y)^2+a)}\right)-\left(\frac{1}{y}+\frac{1}{x}\right)\left(\frac{2a(x+y)}{(x+y)^2((x+y)^2+a)}\right) \\
&= -\left(\frac{1}{x^2}+\frac{1}{y^2}\right)\left[\log\left(\frac{(x-y)^2}{(x+y)^2}\right)+\log\left(\frac{(x+y)^2+a}{(x-y)^2+a}\right)\right] \\
&-\frac{2a}{xy((x-y)^2+a)}-\frac{2a}{xy((x+y)^2+a)}.
\end{aligned}
$$

Now our aim is to prove the positivity of $G(x, y, a)$. Notice that when $a = 0$, $G(x, y, a) = 0$, as a consequence, to prove the positivity of $G(x, y, a)$, the only thing we need to show is this function is increasing in a for any x, y in the domain. On the other hand,

$$\begin{aligned}\partial_a G(x, y, a) = &-\left(\frac{1}{x^2}+\frac{1}{y^2}\right)\left(\frac{1}{(x+y)^2+a}-\frac{1}{(x-y)^2+a}\right)\\ &-\frac{2}{xy}\left[\frac{(x-y)^2}{((x-y)^2+a)^2}+\frac{(x+y)^2}{((x+y)^2+a)^2}\right].\end{aligned}$$

As a conclusion,

$$\begin{aligned}&((x-y)^2+a)^2((x+y)^2+a)^2\partial_a G(x, y, a)\\ &=\left(\frac{1}{x^2}+\frac{1}{y^2}\right)((x+y)^2-(x-y)^2)((x+y)^2+a)((x-y)^2+a)\\ &-\frac{2}{xy}\left[(x-y)^2((x+y)^2+a)^2+(x+y)^2((x-y)^2+a)^2\right]\end{aligned}$$

It is easy to see this is a quadratic polynomial in a. Let's call the coefficient of the second order term A_2 , then

$$\begin{aligned}A_2 &= \left(\frac{1}{x^2}+\frac{1}{y^2}\right)((x+y)^2-(x-y)^2)-\frac{2}{xy}[(x-y)^2+(x+y)^2]\\ &=\left(\frac{1}{x^2}+\frac{1}{y^2}\right)\cdot 4xy-\frac{2}{xy}[2x^2+2y^2]\\ &=\frac{4}{x^2y^2}((x^2+y^2)xy-xy(x^2+y^2))\\ &=0.\end{aligned}$$

Similarly, for coefficient of the first order term A_1, we have

$$\begin{aligned}
A_1 &= \left(\frac{1}{x^2}+\frac{1}{y^2}\right)(4xy)((x+y)^2+(x-y)^2) - \frac{2}{xy}[2(x-y)^2(x+y)^2+2(x+y)^2(x-y)^2] \\
&= \frac{1}{x^2y^2}[(x^2+y^2)^2\cdot 8xy - 8xy(x^2-y^2)^2] \\
&\geq 0.
\end{aligned}$$

Lastly, for the coefficient of the constant term A_0, we have

$$\begin{aligned}
A_0 &= \left(\frac{1}{x^2}+\frac{1}{y^2}\right)(4xy)(x+y)^2(x-y)^2 - \frac{2}{xy}[(x-y)^2(x+y)^4+(x+y)^2(x-y)^4] \\
&= \frac{(x+y)^2(x-y)^2}{x^2y^2}[(x^2+y^2)\cdot 4xy - 2xy((x+y)^2+(x-y)^2)] \\
&= 0.
\end{aligned}$$

In all, we have $\partial_a G(x,y,a) \geq 0$ for $x, y > 0$. □

From this lemma, one can do the same argument to get the blow up result, which is the following theorem:

Theorem 2.8.2 *There exists initial data such that solutions to* (2.24) *and* (2.25)*, with velocity given by* (2.35)*, and* $F(x,y,a)$ *defined by (2.62), blow up in finite time.*

In fact, we can prove the following type of initial data will lead to blow up:

- θ_{0x}, ω_0 smooth odd and are supported in $[-1,1]$.
- $\theta_{0x}, \omega_0 \geq 0$ on $[0,1]$.
- $\theta_0(0) = 0$.
- $\|\theta_0\|_\infty \leq M$.

