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Cooperation under Imperfect Monitoring and External Threat

by

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ABSTRACT

Cooperation under Imperfect Monitoring and External Threat

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Cooperation among players is often a good deed to pursue. The famous "Prisoner’s Dilemma" in game theory has long been an example of showing how non-cooperation results because of conflict of interest among agents. My thesis investigates how factors like imperfect public monitoring, emergence of an external threat influences the cooperative behaviors among players in dynamic environments.

Chapter 1 investigates bidder collusion in repeated procurement auctions without communication or side payments, focusing on the case of bidders having identical costs under imperfect public monitoring where only winners, not bids, are publicly observed. It presents a simple bid rotation scheme, in which bidders take turns entering bids equal to the auction’s reserve price. This behavior is supported by the threat of a bidding war, i.e. if an auction is won by the wrong bidder, all bidders will enter bids equal to their common production cost in all future auctions. Bid rotation maximizes the bidders’ joint profits and is a perfect public equilibrium if and only if the discount factor is greater than or equal to a critical value, call it $\delta^*$. This paper presents two main results. First, except for a measure zero set of discount factors, joint profit maximization cannot be achieved for discount factors below $\delta^*$ by any profile of bidding strategies. Second, in the case of two bidders, there is no profile of bidding strategies that achieves joint profit maximization for discount factors less than or equal to $\delta^*$. 
Chapter 2 investigates a situation where firms selling different products refer misallocated customers to one another. Under monitoring over each other’s sales in every period, we analyze firm referrals in an infinitely repeated game with by looking at a class of “$k+1$ punishment schemes”, in which players “forgive” the first $k$ bad signals, and “punish” each other forever after the $k+1$’s bad signal. We characterize the unique optimal $k$ in this class of schemes.

Chapter 3 is an empirical paper that models the impacts of telemarketing calls for selling bank long-term deposits. We use a dataset from a Portuguese retail bank from 2008 to 2013. This dataset contains features related to the calls and customers. We model a binary response of the outcome of the telemarketing call (yes or no) using those features. In second part of the paper, we propose methods to model price elasticity of demand (PED), which measures sensitivity of the long-term deposits resulting from changing interest rates. In estimating the PED’s, propensity-score-matching is used to adjust for potential group differentiation.

Chapter 4 studies alliance behavior under external threat. When an alliance faces danger of appropriation from an external enemy, it is optimum for its members to jointly invest in their defense. For members to behave collusively in the subgame perfect Nash equilibrium (SPNE), each of them has to be allocated with some minimum share of that resource. Under proportion-to-share rule of cost contribution and profit earning, this paper looks at how that minimum share requirement changes after the emergence of the external threat. We find that two factors, alliance size and cost factor contribute to the stability of the alliance in opposite directions. Further, force from more costly investment outweighs the force from increasing alliance size, which makes the alliance easier to maintain.
Chapter 1

Bid Rotation in Repeated Procurement Auctions

1.1 Introduction

According to the antitrust division of United States Department of Justice, “One of the most common violations the Division prosecutes is bid rigging. Common schemes of bid rigging include bid suppression, complementary bidding, and bid rotation, among others.”¹ For any of such bid rigging schemes to work when procurement auctions are held regularly over time, bidders must possess a certain level of patience for them to be better off staying in the scheme and waiting for their turns than deviating from the scheme and starting a bidding war. This paper asks the question of which collusive scheme maximizes bidders’ joint profit, while requiring the lowest level of patience from the bidders.

Most of the existing analysis of collusion in repeated auctions focuses on whether and when particular profit-maximizing collusive schemes can be supported as equilibria, in terms of the discount factor which measures bidders’ level of patience.² This paper looks for necessary conditions: For any profit-maximizing collusive scheme to be supported as an equilibrium, what is the minimum level of patience that bidders must possess. Finding the lower bound of discount factor to support any equilibria is very much an unexplored area. Only Harrington Jr. (1989) studies a repeated Bertrand model with firms having different discount factors, under perfect monitoring over prices, and allowing transfers. It finds that collusion can be achieved, it is necessary that the average discount factor in the grand coalition is higher than some lower bound.

¹Bid rigging is an agreement among competitors as to who will be the winning bidder. Bid suppression: one or more bidders agree not to bid and receive compensation after the auction. Complementary bidding: some bidders bid but purposely lose the auction to create an appearance of competition. Bid rotation: bidders take turns to win contracts in repeated auctions.

²See Section 1.1.1 for more details.
We consider a model of infinitely repeated procurement auctions with \( n \) contractors who have access to the same factor markets. In every period, these contractors compete in a first-price sealed-bid auction — a popular form of auction in both public and private procurement. The contractor who bids at the lowest price gets the contract, as long as that winning bid is below the publicly announced reserve price — the maximum amount of money that the government or firm is willing to pay for the job. Ties among bidders entering the lowest bid are broken randomly, and the only information revealed to the bidders at the end of the auction is the identity of winner.

Bid rotation constructed in this paper calls for bidders to take turns entering bids equal to the auction’s reserve price. This behavior is supported by the threat of a bidding war, i.e. if an auction is won by the wrong bidder, all bidders will enter bids equal to their common production cost in all future auctions. As an example of bid rotation, in the 1950s, 29 suppliers of industrial electrical generators and equipment colluded in first-price sealed-bid procurement auctions (Smith (1961)).\(^3\) This “Electrical Conspiracy” used a bid-rotation scheme in which bidders took turns to bid meaningfully in the auctions, using a formula tagged as “phase of the moon”\(^4\).

The main result of the paper is to show that generically bid rotation is the profit-maximizing scheme that requires the lowest level of patience from the bidders. Formally we show that except for a measure zero set of discount factors, joint profit maximization cannot be achieved by any other colluding schemes for discount factors lower than the minimum level required by bid rotation. And in the case of two bidders, there is no profile of bidding strategies that achieves joint profit maximization for discount factors lower than the minimum level required by bid rotation. This result may be interpreted as a prediction of the model: In most cases where bidders want to maximize their joint profits, if bid rotation cannot be sustained as an equilibrium, no other collusive schemes

\(^3\)More on bid rotation, Kwasnica (2000) designed an experimental environment where multi-object, simultaneous sealed auctions are held repeatedly. By evaluating the strategies used by the bidders in these auctions, they find that bid rotation is more likely to occur in repeated auctions: In seven out of ten experiments participants used such a scheme.

\(^4\)The phase of the moon at the time of the auction determined which bidder had the right to bid, free from competition from other bidders in the conspiracy.
The source of imperfect monitoring in this model comes from the auction’s tie-breaking rule: as the winner is selected at random when several bidders bid at the lowest price, a loser who does not get the contract does not know whether or not the winner has undercut him. It is usually the case that a colluding scheme is easier to sustain when bidders are more patient. However this feature does not hold for colluding schemes involving any tie-breaking. Certain discrete levels of the discount factors are needed to balance the expected payoffs between winning and losing the tie. A higher discount factor will break the balance and make either winning or losing more appealing to the bidders, and hence break the equilibrium.

This paper has certain limitations. First, it only considers colluding schemes that maximize bidders’ joint profit. Although it discusses some particular forms of non profit-maximizing schemes, in general it might be possible to find a non profit-maximizing scheme that requires a patience level lower than bid rotation does. This paper also assumes that bidders have fixed costs of doing the jobs, which applies to procurement auctions over standardized goods, or to cases where bidders have access to the same labor or capital market. That being said, this paper represents an important benchmark for the study of collusion in repeated auctions with imperfect monitoring, and is to my knowledge the first paper that gives a sharp statement about the set of discount factors for which perfect collusion is possible.

1.1.1 Related Literature

The literature on collusion in auctions or cartels has been focusing on whether and when profit-maximizing schemes can be supported as equilibria, based on different assumptions on the level of monitoring and communication among bidders/firms. Fudenberg et al. (1994) studies a family of repeated games with imperfect public monitoring over signals. They showed that as long as players’ deviations can be distinguished statistically, profit-maximizing can be obtained when the discount factor approaches to 1. Skrzypacz and Hopenhayn (2004) studies repeated auctions with monitoring only on the identities of
winners, in the case of two bidders having i.i.d values. They found that perfect collusion cannot be sustained as an equilibrium even for any discount factors arbitrarily close to 1, as any equilibrium strategies of profiles that is different from bidding war in every period necessarily involves non-trivial payments to the auctioneer. Athey and Bagwell (2001) considers a repeated Bertrand game with two firms having identical distribution over two types of costs: high and low. In this setting, the authors showed that first-best payoffs can be achieved even if firms are not infinitely patient. Harrington and Skrzypacz (2007) consider a multi-unit market and finds an impossibility result that no strongly symmetric equilibrium exists even if the discount factor is 1. Harrington and Skrzypacz (2011) argues that in some industries, prices are privately negotiated between firms and buyers, while quantity is publicly observed. They studied a specific collusive scheme used in markets including citric acid, lysine and vitamins and show that when the discount factor is high enough, that scheme can be supported as a PPE.⁵

Closely related to this work, Aoyagi (2003) studies a repeated auction environment in which two bidders, whose actions are coordinated by a third party called “center”, with independent private values (IPVs) on a single product. It proposes a dynamic bid rotation scheme (BRS) as follows. Bidders are instructed to report their values truthfully (called phase $S$) in the first period, and the bidder with the higher value receives the product. In the next phase $A_j, j = 1, 2$, bidder $j$ who does not get the product in the first period is favored: As long as $j$'s value is higher than the reservation price, $j$ gets the products no matter what values $i$ has. After $A_j$, it returns to $S$ where the product is allocated to a bidder with a higher reported value. So a particular pattern of the BRS can look like this: $S - A_1 - S - A_2 - S - A_2 - S - \ldots$. Aoyagi argued that when the common discount factor is asymptotically close to 1, this scheme can be supported as a perfect public equilibrium (PPE). This model is different from Aoyagi (2003) in the following aspects. First, we consider a more general $n$ bidder setting. Second, we assume that bidders have identical costs rather than IPVs over the contracts. This simplifying assumption has many real-life applications. Gale et al. (2000) argues

⁵In term of techniques in analyzing repeated games with imperfect public monitoring, Mailath and Samuelson (2006) provides a thorough exploration.
that identical or perfectly correlated costs on goods can often be observed in procurement of goods and services for standard goods, like a bridge to be rebuilt or a stretch of road to be repaired for a company by a municipality, or government agency seeking health care or insurance for its work force, especially when bidders have the same access to the labor or capital markets for doing the job. Third, different from Aoyagi (2003), we are interested in a setting where bidders cannot communicate. That is, we do not allow the existence of a “center” to coordinate bidders’ bids. In most procurement auctions, communication among bidders is illegal. And when procurement takes place in high frequency or under supervision, communication during the auction may not be feasible (Skrzypacz and Hopenhayn (2004)).

1.1.2 Outline of Results

The main result (Theorem 1.1) of this paper consists of two parts. First, Proposition 1.2 shows that among all profit-maximizing bidding schemes that do not involve tie-breaking at any period, if for some discount factor bid rotation cannot be supported as an equilibrium, then no other bidding scheme can be supported as one. The proof of this proposition consists of several arguments: It first establishes that if a certain discount factor can support some scheme as an equilibrium, then all higher discount factors also support that scheme as an equilibrium. Next, to make the problem more tractable, we prove that it is without loss of generality to consider allocation patterns that repeat after a finite number of periods. Then we show that any of such schemes is “behind” bid rotation in payoffs. Roughly speaking, this means that for a finite number of periods, identical payoffs in other schemes come after bid rotation. As a result, bidders wait for a shorter period of time for the same payoff under bid rotation than under other schemes, therefore the patience level required under bid rotation is lower than that under other schemes.

The second part of Theorem 1.1 is Proposition 1.3. It says that any equilibrium bidding scheme \( p \) that involves tie-breaking in some periods necessarily requires the discount factor \( \delta \) to be in some finite set \( \Delta(p) \). This result is true because for bidders to submit
equal bids at any period, the expected payoff of winning and losing the stage auction must be identical. This generates an equation of a power series in a single variable, which gives at most a finite number of solutions of $\delta$. From this result, we can argue that in general bidding schemes involving tie-breaking are less useful than those without equal bidding: although it might be the case that some bidding scheme can be supported as an equilibrium for some discount factor that is lower than the critical value $\delta^*$ required by bid rotation, such discount factors occupy a set of measure 0 compared to the range of discount factors supported by bid rotation, which is $[\delta^*, 1]$.

To compare with a well-known paper on bid rotation and repeated auctions by Aoyagi (2003), we study the special case of two bidders. This case is interesting to consider for two reasons. First, when we have two bidders, even if we allow bidders to bid equally in certain periods, the alternating bidding scheme is still the one that requires the least level of patience (Theorem 1.2). This is the result that we could not get in the general $n$-bidder case. And second, the proof the result in the two bidder case is different from the one we use in the $n$-bidder case. Here we show that with two bidders, any scheme different from bid rotation requires a higher discount factor because otherwise the total market surplus is not enough to be split between the bidders. Extensions — a model with a center and non profit-maximizing colluding schemes — are also discussed.

The rest of the paper is organized as follows. Section 1.2 introduces the model. Section 1.3 proves the main results of the paper involving $n$ bidders. Section 1.4 consider the special case of two bidders. Section 1.5 concludes the paper with future research directions.

1.2 Model

There is a set $N = \{1, 2, \ldots, n\}$, $n \geq 2$, of symmetric, risk-neutral bidders participating in an infinite sequence of procurement auctions held by a single buyer. In each period, the same indivisible job contract is awarded to one of the bidders in the form of a first-price
sealed auction. The reservation price, which is the maximum price that the buyer is willing to pay for the job, is known to all bidders and fixed at $r > 0$ across all periods. As bidders have access to the same factor markets, they have the same cost $c < r$ of doing the job. Normalize $r - c$, the maximum payoff to each bidder at a stage auction, to 1.

In each period $t = 1, 2, 3, \ldots$, all bidders simultaneously and privately submit bids to the buyer, and the contract goes to the bidder who submits the lowest bid, as long as that bid is less than or equal to $r$. In the case of several bidders submitting the same lowest bid, the contract goes to each low bidder with equal probability. The bidder who gets the contract at $t$ is called the *winner (at $t$)*, all others are called *losers (at $t$)*. The winner gets paid by her bid and pays the cost, while all losers get and pay nothing.

Each bidder $i$’s bid at period $t$ is $p_i^t \in \mathbb{R}_+$. Let $p^t = (p_i^t)_{t=1,2,3,\ldots}$ and $p_t = (p_i^t)_{i \in \mathbb{N}}$. Let $M_t \subset \mathbb{N}$, with $|M_t| = m_t$, be the set of potential winners at $t$, who bid equally at the lowest level at period $t$. The stage payoff $\pi_i^t(p_t)$, generated from $p_t$, for bidder $i$ in period $t$ can be summarized as follows:

$$
\pi_i^t(p_t) = \begin{cases} 
\frac{1}{m_t}(p_i^t - c), & \text{if } i \in M_t \text{ and } p_i^t \leq r \\
0, & \text{otherwise}
\end{cases}
$$

At the beginning of each period $t$, a *public history* of identities of winners is observed by all bidders and is defined as $s_{1} = (s_0, s_1, s_2, \ldots, s_{t-1})$, where for each $k \geq 1$, $s_k \in \mathbb{N} \cup \{0\}$ denotes period $k$’s auction outcome: $s_k = i$ means that bidder $i$ is the winner at period $k$, whereas $s_k = 0$ means that no bidder is a winner at period $k$. Define $s_0 = 0$. Let $S_{t-1}$ be the set of all public histories up to $t - 1$.

Let $\delta \in (0, 1)$ be the common discount factor of all the bidders. Let $\nu_i^t(p, \delta)$ be bidder $i$’s discounted payoff, starting from period $t$, following strategy profile $p$, given the

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$^6$First-price sealed auction is common in private sector procurement. The problem is much simpler to solve when bids are revealed publicly, like in the case of public procurement.
discount factor $\delta$. Specifically, for all $i$,

$$v^i_t(p, \delta) = \sum_{k=t}^{\infty} \pi_k(p_k) \delta^{k-t}$$

In this paper, we restrict our attention to public strategies where the current period’s bid depends only on past periods’ public histories, and not on bidders’ private strategies in the past. A public strategy for bidder $i$ is a sequence of functions $p^i_t : S^{t-1} \rightarrow \mathbb{R}$, for $t = 1, 2, 3, \ldots$.

A strategy profile $p$ is a perfect public equilibrium (PPE), if there does not exist $p'^i$, $i \in N$, such that for some $t$, $v^i_t((p'^i, p^{-i}), \delta) > v^i_t(p, \delta)$.

Since we are interested in finding the lower bound of the discount factor that supports any PPE, we focus on grim-trigger strategies because they have the harshest form of punishment and hence characterize the set of equilibria in terms of the discount factor. Let $P$ be the set of all grim-trigger strategy profiles. $p \in P$ if it starts with a collusive phase, in which $p$ describes $p^i_t$ for all $t = 1, 2, 3, \ldots$, and all $s^{t-1}$ resulting from $p$. Any observable deviation from the collusive phase, that is if $p_t$ results in $M_t$, but $s_t \notin M_t$, reverts $p$ to the punishment phase forever, in which all bidders bid at $c$ in all future periods, which are the one-shot Nash equilibria of the stage auctions.

Next I define the set of profit-maximizing strategies.

**Definition 1.1** ($\hat{\Gamma}$). A strategy profile $p \in P$ is profit-maximizing if for all periods $t$ all and all possible histories $s^{t-1}$ resulting from $p$ in the collusive phase, $\min\{p^i_t\}_{i \in N} = r$. Let $\hat{\Gamma} \subset P$ be the set of all profit-maximizing strategy profiles.

**Definition 1.2** ($\Gamma$). Let $\Gamma = \{p \in \hat{\Gamma} \mid m_t = 1 \text{ for all } t \text{ in the collusive phase}\}$.

That is, for any profit-maximizing bidding scheme $p \in \hat{\Gamma}$, jobs are procured to

---

7This is WLOG in finding all the equilibria, when pure strategy profiles are considered. For details, see Mailath and Samuelson (2006), Chap 7, page 225-272.

8To see this, the incentive compatibility constraint for grim trigger strategies takes the form $v^i_t(p, \delta) \geq 1$ in this setup. For non grim trigger strategies, as the punishment periods are finite, the deviation payoff is higher than 1. Therefore the required $\delta$ in non grim trigger strategies is also higher.

9Remember that $M_t$ is the set of potential winners at $t$, and $|M_t| = m_t$. And when $m_t = 1$ for all $t$, there is no need to describe histories in $p$. 
bidders at the reservation price in all periods of the collusive phase. $\Gamma$ is the set of profit-maximizing strategy profiles that each profile in it assigns deterministic winners in the collusive phase. It can be viewed as the set of all bid rigging strategies where in each round a single bidder is selected to win the contract. For example, in the collusive phase, when $n = 2$,

$$(p^1, p^2) = ((r, r + 1, r, r + 1, \ldots), (r + 1, r, r + 1, r, r + 1, \ldots)) \in \Gamma \subset \hat{\Gamma},$$

and

$$(p^1, p^2) = ((r, r, r, r, \ldots), (r, r, r, r, \ldots)) \in \hat{\Gamma} \setminus \Gamma.\textsuperscript{10}$$

Now we introduce the key bidding scheme to be discussed in this paper.

