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Essays on Information Asymmetry, Strategic Trading, Liquidity, and Heterogeneity

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ABSTRACT

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Asset markets are imperfect. However, it is common in Finance and Economics to work under a set of assumptions that yields perfect markets. I examine the equilibrium implications of relaxing some of the assumptions made when considering perfect markets. Specifically, I study economies where one or more of the following holds: (1) Incomplete or asymmetric information; (2) Illiquid markets; Non-Competitive markets; (4) Irrational and heterogeneous beliefs.

In the first chapter, I study the strategic interaction between large investors when one of the investors is in distress and markets are illiquid. Such non-fundamental demand/supply shocks due to distress liquidation are ubiquitous in assets markets and they lead to predictable price movements in illiquid markets. Arbitrageurs, which I refer to as potential predators, can profit from this predictability by either providing liquidity, which stabilizes markets, or engaging in predatory trading, which destabilizes markets. I show that the presence of information asymmetry increases the probability that a potential predator will provide liquidity rather than engaging in predatory trading during liquidation by a distressed trader. More information
asymmetry is associated with lower expected losses from liquidation for the distressed trader in illiquid markets. There is a negative correlation between the degree of information asymmetry and the returns from predatory trading, which is consistent with empirical findings. These results imply that strategic traders are more likely to stabilize markets by providing liquidity when information is asymmetric. These findings highlight a cost associated with disclosure and can explain the documented rarity of illiquidity episodes in financial markets.

In the second chapter, I examine the implications of irrational and heterogeneous beliefs for both portfolio choices and asset prices. There is a large literature challenging the common assumption of rational belief by the “economic man”. I propose a theoretical model for one irrational belief, the belief in the Law of Small Numbers. My model generates both beliefs in short-term reversal and long-term extrapolation. I embed this model into a Lucas economy with multiple trees and derive its implications. I find that irrational investors exhibit the disposition effect when making portfolio choices. Moreover, asset prices dynamics are such that momentum strategies are profitable.

The third chapter is an empirical study exploring the causal link between volume and volatility. I use a natural experiment to identify exogenous variation in volume. This experiment is the set of blizzards in Manhattan, New York, New York, where the New York Stock Exchange is located. I find that blizzards cause a significant drop in trading volume. This drop is also associated with a reduction in volatility. I find indirect evidence that the channel through which volume affects volatility is the trading by institutional investors.
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\[
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Chapter 1

Asymmetric Information and Liquidity Provision

Non-fundamental demand/supply shocks are ubiquitous in assets markets and they lead to predictable price movements when markets are illiquid. Institutional investors (or arbitrageurs) can profit from this predictability by either providing liquidity, which stabilizes markets, or engaging in predatory trading, which destabilizes markets and amplifies the original shocks. Both liquidity provision and predatory trading have been documented empirically. However, we do not have a clear understanding as to why do institutional investors at times stabilize markets by providing liquidity and at other times destabilize markets by engaging in predatory trading.

We examine the determinants of the choice between providing liquidity and engaging in predatory trading when another large trader is forced to sell or buy a risky asset. This choice has important implications for financial markets. Predatory trading reduces liquidity, usually at times when it is needed the most, and increases transaction costs for large traders. Moreover, Brunnermeier and Pedersen (2005) argue that predatory trading increases the risk of a financial crisis, amplifies financial

\footnote{Trading is non-informed, that is independent of risky asset’s fundamental value. Vayanos (2001) argues that non-informed trading must represent a large subset of the trading activity in financial markets. Examples of non-informed trading include trading resulting from either index reconstitution or flow of fund to/from institutional investors. The extant literature documents significant non-informed trading activity in financial markets (see Chen et al. (2004), Coval and Stafford (2007), Zhang (2010b), Petajisto (2011), and Bessembinder et al. (2014)).}
contagion, and affects institutional investors’ risk management strategies.

We argue that a key variable affecting this choice is the presence of information asymmetry between the distressed trader and her potential predators. A natural source of information asymmetry is the amount of assets to be sold or bought, which we assume is only known by the distressed trader. This argument is supported by anecdotal evidence. For example, Lowenstein (2000) notes that the head of Long Term Capital Management “... bitterly complained to the Fed’s Peter Fisher that Goldman, among others, was front-running, meaning trading against it on the basis of inside knowledge.” Also, Wermers (2001) argues that “more frequent portfolio disclosure would enable increased front running by professional investors and speculators.” Along the same line, the International Association for Quantitative Finance (IAQF) recommends that large institutional investors “limit granularity of reporting sufficiently to protect Investors against predatory trading against the Managers positions.”

We model the interaction between large traders in an illiquid market as a two-player nonzero-sum stochastic differential game. Illiquidity means that the price of the risky asset is a function of both the large traders’ aggregate holding of this asset (long-term impact) and their aggregate trading rate of this asset (short-term impact). A distressed trader needs to liquidate the single risky asset. A potential predator

\[^2\text{We refer to the amount to be liquidated as the liquidation size.}\]


\[^4\text{See Table I in Brunnermeier and Pedersen (2005) for additional anecdotal evidence relating information asymmetry to predatory trading.}\]
can either provide liquidity or engage in predatory trading but does not know the liquidation size, only the distribution it is drawn from. Profitable predation requires racing to sell the risky asset while its price is high, ahead of the distressed trader’s price impact, and then buying it at a lower price later on. We consider closed loop equilibria to allow learning about the liquidation size through changes in the price of the risky asset. These changes are a function of changes in the asset’s fundamental value and aggregate trading by the large traders. We provide closed-form solutions of the game under some parameter restrictions and use numerical techniques to solve for the equilibria without restrictions.

Our main finding is that information asymmetry reduces the probability that predatory trading occurs in illiquid markets. The intuition is that the potential predator faces higher losses when engaging in predatory trading relative to providing liquidity, losses due to errors made while estimating the liquidation size.\(^5\) Predatory trading is associated with higher losses because it requires more aggressive trading to race the distressed trader to the market, which leads to more estimation errors and higher trading costs. Moreover, the distressed trader can partially forecast the potential predator’s error in estimating the liquidation size. This forecast can lead to further losses to the potential predator when predation occurs.

We also find that market illiquidity affects the probability of predatory trading occurring. Predatory trading is less likely when the long-term price impact takes low values and is negligible when the long-term price impact is zero. This result is intu-

\(^5\)These losses are increasing in the degree of information asymmetry. The potential predator never incurs losses in equilibrium in models with complete information. See Brunnermeier and Pedersen (2005), Carlin et al. (2007), and Schöneborn and Schied (2007).
itive. As Brunnermeier and Pedersen (2005) note, “the predator derives profit from the price impact of the prey”. Higher long-term price impacts lead to higher profits from predation. In addition, the resolution of uncertainty about the liquidation size, which reduces the degree of information asymmetry, is faster when the long-term price impact is high. The reason is that trading by the distressed trader explains a higher percentage of the changes in the price of the risky asset when the long-term price impact is high.