And similarly, for general pertubation (analogue of theorem 2.7.1), we also have the similar blow up result.

Assume the velocity u is given by the following choice of Biot-Savart Law

$$u(x) = \frac{1}{\pi}\int_{-1}^{1} \left(\log|(x-y)]| + f(x,y)\right)\omega(y)\,dy, \tag{2.63}$$

$$\tag{2.64}$$

where f is a smooth function whose precise properties we will specify later. We view f as a perturbation and we will show solutions to the system (2.24) and (2.25) can still blow up in finite time.

Theorem 2.8.3 *Let $f \in C^2$ be supported on $[-1,1]$, such that $f(x,y) = f(-x,-y)$ for all y. Then there exists initial data ω_0, θ_0 such that solutions of* (2.24) *and* (2.25), *with velocity given by (2.63), blow up in finite time.*

Again we can prove the following type of initial data will form finite time singularity:

- θ_{0x}, ω_0 smooth odd and are supported in $[-1,1]$.
- $\theta_{0x}, \omega_0 \geq 0$ on $[0,1]$.
- $\theta_0(0) = 0$.
- $\operatorname{supp}\omega_0 \subset [0,\epsilon]$.
- $\|\theta_0\|_\infty \leq M$.

We leave the proofs of these theorems as exercises for interested reader.

Bibliography

[1] D. Barbato, F. Morandin, and M. Romito. Smooth solutions for the dyadic model. *Nonlinearity.*, 24:3083–3097, 2011.

[2] A. Cheskidov, S. Friedlander, and N. Pavlovic. An inviscid dyadic model of turbulence: the fixed point and onsager's conjecture. *J. Math. Phys.*, 48, 2007.

[3] K. Choi, T. Hou, A. Kiselev, G. Luo, V. Sverak, and Y. Yao. On the finite-time blowup of a 1d model for the 3d axisymmetric euler equations. *to appear in Comm. in Pure and Appl. Math.*

[4] K. Choi, A. Kiselev, and Y. Yao. Finite time blow up for a 1d model of 2d boussinesq system. *Comm. Math. Phys.*, 334(3):1667–1679, 2015.

[5] P. Constantin, P.D. Lax, and A. Majda. A simple one-dimensional model for the three-dimensional vorticity equation. *Comm. Pure Appl. Math.*, 38:715–724, 1985.

[6] P. Constantin, A. Majda, and E. Tabak. Formation of strong fronts in the 2-d quasi-geostrophic thermal active scalar. *Nonlinearity*, 7(6):1495–1533, 1994.

[7] P. Constantin, A. Tarfulea, and V. Vicol. Long time dynamics of forced critical sqg. *Comm. in Math. Phys.*, 335(1):93–141, 2015.

[8] P. Constantin and J. Wu. Regularity of holder continuous solution of the supercritical quasi-geostrophic equation. *Ann. Inst. H. Poincare Anal. Non Linearie*, 25:1103–1110,

2008.

[9] A. Cordoba and D. Cordoba. A maximum principle applied to quasi-geostrophic equations. *Comm. Math. Phys.*, 249:511–528, 2004.

[10] A. Cordoba, D. Cordoba, and M. Fontelos. Formation of singularities for a transport equation with nonlocal velocity. *Ann. of Math.*, 162(2):1377–1389, 2005.

[11] M. Dabkowski, A Kiselev, L. Silvestre, and V. Vicol. Global well-posedness of slightly supercritical active scalar equations. *Anal. and PDE*, 7(1):43–72, 2014.

[12] M. Dabkowski, A. Kiselev, and V. Vicol. Global well-posedness for a slightly supercritical surface quasi-geostrophic equation. *Nonlinearity*, 25(5):1525–1535, 2012.

[13] T. Do. On a 1d transport equation with nonlocal velocity and supercritical dissipation. *Journal of Differential Equations*, 256(9):3166–3178, 2014.