- **Bid rotation scheme ($p^A$):**

  - $p^A$ starts with the collusive phase: in each $t = 1, 2, 3, \ldots, m_t = 1$, $\min\{p^i_t\}_{i \in N} = r$, and each $i \in N$ gets a contract every $n$ periods.\textsuperscript{11}

  - If any contract goes to the wrong bidder at any periods, $p^A$ enters the punishment phase and all bidders bid $c$ thereafter.

Hence bid rotation assigns each bidder a contract in every $n$ periods, by letting one bidder bid at the reservation price, while all other bidders bid (meaninglessly) above the reservation price (hence $p^A \in \Gamma$). Any deviation can be publicly observed and results in a bidding war where all bidders bid at their cost in all future periods.

### 1.3 Main Results

We first characterize $p^A$ as a PPE in terms of $\delta$.

**Proposition 1.1.** $p^A$ is a PPE if and only if $\delta \geq \delta^*(n)$, where $\delta^*(n)$ is the unique solution to $\delta^{n-1} + \delta^n = 1$.

\textsuperscript{10}Bidders bid at $r$ in all periods, regardless of history.

\textsuperscript{11}It does not matter which bidder gets a contract first. $p^A$ represents a class of schemes of this kind.
Proof. First we should note that bidder $i$ only has incentive to deviate in periods when $i$ is not allocated with a contract. By deviating from $p^A$, by bidding at $r - \epsilon$ for small $\epsilon > 0$, $i$ can at most get $1 - \epsilon$ ($1 - \epsilon$ in that period of deviation, 0 thereafter). Looking at all periods where $i$ is a loser according to $p^A$, the incentive compatibility constraint that is the hardest to satisfy is the period $t'$ right after $i$ is awarded a contract. Thus the binding IC constraint for $i$ is:

$$v^i_t(p^A, \delta) \geq 1 \iff \delta^{n-1} + \delta^{n-1+n} + \delta^{n-1+2n} + \ldots \geq 1$$
$$\iff \frac{\delta^{n-1}}{1 - \delta^n} \geq 1$$
$$\iff \delta^{n-1} + \delta^n \geq 1.$$ 

For any $n \geq 2$, $\delta^{n-1} + \delta^n = 1$ has a unique solution $\delta^*(n) \in (0, 1)$. To see this, for any $n$, $\delta^{n-1} + \delta^n$ is strictly increasing in $\delta$ in $(0, 1)$. And as we can always find some small $\epsilon(n) > 0$, such that $\epsilon(n)^{n-1} + \epsilon(n)^n < 1$, and $(1 - \epsilon(n))^{n-1} + (1 - \epsilon(n))^n > 1$, there must exists a unique $\delta^*(n) \in (\epsilon(n), 1 - \epsilon(n))$, such that $\delta^*(n)^{n-1} + \delta^*(n)^n = 1$. It then follows that for all $\delta \geq \delta^*(n)$, $p^A$ is a PPE.

If there is no confusion, we write $\delta^*$ instead of $\delta^*(n)$ for the remaining part of the paper.\footnote{Except in Section 4 where we consider a two bidder model.}

Next, we consider $\Gamma$, the set of profit-maximizing strategy profiles that does not allow more than 1 bidder bidding at $r$ at any period. We can think of this set as the set of all bid rigging colluding schemes, which we mention earlier in the introduction, that bidders pre-designate winners before going to the auctions to avoid competition.

**Proposition 1.2.** For any $p \in \Gamma$ to be a PPE, it is necessary that $\delta \geq \delta^*$. 

This proposition says that in the set of $\Gamma$, bid rotation is the equilibrium bidding scheme that requires the lowest level of patience from the bidders. The proof consists of several arguments and a conclusion. We present these arguments as a series of lemmas.

**Lemma 1.1.** For all $p \in \Gamma$, if $p$ is a PPE for some $\delta$, $p$ is also a PPE for all $\delta \geq \delta$. 

Proof. As any deviation in any strategy profile \( p \in \Gamma \) can be detected by all bidders, and as we only consider grim-trigger strategies, any deviation at most gives the deviator payoff of 1. If we look at any IC constraint at any period \( t \) for bidder \( i \), it has the form \( v_i^t(p, \delta) \geq 1 \). As \( v_i^t(p, \delta) \) is strictly increasing in \( \delta \), if \( v_i^t(p, \delta) \geq 1 \), then for all \( \delta \geq \hat{\delta} \), \( v_i^t(p, \delta) \geq 1 \). So IC constraints also hold for all \( \delta \geq \hat{\delta} \).

With Lemma 1, we can have the following definition.

**Definition 1.3.** For each \( p \in \Gamma \), let \( \delta_i^t(p) \) be the smallest of the discount factor, such that for all \( \delta \geq \delta_i^t(p) \) the IC constraint for bidder \( i \) at \( t \) is satisfied under \( p \). Let \( \delta^t(p) = \sup \{ \delta_i^t(p) \}_{i \in T} \), where \( T \) is the set of all periods in the collusive phase; and let \( \delta_t(p) = \max \{ \delta_i^t(p) \}_{i \in N} \). Finally define \( \delta(p) = \sup \{ \delta_t(p) \}_{t \in T} \).

By definition, \( p \in \Gamma \) is a PPE if and only if \( \delta \geq \delta(p) \). Next, we show that to search for the scheme \( p \) that gives the lowest \( \delta(p) \) in \( \Gamma \), it is WLOG to consider bidding patterns that repeat themselves after a finite number of periods.

**Lemma 1.2.** For any \( p \in \Gamma \) with \( \delta(p) \), there exists \( p' \in \Gamma' \) such that \( \delta(p') = \delta(p) \), where \( \Gamma' = \{ p \mid p = (J, J, \ldots) \} \), with \( J \) being some block of bidding strategies defined on a finite number of periods.

Proof. Consider any \( p \in \Gamma \), we will construct a \( p' \in \Gamma' \) such that \( \delta(p') = \delta(p) \). By definition, \( \delta(p) = \sup \{ \delta_t(p) \}_{t \in T} = \lim_{k \to \infty} \sup_k \{ \delta_k(p) \}_{k \in T} \) for some finite \( k \). Then we have a finite sequence \( \delta_J = (\delta_1(p), \ldots, \delta_k(p)) \). Let \( J \) be the part of \( p \) that corresponds to \( \delta_J \) and \( \delta(p') = \min \{ \delta_J \} \). Then \( p' \) is such that \( \delta(p') = \delta(p) \). \( \square \)

**Corollary 1.1.** For any \( p \in \Gamma \), there exists a \( \hat{\delta} < 1 \), such that for all \( \delta \geq \hat{\delta} \), \( p \) is a PPE, if and only if \( p \) is such that for any \( i \) and for any \( t \) under any history \( s^t \) resulting from \( p \), there exists a \( t' > t \) and \( s^{t'}, \) such that \( s_{t'} = i \).

Proof. This is true because from Lemma 1.2, we know that there exists a \( p' \in \Gamma' \) such that \( s(p) = s(p') \). Looking at a block \( J \) of \( p' \), as the continuation payoff from any period in \( J \) is greater than 0, the IC constraint implies that \( \hat{\delta} = s(p') < 1 \). \( \square \)

\(^{13}\)Later in Proposition 1.3 we will see that this property does not hold for any \( p \in \hat{\Gamma} \setminus \Gamma \).
Corollary 1.1 implies that as long as we do not exclude any bidder from getting any contracts forever, any bidding scheme in $\Gamma$ can be supported as an equilibrium, as long as bidders are "patient enough".

Next, to prove the proposition, it helps to consider the problem of a single bidder. Consider all bidding schemes that give $m \geq 1$ contracts to a single bidder $i$ in every $mn$ periods in the same order. For example, if $i$ gets payoff 1 in a period with a contract and 0 without one, and if $m = 2$, $n = 3$, then a particular payoff vector that repeats after 6 periods can look like this: $1, 1, 0, 0, 0, 0, | 1, 1, 0, 0, 0, 0, \ldots$. If we can establish that the IC constraints are the easiest to satisfy under $p^A$, then the only way to lose the IC constraints is to provide more contracts to $i$. We cannot do that as we do not have enough contracts to give to everyone. Therefore $p^A$ requires the smallest discount factor to be a PPE compared to other schemes.

Since we are considering the set $\Gamma$, in which strategy profiles give deterministic winners in all collusive periods, strategy profiles and payoff vectors have a one to one correspondence. Let $x$ be a sequence of payoffs that has $m$ 1’s and $m(n - 1)$ 0’s and let $X^{mn}$ be the set of all such sequences. Let $r(x) \in R^m$ be the corresponding vector of the position of 1’s in $x$. For example, if $x = (0, 1, 1, 0, 0, 1)$, then $r(x) = (2, 3, 6)$.

Given $x, x' \in X^{mn}$, we say $x$ is behind $x'$ if $r(x) \geq r(x')$ if $r_t(x) \geq r_t(x')$ for all $t = 1, \ldots, m$, and $r_t(x) > r_t(x')$ for some $t$. For example, if $x = (0, 1, 1, 0, 0, 1)$ and $x' = (1, 1, 1, 0, 0, 0)$, then $x$ is behind $x'$. Note that for any $\delta$, if $x$ is behind $x'$, $x$ generates less discounted payoff than $x'$, and the binding IC constraint is harder to satisfy in $x$ than in $x'$.

Given any $x = (x_1, x_2, \ldots, x_{mn}) \in X^{mn}$, let $x^t = (x_t, x_{t+1}, \ldots, x_{mn}, x_1, \ldots, x_{t-1})$. Note that there are $mn$ ways to rewrite a sequence in this way. Let $\Sigma(x) = \{x^t\}_{t=1,\ldots, mn}$.

For example, if $x = (0, 1, 1, 0, 0, 1)$, then $x^1 = x$, $x^2 = (1, 1, 0, 0, 1, 0)$ etc.

Define $x^A_{mn} = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0, 1)$ to be the vector that has a 1 in every $n$ periods (resulting from $p^A$). Then $r(x^A_{mn}) = (n, 2n, \ldots, mn)$. The next lemma is the key argument to the proof of Proposition 1.2, that the IC constraints $p^A$ generates is the easiest to satisfy, compared to all other bidding schemes in $\Gamma$. 
Lemma 1.3. For each $m$ and $n$, for all $x \in X^{mn}\setminus \Sigma(x_{mn}^A)$, there exists an $x^i \in \Sigma(x)$, such that $x^i$ is behind $x_{mn}^A$.

Proof. The proof is by induction on $m$.

Step 1. $m = 1$. For all $n$, this case is trivial, as $X^{1n}\setminus \Sigma(x_{1n}^A) = \emptyset$.

Step 2. $m = 2$. This is the base case. Given any $n$, if $x \notin \Sigma(x_{2n}^A)$, with $r(x) = (a, b)$ then either $x'$ with $r(x') = (2n - (b - a), 2n)$ or $x''$ with $r(x'') = (b - a, 2n)$ is behind $x_{2n}^A$.

Step 3. Assume that the lemma is true for $m$, for any $n$. We want to show that it is also true for $m + 1$, for any $n$.

Given any $x \in X^{(m+1)n}\setminus \Sigma(x_{(m+1)n}^A)$, we can find a $x' \in \Sigma(x)$, such that there is an $n$-tuple vector $A$ in $x'$, such that $A = (0, 0, \ldots, 1)$. Such an $A$ can always be found, otherwise there are more than $m + 1$ 1’s in $x$. Remember the position of $A$ in $x'$ and take $A$ out of $x'$. Now $x'$ becomes an $mn$ vector and call it $x''$. Note that $x'' \notin \Sigma(x_{mn}^A)$, otherwise $x \in \Sigma(x_{(m+1)n}^A)$.

By the assumption in the inductive step, there exists an $x^i \in \Sigma(x'')$, such that $x^i$ is behind $x_{mn}^A$. Put $A$ back to its original location in $x^i$ (if $A$ can be put either at the very start or the very end, put it at either place), and call the resulting sequence $x^*$ (Note that $x^* \in \Sigma(x)$). It remains to show that $x^*$ is behind $x_{(m+1)n}^A$.

If $A$ is put either at the very start or the very end of $x^i$, as $x^i$ is behind $x_{mn}^A$, $x^*$ is also behind $x_{(m+1)n}^A$ as adding the same $A$ to both sequences in front or at behind does not change their relative positions.

If $A$ is put neither at the very start nor the very end, let $j$ be the position of the “1” in $A$, appearing in $x^*$. We know that $r_{j-1}(x_{mn}^A) \leq r_{j-1}(x^i)$ from the induction step. This implies that $r_{j}(x_{(m+1)n}^A) \leq r_{j}(x^*)$ because we are adding the same $A$ to $x^*$ (somewhere after the $j - 1$’s “1”) and to $x_{mn}^A$ (right after the $j - 1$’s “1”)

By observation, relative positions of the remaining “1”s in $x^*$ and $x_{(m+1)n}^A$ stay the same as in $x^i$ and $x_{mn}^A$. Therefore, $x^*$ is behind $x_{(m+1)n}^A$, and we have shown the induction step, hence the proof.

Example 1.1 (An illustration of Lemma 1.3). Let $m = 3$, $n = 3$, $x_{3\times3}^A = (0, 0, 1, 0, 1, 0, 0, 1, 0)$,

\footnote{It helps to imagine that $x_{(m+1)n}^A$ is obtained from $x_{mn}^A$, by adding $A$ after the $j - 1$’s “1” in $x_{mn}^A$.}
and \( x = (0, 1, 0, 0, 0, 1, 0, 0) \). The first step is to find a \( x' \) that has a \((0, 0, 1)\) in it. Note that \( x' = x \), there is no need to re-arrange \( x \) to find \( x' \).

Take out \( A = (0, 0, 1) \) in the middle of \( x' \) to form \( x'' \). Now \( x'' = (0, 1, 0, 1, 0, 0) \notin \Sigma(x_{mn}^A) \), and \( x'' \) is not behind \( x_{2x3}^A = (0, 0, 1, 0, 0, 1) \). The induction step assumes that we can find \( x^t \in \Sigma(x'') \), such that \( x^t \) is behind \( x_{2x3}^A \). In this case, \( x^t = (0, 0, 0, 1, 0, 1, 0, 1) = x^* \).

Compare \( x^* \) and \( x_{3x3}^A \)

\[
x^* = (0, 0, 0, 1, 0, 0, 0, 1, 1)
\]
\[
x_{3x3}^A = (0, 0, 1, 0, 0, 1, 0, 0, 1).
\]

We can see that the \( r_1 \) values are not affected, \( r_2(x^*) > r_2(x_{3x3}^A) \). Relative positions of \( r_3(x^*) \) and \( r_3(x_{3x3}^A) \) remain the same as they are both pushed away to the right by 3 periods. As \( r(x^*) = (4, 8, 9) \geq (3, 6, 9) = r(x_{3x3}^A) \), \( x^* \) is behind \( x_{3x3}^A \).

Now the following result is easy to show:

**Lemma 1.4.** For all \( p \in \Gamma' \) with \( p \neq p^A \) that gives bidder \( i \) contracts in \( mn \) periods, \( \delta^i(p) > \delta^* \).

**Proof.** Using the terminology in Lemma 3, if \( x \) is different from \( x^A \), the first period IC constraint in \( x^* \) is more difficult to satisfy than the one in \( x^A \). To see this, “1”s in \( x^* \) and \( x^A \) appear in the following positions:

\[
x^* : r_1(p^*), r_2(p^*), \ldots, r_m(p^*), \ldots
\]
\[
x^A : n, 2n, \ldots, mn, \ldots
\]

With \( r(x^*) \geq r(x^A) \), it is obvious to see that for all \( \delta \),

\[
v_1(x^A) > v_1(x^*).
\]
Therefore
\[ \delta^* < \delta(x^*) = \delta^i(p). \]

\[ \square \]

**Proof of Proposition 1.2.** From Lemmas 1.1–1.4, we are able to show that, for one bidder, given fixed number of contracts in every fixed number of periods in a fixed way, evenly splitting the contracts makes this bidder’s IC constraint the easiest to satisfy. In bid rotation for \( n \)-bidders, every bidder has a contract in every \( n \) period. From the analysis above, we cannot find another scheme that allows the bidders to collude with a smaller-than-\( \delta^* \) discount factor, unless we provide each bidder with more contracts which we do not have. Hence Proposition 1.2 is proven. \[ \square \]

With Proposition 1.2, we know that if we do not allow any tie-breaking at \( r \), no bidding scheme out-performs bid rotation in terms of the required discount factor: bid rotation requires the least level of patience from the bidders. However, if we allow schemes to involve equal bidding at certain periods, we can only get the following weaker result.

**Definition 1.4.** Given any \( p \in \hat{\Gamma} \backslash \Gamma \), let \( \Delta(p) \) be the set of \( \delta \)'s under which \( p \) is a PPE.

**Proposition 1.3.** For any \( p \in \hat{\Gamma} \backslash \Gamma \), \( \Delta(p) \) is finite.

**Proof.** Consider any \( p \in \hat{\Gamma} \backslash \Gamma \). So \( p \) consists of at least one period \( t \), associated with an \( s^{t-1} \), at which a subset \( M_t \), with \( m_t \geq 2 \), of the \( n \) bidders bid equally at \( r \), while other bidders bid above \( r \). Let the expected payoffs of \( i \in M_t \) winning (losing) at \( t \) be \( w^i \) (\( l^i \)). \( w^i \) (\( l^i \)) are generated from different continuation strategies, depending on whether \( i \) winning or losing at \( t \). \( w^i \) and \( l^i \) can be written as

\[
    w^i = \sum_{k=t}^{\infty} \alpha^i_k \delta^{k-t}, \quad l^i = \sum_{k=t}^{\infty} \beta^i_k \delta^{k-t},
\]

where \( \alpha^i_k \) (\( \beta^i_k \)) is the expected stage payoff for \( i \) at period \( k \), given that \( i \) wins (loses) at
Note that all $\alpha^i_k, \beta^i_k \in [0, 1]$. For $i$ not to underbid or overbid at $t$, it is necessary that

$$w^i = l^i \iff \sum_{k=t}^{\infty} \alpha^i_k \delta^{k-t} = \sum_{k=t}^{\infty} \beta^i_k \delta^{k-t} \iff \sum_{k=t}^{\infty} (\alpha^i_k - \beta^i_k) \delta^{k-t} = 0. \quad (1.1)$$

As $(\alpha^i_t, \alpha^i_{t+1}, \ldots) = \alpha^i \neq \beta^i = (\beta^i_t, \beta^i_{t+1}, \ldots)$, which is because $\alpha^i_1 = 1 \neq 0 = \beta^i_1$, Rudin (1976),\(^{15}\) proved that (in the context of this paper) if $\alpha^i \neq \beta^i$, then the solution set of equation (1.1) does not contain a limit point. The next lemma helps us prove the proposition.

**Lemma 1.5.** Let $\Delta \subset [0, 1]$ be the set of solutions to (1.1), if $\Delta$ does not a limit point, then $\Delta$ is finite.