Our work highlights the welfare benefits of information asymmetry during crises. We show that an increase in information asymmetry generally benefits distressed traders, and hurts predators, and increases the large traders’ aggregate wealth. These findings imply that there may be a cost associated with implementing recent policies requiring more transparency for institutional investors.

Several of our predictions are consistent with existing empirical evidence. Parida and Teo (2011) provide evidence that funds reporting semiannually outperform funds reporting quarterly. Moreover, this difference in performance disappear when all funds are required to report quarterly. These results are consistent with our model’s prediction that greater information asymmetry is associated with higher returns for the distressed trader. The model’s prediction that a higher degree of information asymmetry leads to lower returns for the potential predator is consistent with the findings of Shive and Yun (2013). They show that returns from predatory trading are higher when mutual funds are required to have more frequent disclosure. Bessem-
binder et al. (2014) find empirical evidence that strategic traders provide liquidity when markets are resilient. They define market resiliency as the degree to which “some or all of the immediate price impact of trades is subsequently reversed.” Their finding is consistent with our prediction that the potential predator provides liquidity when the long-term price impact is low. We predict that the potential predator’s value is higher when the permanent price impact is higher. Both Shive and Yun (2013) and Arif et al. (2014) present empirical evidence consistent with this prediction. These papers find that the potential predators’ values are higher when they trade in less liquid assets.

Related Literature

Our research is related to several strands of literature including models of liquidity crises, competition among strategic traders, and distressed liquidation of risky assets. The nature of the information structure makes our model unique. In our model, one agent is better informed than the other and the private information is about asset allocation and not the asset’s fundamental value.

Our model is an extension of the first stage game in Carlin et al. (2007), who explain the puzzling fact that illiquidity is rare and episodic in financial markets. They model predation as a breakdown in cooperation between institutional investors in a repeated game. We complement their work by showing that asymmetric information provides an alternative explanation for the episodic illiquidity. Our model applies to

7Infrequent and episodic illiquidity was puzzling because Brunnermeier and Pedersen (2005) showed that predatory trading is the equilibrium strategy during forced liquidations and forced liquidations are frequent in financial markets.
important types of interaction between institutional investors in which cooperation as described by Carlin et al. (2007) does not apply. These instances include interactions between high frequency traders/hedge-funds and mutual funds. All traders in their model must be able to execute a punishment strategy for the equilibrium to hold in their repeated game. In financial markets, mutual funds are unlikely to engage in predatory trading against high frequency traders and hedge-funds in part because of regulatory requirements and inferior technological sophistication.

Other research related to our model analyzes predatory trading under complete information. Predatory trading always occurs in equilibrium in Brunnermeier and Pedersen (2005). Schöneborn and Schied (2007) study predatory trading in a two-stage-game extension of the first stage game in Carlin et al. (2007). Carmona and Yang (2011) consider a similar two-stage extension but allow both strategic players to follow closed-loop strategies. Liquidity provision occurs in the models of both Schöneborn and Schied (2007) and Carmona and Yang (2011) when the permanent price impact is low, consistent with our results. Bessembinder et al. (2014) extend Brunnermeier and Pedersen’s model to include resiliency. In their model predatory trading only occurs when markets are not resilient. In reality, it is often impossible to know the exact liquidation need of a trader even in nonanonymous markets. We complement this literature by highlighting the role of asymmetric information in determining the equilibrium outcome of the interaction between strategic traders when one trader is in distress.

Competition among strategic traders has been explored in extensions of Kyle (1985). Foster and Viswanathan (1996) and Back et al. (2000) characterize the trad-
ing behavior of informed strategic traders. There are two main differences between their models and ours. First, strategic traders have symmetric information \textit{ex-ante} in their models. Second, the trading motives in their models are related to the risky asset’s fundamental value. Choi et al. (2015) and Vayanos (2001) study the effect of non-informational trading by large traders. Vayanos investigates competition among large traders when trades are the result of risk-sharing needs. Choi et al. examine the equilibrium outcome of competition among two traders when one has a trading target and the other has private information about the value of the asset in a multi-period Kyle model. Predatory trading does not occur in the model of Choi et al. (2015). We complement this strand of the literature by studying the effect of information asymmetry on liquidity provision during distress liquidation.

Our paper is related to the literature on the Scholes liquidation problem, which is concerned with the optimal way to liquidate an illiquid asset (see Bertsimas and Lo (1998), Huberman and Stanzl (2005), Moallemi et al. (2012), Carmona and Yang (2011), Obizhaeva and Wang (2013) and references therein). Moallemi et al. (2012) study the liquidation problem with asymmetric information in a discrete time setting. Carmona and Yang (2011) study the role of noise traders on predatory trading when both strategic traders follow closed-loop strategies. We complement this literature by focusing on the interaction between the degree of information asymmetry and the likelihood of predatory trading occurring and thus the implications of portfolio disclosure.
1.1 Basic Model

Our model is an extension of the first-stage game in Carlin et al. (2007). We consider a continuous time economy with two assets: a riskfree asset with zero return and a risky asset. There are two types of traders interacting in the market: long-term investors and strategic traders. Long-term investors have three key characteristics: (i) They are price takers, (ii) they have downward sloping demand curves, and (iii) their demand is a function of the margin of safety, that is the difference between the asset’s fundamental value and its price.

Strategic traders are large, risk-neutral agents. Their trades affect the risky asset’s price. Examples of strategic traders include hedge funds and proprietary trading firms. Both Brunnermeier and Pedersen (2005) and Carlin et al. (2007) assume that trades by each strategic trader are observable. An important departure of our model from this work is that we adopt the realistic assumption that trades by a strategic trader are her private information. A strategic trader can use changes in the price of the risky asset to estimate the trades/ asset holding of other strategic traders. Changes in price are a function of the strategic traders’ trading rate and asset holding (because of illiquidity), and changes in the risky asset’s fundamental value, none of which is observable. Thus, prices are not fully revealing in our model.

We assume that there are two strategic traders. The first strategic trader is the distressed trader who is required to sell (or buy) a certain amount of the risky asset between time 0 and time $T > 0$. We take as exogenous both the amount of assets to be sold and the time $T$ by which she has to sell them. Distressed liquidation
does not affect the risky asset’s fundamental value and can arise as a result of risk management, regulatory requirements, or margin calls.

The second strategic trader is the *potential predator* who optimally buys (or sells) the risky asset in response to the distress event. The potential predator’s optimal behavior has significant implications for the economy. She can either reduce liquidity by by engaging in predatory trading or supply additional liquidity to the market at a time when it is needed.