[14] T. Do. A discrete model for nonlocal transport equations with fractional dissipation. *Comm. Math. Sci*, 15(2):289–303, 2017.

[15] T. Do, V. Hoang, M. Radosz, and X. Xu. One dimensional model equations for hyperbolic fluid flow. *Nonlinear Analysis*, 140:1–11, 2016.

[16] T. Do, A. Kiselev, and X. Xu. Stability of blow-up for a 1d model for axisymmetric euler equations. *to appear in Nonlinear Sci.*

[17] H. Dong. Well-posedness for a transport equation with nonlocal velocity. *J. Funct. Anal*, 255:3070–3097, 2008.

[18] H. Dong and D. Du. Global well-posedness and a decay estimate for the critical dissipative quasi-geostrophic equation in the whole space. *Discrete Contin. Dyn. Syst.*, 21(4):1095–1101, 2008.

[19] H. Dong and D. Li. On a one-dimensional α-patch model with nonlocal drift and fractional dissipation. *Trans. Amer. Math. Soc.*, 366(4):2041–2061, 2014.

[20] H. Dong and N. Pavlovic. Regularity criteria for the dissipative quasi-geostrophic in holder spaces. *Comm. in Mathematical Physics*, 290:801–812, 2009.

[21] T. Hou and P. Liu. Self-similar singularity of a 1d model for the 3d axisymmetric euler equations. *Research in Math. Sci.*, 2(5):1–26, 2015.

[22] T. Hou and G. Luo. Potentially singular solutions of the 3d axisymmetric euler equations. *PNAS*, 111(36):12968–12973, 2014.

[23] T. Hou and G. Luo. Toward the finite=time blowup of the 2d axisymmetric euler equations: A numerical investigation. *Multiscale Mod. Sim.*, 12(4):1722–1776, 2014.

[24] N. Katz and N. Pavlovic. Finite time blow-up for a dyadic model of the euler equations. *Trans of AMS.*, 357:695–708, 2005.

[25] A. Kiselev. Regularity and blow up for active scalars. *Math. Model. Math. Phenom.*, (5):225–255, 2010.

[26] A. Kiselev. Non-local maximum principles for active scalars. *Adv. in Math.*, 227:1806–1826, 2011.

[27] A. Kiselev, F. Nazarov, and R. Shterenberg. Blow up and regularity for fractal burgers equation. *Dyn. Partial Diff. Eq.*, 5(3):211–240, 2008.

[28] A. Kiselev, F. Nazarov, and A. Volberg. Global well-posedness for the critical 2d dissipative quasi-geostrophic equation. *Inventiones Math.*, 167:445–453, 2007.

[29] A. Kiselev, L. Ryzhik, Y. Yao, and A. Zlatoš. Finite-time singularity formation for the modified sqg equation. *to appear in Annals of Math.*

[30] A. Kiselev and V. Šverák. Small scale creation for solutions of the incompressible two dimensional euler equation. *Annals of Math.*, 180(3):1205–1220, 2014.

[31] A. Kiselev and A. Zlatos. On discrete models of the euler equation. *IMRN.*, 38:2315–2339, 2005.

[32] D. Li and J. Rodrigo. Blow-up of solutions for a 1d transport equation with nonlocal velocity and supercritical dissapation. *Adv. Math.*, 217:2563–2568, 2008.

[33] A. Majda and A. Bertozzi. Vorticity ad incompressible flow. *Cambrige University Press*, 27, 2002.

[34] L. Silvestre. Eventual regularization for the slightly supercritical quasi-geostrophic equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 27:693–704, 2010.

[35] L. Silvestre and V. Vicol. On a transport equation with nonlocal drift. *Trans. of AMS*, 368(9):6159–6188, 2016.

[36] H. Yang and S. Goodchild. Local well-posedness of a nonlocal burgers equation. *Involve*, 9(1):67–82, 2016.

[37] V. Yudovich. Eleven great problems of mathematical hydrodynamics. *Moscow mathematical journal*, 3:711–737, 2003.