*Proof.* Suppose on the contrary that $\Delta$ is infinite. Then we can find a sequence $(\delta_n)$ contained in $\Delta$ such that $\delta_i \neq \delta_j$ whenever $i \neq j$. Since $(\delta_n)$ is a bounded sequence, the Bolzano-Weierstrass Theorem implies that it contains a convergent subsequence $\delta_{n_k}$. Let $L$ be the limit of $\delta_{n_k}$. Since the $\delta_{n_k}$ are all distinct, at most one term can be equal to $L$. Form the subsequence $\delta_{n_{k_l}}$ from $\delta_{n_k}$ by deleting the term equal to $L$ if it exists. Thus, $\delta_{n_{k_l}}$ is a sequence contained in $\Delta$ and converging to $L$ with $\delta_{n_{k_l}}$ for all $l$. Therefore, $L$ is a limit point of $\Delta$ and a contradiction occurs. \(\square\)

As $\Delta(p) \subset \Delta$, $\Delta(p)$ is also finite, hence Proposition 1.3 is proven. \(\square\)

Next, let $\Delta(\hat{\Gamma})$ be the set of discount factors $\underline{\delta}$, with $\underline{\delta} < \delta^*$ (lower bound of discount factor from bid rotation), and there exists a strategy profile $p \in \hat{\Gamma}$, such that $p$ is PPE when $\delta = \underline{\delta}$. That is, $\Delta(\hat{\Gamma})$ is the collection of discount factors smaller than $\delta^*$, for which bidders’ joint profits can be maximized for *some* profile of bidding strategies.

**Corollary 1.2.** $\Delta(\hat{\Gamma})$ is at most countable.

*Proof.* Since strategy profiles we consider must be profit-maximizing, in each period $t$, there is a finite number of ways to assign $M_t$, with $m_t \geq 1$, as the set of potential winners with them all bidding at $r$ (while others bidding above $r$). So counting all periods

---

\(^{15}\)Rudin (1976) (Principle of mathematical analysis Theorem 8.5)
ways of assigning potential winners are countable. This implies that the set of PPE payoffs that are profit-maximizing is countable. From Proposition 1.3, for each \( p \in \hat{\Gamma} \setminus \Gamma \), at most countable \( \delta \)'s below \( \delta^* \) can support \( p \) as a PPE. The statement is correct because the countable union of countable sets is still countable.

From Proposition 1.3, we know that if we search in the set of all profit-maximizing collusion schemes, it may be possible that for some \( \delta < \delta^* \), \( p' \in \hat{\Gamma} \setminus \Gamma \) is a PPE. However, such \( \delta \)'s are finite, which has measure 0 in the interval \((0,1)\). And from Corollary 1.2, even if we use different strategies for different \( \delta \)'s below \( \delta^* \), such \( \delta \)'s still has measure 0.

This result also shows that unlike bidding schemes considered in \( \Gamma \) where in each period a single bidder is selected to be a winner, bidding schemes in \( \hat{\Gamma} \setminus \Gamma \) require expected payoffs of winning and losing at the tie to be identical. Only finite number of \( \delta \)'s can make these payoffs equal. A higher or lower discount factor will break the balance and make either winning or losing more appealing to the bidders, and hence break the equilibrium.

It is difficult to find \( \delta(p) \) like Proposition 1.2 does when equal bidding is allowed. This is because given any \( p \in \hat{\Gamma} \setminus \Gamma \), not only must the IC constraints be satisfied in periods \( t \)'s where \( m_t = 1 \), which is an easy part to consider as it generates inequalities like \( \delta \geq \delta^t(p) \), and \( \delta^t(p) < \delta^* \). However, when \( m_t > 1 \), we must then have \( w_t^i = l_t^i \) which gives discrete solutions of \( \delta \). It is hard to exhaust all \( \delta \)'s such that both conditions are met. We conclude this section with the main theorem (and the take home message) of this paper.

**Theorem 1.1.** \( p^A \) is the only \( p \in \hat{\Gamma} \), such that for all \( \delta \geq \delta^* \), \( p \) is a PPE.

**Proof.** The proof follows from Propositions 1.1–1.3.

### 1.4 A Model with Two Bidders

In this section, we draw reference to Aoyagi (2003), and focus on the case of two bidders. As stated in the introduction, the result from the two bidder case is cleaner than the case of \( n \) bidders. Also the idea in the proof is different from the \( n \) bidder case: If another
schemes requires a lower-than-$\delta^*$ discount factor, there will not be enough total surplus to share between the two bidders.

**Theorem 1.2.** With 2 bidders, there exists a PPE profile $p \in \hat{\Gamma}$ if and only if $\delta \geq \delta^*$.\(^{16}\)

Like the proof of Theorem 1.1, the *only if* part of the theorem consists of the following two lemmas.

**Lemma 1.6.** For any profile $p \in \Gamma$ to be a PPE, it is necessary that $\delta \geq \delta^*$.\(^{17}\)

**Proof.** Suppose on the contrary that for some $a < \delta^*$, $p' \in \Gamma$ is a PPE for all $\delta \geq a$. From Proposition 1.1, we know that $p' \neq p^A$.

So in the collusive phase of $p'$, there exists a period $t$, such that one bidder has at least two consecutive contracts. WLOG let bidder 1 has two or more consecutive wins from $t$. For $p'$ to be a PPE, there also needs to exist a period $t' > t + 1$, such that bidder 2 is the winner at $t'$, otherwise bidder 2 cheats at period $t$. At $t + 1$, bidder 1 has expected payoff

$$v_{t+1}^1(p', \delta) = 1 + \delta + \delta^2 + \ldots + \delta^{t'-t-1}v_{t'}^1(p', \delta) \geq 1 + \delta v_{t'}^1(p', \delta),$$

(1.2)

where the inequality is true by observing that $v_{t+1}^1(p', \delta)$ is higher if we give more consecutive wins to bidder 1 before period $t'$, while keeping the rest of $p'$ from $t'$ onwards identical.

Next we look at bidder 1’s IC constraint at period $t'$. We know by assumption of $p'$, at $t'$, $v_{t'}^1(p', \delta) \geq 1$ holds for all $\delta \geq a$. Specifically, $v_{t'}^1(p', a) \geq 1$. Since $v_{t'}^1(p', \delta)$ is strictly increasing in $\delta$, $v_{t'}^1(p', \delta^*) > v_{t'}^1(p', a) \geq 1$.

The IC constraint at $t + 1$ for bidder 2 is $v_{t+1}^2(p', \delta^*) \geq 1$. Adding up two bidders’ expected payoff at period $t + 1$ when $\delta = \delta^*$, we have

$$v_{t+1}^1(p', \delta^*) + v_{t+1}^2(p', \delta^*) \geq 1 + \delta^* v_{t'}^1(p', \delta^*) + v_{t+1}^2(p', \delta^*) > 1 + \delta^* + 1 = 2 + \delta^*,$$

\(^{16}\)With an abuse of notation, we refer $\delta^*$ as $\delta^*(n)$ when $n = 2$ in this section.

\(^{17}\)This result can be directly obtained from Proposition 1.2 setting $n = 2$. The idea of the proof is different, therefore shown here.
where the first inequality generates from (1.2), the second generates from $v_1^t(p', \delta^*) > v_1^t(p', a) \geq 1$ and $v_2^t(p', \delta^*) \geq 1$. However, the total payoff available in the market is always $1 + \delta + \delta^2 + \ldots = \frac{1}{1-\delta}$. When $\delta = \delta^*$, $\frac{1}{1-\delta^*} = 2 + \delta^*$. As

$$v_1^{t+1}(p', \delta^*) + v_2^{t+1}(p', \delta^*) > 2 + \delta^*$$

we reach a contradiction, because $v_1^{t+1}(p', \delta) + v_2^{t+1}(p', \delta)$ exceeds the total market payoff, when $\delta = \delta^*$.

Lemma 1.7. For any profile $p \in \hat{\Gamma} \setminus \Gamma$ to be a PPE, it is necessary that $\delta \geq \delta^*$.

Proof. Given any PPE profile $p \in \hat{\Gamma} \setminus \Gamma$, if both bidders bid at $r$ in all periods of the collusive phase, that is, for all $k$, $p_1^k = p_2^k = r$ for all $s^{k-1}$, then $p$ is not a PPE for any $\delta > 0$, as either bidder has an incentive to undercut in the first period. If there exists $K$, such that for all $k \leq K$, $p_1^k = p_2^k = r$ and punishment starts after some history $s^K$, then due to the positive probability to reach this triggering history by following $p$, $p$ is not profit-maximizing. So for $p$ to be both a PPE and profit-maximizing, there must exist a period at which bidders set different bids under some history. Also, if after that period, bidders always set different bids, then the part of $p$ starting that period belongs to $\Gamma$. From Lemma 1.6 we know that $\delta \geq \delta^*$, and we are done. As a result, there exists a period at which bidders set the same bid again. So WLOG, there exists periods $t - 1$ and $t$, and part of $p$ starting at $t - 1$ can be written as follows:

$$
p_{t-1}^1 = r' > r \quad p_t^1 = r \\
\ldots \quad p_{t-1}^1 = r \quad p_t^2 = r \quad \ldots \\
t - 1 \quad t
$$

For $j = 1, 2$, let $p(t+1 \mid j)$ be the part of $p$ from period $t + 1$ onwards, when $s_t = j$. So $p$ from $t$ onwards looks like the following:

$$p_t^1 = p_t^2 = r, \quad \begin{cases} p(t+1 \mid 1) & \text{if } s_t = 1 \\ p(t+1 \mid 2) & \text{if } s_t = 2 \end{cases}$$
For simplicity, denote $v^i_j := v^i_{t+1}(p(t+1 | j), \delta)$, as the continuation payoff of bidder $i$ following $p$, starting from period $t+1$, when $s_t = j$. As $p \in \hat{\Gamma}$, $p$ is profit-maximizing.

We have, at $t+1$

$$v^1_1 + v^2_1 = v^1_2 + v^2_2 = 1 + \delta + \delta^2 + \ldots = \frac{1}{1-\delta}$$

We have another relationship of the $v^i_j$’s from the IC constraint at period $t$: Given any PPE profile $p \in \hat{\Gamma} \Gamma$ that has the pattern described in (1.1), for any bidder not to deviate at $t$,

$$v^1_2 - v^1_1 = v^2_2 - v^2_1 = \frac{1}{\delta}.$$

To see this, at $t$, for bidder 1 not to underbid, by bidding at $r - \varepsilon$, $\varepsilon > 0$,

$$\frac{1}{2}(1 + \delta v^1_1) + \frac{1}{2}(0 + \delta v^1_1) \geq 1 - \varepsilon + \delta v^1_1 \implies v^1_2 - v^1_1 \geq \frac{1}{\delta} - \frac{\varepsilon}{2}$$

At $t$, for bidder 1 not to overbid, by bidding at $r' > r$,

$$\frac{1}{2}(1 + \delta v^1_1) + \frac{1}{2}(0 + \delta v^2_1) \geq \delta v^1_2 \implies v^1_2 - v^1_1 \leq \frac{1}{\delta}$$

Allowing $\varepsilon \to 0$, the above two inequalities imply $v^1_2 - v^1_1 = \frac{1}{\delta}$. Similarly for bidder 2, $v^2_2 - v^2_1 = \frac{1}{\delta}$.

Notice that the equations $v^1_1 + v^2_1 = v^1_2 + v^2_2 = \frac{1}{1-\delta}$ and $v^1_2 - v^1_1 = v^2_2 - v^2_1 = \frac{1}{\delta}$ do not uniquely determine values of the $v^i_j$’s. However if we let one of them to be $x$, the other three can be uniquely expressed. For example if we assign $x = v^1_1 \geq 0$, then

$$v^1_1 = x \quad v^2_1 = \frac{1}{1-\delta} - x$$
$$v^1_2 = x + \frac{1}{\delta} \quad v^2_2 = \frac{1}{1-\delta} - \frac{1}{\delta} - x.$$
WLOG, we can consider the following pattern

\[
p_{t-1}^1 = r' < r \quad p_t^1 = r \quad \left\{ \begin{array}{ll}
p(t + 1 | 1), & \text{where} \\
\begin{array}{ll}
v_1^1 &= x \\
v_2^1 &= \frac{1}{1-\delta} - x
\end{array} & \text{if } s_t = 1
\end{array} \right.
\]

\[
\cdots \\
p_{t-1}^2 = r \quad p_t^2 = r \quad \left\{ \begin{array}{ll}
p(t + 1 | 2), & \text{where} \\
\begin{array}{ll}
v_1^2 &= x + \frac{1}{\delta} \\
v_2^2 &= \frac{1}{1-\delta} - \frac{1}{\delta} - x
\end{array} & \text{if } s_t = 2
\end{array} \right.
\]

(1.3)

Now let us consider bidder 1’s IC constraint at \( t - 1 \). We have

\[
\frac{1}{2}(\delta + \delta^2 x) + \frac{1}{2}(0 + \delta^2(x + \frac{1}{\delta})) \geq 1,
\]

which reduces to

\[
\delta + \delta^2 x \geq 1.
\]

Note that from the above inequality, the required minimum \( \delta \) decreases as \( x \) increases.

From the form of the \( v_i^j \)'s, we see that \( x \leq \frac{1}{1-\delta} - \frac{1}{\delta} \) (Otherwise, one of the \( v_2^2 < 0 \)). So take \( x = \frac{1}{1-\delta} - \frac{1}{\delta} \), and

\[
\delta + \delta^2 \left( \frac{1}{1-\delta} - \frac{1}{\delta} \right) \geq 1 \implies \delta \geq \frac{\sqrt{5} - 1}{2} = \delta^*.
\]

which is the desired result and the lemma is proven.

The proof of the above lemma does not generalize to the proof in Proposition 1.2 for the general \( n \) bidder case. This is because discounted payoffs after equal bidding cannot be written in the form of (1.2), when in the general case of \( n \geq 2 \). We conclude this part by the proof of Theorem 1.2.

*Proof of Theorem 1.2.* For the if part, \( p^A \) does the job through Proposition 1.1. And from Lemmas 1.6 and 1.7, the only if part is proven. Therefore Theorem 1.2 is proven. \( \square \)
1.4.1 Non Profit-Maximizing Collusion

So far we only consider \( \hat{\Gamma} \), the set of profit-maximizing strategy profiles. We would like to know whether non profit-maximizing collusion is possible for \( \delta \)'s below \( \delta^* \). We present some partial results here. We first note a result from Obara and Zincenko (2011).

**Proposition 1.4** (Obara and Zincenko (2011)). For any form of collusion (profit-maximizing or not) to occur, it is necessary that \( \delta \geq 0.5 \).

For the proof, see Theorem 3.1 of their paper.

Therefore we know that no collusion (profit-maximizing or not) can be sustained when \( \delta < 0.5 \). The following result strengthens our results from Theorem 1.2. Let \( \alpha^1 \in [0, 1] \) be bidder 1’s payoff for all the odd periods. Let \( \alpha^2 \in [0, 1] \) be bidder 2’s payoff for all the even periods. So payoff of the bidders \( \pi(\alpha^1, \alpha^2) \) looks like the following\(^\text{18}\)

\[
\pi(\alpha^1, \alpha^2) = \begin{pmatrix}
\alpha^1 & 0 & \alpha^1 & 0 & \alpha^1 & \ldots \\
0 & \alpha^2 & 0 & \alpha^2 & 0 & \ldots
\end{pmatrix}
\]

**Proposition 1.5.** For any strategy profile \( p \) that generates payoff in the form of \( \pi(\alpha^1, \alpha^2) \), for \( p \) to be a PPE, it is necessary that \( \delta \geq \delta^* \).

**Proof of Proposition 1.5.** For bidder 1 not to cheat in even periods,

\[
\alpha^1(\delta + \delta^3 + \delta^5 + \ldots) \geq \alpha^2.
\]

This reduces to

\[
\frac{\alpha^1}{\alpha^2} \geq \frac{1 - \delta^2}{\delta}.
\]

For bidder 2 not to cheat in odd periods,

\[
\alpha^2(\delta + \delta^3 + \delta^5 + \ldots) \geq \alpha^1.
\]

\(^{18}\)Note that it is easy to construct bidding schemes to achieve this payoff pattern. For example, \( p^1_t = \alpha^1 < p^2_t \leq r \) for all \( t = 1, 3, 5 \ldots \); and \( p^2_t = \alpha^2 < p^1_t \leq r \) for all \( t = 2, 4, 6 \ldots \).
This reduces to
\[ \frac{\alpha^1}{\alpha^2} \leq \frac{\delta}{1 - \delta^2}, \]
which means
\[ \frac{1 - \delta^2}{\delta} \leq \frac{\alpha^1}{\alpha^2} \leq \frac{\delta}{1 - \delta^2}, \]
and
\[ \frac{1 - \delta^2}{\delta} \leq \frac{\delta}{1 - \delta^2} \implies \delta \geq \delta^*. \]

1.4.2 A Model with a “Center”

In this subsection, like Aoyagi (2003), we extend the 2-bidder model by involving a center, an external device that suggests bidding strategies (probably mixed) based on the history of winners from previous auctions. Rigorously in every period the center assigns bidder 1 as the winner with probability \( q_t : S^{t-1} \to [0, 1] \times \emptyset \), where \( \emptyset \) denotes that the center assigns no winner and the game enters and stays in the punishment phase forever. Therefore, a perfect public equilibrium with a center is a pair of strategies and instruction rules \((p, q)\), such that for all \( i = 1, 2, v_t^i(p, q, \delta) \geq v_t^i((p^0, p^{-1}), q, \delta)\), for all \( t = 1, 2, \ldots \) and all \( p^0 \). Let \( Q \) be the set of all possible \( q \)'s. Note that we only let the center assign winners, not bids. Therefore, the center can be viewed as a (biased) coin which in expected terms, assigns partial surplus to each bidder in each round. Also note that \( P \subset Q \), as every \( p \in P \) can be translated to a \( q \in Q \), by specifying the corresponding histories resulting from \( p \) into \( q \). For example bid rotation (assuming \( s_1 = 1 \)) can be translated as, for \( t \geq 2 \),

\[
q^A_t = \begin{cases} 
1 & \text{if } s^{t-1} = (1, 2, 1, 2, \ldots) \text{ and } s_{t-1} = 2, \\
2 & \text{if } s^{t-1} = (1, 2, 1, 2, \ldots) \text{ and } s_{t-1} = 1, \\
\emptyset & \text{if } s^{t-1} \neq (1, 2, 1, 2, \ldots). 
\end{cases}
\]

New strategies are added in \( Q \). For example \( q_t = 0.3 \) could not be achieved in any \( p \in P \). With new strategies added in \( Q \), we should potentially obtain an lower bound to
support any profit-maximizing collusion not higher than the $\delta^*$ obtained in Theorem 1.2. Our next result shows that $\delta^*$ is still the lower bound in the new strategy set. In other words, adding a center does not help finding new strategies that permit a lower $\delta$. We first introduce another desirable feature.

**Definition 1.5.** A strategy profile $p$ is (ex-ante) fair if $v_1^1(p, \delta) = v_1^2(p, \delta)$.

That is, a fair strategy profile ensures that two players get the same ex-ante expected payoff before going to the auctions.

**Theorem 1.3.** With a center, there exists a profit-maximizing, fair PPE profile if and only if $\delta \geq \delta^*$.