We model the interaction among strategic traders as a differential game; that is, a continuous time game. The game takes place in the time interval $[0, T]$. Let $\tilde{\Delta}x$ denote the amount of the risky asset that the distressed trader must sell. We assume that $\tilde{\Delta}x$ is a normal random variable with mean $\mu$ and variance $\sigma^2$. Our model applies to both positive and negative values of the mean $\mu$. However, we shall discuss our result under the assumption that $\mu$ is negative and perform simulations under the same assumption, which is the more relevant case empirically. The case of positive $\mu$ is symmetric.

Our first key extension of the first-stage game in [Carlin et al. (2007)] is that we assume that Nature picks a realization $\Delta x$ of $\tilde{\Delta}x$ at $t = 0$ and announces it to the distressed trader, but not to the potential predator. We also assume that the potential predator receives a private signal $\tilde{S}$ that can contain information about the realization $\Delta x$. Formally, we assume that

$$\tilde{S} = \tilde{\Delta}x + \tilde{\epsilon}$$  \hspace{1cm} (1.1)
where $\tilde{\epsilon}$ is a normal random variable with mean zero and variance $\sigma_0^2$ independent of $\tilde{\Delta}x$.

Let

$$ \kappa \equiv \frac{\sigma_0^2}{\sigma^2 + \sigma_0^2} \quad \text{and} \quad R^2 \equiv 1 - \kappa. $$

$R^2$ is the R-squared of the regression of $\tilde{\Delta}x$ on $S$. It measures the quality of the signal $S$ in predicting the realization of the random variable $\Delta x$. The information asymmetry between the distressed trader and the potential predator in our model is captured by $\kappa$. We refer to $\kappa$ as the degree (or percentage) of information asymmetry between the distressed trader and the potential predator. There is no hierarchical information structure as in Townsend (1983) because the distressed trader does not observe $S$ and thus no trader has strictly superior information to the other.

We denote the amounts of the risky asset held by the distressed trader and the potential predator at time $t$ by $X^d_t$ and $X^\ell_t$ respectively. Similarly, $Y^d_t$ and $Y^\ell_t$ denote the rates at which strategic traders are buying or selling the risky asset at time $t$; that is,

$$ dX^d_t = Y^d_t \, dt \quad \text{and} \quad dX^\ell_t = Y^\ell_t \, dt. $$

The main state variable in the economy is the price $P$ of the risky asset. $P$ is the only variable (other than time $t$) that is observed by both strategic traders. Following Carlin et al. (2007) we assume that the price evolves as

$$ dP_t = dF_t + \gamma dX_t + \lambda dY_t, \quad (1.2) $$
where $F$ is the fundamental value of the risky asset, and we assume that $P_0 = F_0 + \lambda Y_0$. $X_t$ is the sum of $X^d_t$ and $X^\ell_t$, and $Y_t$ is the sum of $Y^d_t$ and $Y^\ell_t$. Following Carlin et al. (2007) we model $F$ as a driftless Brownian motion with constant volatility $\sigma^2_F = 1$. It is natural to assume that the fundamental value of the risky asset cannot be (perfectly) observed by all market participants independently of its price. The theoretical literature on informed trading relies on this assumption.

The strategic trader $i \in \{d, \ell\}$ knows both $X^i_t$ and $Y^i_t$. Therefore, this strategic trader can estimate the following quantity when observing price’s changes:

$$dZ_t = dF_t + \gamma dX^{-i}_t + \lambda dY^{-i}_t,$$

where $\{-i\} = \{d, \ell\}\{i\}$. The price reveals neither $X^{-i}_t$ nor $Y^{-i}_t$ to the strategic trader $i$ because the fundamental value is not observable.

Following Carlin et al. (2007), the constants $\gamma$ and $\lambda$ are called the permanent price impact and the temporary price impact respectively. To understand these definitions note that $P$ satisfies

$$P_t = F_t + \gamma (X_t - X_0) + \lambda Y_t$$

in the partial equilibrium we consider; that is, when both $\gamma$ and $\lambda$ are constant. The price impact $\lambda Y_t$ is called temporary because it vanishes if the traders’ aggregate trading rate is zero (that is, if $dX = Ydt = 0$). The price impact $\gamma X_t$ is called permanent because it persists as long as the traders aggregate holding of the risky
asset is non-zero.

The distressed trader’s optimization problem is a trade-off between her desire to sell slowly to reduce trading costs and her need to sell faster to reduce the adverse effects of trades by the potential predator. She solves the following problem:

$$\max_{Y^d} \mathbb{E}^d \left[ \int_0^T -P_t Y^d_t dt \right]$$

subject to

$$\begin{align*}
  dP_t &= \gamma dX_t + \lambda dY_t + dF_t \\
  X^d_0 &= 0 \\
  X^d_T &= \Delta x \\
  dX^d_t &= Y_t dt.
\end{align*}$$

The potential predator faces a slightly different optimization problem. Following Brunnermeier and Pedersen (2005), we do not impose the restriction that the potential predator has zero excess holding of the risky asset at the end of the game, a restriction present in Carlin et al. (2007)’s first-stage game. However, we require that the potential predator liquidate her excess holding of the risky asset within a certain period after the game. Our modeling choice is more realistic than that of Carlin et al. (2007) since the potential predator is not in distress.

We assume that the potential predator starts with zero shares of the risky asset. This assumption is without loss of generality. Her excess holding at the end of the

\footnote{We show that the first-stage game with restriction in Carlin et al. (2007) is a limit of the model that we consider.}
game is $X^\ell_T$. We model her return from liquidating her excess holding of the risky asset at the end of game by assuming that she has the following payoff at the end of the game

$$X^\ell_T (F_T + \gamma X^d_T) - \frac{C}{2} \gamma (X^\ell_T)^2,$$

(1.4)

where $C > 0$. This terminal payoff is the gain/loss from optimally liquidating $X^\ell_T$, the excess assets bought/sold by the potential predator during the game, over a fixed period of time following the end of the game.

The potential predator solves the following problem:

$$\max_{Y_t} \mathbb{E}^\ell \left[ \int_0^T -P_t Y_t dt + X^\ell_T (F_T + \gamma X^d_T) - \frac{C}{2} \gamma (X^\ell_T)^2 \right]$$

subject to

$$\begin{cases}
  dP_t = \gamma dX_t + \lambda dY_t + dF_t \\
  X^\ell_0 = 0 \\
  dX^\ell_t = Y_t^\ell dt.
\end{cases}$$

(1.5)

Next we define the set of feasible strategies. Learning about the distressed trader’s liquidation is important for the potential predator. Therefore, we assume that the