**Proof.** The proof is essentially the same as Theorem 1.2. The only difference is that now we can assign $q_t \in [0, 1]$, compared to $q_t \in \{0, \frac{1}{2}, 1\}$. However, the relationship of $v_2^1 - v_1^1 = \frac{1}{\delta}$ stays unchanged. To see this, at $t$, for bidder 1 not to underbid, by bidding at $r - \varepsilon$, $\varepsilon > 0$,

$$q_t(1 + \delta v_1^1) + (1 - q_t)(0 + \delta v_2^1) \geq 1 - \varepsilon + \delta v_1^1 \implies v_2^1 - v_1^1 \geq \frac{1}{\delta} - \frac{\varepsilon}{2}$$

At $t$, for bidder 1 not to overbid, by bidding at $r' > r$,

$$q_t(1 + \delta v_1^1) + (1 - q_t)(0 + \delta v_2^1) \geq \delta v_2^1 \implies v_2^1 - v_1^1 \leq \frac{1}{\delta}$$

And the rest of the argument is identical to Theorem 1.2. \hfill $\Box$

We can see that even if we allow coin tosses before each auction, it does not help us find new schemes allowing less patient bidders to collude. In this sense result obtained in Theorem 1.2 is robust.

### 1.5 Concluding Remarks

This paper justifies bid rotation as a natural, simple collusion scheme which generically requires the minimum level of patience from the bidders. For future research, for the $n$
bidder case, finding the minimum required $\delta$ in the set of profit-maximizing schemes that allow tie-breaking is an interesting topic to pursue. Another unexplored area is to allow bidders to have uncertain costs during the auctions.
Chapter 2

Referring Customers under Imperfect Public Monitoring

2.1 Introduction

2.1.1 Motivation

Consider the situation when you send your car to a body shop (say store A). If store A cannot fix your problem, it will probably refer you another body shop (B). Then, you bring your car to B.

Suppose you want to buy children shoes, but you do not know where to get it. The natural thing to do is to search on the E-store (let’s say store A) you usually shop. If the store does not sell children shoes, it will probably pop out some suggested store (let’s say store B) based on your search.

Similar examples can be found everywhere in real life. Curious enough, people may wonder why these stores want to refer each other, what they can gain by mutual recommendation. In this paper, we wants to study the motivation of mutual recommendation.

We model such “return of favors” in a repeated game setting. Instead of looking at all strategy profiles players may adopt, we focus on public strategies, in which players make decisions in each period using only information from the commonly-observed history. That is, their private, past actions do not affect how they act in the current period. As we are only interested in pure strategy equilibria, such restriction on strategy profiles is without loss of generality (Mailath and Samuelson, 2006). We define history as the number of customers arrived in each store. Although the history is public, that is, all players observe the same history, the information on strategies that history reflects is not accurate. Looking at a day in which no customer arrived at one store, it might be

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1This is joint work with Minyan Zhu, graduate student from Rice University.
because a customer arrived at the other store but that store did not recommend him. Or, it might be simply a “bad day”, in which the customer did not want the store’s product in the first place and arrived at the other store. Solution concept used in this exercise is perfect public equilibrium (PPE), which requires public strategies to constitute a Nash equilibrium under any history for the repeated game (a counterpart of SPNE, with public strategies).

Beyond the context of stores recommending customers to one another, our model best describes the following more general scenario: An opportunity is randomly passed onto one of the players’ hand (say player 1). Because of the nature of this opportunity, only one player can take and enjoy this opportunity. In the case of player 1 not able to take the opportunity, he has the choice of passing the opportunity down to player 2. When an opportunity is not passed to player 2, player 2 has no knowledge whether player 1 took the opportunity himself, or player 1 could not take the opportunity but did not pass it to player 2 either, as passing down the opportunity incurs cost. Our model discovers how two players, without direct business connection can potentially benefit each other from these “misallocated opportunities”.

2.1.2 Related Literature

Literature on public or private monitoring in the Industry Organization context has been focusing on firms within the same industry. In the classical work of Green and Porter (1984), firms monitor a single market price while choosing their own quantities. That paper discovered a punishment scheme, in which punishment only takes place for a finite number of periods. In their model, firms are within the same industry. Their signal (market price) is a continuous variable. The richness of their signal in every period ensures that their trigger strategy from last period market price covers all symmetric sequential equilibria. In our model, firms are in different industries. Our signal (sale) is a binary variable. As our signal is discrete and limited, strategy considered in GP does not apply to our story. Our $k + 1$-punishment scheme makes more sense here: when the signal set is not rich, players want to wait and observe enough signals to implement
the punishment, as punishment is not optimal. Harrington and Skrzypacz (2007) argues that in some industries, prices are privately negotiated between firms and buyers. In case of an inelastic demand market, it establishes a strong impossibility result, that if relative shares of the market is not sensitive to price cuts, that is, price cuts cannot be observed by other firms, then no collusion schemes can be supported in equilibrium. It also provides an alternative modeling, arguing that transfer among firms can help sustaining collusion. In Harrington and Skrzypacz (2011), the authors completed their ideas in the 2007 paper. Its main finding is that given stochastic total demand, stochastic individual firms’ demands, with transfers, under some requirement of the parameters, collusion can sustain forever. In term of techniques in solving repeated games, Mailath and Samuelson (2006) provides a thorough exploration.

Although conveying similar ideas of how firms cooperate to achieve a long term collusive payoff, our paper is in the setting of inter-industry. A fundamental difference from an intra-industry set up lies in the incentive perspective. In our setting, firms do not compete for customers in a straight-forward sense. Store A may not want a customer who walks into Store B, if that customer does not want Store A’s product. Therefore, the sacrifice each store makes to sustain a cooperative relationship is by helping to recommend customers who walk into the wrong store. This is in contrast with the case in the intra-industry setting, in which firms maintain collusion by giving up customers which otherwise could be gotten by undercutting price or overproducing output. Our model is more general than previous models mentioned in the literature in the sense that we do not make firms cooperate or not cooperate explicitly through setting prices or quantities. “Recommending customers” can be viewed in general as firms doing favors to each others. It can be interpreted as firms charging a collusive price, or setting collusive quantities within the same industry. The idea is the same: to play collusively, firms scarifies current-period non-collusive profit to achieve a long term collusive profit.

Another strand of literature relevant to our paper is those on favor exchange models. All models that we have looked at (Mobius 2001, HH 2011 etc) assumes that in every period, both players need help. Player A’s problem can only be resolved by Player B,
and vice versa. In each period the probability to solve the other player’s problem is $p$, for both players. At any time, it is not permitted that both players can help each other simultaneously. Our model extends their models by allowing randomness on who needs help at any given time. Our motivation of doing this is not only that it is a more realistic assumption to make, but also more importantly because of the following: By distinguishing who needs help and who does not at a given period, when no help is given (in our story, no reference is done), no help means different things to the players: If A does not need help, and sees no help was delivered (either from A to B or B to A), he is fine with it as he does not need help in the first place. If B needs help, and see no help was delivered, he is less fine with it than A is. Then the trigger in the strategy profile we consider is based on players’ own consecutive bad signals. This strategy cannot be defined in previous models as those models do not distinguish after “bad signals”. Another contribution of our paper is that our model is robust in explaining asymmetric degrees of heterogeneity of the products that firms sell. The products can be very similar that customers have an equal probability to mistakenly walk into either store. Or, it can be the situation where customers are more likely to walk into one store by mistake than the other. An implication of such asymmetry is that the stores bear different costs of cooperation. The different incentives to cooperate can be modeled in our paper by setting different customer type parameters.

We investigate a class of punishment schemes: $k + 1$-signal punishment schemes. This scheme allows for $k$ consecutive bad signals to occur, but starts the punishment when $p$ the $k + 1$-th’s bad signal occurs. As explained earlier, the GP model assumes the set of the signal (price) space is very rich. Their signals provide sufficient information for the players to cooperate or punish. As we use customer arrivals as our signal set and assume in every period, either 1 or 0 customer arrives at a store, their strategy is inefficient compared to our proposed schemes. We characterize the set of PPEs within this class of schemes, prove and find the unique optimal PPE.

The rest of the paper is organized as follows: we introduce the model in section 2. In section 3, we discuss $k + 1$-punishment schemes and a baseline case of $k = 0$. 
which serves to explain the basic intuition in the conflict of interest in maintaining the cooperative relationship and deviation. In section 4, we provide an algorithm to find the optimal $k$, i.e., the number of forgiving periods, given a set of parameters. Moreover, we show that if PPE exists in the model, we can always find the unique $k$ which provides the best cooperative payoffs. In section 5, we discuss some numerical examples for our model and schemes. Section 6 concludes the paper.

2.2 The Model

Two stores A and B in the marketplace sell different products $a$ and $b$. We consider an infinitely repeated game. At the beginning of period $t$, One customer $C_t$ arrives at the marketplace looking for either $a$ or $b$. $C_t$ wants product $a$ with probability $p$ and $b$ with $1-p$. $C_t$ arrives at $A$ with probability $q$ and $B$ with $1-q$. $p$ and $q$ are independent. If $C_t$ arrives at the right store (looking for $a$ and arrives at $A$, or looking for $b$ and arrives at $B$), trade takes place between the respective store and $C_t$. Stores get $\pi_A$ or $\pi_B$ depending on where $C_t$ arrives. If $C_t$ arrives at the wrong store (looking for $a$ but arrives at $B$, or looking for $b$ but arrives at $A$), $A$ (or $B$) has the choice of Recommending (R) $C_t$ to the other store or Not recommending (N). That is, $a'_A, a'_B \in \{R, N\}$. If $a'_A = N$, $C_t$ stops shopping and leaves the market. If $a'_A = R$, $C_t$ goes to store $B$ and trades with $B$, $B$ earns a profit $\pi_B$ while $A$ incurs a cost $c_A$ by recommending $C_t$ to $B$. The same happens when $B$ has the choice of recommendation. We can think cost as the effort $A$ or $B$ makes through recommendation. In the E-store example of the introduction, the cost for recommendation is the space for advertisement, like the opportunity cost. Each E-store sacrifices the opportunity to gain revenue by advertising his own products or other paid firms. More general, the cost can be reputation cost, advertisement cost, effort cost, etc.

The common discount factor is $\delta$. Table 1 is the summary of payoffs of the stage game where $A$ is the row player and $B$ is the column player. To make our setting non-trivial, we assume that $p\pi_A - c_A(1-p)q \geq pq\pi_A$, $(1-p)\pi_B - c_B p(1-q) \geq (1-p)(1-q)\pi_B$, $pq\pi_A - c_A(1-p)q < pq\pi_A$ and $(1-p)(1-q)\pi_B - c_B p(1-q) < (1-p)(1-q)\pi_B$. 

\[ p\pi_A - c_A(1-p)q \geq pq\pi_A, \quad (1-p)\pi_B - c_B p(1-q) \geq (1-p)(1-q)\pi_B, \]
\[ pq\pi_A - c_A(1-p)q < pq\pi_A \quad \text{and} \quad (1-p)(1-q)\pi_B - c_B p(1-q) < (1-p)(1-q)\pi_B. \]
In our model, each player can only observe the number of customers entering each store. Because, under the mutual recommendation, no player can directly observe the other’s action. The only thing he can do is to infer the action through history. Define \( h_t \in H \) as the public signal at period \( t \), where \( H = \{01, 10, 11\} \) is the set of all possible public signals of arrival of the customer at time \( t \). “01” refers to that \( A \) sees no customer and \( B \) sees a customer. “10” refers to that \( B \) sees no customer and \( A \) sees a customer. “11” refers to that both \( A \) and \( B \) see a customer. This happens when one store refers the customer to another store. So implicitly, we assume that recommendation can be observed by both stores. We initialize \( h_0 = \{11\} \), i.e., two stores initially play \( R \) in the period 1. Define \( h_{t-1} = \{h_1, ..., h_{t-1}\} \in H^{t-1} \).

\( A \)'s public strategy at \( t \) is a function \( f_A : H^{t-1} \rightarrow \{R, N\} \) where \( a^t_A = f_A(h^{t-1}) \). Similarly, \( B \)'s public strategy at \( t \) is a function \( f_B : H^{t-1} \rightarrow \{R, N\} \) where \( a^t_B = f_B(h^{t-1}) \).

Let \( a_A = (a^0_A, a^1_A, ...) \) and \( a_B = (a^0_B, a^1_B, ...) \). A public strategy profile \((a_A, a_B)\) is a solution if it induces a perfect public equilibrium (PPE).

The sequence of moves at period \( t \) is summarized below:

- \( A \) and \( B \) both observe \( h^{t-1} \) at the start of period \( t \);
- \( A \) and \( B \) simultaneously choose actions in \( \{R, N\} \);
- Single customer arrives and what happens depends on the customer’s type and what the stores choose.
- \( h_t \in \{11, 01, 10\} \) realizes at the end of period \( t \).

### Table 1. The Stage Game

<table>
<thead>
<tr>
<th></th>
<th>( R )</th>
<th>( N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R )</td>
<td>( p \pi_A - c_A(1-p)q, (1-p)\pi_B - c_B p(1-q) )</td>
<td>( pq \pi_A - c_A(1-p)q, (1-p)\pi_B )</td>
</tr>
<tr>
<td>( N )</td>
<td>( p \pi_A, (1-p)\pi_B - c_B p(1-q) )</td>
<td>( pq \pi_A, (1-p)(1-q)\pi_B )</td>
</tr>
</tbody>
</table>
2.3 $k + 1$ Punishment Scheme

In this section, we introduce the class of schemes of this game. As we explained in the introduction, stores prefer to wait enough periods to punish for the bad signals. This is different from the standard model because of the limited size of the signal sets. However, delaying punishment causes another problem: players can gain by deviation from cooperative behaviors. On the one hand, delaying punishment provides higher equilibrium payoff when both players follow the cooperative actions. On the other hand, delaying punishment causes incentive problems. In order to balance these two, we propose $k + 1$ punishment scheme. Our goal is to find the largest number of forgiving periods with incentive compatibility and best equilibrium payoff we can do.

Consider the following event:

*Event A:* signal $(10)$ has occurred for $k$ consecutive periods, or $(01)$ has occurred for $k$ consecutive periods.

Event $A$ is the definition of $k$ bad signals. Here we define bad signals as the consecutive occurrences of either $(10)$ or $(01)$. This is because, the same signal has different meanings to different stores. Consider signal $(10)$, it is a good signal to store $A$ since store $A$ has one customer entering into its store; but it is a bad signal to store $B$ since store $B$ has no customer arrival at this period. If store $B$ always observe $(10)$, it may think that store $A$ does not play cooperative actions. However, if $(10)$ and $(01)$ occurs alternatively, store $B$ may think this situation is reasonable. Therefore, only the consecutive occurrences of either $(10)$ or $(01)$ will cause the problem. Thus, we define them as bad signals.

**Definition 1** ($k + 1$-signal punishment scheme): Players both play $R$ in the first $k$ periods. At any period $m \geq k + 1$: if Event $A$ occurs, players both play $N$ forever from period $m$. If Event $A$ did not occur, they both play $R$ at period $m$. 
To provide some intuition, we first investigate the baseline case when $k = 0$: the simplest form of punishment scheme that does better than noncooperation. This is a special case of the $k + 1$-signal punishment scheme with $k = 0$. Also this is the standard punishment scheme we use in the general model.

- **The single-signal punishment scheme**: Both stores play $R$ in the first period, and continue playing $R$ until either state (01) or (10) occurs, after which, both stores play $N$ forever.

Let $V^A(11), V^A(10)$ and $V^A(01)$ be the continuation value of $A$ when the previous signal is (11), (10) and (01), respectively. Based on the single-signal punishment scheme $RR$ is played when (11) occurs. For the other two cases, both players will play NN. Therefore, we get the following equation for $V^A(11), V^A(10)$ and $V^A(01)$.

\[
(1) \quad V^A(11) = (1 - \delta)(\pi_A - c_A(1 - p)q) + \delta[(p(1 - q) + (1 - p)q)V^A(11) + (1 - p(1 - q) - (1 - p)q)V^A(10)]
\]

\[
(2) \quad V^A(10) = V^A(01) = (1 - \delta)pq\pi_A + \delta V^A(01)
\]

From (2), we get $V^A(01) = V^A(10) = pq\pi_A$. Substituting (2) to (1), we get

\[
V^A(11) = (1 - \delta)(\pi_A - c_A(1 - p)q) + \delta\{[p(1 - q) + (1 - p)q]V^A(11) + (1 - [p(1 - q) + (1 - p)q])pq\pi_A\}
\]

This implies that

\[
V^A(11) = \frac{(1 - \delta)(\pi_A - c_A(1 - p)q) + \delta(1 - [p(1 - q) + (1 - p)q])pq\pi_A}{1 - \delta[p(1 - q) + (1 - p)q]}
\]

Now consider the only potentially beneficial one-shot deviation of firm $A$: firm $A$ plays $N$ given the previous history (11) at period $t$ and plays $R$ if (11) later occurs, or $N$ otherwise in the subsequent period. The continuation value of deviation is expressed as
\[ V_d^A = (1 - \delta)p\pi_A + \delta\{p(1 - q)V^A(11) + (1 - p(1 - q))V^A(10)\} \]

For the punishment scheme to be a PPE, we need \( V^A(11) \geq V_d^A \). That is,

\[ V^A(11) \geq \frac{(1-\delta)p\pi_A + \delta(1-p(1-q))pq\pi_A}{1-\delta p(1-q)} \]

which is equivalent to

\[ \frac{(1-\delta)(p\pi_A-c_A(1-p)q)+\delta(1-p(1-q)-(1-p)q)pq\pi_A}{1-\delta[p(1-q)+(1-p)q]} \geq \frac{(1-\delta)p\pi_A + \delta(1-p(1-q))}{1-\delta p(1-q)} \]

After simplifications, we can get the following condition:

\[ \frac{c_A}{\pi_A} < \frac{\delta p(1-q)}{1-\delta p(1-q)} \]

That is, the punishment scheme is easier for A to sustain when \( \frac{c_A}{\pi_A} \) is small, \( p(1-q) \) is big, and when \( \delta \) is big.

Use the same logic to solve for B, we get

\[ \frac{c_B}{\pi_B} < \frac{\delta q(1-p)q}{1-\delta(1-p)q} \]

That is, the punishment scheme is easier for B to sustain when \( \frac{c_B}{\pi_B} \) is small, \( q(1-p) \) is big and when \( \delta \) is big.

We can see that when the cost of recommendation is relatively small to profit from trade, and store owners are patient, this single-signal punishment scheme is easier to sustain. In an extreme case of \( c_A = 0 \), stores have no problems of recommending each other. And return of favors can sustain forever. However, the probabilities of the arriving customers’ types have different effects towards each store in maintaining cooperative relationship. These findings and intuitions apply to later schemes as well. In this single-signal punishment scheme, as soon as (01) or (10) takes place, punishment starts. In the
next section, we explore a more general forgiving scheme in which we forgive the other store when at most $k$ bad signals appear.

The probabilities $p(1-q)$ and $q(1-p)$ are worth mentioning. They are the probabilities that a customer walks into the wrong store. The difference between these probabilities stands for the relative responsibilities of the stores. In the cooperation phase, Store A wants more customers who need $a$, but arrive at $B$, so that although $A$ is obligated to help $B$, she does not have to actually do it very often. And Store $B$ wants the exact opposite probability distribution from what $A$ wants them to be. But when $p(1-q)$ is large, $(1-p)q$ is small. Thus in order to sustain the single-punishment scheme, $p, q$ cannot extremely favor or disfavor one store when $\frac{c_B}{\pi_B}$ and $\frac{c_A}{\pi_A}$ are similar.