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9Suppose that the potential predator wants/needs to liquidate $X^\ell_T$ within a time period $\Delta T$. Her optimal strategy is to liquidate at constant rate $X^\ell_T/\Delta T$ (see Carlin et al. (2007)). The resulting return is

$$\frac{X^\ell_T}{\Delta T} \int_T^{T+\Delta T} P_t dt = X^\ell_T (F_T + \gamma X^d_T) - \frac{1}{2} \gamma \left[ -1 + \frac{T}{\Delta T} + \frac{2\lambda}{\Delta T} \right] \left( \frac{X^\ell_T}{\Delta T} \right)^2,$$

given that $P_T = F_T + \gamma (X^\ell_T + X^d_T) + \lambda X^\ell_T$. Note that the cost function evaluated at 0 is zero. Thus the potential predator’s value reduces to that considered in Carlin et al. (2007) when $X^\ell_T = 0$. 
potential predator follows a closed-loop strategy, which is a departure from the ex-
tant predatory trading literature\textsuperscript{10}. The potential predator updates her beliefs by
observing the price dynamics and using Bayes’ rule. The price dynamics generate a
filtration $\{\mathcal{F}(t), 0 \leq t < T\}$. The potential predator learns about $\Delta x$ through this
filtration. The potential predator’s time $t$ estimate of $\Delta x$ is

$$\hat{X}_t \equiv E[\Delta x | \mathcal{F}(t); \mathcal{S}].$$

We consider strategies of the form\textsuperscript{11}

$$Y^\ell_t \equiv \phi^\ell(t, X^\ell_t, \hat{X}_t).$$

For simplicity and following most of the predatory trading literature, we assume that
the distressed trader follows a time-dependent (open-loop) strategy:

$$Y^d_t \equiv \phi^d(t).$$

We require that both $\phi^\ell$ and $\phi^d$ be differentiable for feasible strategies.

\begin{definition}
An equilibrium is a set of feasible strategies $\{Y^d, Y^\ell\}$ such that $Y^d$ is a
solution of the optimization problem (1.3) given $Y^\ell$ while $Y^\ell$ solves the optimization
problem (1.5) given $Y^d$.
\end{definition}

\textsuperscript{10}Carlin et al. (2007) discuss the closed-loop equilibrium of their first-stage model.

\textsuperscript{11}We also solve the game under the assumption that the potential predator follows open-loop
strategies. The results are qualitatively similar.
The presence of the potential predator has an effect on the losses the distressed trader faces due to distress liquidation in an illiquid market. We study this effect by comparing the distressed trader’s value from distress liquidation in the presence of the potential predator to that in its absence. We say that the potential predator engages in predatory trading in a given state of the world if the distressed trader loses more than she would have lost in the absence of the potential predator. Otherwise we say that the potential predator provides liquidity.\footnote{This definition is consistent with that of \cite{brunnermeier2005}, who define predatory trading as “trading that induces and/or exploits the need of other investors to reduce their positions”. We considered the following alternative definitions of predatory trading: (1) A potential predator predates if the excess holding of the risky asset by the potential predator at time $T$ is negative, that is, if $X_T^\ell < 0$. (2) A potential predator predates if the aggregate amount of time she trades in the same direction as the distressed trader during the game is greater than $T/2$. Our results did not change qualitatively under these alternative definitions of predatory trading/providing liquidity.}

\section*{Discussion of assumptions}

Our model adopts two features shared by models in the theoretical literature concerned with the interaction between strategic traders in illiquid markets (see \cite{kyle2001,xiong2001,pritsker2009,shin2004,attari2005,attari2005b,huberman2004,ohmke2014}, and references therein). The first feature is the presence of long-term (non-strategic) traders who have a downward sloping demand curve. The demand curve is a function of Graham (1973)’s safety margin, that is, the difference between the asset’s fundamental value and its price. The assumption that long-term investors have downward sloping demand curves is motivated by the fact that long-term investors need to be rewarded with higher...
returns to change their long-run equilibrium holding of the risky asset, possibly because of risk-aversion. Higher returns are achieved through lower prices. Long-term investors do not take advantage of short-term opportunities unrelated to the risky asset’s fundamental value, such as asset fire sales. They provide liquidity to the market by buying (selling) the risky asset when its price is below (above) its fundamental value. Examples of long-term investors include retail investors. Kaniel et al. (2008) find empirical evidence that individual investors provide liquidity to strategic traders which enables the latter to trade more frequently. Shleifer (1986), Wurgler and Zhuravskaya (2002), and Krishnamurthy and Vissing-Jorgensen (2012) find empirical support for the assumption of downward sloping demand curves.

The second feature is that the changes in the price of risky asset are a linear function of the changes in the asset’s fundamental value, the strategic traders’ aggregate order flow, and the aggregate changes in their order flow. These price dynamics reflect the notion that changes in price are due to both changes in the asset’s fundamental value and market frictions. That is, the risky asset is illiquid and trading by strategic traders is associated with a price impact. The price impact has two components: a long-term component related to the aggregate holding of the risky asset by the strategic traders and a short-term component related to the strategic traders’ aggregate order flow. Both Kyle (1985) and Pritsker (2009) present models that endogenously generate permanent price impacts. Madhavan and Cheng (1997), Glosten and Harris (1988), and Sadka (2006) find empirical evidence supporting the assumption that large trades have distinct permanent and temporary price impacts.

We relax Carlin et al. (2007)’s assumption that the potential predator finishes
the game with zero excess liquidity. The potential predator is not in distress, thus it is more realistic to allow her to choose her excess holding of the risky asset at the end of the game. This modeling choice is made by both Brunnermeier and Pedersen (2005) and Schöneborn and Schied (2007). We assume that the potential predator liquidate her excess holding of the risky asset within a certain period after the game. The potential predator deviates from her long-run holding of the risky asset to take advantage of distress liquidation so it natural to assume that she eventually returns to her initial holding of the risky asset. The time by which the potential predator returns to her initial holding of the risky asset, which is $T + \Delta T$ in our model, determines the constant $C$. In practice, this time depends on several factors (e.g.: Regulation, disclosure of information about the firm, etc). We obtain our theoretical results for arbitrary but finite $C$. Our main numerical results are derived under the assumption that

$$C \equiv 1.$$  

We also consider the equilibrium when

$$C \to \infty \iff \Delta T \to 0$$

in the Appendix. The potential predator does not have time to liquidate her excess holding of the risky asset at the end of the game in the later limit. We show that she

---

$^{13}$The time at which the distressed trader completes liquidation is endogenously determined in Brunnermeier and Pedersen (2005). However, this time is known at the start of the game because they assume perfect information.

$^{14}$We implicitly assume that $\Delta T > 0$. 

has zero excess holding of the risky asset almost surely in this case, which coincides with the first-stage game of Carlin et al. (2007)\textsuperscript{15} The results are qualitatively similar in both cases.

1.2 Equilibrium

Equilibria in the game are determined by trade-offs faced by the players. The distressed trader faces a trade-off between two forces. Trading costs lead her to try to liquidate the risky asset at a slow rate. On the other hand, trades by the potential predator have a price impact. This impact reduces the distressed trader’s value when the potential predator is trading in the same direction as the distressed trader, ceteris paribus. Thus, the distressed trader try to sell at a higher rate in response to the potential trader racing to the market.

The potential predator generates profits from the distressed trader’s price impact by selling high and buying low. She can first race the distressed trader to the market and sell the risky asset when the price is high. We call this strategy sell-first. For this strategy to be profitable, she needs trades by the distressed trader to have permanent price impact so she can buy the risky asset at a lower price later on. Alternatively, the potential predator can first buy the risky asset. We call this strategy buy-first. This strategy is profitable if the potential predator can sell the risky asset at a higher price.

\textsuperscript{15}Either one of the following two conditions can lead the potential predator to find it optimal to have non-zero excess holding risky asset even if she cannot liquidate at the end of the game under two conditions: (1) Information about the fundamental value of the risky asset is released during the game; (2) The potential predator’s initial holding of the risky asset was sub-optimal. We rule out both of these possibilities.
later on, which can occur when prices recover following the distressed trader’s exit from the market. The first strategy is associated with faster trading by the potential predator because of the need to race to the market. This difference in trading speed affects the choice between the two strategies when trading costs, captured by the temporary price impact, are non-zeros.

In the presence of asymmetric information, the potential predator can incur losses in some states of the world because she can sell too much or too little of the risky asset due to the fact that she does not know the liquidation size. These losses depend on the rate at which the potential predator is trading. The potential predator’s value is non-linear in her estimate of the liquidation size and thus the potential predator is not risk-neutral with respect to uncertainty about the liquidation size. That is, the degree of information asymmetry matters. As a result, information asymmetry and learning are important in determining the equilibrium in our model and affect the probability that the potential predator chooses to provide liquidity.

We start by solving for the equilibrium in the special case of no permanent price impact, a case where we can solve the game in closed-form.

1.2.1 No permanent price impact case

The following proposition characterizes the equilibrium in the absence of a permanent price impact. The closed-form solutions will help build the intuition for the general

\[^{16}\text{We consider the problem where } \gamma = 0 \text{ but the cost function is of the form} \]

\[X_T^F F_T - \frac{C}{2} \lambda (X_T^L)^2,\]
case we present in the next subsection.

**Proposition 1** Suppose that there is no permanent price impact, that is $\gamma = 0$. Then there exists a unique equilibrium $(Y^d_t, Y^\ell_t)$ with

$$Y^\ell_t = -\frac{1}{2T} \hat{X}_t = -\frac{1}{2T} \left[ \mu + (1 - \kappa)(\tilde{S} - \mu) \right]$$

$$Y^d_t = \frac{1}{T} \Delta x.$$  

The distressed trader’s equilibrium expected value is

$$V^d = V^{0,d} + \frac{\lambda}{2T} \left[ (1 - \kappa) \sigma^2 + \mu^2 \right], \quad (1.6)$$

where $V^{0,d}$ is the value obtained by the distressed trader in the absence of the potential predator. The probability that the potential predator provides liquidity in equilibrium is

$$\Pr = \int_0^\infty \Phi \left( \frac{1}{\sigma} \left[ \sqrt{\frac{\kappa}{1 - \kappa}} \mu + \sqrt{\frac{1 - \kappa}{\kappa}} x \right] \right) \phi \left( \frac{x - \mu}{\sigma} \right) dx$$

$$+ \int_{-\infty}^0 \Phi \left( -\frac{1}{\sigma} \left[ \sqrt{\frac{\kappa}{1 - \kappa}} \mu + \sqrt{\frac{1 - \kappa}{\kappa}} x \right] \right) \phi \left( \frac{x - \mu}{\sigma} \right) dx$$

where $\Phi (\phi)$ is the cumulative distribution (probability density) function of the standard normal distribution.

Numerical solutions are qualitatively similar to the proposition.
Proof 1.1 See \[A.1\]

The results in Proposition 1 are consistent with the empirical evidence in \[\text{Bessembinder et al.} (2014).\] They find that potential predators provide liquidity when markets are resilient. They define resilient markets as markets where “some or all of the immediate price impact of trades is subsequently reversed”. Resilient markets can be viewed as markets with negligible permanent price impacts.

Proposition 1 also provides a new rationale for “sunshine trading”, the practice of pre-announcing order size. It shows that it is optimal for the distressed trader to reduce information asymmetry about the liquidation size when the permanent price impact is negligible (see Equation 1.6). The mechanism is in our paper is complementary to that in \[\text{Admati and Pfleiderer (1991).} \] Liquidation by the distressed trader has no permanent price impact when \(\gamma = 0\), which implies that prices recover quickly after trades by the distressed trader. Thus, a sell-first strategy is not profitable. In equilibrium, the potential predator follows a buy-first strategy based on her estimate of the liquidation size. The buying reduces the magnitude of the aggregate price impact of strategic traders (\(\lambda|\Delta x - \bar{X}/2|/T\)) relative to the case without the potential predator (\(\lambda|\Delta x|/T\)). Therefore, the price at which the distressed trader liquidates the risky asset (\(P_t = F_t + \lambda(\Delta x - \bar{X}_t/2)/T\) is higher on average relative to the case without the potential predator (\(P_t = F_t + \lambda\Delta x/T\)). The higher price means lower losses from distress liquidation for

\[\text{Admati and Pfleiderer (1991)}\] argue that sunshine trading improves liquidity by reducing adverse selection, which lead to higher value for the distressed trader.
the distressed trader. Hence, in expectation, the potential predator provides liquidity when $\gamma = 0$.

The proposition highlights that a key force determining the occurrence of predatory trading is whether or not trading by the potential predator amplifies the distressed trader’s price impact. We define the notion of gap to study this key force. A quantity’s gap is the difference between the quantity’s value when the potential predator is present in the market from the value when the potential predator is absent from the market. In the case $\gamma = 0$, the strategic traders’ aggregate trading rate gap is $\Delta x/T - \hat{X}_t/(2T) - \Delta x/T = -\hat{X}_t/(2T)$. This gap is positive on average. As a result, the price gap is positive on average because it is linear and increasing in the strategic traders’ aggregate trading rate gap. That is, the distressed trader sells the risky asset at a higher price in the presence of the potential predator. Hence, the distressed trader’s expected value gap is positive since her trading rate gap is zero.

In general, the price gap is a linear function of both the strategic traders’ aggregate holding and trading rate gaps. We shall study how information asymmetry and market illiquidity affect these gaps to understand how they impact the choice between predatory trading and providing liquidity in the general case.

The potential predator’s presence in the market can reduce the distressed trader’s value through two channels: (1) A direct channel which occurs when the potential predator attempts to trade in the same direction as the distressed trader for the majority of the game and is successful in doing so. (2) An indirect channel which occurs when the potential predator attempts to trade in the opposite direction as the distressed trader for the majority of the game but fails to do so because her
estimate of the liquidation size has the opposite sign as the true realization\textsuperscript{18} The direct channel is absent when $\gamma = 0$. We shall see that the direct channel is present and dominates in the general case for higher values of $\gamma$.