2.4 Main Results

In this section, we introduced our main results. Proposition 1 provides the continuation value of cooperation for the firms and the condition in which $k+1$-signal punishment is an equilibrium. In the following, we assume $\pi_A = \pi_B = 1$.

Recall $V^A(11)$ is the continuation value when (11) is the last period signal. Define $V^A = V^A(11)$ and $V^A(r,s)$, $0 \leq r, s \leq k$ as the $A$'s continuation value of both stores playing $R$, given that signal (10) appears in the last $r$ consecutive periods and signal (01) appears in the last $s$ consecutive periods. Notice that for $V^A(r,s)$ to be well defined, at least one of $r$ and $s$ has to be 0. Let $C = (1-\delta)(p-c_a(1-p)q)$.

**Lemma 1** $V^A(m,0) < V^A(m-1,0)$ for all $1 \leq m \leq k$, where $V^A(0,0) = V^A$.

**Proof.**

As the recursive transformation formula of $V^A(m-1,0)$ is linear and increasing in $V^A(m,0)$, we only need to show that $V^A(k,0) > V_{NN} = pq$. This is true, because from $V^A(k,0) = C + \delta[pq + (1-p)(1-q)V^A(0,1) + (1-\alpha)V^A]$, we can see that every component in the second part of R.H.S is greater than or equal to $pq$. So the next period payoff is greater or equal to $pq$. We also know that $C > (1-\delta)pq$ (from our model.
Proposition 1. Under the $k+1$-signal punishment, $A$’s continuation value of cooperation is

$$V_A = C(1 + ABDE + AD + BE) + pq(A^{k+1} + B^{k+1} + AB^{k+1}D + A^{k+1}BE)$$

when $k \geq 2$, where $A = \delta pq$, $B = \delta(1-p)(1-q)$, $D = \frac{1-(\delta pq)^k}{1-\delta pq}$, $E = \frac{1-(\delta(1-p)(1-q))^k}{1-\delta(1-p)(1-q)}$.

Moreover, $k+1$-signal punishment is a PPE if and only if the followings are satisfied:

$$\frac{\delta c_A}{1-\delta} \leq V^A - V^A(1,0)$$

$$\frac{\delta c_B}{1-\delta} \leq V^B - V^B(1,0)$$

Where $V^A(1,0)$ and $V^B(1,0)$ are in terms of the parameters and defined in the appendix.

Proof of Proposition 1.

We first write down each $V^A(r, s)$ when $0 \leq r, s \leq k$.

Recall $C = (1-\delta)(p - c_A(1-p)q)$, $\alpha = 1 - pq - (1-p)(1-q)$,

$$V^A(k, 0) = C + \delta [pq pq + (1-p)(1-q)V^A(0,1) + (1-\alpha)V^A]$$

The continuation value of consecutive $k$ bad signals is composed of two parts: the payoff of the current period and the payoff of the future periods. The payoff of future periods is further determined by three possible signals: if (10) occurs in the next period, then both players play $N$ and can only get non-cooperative payoff; if (01) occurs, then the payoff changes to $V^A(0,1)$, since only 1 bad signal (01) occurs; if (11) occurs, then
the game is reset to the initial game, thus the payoff is $V_A$. The probabilities of these three signals are $pq, (1-p)(1-q)$ and $(1-\alpha)$ respectively. Following the same logic, we can write down all the value functions:

$$V_A(k - 1, 0) = C + \delta[pqV_A(k, 0) + (1-p)(1-q)V_A(0, 1) + (1-\alpha)V_A];$$

$$V_A(k - 2, 0) = C + \delta[pqV_A(k - 1, 0) + (1-p)(1-q)V_A(0, 1) + (1-\alpha)V_A];$$

$$\vdots$$

$$V_A(1, 0) = C + \delta[pqV_A(2, 0) + (1-p)(1-q)V_A(0, 1) + (1-\alpha)V_A];$$ and

$$V_A = C + \delta[pqV_A(1, 0) + (1-p)(1-q)V_A(0, 1) + (1-\alpha)V_A].$$

Recursively,

$$V_A(m - 1, 0) = C + \delta[pqV_A(m, 0) + (1-p)(1-q)V_A(0, 1) + (1-\alpha)V_A]$$

for $1 \leq m \leq k$.

Therefore,

$$V_A(1, 0) = (\delta pq)^{k-1}V_A(k, 0)$$

$$\quad + \sum_{i=0}^{k-2} (\delta pq)^i[\delta(1-p)(1-q)V_A(0, 1) + \delta(1-\alpha)V_A + C]$$

$$= (\delta pq)^{k-1}\{C + \delta[pq + (1-p)(1-q)V_A(0, 1) + (1-\alpha)V_A]\}$$

$$\quad + \sum_{i=0}^{k-2} (\delta pq)^i[\delta(1-p)(1-q)V_A(0, 1) + \delta(1-\alpha)V_A + C]$$

$$= (\delta pq)^k pq + \sum_{i=0}^{k-1} (\delta pq)^i[\delta(1-p)(1-q)V_A(0, 1)$$

$$\quad + \delta(1-\alpha)V_A + C] \forall k \geq 2$$

$$V_A(1, 0) = \frac{1-(\delta pq)^k}{1-\delta pq} \delta(1-p)(1-q)V_A(0, 1)$$
\[ + \frac{1 - (\delta pq)^k}{1 - \delta pq} (\delta(1 - \alpha)V^A + C) + (\delta pq)^k pq. \]

Similarly,

\[
V^A(0, 1) = \frac{1 - (\delta(1 - p)(1 - q))^k}{1 - \delta(1 - p)(1 - q)} \delta pq V^A(1, 0)
+ \frac{1 - (\delta(1 - p)(1 - q))^k}{1 - \delta(1 - p)(1 - q)} (\delta(1 - \alpha)V^A + C) + (\delta(1 - p)(1 - q))^k pq.
\]

Solve for Equations for \(V^A(1, 0)\) and \(V^A(0, 1)\), let \(A = \delta pq\), \(B = \delta(1 - p)(1 - q)\), \(D = \frac{1 - (\delta pq)^k}{1 - \delta pq}\), \(E = \frac{1 - (\delta(1 - p)(1 - q))^k}{1 - \delta(1 - p)(1 - q)}\), we have

\[
V^A(1, 0) = \frac{D(EB + 1)(\delta(1 - \alpha)V^A + C) + pq(A^k + DBk + 1)}{1 - ABDE}
\]

\[
V^A(0, 1) = \frac{E(DA + 1)(\delta(1 - \alpha)V^A + C) + pq(B^k + EA k + 1)}{1 - ABDE}
\]

Substitute the previous two equations into \(V^A\), we have

\[
V^A = \frac{C(1 + ABDE + AD + BE) + pq(A^{k+1} + B^{k+1} + AB^{k+1} D + A^{k+1} BE)}{1 - ABDE - \delta(1 - \alpha)(1 + ABDE + AD + BE)}
\]

For the incentive constraints part, we consider all the deviation payoffs under different histories. Define \(V^d_A(r, s)\) as the deviation payoff given \((r, s)\). Define \(C' = (1 - \delta)p\) as the payoff under one-shot deviation.

\[
V^d_A = C' + \delta[qV(1, 0) + (1 - p)(1 - q)V(0, 1) + (1 - q - (1 - p)(1 - q))V^A]
\]

\[
V^d_A(1, 0) = C' + \delta[qV(2, 0) + (1 - p)(1 - q)V(0, 1) + (1 - q - (1 - p)(1 - q))V^A]
\]

\[
V^d_A(0, 1) = C' + \delta[qV(1, 0) + (1 - p)(1 - q)V(0, 2) + (1 - q - (1 - p)(1 - q))V^A]
\]
\[ V_d^A(2, 0) = C' + \delta [q V(3, 0) + (1-p)(1-q)V(0, 1) + (1-q-(1-p)(1-q))V^A] \]

\[ V_d^A(0, 2) = C' + \delta [q V(0, 1) + (1-p)(1-q)V(0, 1) + (1-q-(1-p)(1-q))V^A] \]

Comparing these deviation payoffs with continuation payoffs obtained earlier, we have

\[ V^A \geq V_d^A \iff \frac{\delta c_A}{1 - \delta} \leq V^A - V^A(1, 0) \]

\[ V^A(1, 0) \geq V_d^A(1, 0) \iff \frac{\delta c_A}{1 - \delta} \leq V^A - V^A(2, 0) \]

\[ V^A(0, 1) \geq V_d^A(0, 1) \iff \frac{\delta c_A}{1 - \delta} \leq V^A - V^A(1, 0) \]

\[ V^A(2, 0) \geq V_d^A(2, 0) \iff \frac{\delta c_A}{1 - \delta} \leq V^A - V^A(3, 0) \]

\[ V^A(0, 2) \geq V_d^A \iff \frac{\delta c_A}{1 - \delta} \leq V^A - V^A(1, 0) \]

From Lemma 1, we know that \( V^A(m, 0) < V^A(m - 1, 0) \) for all \( 1 \leq m \leq k \). Therefore, the only binding inequality for the scheme to be an PPE is

\[ \frac{\delta c_A}{1 - \delta} \leq V^A - V^A(1, 0) \]

Similarly, for B, we have

\[ \frac{\delta c_B}{1 - \delta} \leq V^B - V^B(1, 0) \]

We next establish the existence of the unique \( k \) such that the \( k + 1 \) punishment scheme yields the largest payoff for player A.

**Theorem 1:** If the single punishment scheme is a PPE, then there exists a unique
$K^* \geq 1$, such that the $K^* + 1$-punishment scheme is the unique optimal PPE in the class of $k + 1$-punishment schemes.

In order to prove the theorem, we would like to consider 2 schemes in the following 2 lemmas: $(k - 1) + 1$-punishment and $k + 1$-punishment. To distinguish $V^A(m, 0)$ under different schemes, we use $V^A_k(m, 0), V^A_k$ to represent the continuation payoff under $k$-punishment and use $V^A_{k+1}(m, 0), V^A_{k+1}$ to represent the continuation payoff under $k + 1$ punishment.

**Lemma 2** $V^A_{k+1}$ is increasing in $k$.

**Proof.**

From proposition 1, we know that

$$V^A_{k+1} = \frac{C(1 + ABDE + AD + BE) + pq(A^{k+1} + B^{k+1} + AB^{k+1}D + A^{k+1}BE)}{1 - ABDE - B(1 - \alpha)(1 + ABDE + AD + BE)}.$$

To show $V^A_{k+1}$ is increasing in $k$, we need to show the derivative of $V^A_{k+1}$ with respect to $k$ is positive.

Based on the definition of $D$ and $E$, it is easy to show that the denominator decreases when $k$ increases. Thus, we only need to show the numerator is increasing when $k$ increases.

$$C(1 + ABDE + AD + BE) + pq(A^{k+1} + B^{k+1} + AB^{k+1}D + A^{k+1}BE)$$

$$= C \frac{(1 - \alpha)(1 - B) + AB(1 - A)(1 - B) + A(1 - A)(1 - B) + B(1 - B^k)(1 - A) + pq(A^{k+1} + B^{k+1})(1 - A)(1 - B) + AB^{k+1}(1 - A)(1 - B) + A^{k+1}B(1 - B^k)(1 - A)}{(1 - \alpha)(1 - B)}$$

When taking the derivative, we are only interested in the terms containing $A^k, B^k$ and $A^kB^k$. Grouping them into three parts:
• $-CB A^{k+1} - C(1 - B)A^{k+1} + pq(1 - A)(1 - B)A^{k+1} + pqBA^{k+1} - pqBA^{k+2}$

• $-CA B^{k+1} - C(1 - A)B^{k+1} + pq(1 - A)(1 - B)B^{k+1} + pqAB^{k+1} - pqAB^{k+2}$

• $CA^{k+1}B^{k+1} - 2pqA^{k+1}B^{k+1} + pqA^{k+1}B^{k+2} + pqA^{k+2}B^{k+1}$

Take the derivative with respect to $k$, we separately consider the coefficient of $\ln(A)$ and $\ln(B)$. For $\ln(A)$, we get

$$[-C + pq(1 - A)(1 - B) + pqB]A^{k+1} - pqBA^{k+2} + CA^{k+1}B^{k+1} - 2pqA^{k+1}B^{k+1} + pqA^{k+1}B^{k+2} + pqA^{k+2}B^{k+1}$$

$$= A^{k+1}[-C + pq(1 - A)(1 - B) + pqB - pqAB] + A^{k+1}[CB^{k+2} - 2pqB^{k+1} + pqB^{k+2} + pqAB^{k+1}]$$

The second term $CB^{k+2} - 2pqB^{k+1} + pqB^{k+2} + pqAB^{k+1}$ converges to zero as $k$ increases. Consider the first term:

$$A^{k+1}[-C + pq(1 - A)] \leq A^{k+1}[-C + pq] < 0.$$ Therefore, the coefficient of $\ln(A)$ is negative. Since $A = \delta pq < 1$, $\ln(A) < 0$. Combing these two together, the derivative of $\ln(A)$’s part is positive. Similarly, we can show the derivative of $\ln(B)$’s part is also positive. Therefore, the numerator of $V^A_{k+1}$ increases in $k$. \[\]

**Lemma 3** Given $(p, q, c_A, \delta)$, we can always find an $K^*_A$ such that $V^A_k - V^A_1(1, 0)$ is decreasing in $k$ for all $k \geq K^*_A$. \[2\]

**Proof.**

From the definitions of the value functions, we have the following:

\[
^2\text{One might consider express } V^A_k - V^A_1(1, 0) \text{ in terms of } k \text{ and other given parameters and compute the derivatives. The authors have done that and due to the complexity of that expression, not able to do it directly. We have come up with the following constructive proof.}
\[ V_k^A - V_k^A(1, 0) = (\delta pq)^{k-1}(V_k^A(k - 1, 0) - pq), \text{ and} \]

\[ V_{k+1}^A - V_{k+1}^A(1, 0) = (\delta pq)^k(V_{k+1}^A(k, 0) - pq) \]

Therefore, to show

\[ V_{k+1}^A - V_{k+1}^A(1, 0) < V_k^A - V_k^A(1, 0) \]

We need to show

\[ (3) \quad (\delta pq)(V_{k+1}^A(k, 0) - pq) < V_k^A(k - 1, 0) - pq \]

Also from the definitions,

\[ V_{k+1}^A(k, 0) - V_{k+1}^A(1, 0) = \delta pq(pq - V_{k+1}^A(k, 0)) \]

we can get

\[ V_{k+1}^A(k, 0) = \frac{\delta pq \times pq + V_{k+1}^A(k-1, 0)}{1 + \delta pq}. \]

Thus, (3) is equivalent to

\[ \delta pq \frac{\delta pq \times pq + V_{k+1}^A(k-1, 0) - pq - \delta(pq)^2}{1 + \delta pq} < V_k^A(k - 1, 0) - pq, \]

which is also equivalent to

\[ \frac{V_{k+1}^A(k-1, 0) - V_k^A(k-1, 0)}{V_k^A(k-1, 0) - pq} < \frac{1}{\delta pq}. \]

As \( \delta pq < 1 \), it is sufficient to prove that

\[ \frac{V_{k+1}^A(k-1, 0) - V_k^A(k-1, 0)}{V_k^A(k-1, 0) - pq} < 1, \text{ i.e.,} \]
Define \( V^e = p \pi_A - c_A(1-p)q \). Given any \( 0 < \epsilon \leq (1-\delta pq)(V^e - pq)/2 \), \( \exists K_A^* > 0 \), such that \( \forall k \geq K_A^* \), the following inequalities are all satisfied:

(4) \[ \epsilon < (1-\delta pq)(V^A_k(k-1) - pq) \]

(5) \[ V^A_{k+1}(01) - V^A_k(01) < \frac{\epsilon}{\delta(1-pq)(1+\delta pq)} \]

(6) \[ V^A_{k+1} - V^A_k < \frac{\epsilon}{\delta(1-pq)(1+\delta pq)} \]

Recall that \( V^A_k(01) \) stands for the continuation value for A under \((k-1) + 1\) punishment scheme, and the previous signal was \((01)\).

(4), (5), and (6) are true because \( V^A_k(k-1,0) \to V^e \). \( V^A_k = V^A_{k+1} \), and \( V^A_k(01) = V^A_{k+1}(01) \) as \( k \to \infty \). From

\[
V^A_{k+1}(k-1,0) = C + \delta[pqV^A_{k+1}(k,0) + (1-p)(1-q)V^A_{k+1}(0,1) + (1-\alpha)V^A_{k+1}] \quad \text{and} \\
V^A_k(k-1) = C + \delta[pq\pi + (1-p)(1-q)\pi_A(0,1) + (1-\alpha)\pi_A],
\]

we can get

\[
V^A_{k+1}(k-1,0) - V^A_k(k-1,0) < \delta pq(V^A_{k+1}(k) - pq) + \frac{\delta(1-pq)\epsilon}{\delta(1-pq)(1+\delta pq)}.
\]

Moreover from

\[
V^A_{k+1}(k,0) = C + \delta[pq\pi + (1-p)(1-q)\pi_A(01) + (1-\alpha)\pi_A] \quad \text{and} \\
V^A_k(k-1) = C + \delta[pq\pi + (1-p)(1-q)\pi_A(01) + (1-\alpha)\pi_A],
\]

we can get
0 < V_{k+1}^A(k, 0) - V_k^A(k-1, 0) < \frac{\delta(1-pq)\epsilon}{\delta(1-pq)(1+\delta pq)}. This implies

V_{k+1}^A(k, 0) - pq - \frac{\delta(1-pq)\epsilon}{\delta(1-pq)(1+\delta pq)} < V_k^A(k-1, 0) - pq.

Therefore,

V_{k+1}^A(k - 1, 0) - V_k^A(k - 1, 0)

< \delta pq(V_k^A(k - 1, 0) - pq) + \frac{\delta pq\delta(1-pq)\epsilon}{\delta(1-pq)(1+\delta pq)} + \frac{\delta(1-pq)\epsilon}{\delta(1-pq)(1+\delta pq)}

= \delta pq(V_k^A(k - 1, 0) - pq) + \epsilon

< (\delta pq + 1 - \delta pq)(V_k^A(k - 1, 0) - pq). \blacksquare

Proposition 1, Lemma 1-3 are shown for store A, we can get the similar results for store B. In the following, we provide a proof for theorem 1.

**Proof of Theorem 1.**

We need to find the largest $k$ such that the resulting $k + 1$ punishment is still a PPE. From Lemma 3, we know that for all $k \geq K_A^*$, $V_k^A - V_k^A(1, 0)$ is decreasing in $k$. We start by testing $K = K_A^*$, if the incentive constraint is satisfied for the $K_A^* + 1$-punishment scheme, then test $K = K_A^* + 1$, etc until we find a $K_A^{**}$ such that $K_A^{**}$ is a PPE but $(K_A^{**} + 1)$ is not. By Lemma 3, we know that from $K_A^{**} + 1$ onwards, the incentive constraint will all be violated, thus $K_A^{**}$ is the largest $k$ we look for.