1.2.2 General case

We characterize the equilibrium strategies in terms of a system of differential equations:

Theorem 1.1

Given a set of time-dependent functions $(c_1, c_2, c_3, a_1, a_2)$ satisfying the system of first-order differential equations (A.34)–(A.40) in A.1 a linear equilibrium $(Y^d, Y^t)$ is defined by

\begin{align*}
Y^t_t &= c_1(t)X^t_t + c_2(t)\hat{X}_t + c_3(t) \\
Y^d_t &= a_1(t) + a_2(t)\Delta x.
\end{align*}

In this equilibrium, the amount of the risky asset held by the potential predator at time $t$ is

\begin{align*}
X^t_t &= \int_0^t \frac{c_1(s)}{c_1(t)} \left(c_2(s)\hat{X}_s + c_3(s)\right) ds. \tag{1.7}
\end{align*}

\textsuperscript{18}The indirect channel is possible in equilibrium because we model the liquidation size as a normal random variable, allowing for both positive and negative realizations of $\Delta x$ for any choice of $\mu$ and $\sigma \neq 0$. However, the probability of the indirect channel occurring in equilibrium in our model is negligible for realistic families of parameters.
Moreover, the uncertainty faced by the potential predator about the realization of the liquidation size $\tilde{\Delta}x$ is a decreasing function of time.

**Proof 1.2** See [A.1]

We solve the system of first-order differential equations characterizing $c_2, c_3, a_1$ and $a_2$ numerically. See [A.2] for details. We obtain the closed-form solution for $c_1$:

$$c_1(t) = -\frac{\rho(C + 1)}{2 + \rho(C + 1)(T - t)}$$  \hspace{1cm} \text{where} \hspace{1cm} \rho = \frac{\gamma}{\lambda}.$$

c_1 is negative and a decreasing function of $t$. Therefore, the potential predator reduces her excess holding of the risky asset toward the end of the game, *ceteris paribus* (assuming that $\gamma \neq 0$). The potential predator reduces her excess holding of the risky asset to zero when she has no time to liquidate at the end of the game:

**Corollary 1.1**

The potential predator has zero excess holding of the risky asset almost surely when she has no time to liquidate the risky asset at the end of the game. That is,

$$X_T^\ell = 0 \text{ a.a.} \quad \text{in the limit} \quad \Delta T \to 0.$$

The corollary shows that Carlin et al. (2007)’s requirement that the potential predator holds zero excess return is a limit of the game we consider. The term $c_1$
satisfies
\[
\lim_{C \to \infty} c_1(t) = -\frac{1}{T-t}.
\]
Thus,
\[
dX_t^t = -\frac{1}{T-t} X_t^t + c_2(t) \hat{X}_t + c_3(t)
\]
which implies that \(X_t^t\) is a Brownian bridge.

We fix the value of the constant \(C\) in the cost function (see Equation (1.4))
\[
C \equiv 1
\]
for the remainder of the paper.

### 1.2.3 Properties of the Equilibrium

We examine how information asymmetry and market illiquidity affect the linear equilibrium. The trade-offs faced by the traders determine the equilibrium and the occurrence of either liquidity provision or predatory trading.

The distressed trader liquidates faster in response to the potential predator racing to the market. Thus, both traders trade faster and in the same direction on average in equilibria where the potential predator chooses a sell-first strategy characterized by racing to the market for a long period. In such case, the strategic traders’ aggregate trading rate gap is negative on average, which leads to a negative aggregate holding gap and thus a negative price gap on average. A lower price gap means that the distressed trader liquidates the asset at a lower price. Thus, the distressed trader’s
value is lower when the price gap is lower. Therefore, predatory trading occurs when the potential predator’s presence results in a negative price gap on average.

We illustrate the linear equilibrium and the effects of information asymmetry in Figure 1.1. We simulate 100 × 100 equilibrium paths of the game for 100 realizations of the liquidation size $\Delta x$ and 100 paths of the risky asset’s fundamental value for each of two values of the degree of information asymmetry. We plot the average equilibrium strategies for both the distressed trader (Fig 1.1 (a)) and the potential predator (Fig 1.1 (b)). We also plot the strategic traders’ aggregate holding gap of the risky asset (Fig 1.1 (c)) and the price gap (Fig 1.1 (d)). Finally, we plot the dynamics of both the distressed trader’s expected value gap (Fig 1.1 (e)) and the potential predator’s expected value gap (Fig 1.1 (f)).

Figure 1.1 (e) indicates that the distressed trader’s expected value is higher with a higher degree of information asymmetry. It follows that liquidity provision is more likely to occur with a higher degree of information asymmetry. When information asymmetry is lower, the potential predator faces lower losses from estimation errors. Thus, she can follow a more aggressive sell-first strategy, which results in a lower price gap (see Fig 1.1 (d)) through faster trading on average by both traders (see Fig 1.1 (a) and (b)) and a lower aggregate holding gap (see Fig 1.1 (c)).
Figure 1.2 (e) shows that predatory trading occurs in markets with high permanent price impacts and liquidity provision occurs in markets with low permanent price impacts. The intuition is as follows. A higher permanent price impact increases the profits to the potential predator of racing to the market. As a result, the potential predator chooses a more aggressive self-first strategy which leads to a negative price gap (see Fig 1.2 (d)) through faster trading on average by both traders (see Fig 1.2 (a) and (b)) and a negative aggregate holding gap (see Fig 1.2 (c)). The case with low permanent price impact follows from a similar argument.
We now examine the learning dynamics in the equilibrium. The potential predator learns about the liquidation size by observing fluctuations in the price of the risky asset. Her estimate of the liquidation size is

\[ \hat{X}_t \equiv \mathbb{E} \left[ \tilde{\Delta} \tilde{x} \, | \, \tilde{y}(t) ; \tilde{S}_1 \right]. \]

The degree of uncertainty about the liquidation size is characterized by the variance of the random variable \( \hat{X}_t \), denoted \( \Omega(t) \). A sufficient statistic for learning in our model is the percentage of uncertainty left at time \( t \), which we denote \( \delta(t) \):

\[ \delta(t) \equiv \frac{\Omega(t)}{\Omega(0)} = \frac{\Omega(t)}{\kappa \sigma^2}. \]

Learning by the potential predator is a function of changes in the price of the risky asset resulting from the distressed trader’s action. Therefore, it follows from Equation (1.2) that learning is driven by two forces in our model. The first is the rate (and acceleration) at which the distressed trader trades. The second is the set of price impacts. Learning in equilibrium depends on how these two forces interact.

Figure 1.3 (a) shows that learning is faster when the permanent price impact is higher. The reason is two-fold. First, a higher permanent price impact means that changes in price are more informative about changes in the distressed trader’s holding of the risky asset. Second, a higher permanent price impact is associated with greater changes in the rate at which the distressed trader trades in equilibrium (see Figure 1.2 (a)). These two forces combine to improve learning when there is a
higher permanent price impact.