On the other hand, If the incentive constraint for A violates in the $K_A^* + 1$-punishment scheme, we know that all $k + 1$-punishment schemes with $k \geq K_A^*$ cannot be a PPE. Then we start testing $K = K_A^* - 1$ to see if it satisfies incentive constraint. If not, we try $K = K_A^* - 2$, etc until we find the first $k$, denote it as $K_A^{**}$. This $K_A^{**}$ is the largest $k$ we look for. From Lemma 2, we also know $V_k^A$ is increasing in $k$, therefore $K_A^{**}$ we find
is largest payoff for A in the cooperative equilibrium.

Similarly we can find $K_B^{**}$. Take the minimum, $K^* = \min\{K_A^{**}, K_B^{**}\}$ is the unique optimal $k$ that satisfies incentive constraints for both players A and B. It also provides the largest equilibrium payoff for both players.

Next, we provide an algorithm to find the optimal $K^*$.

**Algorithm 1.**

- Set $\epsilon_A = (1 - \delta pq)(V_A^e - V_{NN})$, and let $K_A^*$ be the minimum value such that inequalities (4), (5), and (6) are all satisfied.

- If the resulting $K_A^* + 1$ punishment scheme satisfies the incentive compatibility constraint for Store A, then increase $K_A^*$ one by one until $K_A^{**}$, the largest number such that the incentive compatibility constraint for Store A is satisfied;

- If the resulting $K_A^* + 1$ punishment scheme does not satisfy the incentive compatibility constraint for Store A, then decrease $K_A^*$ one by one until $K_A^{**}$, the largest number such that the incentive compatibility constraint for Store A is satisfied.

- Do the same for Store B and find $K_B^{**}$.

- $K^* = \min\{K_A^{**}, K_B^{**}\}$.

**Proposition 2.** $K^*$ found in Algorithm 1 is the unique k that defines the optimal PPE in the class of $k + 1$-punishment schemes.

### 2.5 Numerical Examples

By Theorem 1 and Proposition 2, given any set of parameters $p, q, \delta, c_A, c_B$, we can always find an optimal PPE for the class of $k + 1$-punishment schemes. In this section, we want to provide some numerical examples to discuss our $k + 1$-punishment schemes. We assume $c_A = c_B$. 
2.5.1 Symmetric Case When $p = q = 0.5$

We first consider a symmetric case, where $p = q = 0.5$. In this case, we assume customers want and enter in each store with the same probability. Therefore, the stand-alone payoff for both store is 0.25 assuming $\pi_A = \pi_B = 1$. However, if both store recommends each other, and no one deviates, the payoffs for the two stores are $0.5 - 0.25c_A$ and $0.5 - 0.25c_B$.

Now we consider different combination of $c_A, c_B, \delta$ for finding the optimal $k$.

The left table depicts the optimal $k$, the number of largest forgiving periods under the different combinations of $\delta$ and $c_A$. The right figure depicts the graphs of the table when the cost is 1%, 4%, 8% and 20% of the profit. When players are patient and the cost is small, they would like to forgive more periods compared to the other cases. With $\delta$ decreases and $c_A$ increases, player’s choices are more and more close to the standard punishment: forgive a small number of periods and punish as soon as possible. This is because both players care more about the current payoffs and the cost for recommendation is large. Also note, if the cost is 15% of the profit $\pi_A(\pi_B)$, equilibrium exists with the extremely patient players, which can be seen from the last two columns of the left table. In this case, two players have the same optimal $k$.

2.5.2 General Case

When we consider different values of $p, q$, we assume $c_A = c_B = 0.02$, that is cost is only 2% of the profit. From section 5.1, we know that $k$ increases when $c_A, c_B$ decreases. Table 2 depicts the case of $\delta = 0.99$, i.e., when players are extremely patient. The first row
represents the value of \( p \) and the first column represents the value of \( q \). The table reads as follows: suppose the pair of \( (p, q) \) is \((0.35, 0.4)\), the maximum \( k \) we can find to support equilibrium is 2. We can see that when \( p \) and \( q \) are near 0.5, \( k \) is largest and when \( p, q \) are near 0, \( k \) is smallest. This is because, if \( p, q \) strongly favor one player, this player has larger \( k \) to support the equilibrium than the other player. However, in order to find the equilibrium, we need to take the minimum of the two \( k \) values, like the algorithm we provide from the previous section. If \( p, q \) are near 0 or 1, one player has larger \( k \) while the other has very small \( k \), which makes the minimum of \( k \) even less. On the other hand, we can see \( k \) is the same under \( (p, q) \) and \( (1-p, 1-q) \), which is always true by our game.

Table 3 consider the case when \( \delta = 0.9 \). Comparing Table 2 and Table 3, players become impatient. If players are impatient, they care more about the current payoff, thus intolerant to the bad signals. That is why in general, \( k \) decreases when \( \delta \) decreases.

### 2.6 Concluding Remarks

In this paper, we looks at a market place consisting of two firms. Each firm can choose refer or not refer the other firm when a customer mistakenly enter into the store. Due
to the imperfect observation of firms’ actions, they can only infer the action through a
signal. The main difference between our model and other models is the limited size of the
signal set. We only consider the binary signals and because of which, players may want to
wait for enough periods to implement punishment. We mainly discuss a class of schemes,
general $k + 1$-punishment schemes, when firms forgive the first $k$ bad signals and start to
punish the other forever after the reveal of the $k + 1$th bad signal. We first provide the
continuation value for the players and the condition to satisfy incentive compatibility.
Moreover, we show that for any given parameters, we can always find an optimal $k$ which
supports the cooperative equilibrium and provides the highest payoff. This payoff is very
close to the efficient payoff, which is significantly better than our baseline model.
Chapter 3

Modeling Success of Bank Telemarketing and Elasticity of Demand

3.1 Introduction

Price Elasticity of Demand (PED) measures how sensitive market demand is to price changes. With the knowledge of PED, policy makers can make predictions on demand impact on the market. The following chart is an illustration of the policy making process of changing country-wide and state level prices. Suppose that a policy maker wants to increase the country-wide price of the long term deposit by 8%. If we know, through statistical modeling, that the country wide PED is -1.3, we can infer that demand country-wide is expected to drop by 10.4%. To achieve this 8% increment country-wide, each state needs to play its part. If, for example, it is decided that TX needs to raise price by 7%, NY by 10% so on and so forth, with the knowledge of state level PED’s, we can infer demand impact from price changes at a state level.

To draw inferences on PED’s at different levels, we start with modeling propensity, the likelihood that an individual buys long-term deposit. We use a Portuguese banking dataset from 2008 to 2013. Next we propose methods so that we can draw inferences on PED’s. Since we do not have data on interest rate changes, this part mainly provides a direction for future work. Given the price change data, for each observed price change, we find relevant customers before and after the price change, and calculate the respective PED for that price change. For our dataset, each individual is only recorded whether the call was successful or not. Therefore it is possible that the before-price-change-group and after-price-change-group are different in certain aspects, say gender or credit distribution. This group differentiation can potentially bias our analysis of PED from price changes: For an extreme case, if for a certain price change, say 5%, only men are quoting before the
change, and only women are quoting after the change, then any observed demand impact is more likely to result from gender differences, than a 5% price increase. To counter for this problem from potential group differentiation, we use Propensity-Score-Matching to select two subgroups from the before and after groups, so that the selected subgroups are more similar (defined later in Section 5) than the original groups. The rest of the paper is organized as follows. Section 2 gives a review of PED. Section 3 introduces the dataset. Section 4 describes the modeling procedure on propensity to bind. Section 6 discusses how we do propensity score matching on different groups before and after price change. Section 7 describes methods to draw inferences on PEDs and some simulation results. Section 8 concludes the paper with future directions.

### 3.2 Background of Price Elasticity of Demand

Price Elasticity of Demand (PED) measures how sensitive market demand is to price changes. There are many ways to define PED. In this project, we use a common definition:

\[
PEDat(Q_b, P_b) = \frac{\text{percentage in quantity demanded}}{\text{percentage change in price}} = \frac{(Q_a - Q_b)/Q_b\%}{(P_a - P_b)/P_b\%}
\]
where $Q_b(Q_a)$ is the market demand (Number of Binds) before (after) the price change; $P_b(P_a)$ is the price before (after) the price change. For example, if $Q_b = 100, Q_a = 80, P_b = 20, P_a = 30$, then

$$PED_{at}(100, 200) = \frac{(800 - 100)/100}{(30 - 20)/20} = -0.4$$

Note that PED is defined at a particular demand-price point. Usually, we say that the market is elastic if PED<-1. In this case when price increases, revenue (=price*demand) decreases. The market is inelastic if PED>-1. In this case, when price increases, revenue increases as well.

### 3.3 Dataset

The dataset used to model conversion consists of 44307 telemarketing calls from 2008 to 2013. I split it to a training (90%) and a testing dataset (10%).

It consists of 21 variables. The response variable of interest is $y$, which is 1 or 0, indicating whether the customer purchased the long term deposit. It is relatively a very clean dataset, with categories clearly defined and very few missing values. I group these variables into 3 broad categories.

- **Personal**: variables that are particular to the potential customer. Examples include customer’s age, marital status, education level, etc.

- **Call Information**: variables that describe the details of telemarketing call. For example, it includes date and time of the call, duration of the call etc.

- **Economic variables**: variables that relate to economic conditions. These include employment variation rate, consumer price index, etc.

A detailed list of the variables can be found in the following website address [http://archive.ics.uci.edu/ml/datasets/Bank+Marketing](http://archive.ics.uci.edu/ml/datasets/Bank+Marketing).
3.4 Modeling Procedure

This section discusses procedures in modeling conversion. We use a logistic regression to model the binary outcome \( y \) (success or fail). For variable selection, we first use our intuition to screen off variables that we think not relevant to the outcome, regrouping variables to be more representative to the characteristic it represents (say pdays). Then we use a backward selection to select the rest of the variables. The complete model selection and resulting outcome summarized in Table 3.4. Figure 3.4 plots the validation and univariate plots from the final model.

We see that from the ROC curves and AUCs across the training (90%) and validating (10%) dataset, we are confident that our prediction on \( y \) is accurate and consistent for different datasets. The next plot below is a decile chart on the validating set (10%). On the horizontal axis, we rank the observation according their predicted probability to bind. We see a consistent prediction across different deciles.

Figure 3.4 plots actual vs predicted probability across different age groups. We see good predictions across all age groups except for the group 33.1-35. For this group of people, I suspect that most of them are from working classes, and perhaps are too busy to take calls seriously. Also, we do not observe a monotone relationship between conversion rate and age. We see that conversion rate decreases until age 34, and starts to increase from there. Conversion rate is considerably higher at high age groups (55-88), indicating that old people are more likely to be converted into business.

3.5 PED Inference and Propensity Score Matching (PSM)

In this section, we describe the method to draw inferences on PED from results from Section 4. As we do not have data on price (interest rate), we use similar work on a separate project in modeling price elasticity of demand for an insurance market in the USA. The data is modified in order to protect the company’s business secret. The point of this section is to illustrate the methodology of modeling PED when we have the data on price change.

In 3.5.1, we first describe how we draw inferences from price change data and
|                          | Estimate | Std. Error | z value | Pr(>|z|) |
|--------------------------|----------|------------|---------|----------|
| (Intercept)              | -107.68  | 4.81       | -22.4   | 0.00     |
| jobblue-collar           | -0.21    | 0.06       | -3.8    | 0.00     |
| jobentrepreneur          | -0.08    | 0.11       | -0.8    | 0.44     |
| jobhousemaid             | -0.14    | 0.12       | -1.2    | 0.24     |
| jobmanagement            | -0.05    | 0.07       | -0.7    | 0.49     |
| jobreitired              | 0.19     | 0.08       | 2.6     | 0.01     |
| jobself-employed         | -0.06    | 0.10       | -0.6    | 0.55     |
| jobservices              | -0.17    | 0.07       | -2.4    | 0.02     |
| jobstudent               | 0.19     | 0.09       | 2.1     | 0.03     |
| jobtechnical             | -0.04    | 0.06       | -0.7    | 0.47     |
| jobunemployed            | -0.06    | 0.11       | -0.6    | 0.55     |
| jobunknown               | -0.15    | 0.21       | -0.7    | 0.47     |
| defaultunknown           | -0.25    | 0.06       | -4.3    | 0.00     |
| contacttelephone         | -0.59    | 0.06       | -9.9    | 0.00     |
| monthaug                 | 0.18     | 0.09       | 2.0     | 0.05     |
| monthdec                 | 0.33     | 0.18       | 1.9     | 0.06     |
| monthjul                 | 0.14     | 0.08       | 1.7     | 0.09     |
| monthjun                 | -0.23    | 0.08       | -2.9    | 0.00     |
| monthmar                 | 1.19     | 0.11       | 11.2    | 0.00     |
| monthmay                 | -0.53    | 0.07       | -8.1    | 0.00     |
| monthnov                 | -0.36    | 0.08       | -4.3    | 0.00     |
| monthoct                 | -0.02    | 0.11       | -0.2    | 0.84     |
| monthsep                 | -0.08    | 0.12       | -0.6    | 0.52     |
| day_of_weekmon           | -0.21    | 0.06       | -3.7    | 0.00     |
| day_of_weekthu           | 0.07     | 0.06       | 1.2     | 0.22     |
| day_of_weektue           | 0.05     | 0.06       | 0.9     | 0.37     |
| day_of_weekwed           | 0.15     | 0.06       | 2.6     | 0.01     |
| campaign                 | -0.04    | 0.01       | -4.9    | 0.00     |
| pdays                    | -0.00    | 0.00       | -5.5    | 0.00     |
| poutcomenonexistent      | 0.54     | 0.06       | 9.5     | 0.00     |
| poutcomesuccess          | 0.81     | 0.19       | 4.3     | 0.00     |
| emp_var_date             | -0.76    | 0.02       | -36.7   | 0.00     |
| cons_px_index            | 1.15     | 0.05       | 22.3    | 0.00     |
| cons_conf_index          | 0.03     | 0.00       | 6.3     | 0.00     |

Table 3.1: Coefficients Estimates of the Final Model
Figure 3.2: ROC for training (left: AUC=0.794) and validating (right: AUC=0.785)

Figure 3.3: Decile Chart on Testing Dataset
regression results obtained from the previous section. We then introduce PSM to deal with potential group heterogeneity problem in 3.5.2. Then we report country-wide and state level PED results for the landlord new business market in 3.5.3.

### 3.5.1 PED Inference

Figure 3.5.3 summarizes how we draw inferences from PL rate change data and regression results. For price change data, we have documented premium rate changes (%) in all states at different times. We use the following rules (in order) to subset the relevant rate changes for our PED calculations:

- Rule 1: If two rate changes occurred within 1 month, combine them and use the NB effective date from the first rate change;
- Rule 2: If a prior rate change is within 5 months, drop the current rate change.

These rules are here to find the more relevant price changes. We do not want to have too many price changes to blur our view on market impacts. As a result of these rules, maximum only three price changes per year may happen. For example, if for the same year, we have rate changes occurring at 1/1, 4/1, 7/1, 10/1 and 12/31, then we see that first rule does nothing as we do not have any consecutive rate changes within 1 month. However the second rule says keep 1/1 and throw away 4/1. Then 7/1 is more than 5 months away from 1/1 (4/1 is already thrown away) and kept. Similarly 10/1 is thrown away and 12/31 is kept. So the resulting rate changes left for this year are 1/1, 7/1 and 12/31. After applying these rules to all rates changes, 191 rate changes are kept (most states have 2 rate changes per year).

One concern of these rules is that they are somehow arbitrary. The 'maximum 3 relevant changes per year' lacks scientific validation. Although these rules come from practical experience, vigorous future market research on the relevant interval is preferred. From the regression results, for each individual 1, 2... we have predicted probability of that individual binding with us: p1, p2....

We look at one rate change that took place in Texas, 8/1/2015, for a 10% increase for instance. We can draw a time line (as shown in the above graph), centering at 8/1/2015.
Figure 3.4: Conversion on Age on Testing Dataset

- **Price change data**
  - TX 8/1/2015 10%
  - NY 4/1/2014 -5% ...

- **Regression Results**
  - $p_1 = 0.875$
  - $p_2 = 0.678$ ...

\[
Q_b = \sum p_i \quad \text{90 days before} \\
Q_a = \sum p_i \quad \text{90 days after}
\]

8/1/2015
TX

\[
\text{PED (TX 8/1/2015 10%)} = \frac{(Q_a - Q_b)}{Q_b} \times 10\%
\]

Figure 3.5: PED Calculation
Then we look for a 3-month window before and after 8/1/2015 to define the 'before-change' and 'after-change' quote groups. The actual demand before the change is the number of people who bought insurance with us up to 3 months before the rate change date. The predicted demand before the change is the summed probabilities of each individual binding with us. Similarly the actual and predicted demand after the rate change can be calculated. Then the actual and predicted PED’s for that rate change can be calculated as follows:

\[
\text{Actual PED (TX 8/1/2015 10\%) } = \frac{(B_a - B_b) / B_b\%}{10\%} \\
\text{Predicted PED (TX 8/1/2015 10\%) } = \frac{(Q_a - Q_b) / Q_b\%}{10\%}
\]

Where \(B_a\) (\(B_b\)) is the number of binds after (before) the rate change.

### 3.5.2 Propensity Score Matching

Before we proceed and calculate PED’s according to the above formulae, we should take caution of the potential differences between the before and after groups. Although we control for such differences in modeling the likelihood to bind (in logistic regression in Section 6), we have no control over who quote from us. It might be well possible, for an extreme case, that for a given price change (say +5\%), only men are quoting before the price change and only women are quoting after the price change. In that case any PED inference we draw may come from the gender difference, rather than a 5\% increase in premium.

Another common scenario we observe is that the number of people quoting before and after a price change can be very different. For example, for NY, 5/1/2012, -2.6\%, we see about 300 people quoting from us before the price change (3 month window) and 1100 people quoting from us after the price change. Although the conversion rates are almost the same (33\% and 35\%), PED as we discussed from Section 2 (comparing PEC) gives
us PED=-111.11, rather than a -2.6 calculated by PEC. Therefore we definitely should control for scales of people quoting from us.

Propensity Score matching solves the above two problems at the same time. Figure 7 provides an illustration of how PSM works. For example, we have 700 quotes before the rate change (Group A), and 300 quotes after the rate change (Group B). We first create a variable $y$ and label all 700 in A as ‘1’, all 300 in B as ‘0’. Then we model $y$ against demographic variables (x’s) using a logistic regression. Then for each individual, we have predicted probabilities of that individual quoting before the rate change. These are the ?propensity scores? to be matched on.

We then, for each individual in the A group, find an individual in the B group that has the closest propensity score. This is the first matched pair and is put outside. Then we continue doing the same matching work, until either everyone is the smaller group is matched with or the difference in the scores are too much (greater than a cutoff point, say 0.1). In the example in Figure 6, 250 people are matched. This means that the remaining 50 people in the B group is too different from anyone in the A group, therefore left unmatched.

To see whether or not the before and after groups are different, we take samples of the price changes and compare key variables for the before and after groups. The following tables summarize comparison results from two samples we take.