Figure 1.3 (b) illustrates that the effect of the temporary price impact on learning is ambiguous. The two forces driving learning can work in opposite directions in equilibrium when varying the temporary price impact. Increasing the temporary price impact increases the learning rate, ceteris paribus. However, in equilibrium, both traders trade less aggressively in markets with higher temporary price impacts. A lower trading rate by the distressed trader decreases the rate at which the potential predator learns about the liquidation size. This effect can dominate the positive direct effect of a higher temporary price impact on learning. Therefore, in equilibrium, learning can be slower at some point in time in markets with higher temporary price impacts.
1.2.4 Predatory trading versus liquidity provision

The previous subsection studied the distressed trader’s value, which is an expectation. Here, we study the probability of predatory trading occurring in equilibrium.

We estimate the probability that the potential predator will predate for several sets of values of the permanent price impact, the temporary price impact, and the degree of information asymmetry and report the results in Table 1.1.

Table 1.1 shows that predatory trading occurs with certainty for higher (lower) values of the permanent (temporary) price impact when the degree of information asymmetry is zero. This result is consistent with the findings of Brunnermeier and Pedersen (2005), and Schöneborn and Schied (2007). We define predation markets as markets where predatory trading occurs with certainty when there is no information asymmetry.19 We shall focus our discussions on predation markets.20

Table 1.1 indicates that the probability of predatory trading occurring decreases as the degree of information asymmetry increases in predation markets. This result is intuitive. Increasing the degree of information asymmetry decreases the marginal value of racing to the market by increasing the expected losses to the potential predator due to estimations errors. Less aggressive racing to the market (or no racing in the case of the buy-first strategy) increases the likelihood of liquidity provision.

---

19Not all markets are predation markets in our model (see Proposition 1 and the first rows of Table 1 (a) and (b)).

20In reality, information asymmetry is mainly relevant to predatory trading in the context of predation markets: The distressed trader will reduce information asymmetry in non-predation markets where predatory trading would otherwise occur. This point is consistent with Proposition 1 and the presence of sunshine trading in financial markets.
We also observe from Table 1.1 that there is a positive relation between the probability of predatory trading occurring and the permanent price impact in predation markets. Increasing the permanent price impact improves learning and increases the profits from racing to the market. Both effects combine to increase the marginal value of racing to the market in equilibrium.
Table 1.1: Probability of Predatory Trading Occurring in the Presence of Information Asymmetry.

We run $100 \times 100$ simulations of the game assuming that each player follows her linear equilibrium strategy and estimate the probability that predatory trading occurs. See A.2.3 for more details on the simulations. Parameters: $\Delta x \sim N(-10, \sqrt{0.5})$ and $T = 1.0$.

(a) Fixed $\lambda = 1$.

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1.2.5 Welfare

We examine the effects of uncertainty about the amount of the asset to be liquidated on the strategic traders’ aggregate and individual values.

In the partial equilibrium that we consider, the permanent price impact is a transfer of wealth from the strategic traders to the long-term investors. This transfer of wealth is a function of the aggregate change in holding of the risky asset by the strategic traders. The transfer occurs because long-term investors have a downward sloping demand curve which is characterized by the permanent price impact in the price function. The downward sloping nature of the demand curve represents the compensation required by long-term investors when strategic traders change their aggregate positions, walking up or down the demand curve. Long-term investors require this compensation because they are risk-averse.

The temporary price impact is a deadweight cost for the strategic traders’ collective trading. This deadweight cost is due to trading costs such as inventory costs, search costs, bid-ask spread, clearing fees, etc.

We will focus on the welfare gains/losses to the strategic traders due to the presence of the potential predator. We use simulated equilibrium paths to compute each trader’s equilibrium expected value. Table 1.2 presents the strategic traders’ aggregate value as a percentage change relative to the case without the potential predator. It also contains the potential predator expected value and the distressed trader expected value as a percentage change over her value in the absence of the potential predator.
Relating Table 1.2 to Table 1.1 we observe that the percentage change in the strategic traders aggregate expected value is negatively related to the probability of predatory trading occurring in predation markets. The strategic traders trade in the same direction and more aggressively on average when predatory trading occurs. As a result, there is a larger transfer of wealth from the strategic traders to the long-term investors and a larger deadweight loss in trading costs when predatory trading occurs. The potential predator’s gains from predation are lower than the distressed trader losses. Hence, information asymmetry improves the strategic traders’ aggregate welfare in predation markets.

Table 1.2 shows that the potential predator’s value decreases as the degree of information asymmetry increases in predation markets. This result is consistent with our argument that the presence of information asymmetry can lead to losses due to estimation errors. We explore the mechanism yielding this relation. The value achieved by the potential predator is:

\[
E^\ell \left\{ - \int_0^T \left[ F_t + \gamma (X_t^d + X_t^\ell) + \lambda (Y_t^d + Y_t^\ell) \right] Y_t^\ell dt + X_T^\ell \left( F_T + \gamma X_T^d \right) - \frac{C}{2 \gamma (X_T^\ell)^2} \right\}
\]

Assume that both players play linear strategies on the form given in Theorem 1.1 (we do not make this assumption when deriving Theorem 1.1). We can use conditional
expectation to write this value as

$$
\int_0^T -E^\ell \left\{ \left[ \left( F_t + \gamma \left[ \bar{a}_1(t) + \bar{a}_2(t) \hat{X}_t + X_t^\ell \right] + \lambda \left[ a_1(t) + (a_2(t) + c_2(t)) \hat{X}_t + c_1(t) X_t^\ell + c_3(t) \right] \right) \right.
\times \left( c_1(t) X_t^\ell + c_2(t) \hat{X}_t + c_3(t) \right) \right\} dt + X_T^\ell \left( F_T + \gamma \left[ \bar{a}_1(T) + \bar{a}_2(T) \right] \hat{X}_T \right) - \frac{C}{2} \gamma (X_T^2)^2
$$

where

$$
\bar{a}_i(t) = \int_0^t a_i(s) ds, \quad i \in \{1, 2\}.
$$

The source of uncertainty is $\hat{X}_t$, potential predator’s estimate of the liquidation size. Focusing on powers of $\hat{X}_t$, the expression inside the expectation is quadratic in $\hat{X}_t$, and the coefficient associated with $\hat{X}_t^2$ is

$$
- \left[ \gamma \bar{a}_2(t) + \lambda (a_2(t) + c_2(t)) \right] c_2(t).
$$

Assuming that $a_2$ (and thus, $\bar{a}_2$) is positive (our simulations indicate that this holds in equilibrium), the expression above is negative if $c_2(t)$ is positive. That is, the coefficient associated with $\hat{X}_t^2$ is negative when the potential predator attempts to trade in the same direction as the distressed trader. An increase in the degree of information asymmetry increases the variance of $\hat{X}_t$ without changing its mean. Thus, by Jensen’s inequality, an increase in uncertainty about the liquidation size decreases the potential predator’s value when she races the distressed trader to the market.