To test for difference, we run Wilcoxon rank tests for continuous variables like Premium, Cov A; and chi-square tests for categorical variables like Education level and
Credit Decile. As we can see from the above table, Sample 1 is taken from a 17.96% price increase taking place in TX, 5/17/2014. We see that Premium and Credit Decile for the before and after groups are statistically different at a 5% level of significance. If we look at another sample for a price decrease in NY (Sample 2) and do the same tests, we see that a different set of variables are statistically different this time.

This testing results for these two samples represent the testing results for most samples we take. It provides an unclear view of 1. Whether the before and after groups are different in general, and 2. Which are the variables that are different across the before and after groups. Therefore we decide to use PSM to prevent any group difference that might arise which could bias our analysis on PED’s.

### 3.5.3 PED Results

Here we present our PED results for both Country-Wide and at state levels. We have, after PSM, 191 actual and predicted PED?s for all the rate changes. And we calculate CW and state PED?s with these individual PED?s. We also present some simulation work in this section.
Country-Wide PED Results

We calculate the Country-Wide PED using the weighted (by bind count) average of all the actual PED's. The following table summarize our PED results. Without PSM, we see that actual PED is -1.07 and predicted PED is -0.95. With PSM, which we think more accurately reflect the direct impact of premium change on demand change, the actual PED is -1.11. We see that our predicted country-wide PED with PSM is very close to the actual PED (-1.10). Therefore, our projection of the country-wide PED for the Landlord new business market is -1.11. This means that given a 10% increase in premium, we expect an 11.1% decrease in demand country-wide.

In comparison with the property market for new business, they use a -2 for the property market PED estimation. Therefore property market is more sensitive to price changes than the landlord market. This makes sense as the LL side has more high-credit customers (84% greater or equal to credit decile 6, compared to 72% from the property side, see Section 6 for the univariate plot). And high credit is positively correlated with high income. A conclusion of this comparison is that LL market has richer people than property market on average, which makes the LL market less sensitive than the property market.

State Level PED Results

Next, we look at PED results by state. The following graph plots predicted and actual PED's of the 10 largest states in terms of bind counts.

There are a few things that we would like to highlight for the above graph. First, we see a relatively accurate prediction of the state level PEDs. Second PED varies across different states.

Third, we notice some unexpected numbers. Namely 2.6 for IL and -15.2 for VA. A positive PED of 2.6 means that demand increases as price increases. This is unintuitive to our basic economic understanding of the market and is likely to be false. A very negative PED of -15.2 in VA is unlikely to be true either. This scale is against our understanding and experience in the VA market.
We attribute these unexpected numbers to factors not considered in our model. For example, we do not incorporate marketing campaign information. If there is a marketing campaign happening in VA, which is very likely to happen, demand is going to increase regard of small premium changes. So if we happen to have a price decrease and a marketing campaign taking place around the same time in VA, then it is very likely that the PED is a positive number.

Geico activation is another rare event that affects our PED interpretation. We know that for a lot of states, we pay commission to Geico for them to refer property or landlord customers to us. In 2012, Geico got involved in VA. So similar to the marketing campaign example, any demand impact from premium change around then will be blurred by the Geico activation.

Therefore one future work is to identify factors outside our current model (say marketing campaigns and Geico Activation) to better explain the binding decision and hence the PED inferences.

**PED Simulations**

Since our PED inference is drawn from the individual propensity model, we can subset any group of interest we want for PED inferences. For such simulations, we assume...
that the same individual are faced with different prices and potential make different bind decisions. The detailed procedures are as follows:

1. For a targeted group (say credit = 10, gender = male, state = TX, time = 8/1/2015), subset the original dataset to find the relevant customers.

2. Score these customers assuming price has risen x%, keeping other variables constant.

3. Compare the score on customers in the targeted group, call the difference in score for customer \(i\), \(\Delta p_i\).

4. The change in demand is the summed differences of \(\Delta p_i\).

5. PED for that price change is percentage change in demand / percentage change in price (x%).

6. Repeat 1-5 for different price changes (x%). \(^1\)

---

\(^1\)Note that our actual 191 rate changes range from -7% to +20%. And this is the range we use for simulation to avoid extrapolation.
The above graph is an example of simulation on a targeted group (state = TX, time = 5/20/2015). We see that for this example (change of) demand is decreasing with rate changes, which makes sense from an economic point of view. And PED is around -0.4 and slightly increasing for the rate change interval. We could definitely explore more interested subsets (say credit). This is left for future work due to time constraint.

3.6 Conclusion and Future Work

In this project, we model PED from a conversion model. We find that the country-wide PED for the landlord new business is -1.11, which is less sensitive than the property market (PED = -2). We also find that PED varies by state. So it makes sense to have different price changes in different states to account for different market reactions. Our model is flexible to predict any targeted groups. For the subgroup we select, we see a downward sloping demand curve and relatively flat PED curve for different price changes.

For future directions, we would like to do the following:

- Model improvement: finding factors (like market campaign and Geico activation) that are currently not considered in the conversion model but having huge impact on demand.

- Model comparison with the property side. Currently property is using PED = -2. We are interested to compare the models they use with our model.

- Extension to retention models. Currently we are modeling PED for new business. This means that each individual only has one observation (bind or not bind). PED is actually defined on the same person receiving different prices. Although our PSM alleviates this problem. PED is best applied to the same individual facing different prices.
Chapter 4

External Threat and Alliance Stability

Unity is strength... when there is teamwork and collaboration, wonderful things can be achieved.

Mattie Stepanek (1990 - 2004), American Teenage Poet

4.1 Introduction

Internal disputes over splitting surplus within a corporation, a tribal society or a military alliance, can be resolved by allocating appropriate shares of the surplus to their shareholders, tribes and membering countries. This can happen because internal conflicts involve costs to the society which results in inefficiency, therefore creating room for cooperation and compromise. Suppose an external threat intends to break down the company, annihilate the tribal society or destroy the military alliance. What changes does that threat bring to the group? Will those changes induce a more united group, or break the group apart? This paper answers these questions.

Literature on dynamic or nested conflicts looks at conflicts over a prize with multiple stages of competition. Due to the complexity from coalition formation with externalities, most papers focus on the case of a 3-player conflict. Konrad and Kovenock (2009) analyzes the alliance formation problem with endogenous effort and capacity constraints. It uses a pay-all auction probability function for which the player putting more effort gets the prize with probability 1. Players can “reload” their capacity after each round of contest. This makes contests in different stages separate problems to solve. A drawback to this model is that it assumes away how effort put in a previous stage affect the ability to invest
in contest in the next stage, which in some war contexts is not natural. Esteban and Sákovics (2003) looks at the same problem with multiple objects to be selected and finds that an alliance is less preferred to fighting alone due to free-riding on similar objectives or lack of interest on different objectives. Skaperdas (1998) characterize the alliance formation problem with different functional forms of the contest success functions (CSF), and finds that additivity of the CSFs characterizes the efficiency of an alliance. Tan and Wang (1997) extends the analysis of dynamic conflict model into a 4-player case, with a non-cooperative approach. Garfinkel (2004) looks at such a model with n identical players and find conditions for stable alliances to form. Except Konrad and Kovenock (2009), all of the above mentioned paper do not endogenize the effort level into the model, such that the probability of success depends only on the strength of the individual/group. None of the above mentioned papers allow peace agreement to be written within the alliance formed during early stages of conflict. None of the papers considers the loss of power between stages of conflicts.

This paper looks at an n-player two-stage conflict model. Players actively choose whether to participate in the alliance and investment levels in effort. In addition to searching for sufficient conditions for efficient outcome to occur, the focus here is on how an external shock changes members’ incentives to maintain the alliance. We look at two potential factors that affect the stability of the alliance in the event of an external threat. “Alliance size”, number of players in the alliance, and “cost of investment”. We find that an alliance is easier to maintain when the alliance size is large, while it is harder to maintain when cost of investment is large. We also compare the relative forces of these factors and find sufficient conditions when one force is stronger than the other. Towards the end, I develop a repeated game model in which external threat comes every period and see how the stability of the alliance is affected.

Cooperation involves making members of the group contribute to the joint investment to fight against the external enemy. Similar to Nash-bargaining and other per-determined decision rules within a group in other models, the decision rule used in this exercise is a proportion-to-share rule. That is, individuals contribute cost and gain profit
according to how much share they occupy on the natural resource. It is a common practice to have members with larger shares to contribute more, smaller shares to contribute less. For example, in helping the poor, richer countries make larger donations; To pay their employees from profit earn by the company, big shareholders effectively pay more. This is to capture a pervasive “fairness” social point of view, in the sense that big shareholders are more responsible in the joint investment than the small ones. As they contribute more, in case of successfully defeating the enemy, big shareholders should also get more than the small ones. Although the only contract players can sign is under which only such a rule can be used, it does not restrict individuals from not signing the contract, or choose not to fulfill their responsibilities once they succeed in defending their resource. In this way of modelling, the main conflict of interest between jointly investing in defense or not is illustrated, while capturing individuals’ self interest by allowing them not to sign the contract or breaking off the contract in between.

The main incentive issue for individuals to decide between maintaining themselves in the alliance or deviating from it, is the distribution and redistribution of surplus before and after the emergence of the external threat. The minimum shares of surplus given to the individuals prevent them from breaking off the alliance. These minimum shares can serve as a measure of easiness to sustain the alliance: The higher the minimum shares, the harder it is to sustain the alliance. This paper attempts to model such internal disputes with an exogenous external threat, and finds out whether such an external threat brings a more united alliance or a more fragile one by measuring minimum shares allocated to its members.

The rest of the paper is organized as follows. Section 2 introduces the model with two illustrative examples. Section 3 analyzes the general game, obtains the equilibria, and compares how alliance size and cost of investment affect the stability of the alliance. Section 4 analyzes the game in a the repeated game setting. Section 5 concludes the paper with some remarks and future directions.
4.2 The Conflict Model

Consider some natural resource $M > 0$, commonly valued and owned by an alliance of risk-neutral players $N = \{1, 2, ..., n\}$, $n \geq 2$ being the alliance size, with shares $r = \{r_1, r_2, ..., r_n\}$ of the resource, with $r_i > 0$ and $\sum_i r_i = 1$. An external threat, represented by a player $k$ outside the alliance, tries to take away the resource from the group by invoking a conflict. Facing such a threat, members of the alliance may choose to fight $k$ as one player, or see $k$ as an opportunity to break the alliance and become the sole owner of the resource.

At any stage of the game described below, in case of a conflict involving $m$ players takes place, players get the resource with probability proportional to their investments. That is, each player investing $c_i$ in the conflict, will get a probability of $\frac{c_i}{\sum_j c_j}$ of winning the entire resource. In case of a conflict between the alliance and player $k$, the alliance invests $x$, contributed by all members of the group. The alliance wins the resource with a probability of $\frac{x}{x + c_k}$. For each player investing or contributing $c_i$, cost incurred to that player is $\lambda c_i^p$, with $p \geq 1$ being the cost factor and $\lambda > 0$. I assume that without $k$, $r$ is such that members willingly join the alliance, compared to a conflict among themselves. Let us call such $r$ a feasible $r$.

The game has two stages. At Stage 1, a post-conflict offer $s = \{s_1, ..., s_n\}$ is provided to members of the alliance. This is the redistribution of shares of $M$ upon winning the conflict as a group. Members simultaneously choose to fight as a group (G) or independently (I) against $k$, $a_i \in \{G, I\}$, for all $i \in N$. If any member chooses I, the alliance breaks down, and players consisting of $N \cup \{k\}$ contest for the resource. One of them gets the resource and the game ends there. If all members of the alliance find the post-war $s$ attractive, they choose G, the alliance keeps alive and chooses a joint investment level $x$ and fight against $k$ as one player. Each member of the alliance contributes $x r_i$ to the joint investment. Then conflict between the alliance and $k$ takes place. If $k$ wins, the game ends. In case of the alliance winning the resource, the game goes to Stage 2.

At Stage 2, again members in $N$ simultaneously choose $a_i \in \{G, I\}$. If all choose $G$, they carry out the proposal and $M$ is redistributed according to $s$. If not, $N$ contests
for the resource. One of the players gets the resource and the game ends. We assume the discount factor between the stages is 1.

By default, if indifferent, players choose to be cooperative (keep the alliance, accept the new shares) if applicable. At any point where the game ends, the payoff to each player is the expected value of $M$, minus her total cost from investment, $\lambda c_i^p$.

Subgame perfect Nash equilibrium (SPNE) is used as the solution concept in this model. As the focus of this model is on the alliance facing and dealing with the external threat, only their strategies and payoffs are considered and discussed in this exercise. Of course, player $k$ plays optimally at all times.

The sequence of moves are summarized as the following:

- $k$ emerges as an external threat, and observes later whether or not an alliance is formed (so that $k$ can choose $c_k$ accordingly);

- Given an offer $s$, players in $N$ simultaneously decide $a_i \in \{G, I\}$
  
  - If at least one player chooses $I$, $n + 1$ players contest for $M$. Each player investing $c_i$ gets $\frac{c_i}{c_1 + \ldots + c_n + c_k}$ chance of winning. One player wins, the game ends;
  
  - If all players choose $G$, then $N$ chooses a joint investment level $x^1$, and contests with $k$, such that each $i$ contributes $r_i$ proportion of the joint investment; $N$ gets $\frac{x}{x + c_k}$ chance of winning.
    
    * If $k$ wins, the game ends;
    
    * If $N$ wins, all players in $N$ simultaneously choose $a_i \in \{G, I\}$
      
      - If at least one player chooses $I$, $n$ players contest for $M$. Each player investing $c_i$ gets $\frac{c_i}{c_1 + \ldots + c_n}$ chance of winning. One player wins, the game ends;
      
      - If all players choose $G$, then $M$ is redistributed to players in $N$ according to their newly negotiated shares $s$.

\[\text{Later we can see that there exists a unique optimal } x.\]
Also, define different moves \( i \in N \) can adopt and their respective payoffs as follows:

- **Stand-alone**: \( i \) chooses \( I \) before fighting \( k \). Payoff is denoted as \( \pi^*_i \);

- **Alliance-Conflict**: \( i \) chooses \( G \) first and \( I \) later. Payoff is denoted as \( \pi^c_i \);

- **Alliance-Peace**: \( i \) chooses \( G \) first and \( G \) later. Payoff is denoted as \( \pi^p_i \).

In this model we only consider two coalitions that members of the alliance can form— the alliance, \( N \), and the individual coalitions \( \{i\}_{i \in N} \). Members have two chances of keeping the alliance \( (a_i \in \{G, I\}) \): one facing the challenge of \( k \), the other after \( k \) is defeated. These are the two times when the shares \( r \) and \( s \) come into play in members’ decisions. \( r \) determines how much each member of the alliance contributes; \( s \) determines how to split the surplus once \( k \) is gone. It is straightforward to see that when \( s_i \) is high, player \( i \) has more incentive to stay in the alliance. Whereas when \( r_i \) is high, player \( i \) is less willing to stay in the alliance, as that would mean contributing more to the joint investment. As \( r \) is exogenously given, we would like to see how \( s \) can be structured so that all members are willing to stay in the alliance. Specifically, we would like to see how the *alliance size* \( (n) \), and the *cost factor* \( (p) \) determine the minimum share, \( \min(s) \), that has to be allocated to the smallest shareholder, such that that player does not want to leave the group. We first look at a 3-player example. This case is simple yet providing much of the intuition the general model carries.

### 4.2.1 A 3-player example

**Example 4.1.** I consider the above game with \( N = \{1, 2\} \) with \( r = \{\frac{1}{4}, \frac{3}{4}\} \). Normalize \( M = \lambda = p = 1 \).

First note that in this example, a general formula for \( n \) players contesting \( M \) is that each player invests \( c_i = \frac{n-1}{n^2} \). Payoff from conflict is \( \pi_i = \frac{1}{n} - \frac{n-1}{n^2} = \frac{1}{n^2} \). Without any external threat, each \( r_i \) must be greater or equal to \( \frac{1}{n^2} \) to prevent the alliance from falling.

\(^2\)To see this, each of the \( n \) players solves the problem \( \max_{c_i} \frac{c_i}{c_1+c_2+...+c_n} - c_i, i = 1, ..., n \). From the first order conditions and symmetry, for each \( i \), \( \frac{(n-1)c_i}{(nc_i)^2} = 1 \), \( c_i = \frac{n-1}{n^2} \), and \( \pi_i = \frac{1}{n} - \frac{n-1}{n^2} = \frac{1}{n^2} \).
apart. In this example, $r$ makes the alliance stable. The question is, after $k$'s intrusion, does such an $r$ still make the alliance stable?

If no alliance is formed, that is, some player $i$ chooses $a_i = I$, each of players 1 and 2 solves the problem $Max \frac{c_i}{c_1 + c_2 + c_k} - c_i$. From the above formula, each $c_i = \frac{2}{9}$ and $\pi_i^p = \frac{1}{9}$.

If an alliance is formed, but upon winning the resource, the alliance breaks down, the alliance first chooses an optimal investment level $x$ against $k$, in solving $Max \frac{x}{x + c_k} - x$. $x = c_k = \frac{1}{4}$. Probability of winning the resource is $\frac{1}{2}$ for the alliance and player $k$. According to initial shares $r = \{\frac{1}{4}, \frac{3}{4}\}$, players contribute $\frac{1}{4} \times \frac{1}{4}$ and $\frac{1}{4} \times \frac{3}{4}$ respectively. Each member of the alliance then solves the problem $Max \frac{1}{2} (\frac{c_i}{c_1 + c_2} - c_i) - \frac{1}{4} r_i$. Each player then invests $c_1 = c_2 = \frac{1}{4}$ and expected payoff from alliance-conflict is $\pi_i^c = \frac{1}{8} - \frac{1}{4} r_i$.

Now looking at $\pi_i^p$ and $\pi_i^c$, the new set of shares $s$ (possibly same as $r$) has to be such that, it makes all players’ payoff from alliance-peace, $\pi_i^p = \frac{1}{2} s_i - \frac{1}{4} r_i$, at least as large as $\pi_i^s$ and $\pi_i^c$.

By solving the two inequalities of $\pi_i^p \geq \pi_i^s$ and $\pi_i^p \geq \pi_i^c$, we get $s_i \geq \frac{1}{4}$ and $s_i \geq \frac{2}{9} + \frac{1}{2} r_i$. Take the maximum of the two, either player has to be given at least $s_1 = \frac{2}{9} + \frac{1}{2} \times \frac{1}{4} = \frac{25}{72}$.

From this example, we can see that the external threat brings in harm and less income inequality to members of the alliance: Even at the most uneven $s$, players get payoffs $(\frac{4}{36}, \frac{5}{36})$ instead of $(\frac{1}{3}, \frac{3}{4})$. It also makes the alliance “harder" to maintain: Without $k$, we only need to have each $r_i \geq \frac{1}{4}$. With $k$, it has to be that each $s_i \geq \frac{25}{72} > \frac{1}{3}$. Does this observation generalize to alliance consisting of more players? It turns out that as long as shares can be anyhow allocated to the shareholders, there is always a way to keep the alliance while fighting against $k$.

### 4.2.2 An n-player case

Now we get to the case where $N = \{1, ..., n\}$ with $r = \{r_1, ..., r_n\}$. Still let $M = \lambda = p = 1$.

We use the following proposition to summarize this subsection:

**Proposition 4.1.** Given alliance size $n$, with the cost factor $p = 1$, for any feasible $r$, there always exists $s$ such that under $s$, members of the alliance willingly fight against $k$ as one
Proof of Proposition 4.1. First note that without \( k \), \( r \) is feasible if \( r_i \geq \frac{1}{n^2} \), for all \( i \in N \). Next, with \( k \) entering the game, similarly as the above example, I compute the payoffs of the members of the alliance under different options:

\[
\begin{align*}
\pi_i^s &= \frac{1}{(n+1)^2}; \\
\pi_i^c &= \frac{1}{2n^2} - \frac{1}{4}r_i; \\
\pi_i^p &= \frac{1}{2}s_i - \frac{1}{4}r_i.
\end{align*}
\]

For \( \pi_i^p \geq \pi_i^s \) and \( \pi_i^p \geq \pi_i^c \), \( s_i \geq \frac{2}{(n+1)^2} + \frac{1}{2}r_i \). As we know \( r_i \geq \frac{1}{n^2} \), simple algebra shows that \( s_i > \frac{1}{n^2} \), for all \( i \in N \).

Given any \( r \) such that \( r_i \geq \frac{1}{n^2} \), does such a new set \( s \) always exist, such that in equilibrium, the alliance fights against the external threat and upon winning, peacefully share the resource?

**Lemma 4.1.** An \( s \) always exist given any \( r \) which satisfies all \( r_i \geq \frac{1}{n^2} \)

**Proof of Lemma 4.1.** Given any such \( r \), it is sufficient to show that there exist an \( s \) such that each \( s_i \geq \frac{2}{(n+1)^2} + \frac{1}{2}r_i \) and \( \sum_i s_i \leq 1 \). Take each \( s_i = \frac{2}{(n+1)^2} + \frac{1}{2}r_i \) and do the summation. \( \sum_i s_i = \frac{2n}{(n+1)^2} + \frac{1}{2} \). The term \( \frac{2n}{(n+1)^2} \leq \frac{1}{2} \), thus the proof.

Following lemma 4.1, Proposition 4.1 is proven.

Our next section analyzes the equilibrium of the general game and see how \( n \) and \( p \) affect \( \min(r) \) and \( \min(s) \).

### 4.3 Equilibrium Analysis of the Model

With the above examples in mind, I analyze the game with \( \lambda > 0 \) and \( p \geq 1 \) in this section. The following lemma helps us with the later calculations.

**Lemma 4.2.** In the general game where \( n \geq 2 \), \( M > 0 \). \( \lambda > 0 \) and \( p \geq 1 \). Different payoffs are as follows:
\[
\pi_i^s = M\left(\frac{1}{n+1} - \frac{n}{(n+1)^2}p\right)
\]
\[
\pi_i^c = M\left(\frac{1}{2}\left(\frac{1}{n} - \frac{n-1}{n^2}p\right) - \frac{1}{4}r_i^p\right)
\]
\[
\pi_i^p = M\left(\frac{1}{2}s_i - \frac{1}{4}r_i^p\right)
\]

**Proof of Lemma 4.2.** We solve \(n\) UMP problems simultaneously, that is,

\[
\text{Max} \frac{c_i}{c_1 + c_2 + \ldots + c_n} M - \lambda c_i^p, \; i = 1, \ldots, n
\]

From concavity of the functions and symmetry, all \(c_i = c = \left(\frac{n-1}{n^2}M\right)^\frac{1}{p}\), probability of winning is \(\frac{1}{n}\), and \(\pi_i = M\left(\frac{1}{n} - \frac{n-1}{n^2}\frac{1}{p}\right)\). We note that \(\lambda\) does not play any role in affecting payoffs from conflict. Also, for an \(r\) to be feasible, \(r\) must be such that each \(r_i \geq \left(\frac{1}{n} - \frac{n-1}{n^2}\frac{1}{p}\right)\). Denote \(r(n, p) = \left(\frac{1}{n} - \frac{n-1}{n^2}\frac{1}{p}\right)\). As \(\min_i(r) \leq \frac{1}{n}\) (otherwise \(\sum_i r_i > 1\)), we know that a feasible \(r\) must be such that \(r(n, p) \leq \min_i(r) \leq \frac{1}{n}\).

As usual, I compute different payoffs of player \(i\) under stand-alone, alliance-conflict and alliance-peace situations.

From the above formula, stand-alone payoff for \(n + 1\) players (N and k) is \(\pi_i^s = M\left(\frac{1}{n+1} - \frac{n}{(n+1)^2}\frac{1}{p}\right)\).

For alliance-conflict payoff, each \(i \in N\) solves the problem of

\[
\text{Max} \frac{1}{2}\left(\frac{c_i}{c_1 + c_2 + \ldots + c_n} M - \lambda c_i^p\right) - \lambda\left[\left(\frac{2}{\lambda p}\right)^\frac{1}{p}\right]r_i^p, \; i = 1, \ldots, n
\]

In which the component in \([\ldots]\) is player \(i\)'s contribution of the optimal investment level for the alliance fighting against \(k\). \(\frac{1}{2}\) is the probability of the alliance winning \(M\). The objective function is a monotonicity increasing linear transformation of the one above, so the solution takes the same form. After simplification, \(\pi_i^c = M\left(\frac{1}{2}\left(\frac{1}{n} - \frac{n-1}{n^2}\frac{1}{p}\right) - \frac{1}{4p}r_i^p\right)\).

For the alliance-peace case, \(\pi_i^p = M\left(\frac{1}{2}s_i - \frac{1}{4p}r_i^p\right)\). \(\square\)
Corollary 4.1. In comparing different payoffs, $M$ and $\lambda$ play no role. The only parameters of interest are $n$ and $p$.

This is obvious from Lemma 4.2. Next, I present the main proposition of this paper.

Proposition 4.2. Given any $n$, $p$ and a feasible $r$, Alliance-Peace can be supported as an SPNE if and only if $\sum_{i\in N} s_i \leq 1$, where for each $i$, $s_i = \frac{1}{2p}r_i^p + \frac{2}{n+1} - \frac{2n}{(n+1)^2} \frac{1}{p}$.

Proof of Proposition 4.2. For players not deviating from $s$, we need $\pi_i^p \geq \pi_i^c$ and $\pi_i^p \geq \pi_i^s$, which requires that each $s_i \geq \frac{1}{n} - \frac{n-1}{n^2} \frac{1}{p} = r(n, p)$ and $\frac{1}{2} s_i = \frac{1}{4p} r_i^p \geq \frac{1}{n+1} - \frac{n}{(n+1)^2} \frac{1}{p} \iff s_i = \frac{1}{2p} r_i^p + \frac{2}{n+1} - \frac{2n}{(n+1)^2} \frac{1}{p}$.

First I need to show that it is always the case that with $p \geq 1$, all $s_i \geq r(n, p)$. This is to show that the first inequality is never a problem.

Lemma 4.3. Let $s_i = \frac{1}{2p} r_i^p + \frac{2}{n+1} - \frac{2n}{(n+1)^2} \frac{1}{p}$. Then all $s_i \geq r(n, p)$, for all $p \geq 1$.

Proof of Lemma 4.3. Again we use the minimum $s_i = \frac{1}{2p} r_i^p + \frac{2}{n+1} - \frac{2n}{(n+1)^2} \frac{1}{p}$. We want to show that $\frac{1}{2p} r_i^p + \frac{2}{n+1} - \frac{1}{n} - \frac{2n}{(n+1)^2} \frac{1}{p} \geq \frac{1}{n} - \frac{n-1}{n^2} \frac{1}{p}$. This is equivalent to show that

$$\frac{1}{2p} r_i^p + \left( \frac{2}{n+1} - \frac{1}{n} \right) + \left( \frac{n-1}{n^2} - \frac{2n}{(n+1)^2} \frac{1}{p} \right) \geq 0.$$ 

We know that

$$\frac{1}{2p} r_i^p + \left( \frac{2}{n+1} - \frac{1}{n} \right) + \left( \frac{n-1}{n^2} - \frac{2n}{(n+1)^2} \frac{1}{p} \right) > \frac{2}{(n+1) - 1} - \frac{n-1}{n^2} - \frac{2n}{(n+1)^2} \frac{1}{p}$$

$$= \frac{n-1}{(n+1)n} + \frac{-n^3 + n^2 - n - 1}{(n+1)^2 n^2}$$

$$= \frac{(n-1)^2 - 2}{n^2 (n^2 + 1)^2} > 0 \text{ if } n \geq 3.$$ 

When $n = 2$, with $p = 1$, this is the example in Section 3. Hence the proof.

By lemma 4.3, we know that if $s$ is such that we give every member $i$ of the alliance at least $s_i = \frac{1}{2p} r_i^p + \frac{2}{n+1} - \frac{2n}{(n+1)^2} \frac{1}{p}$, everyone in the group will stay in the alliance. On
the other hand, for any Alliance-Peace to an SPNE, each $i$ must at least be allocated with share $s_i = \frac{1}{2p}r_i^p + \frac{2}{n+1} - \frac{2n}{(n+1)^2} \frac{1}{p}$. The only remaining question is whether such an $s$ satisfies $\sum_i s_i \leq 1$. Hence the proposition is proven. 

Let us explore how such $s_i$ responds to $p$ and $n$. The next two lemmas establish that $\frac{\partial s_i}{\partial p} > 0$ (most of the times) and $\frac{\partial s_i}{\partial n} < 0$, when $s_i = \frac{1}{2p}r_i^p + \frac{2}{n+1} - \frac{2n}{(n+1)^2} \frac{1}{p}$.

Lemma 4.4. Let $s_i = \frac{1}{2p}r_i^p + \frac{2}{n+1} - \frac{2n}{(n+1)^2} \frac{1}{p}$, then $\frac{\partial s_i}{\partial p} > 0$ for large $p$. When $p = 1$, in the most even case, all $r_i = 1 - (n - 1)\left(\frac{1}{n} - \frac{n-1}{2^p} \frac{1}{p}\right)$ and $r_i = \frac{1}{n} - \frac{n-1}{2^p}$ for all $i \neq 1$, $\frac{\partial s_i}{\partial p} > 0$ when $n = 2$, $\frac{\partial s_i}{\partial p} < 0$ when $n > 2$.

Proof of Lemma 4.4. $\frac{\partial s_i}{\partial p} = \frac{1}{2} r_i^p \ln r_i \frac{-r_i^p}{p^2} - \frac{2n}{(n+1)^2} \frac{1}{p^2} = \frac{1}{2} r_i^p \ln r_i (1 - plnr_i) - r_i^p (1 - plnr_i).$

$r_i^p (1 - plnr_i) > 0$ as $r_i < 1$. $\frac{\partial r_i^p (1 - ln r_i)}{\partial p} = \frac{\partial r_i^p}{\partial r_i} (1 - ln r_i) - \frac{r_i^p}{r_i} ln r_i < 0$. So for large $p$, $\frac{\partial s_i}{\partial p} > 0$.

It is sufficient to check that when $p = 1$, $\frac{\partial s_i}{\partial p} > 0$.

When $p = 1$, $\frac{\partial s_i}{\partial p} = \frac{1}{2} \left[ \frac{4n}{(n+1)^2} - r_i (1 - ln r_i) \right].$ $\frac{\partial r_i (1 - ln r_i)}{\partial r_i} = -ln r_i > 0$. So it is sufficient to show that if for the largest possible $r_i$, $\frac{4n}{(n+1)^2} - r_i (1 - ln r_i) > 0$, then $\frac{\partial s_i}{\partial p} > 0$.

In the most even case, all $r_i = \frac{1}{n}$. Then $\frac{4n}{(n+1)^2} - r_i (1 - ln r_i) = \frac{4n}{(n+1)^2} - \frac{1}{n} (1 - \ln \frac{1}{n}) = \frac{4n}{(n+1)^2} - \frac{1+\ln n}{n}$. By solving it, $\frac{\partial s_i}{\partial p} > 0$ when $n \leq 10$, $\frac{\partial s_i}{\partial p} < 0$ when $n > 10$.

Similar calculations show the rest of the lemma. 

Lemma 4.5. $s_i = \frac{1}{2p}r_i^p + \frac{2}{n+1} - \frac{2n}{(n+1)^2} \frac{1}{p}$, then $\frac{\partial s_i}{\partial n} < 0$ for all $n$, for all $p$.

Proof. $\frac{\partial s_i}{\partial n} = \frac{-2}{(n+1)^2} - \frac{2}{p} \frac{1-n}{(n+1)^2} \frac{n}{(n+1)^4}$

$= \frac{-2}{(n+1)^2} - \frac{1-n}{p (n+1)^3} = \frac{(n+1)p+1-n}{(n+1)^2 (p/2)} = \frac{(p-1)n+1+p}{(n+1)^3 (p/2)} < 0$ as $p \geq 1$.

Now we know that $\frac{\partial s_i}{\partial n} < 0$, and most of the time $\frac{\partial s_i}{\partial p} > 0$ (for cases where $\frac{\partial s_i}{\partial p} < 0$, $\frac{\partial s_i}{\partial p}$ is very close to 0). That is, increase in $p$ makes the alliance less stable; but increase in $n$ makes the alliance more stable. 

Next, let us explore when a feasible $s$ exists and when it does not.

As shown in Lemma 4.2, when the inequality $\pi_i^p = \pi_i^c$ is satisfied, automatically we have $\pi_i^p = \pi_i^c$ satisfied. As long as we know when $\pi_i^p = \pi_i^c$ is true for all $i$, we know a
redistribution of shares $s$ exists. In other words, the question is, given $r$, for what values of $n$ and $p$ do we have the following:

$$\sum_i s_i = \frac{1}{2p} \sum_i r_i^p + \frac{2n}{n+1} - \frac{2n^2}{(n+1)^2} \frac{1}{p} \leq 1$$

This function is hard to analyze in general with the term $\frac{1}{2p} \sum_i r_i^p$. I analyze a few special cases here.

**Case 1.** $p = 1$

As analyzed before in Section 3, $\sum_i s_i \leq 1$ for all $n$ and for all feasible $r$.

**Case 2.** $p = \infty$

As analyzed before in Section 3, $\sum_i s_i > 1$ for all $n$ and for all feasible $r$.

**Case 3.** $n = 2$

As $\frac{\partial s_i}{\partial n} < 0$ for all $i$ shown in Lemma 3, $\frac{\partial \sum_i s_i}{\partial n} = \sum_i \frac{\partial s_i}{\partial n} < 0$. So $n = 2$ is the case where $s$ is the most difficult to construct ($\sum_i s_i$ the biggest). In general for $0 < r_i < 1$, $\sum_i r_i^p$ is the biggest when one of the $r_i$'s takes the maximum value possible and the rest all take the minimum value. To make $s$ the most difficult to find, choose such $r$ to maximize $\sum_i r_i^p$. So WLOG let $r_1 = (\frac{1}{2} - \frac{1}{4p})^p$ and $r_2 = (\frac{1}{2} + \frac{1}{4p})^p$. Then $\sum_i s_i = \frac{1}{2p}[(\frac{1}{2} - \frac{1}{4p})^p + (\frac{1}{2} + \frac{1}{4p})^p] + \frac{4}{3} - \frac{8}{9p}$. After plotting this function, I find that $\sum_i s_i$ is (close to linearly) increasing in $p$. So $\sum_i s_i \leq 1$ if and only if $p \leq 1.69$ (approximately).

**Case 4.** $n = \infty$

We know that a bigger $p$ makes a bigger $s$. For the term $\frac{1}{2p} \sum_i r_i^p$, we know that it goes to 0 as $n \to \infty$ for all $p > 1$. So as $n \to \infty$, $\sum_i s_i \to 2 - \frac{2}{p}$, which is increasing in $p$. When $\sum_i s_i \leq 1$ if and only if $p \leq 2$.

So we can see from the above four cases that $p$ exerts a larger force in making the alliance unstable than the fostering force exerted by $n$. 
4.4 Comparative statistics

4.4.1 Increasing population with a single threat

Now we look at a single threat and see the impact it brings to alliances with different sizes.

Percentage increment of the minimum share is

\[ p(n) = \frac{s_i - r_i}{r_i} = \frac{\frac{2}{(n+1)^2} + \frac{1}{2} r_i - r_i}{r_i}, \]

in which \( r_i \) depend on \( n \), \( r_i = \frac{1}{n^2} \).

After simplification, \( p(n) = 2\left(\frac{n}{n+1}\right)^2 \). In other words, relatively less share has to be re-allocated to the smallest shareholders, making the alliance easier to maintain. Just to run a few examples,

- When \( n = 2 \), \( r_i = 0.2500 \) \( s_i = 0.3472 \)
- When \( n = 3 \), \( r_i = 0.1111 \) \( s_i = 0.1806 \)
- When \( n = 4 \), \( r_i = 0.0625 \) \( s_i = 0.1113 \)

In other words, for the above cases, the biggest shareholder has to give less share to the rest of his members with a single threat, compared with no threat.

**Conclusion 1:** Facing a single threat, an alliance is easier to maintain with more members.

4.4.2 Multiple threats with increasing members

In previous sections, we analyze situations having a one time threat. This threat harms everyone in the alliance and makes the alliance harder to maintain, in the sense that minimum share to keep everyone in the alliance increases after the threat. In the real world companies face obstacles from time to time. What happens to the alliance when external threat comes one after another? This section analyzes such scenarios.

Instead of a single threat \( k \), let us have a sequence of threats \((k_1, k_2, ...)\) takes place in order. After \( k_l \) has taken place, the expected resource in the alliance’s possession shrinks to \( M_l = \frac{1}{2^l} \). In a conflict invoking \( n \) parties, each party invests \( c_i = \frac{1}{2^{l}} \frac{n-1}{n^2} \).

\[ \pi^a_i = \frac{1}{2^l} \frac{1}{(n+1)^2}; \pi^b_i = \frac{1}{2^l} \left( \frac{1}{n^2} - \frac{1}{4} r_i \right); \pi^c_i = \frac{1}{2^l} \left( \frac{1}{4} s_i - \frac{1}{4} r_i \right). \]

So the inequalities stay the same and the binding constraint is still \( s_i \geq \frac{2}{(n+1)^2} + \frac{1}{2} r_i \). This is convergent sequence for all \( n \), which means external threats never destroy the alliance. To run a few examples:

- When \( n = 2 \), \( s_i = 0.4444 \) as \( l \to \infty \)
When $n = 3$, $s_i = 0.2500$ as $l \to \infty$

When $n = 4$, $s_i = 0.1600$ as $l \to \infty$

When $n = 10$, $s_i = 0.0331$ as $l \to \infty$

When $n = 20$, $s_i = 0.0091$ as $l \to \infty$

When $n = 30$, $s_i = 0.0042$ as $l \to \infty$

When $n = 100$, $s_i = 0.0003919$ as $l \to \infty$

**Conclusion 2:** an alliance is harder to maintain as $l \to \infty$, with a fixed $n$.

**Conclusion 3:** an alliance is easier to maintain as $l \to \infty$ and $n \to \infty$.

### 4.5 Concluding Remarks

This paper looks at under proportion-to-share rule, how the stability of an alliance is maintained and how it is affected by the alliance size and the cost factor. Specifically the alliance is easier to maintain with a larger size and a lower cost factor. A future direction following this model can be focusing on coalition formation in the conflict context.
Bibliography