This heuristic argument implies that, in terms of her payoff and equilibrium strategy, the potential predator is averse to uncertainty about the liquidation size.
when engaging in predatory trading.

We observe from Table 1.2 that there is a positive association between the potential predator’s value and the permanent price impact. This relation is driven by the positive association between the permanent price impact and the probability of predatory trading occurring. The relation is consistent with the empirical evidence of both Shive and Yun (2013) and Arif et al. (2014). They find that potential predators earn higher profits when trading in less liquid assets.
Table 1.2: **Degree of Uncertainty and Welfare.**

We run $100 \times 100$ simulations of the game assuming that each player follows her linear equilibrium strategy and estimate each player’s expected value. We present the potential predator’s wealth. We also present the distressed trader’s wealth as a percentage change relative to her wealth in the absence of the potential predator. Finally, we present the strategic traders’ aggregate wealth as a percentage change relative to their aggregate wealth in the absence of the potential predator. See Appendix 2.3 for details of the estimation procedure. Parameters: $\Delta x \sim N(-10, \sqrt{0.5})$, $\lambda = 1$, and $T = 1$.

(a) Distressed Trader.

<table>
<thead>
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<th>$\gamma$</th>
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<td>0.1</td>
<td>0.01</td>
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<tr>
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<td>0.49</td>
<td>0.24</td>
<td>0.08</td>
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</table>

(b) Potential Predator.

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(c) Aggregate Strategic Trader.

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</table>
1.3 Conclusion

We characterized the effect of asymmetric information on a strategic investor’s decision to either provide liquidity or engage in predatory trading when another strategic trader is in distress. The potential predator estimates the liquidation size, that is the amount of assets to be liquidated by the distressed trader, by observing price dynamics. She can either provide liquidity or engage in predatory trading. We define providing liquidity as increasing the value achieved by the distressed trader (relative to the case without the potential predator).

There is a unique equilibrium in the absence of a permanent price impact, that this equilibrium is linear, and we provide the equilibrium strategies in closed-form. In equilibrium, the potential predator always trades in the direction opposite to her estimate of the liquidation size. That is, she attempts to provide liquidity. This result is consistent with empirical evidence.

We provide conditions under which a linear equilibrium exists without parameter restrictions. We find that predatory trading always occurs in the absence of information asymmetry in markets with large permanent price impact. We call these markets predations markets.

Our main finding is that introducing information asymmetry in predation markets increases the probability that the potential predator will provide liquidity. The mechanism driving this result is the fact that the potential predator is adverse to uncertainty about the liquidation size when engaging in predatory trading. We observe that information asymmetry reduces the potential predator’s value. This
result is consistent with the existing empirical evidence that more frequent disclosure by mutual funds is associated with returns from “front-running” mutual funds.

Overall, our results show that, in the presence of information asymmetry, potential predators are more likely to stabilize markets by providing liquidity. Therefore information asymmetry can explain the observed episodic illiquidity in financial markets.

The potential predator’s choice is also a function of market liquidity, represented by both the permanent price impact and the temporary price impact. Our numerical simulations show that the potential predator is more likely to provide liquidity in markets with low (resp. high) permanent (resp. temporary) price impact. The potential predator also achieves higher value for higher permanent price impact, consistent with empirical findings.

This paper highlighted some benefits to having information asymmetry in financial markets. These benefits are relevant when evaluating (recent) policies/regulations requiring more transparency for institutional investors. Understanding the role of information asymmetry before a crisis such as distress liquidation occurs remains an open question.
Figure 1.1

The Effects of Information Asymmetry.

We simulate $100 \times 100$ equilibrium paths of the game for 100 realizations of the liquidation size $\Delta x$ and 100 paths of the risky asset’s fundamental value for two values of the degree of information asymmetry $\kappa$. We plot the average equilibrium strategies for both strategic traders (Panel (a) and Panel (b)). We also plot the corresponding aggregate holding gap for the strategic traders and the price gap (Panel (c) and Panel (d)). Finally, we plot each trader’s expected value gap (Panel (e) and Panel (f)). A quantity’s gap is the difference between that quantity in equilibrium and the same quantity when the potential predator is not in the market. Other parameters: $\Delta x \sim N(-10, \sqrt{0.5})$, $\gamma = 5$, $\lambda = 1$, and $T = 1$. 

(a) Distressed Trader

(b) Potential Predator

(c) Aggregate Holding Gap

(d) Price Gap

(e) Distressed Trader

(f) Potential Predator
Figure 1.2: The Effects of the Permanent Price Impact.
We simulate 100×100 equilibrium paths of the game for 100 realizations of the liq-
 uidation size \(\hat{\Delta}x\) and 100 paths of the risky asset’s fundamental value for two values
of the permanent price impact \(\gamma\). We plot the average equilibrium strategies for
both strategic traders (Panel (a) and Panel (b)). We also plot the corresponding
aggregate holding gap for the strategic traders and the price gap (Panel (c) and
Panel (d)). Finally, we plot each trader’s expected value gap (Panel (e) and Panel
(f)). A quantity’s gap is the difference between that quantity in equilibrium and the
same quantity when the potential predator is not in the market. Other parameters:
\(\hat{\Delta}x \sim N(-10, \sqrt{0.5})\), \(\kappa = 0.1\), \(\lambda = 1\), and \(T = 1\).
Figure 1.3 : The Effects of the Permanent Price Impact.

The remaining uncertainty about the liquidation size $\widetilde{\Delta} x$ is the ratio

$$\delta(t) = \frac{\Omega(t)}{\Omega(0)}.$$

$\Omega(t)$ is the variance of $\hat{X}_t$, the random variable representing the potential predator’s estimate of $\widetilde{\Delta} x$ conditional on her signal and the realizations of the price process. We solve for the equilibrium $\delta(t)$ numerically. Other parameters: $\Delta x \sim N(-10, \sqrt{0.5})$, $\kappa = 0.7$, $T = 1$, $\lambda = 1$ when $\gamma$ is varying, and $\gamma = 2.5$ when $\lambda$ is varying.
Chapter 2

Law of Small Numbers in a Lucas Orchard

The growth rate of an asset is often unobservable and difficult to estimate with great accuracy, unlike other quantities such as its volatility.\footnote{For example, increased sampling frequency does not improve accuracy. Merton (1980) discusses the challenges one faces when estimating the expected return on the market.} Thus, beliefs and specifically beliefs heterogeneity have important implications and have been proposed as a possible explanation of many empirical irregularities in Finance and Economics. A standard assumption in those fields is that investors use Bayes’ rule when updating their beliefs. However, there is extensive empirical evidence that a sizable subset of both individual and institutional investors deviate from Bayes’ rule when forming/updating beliefs about financial assets. A primitive heuristic associated with many of these deviations is the belief in the Law of Small Numbers (LSN), first documented by Tversky and Kahneman (1971). This belief is the mistaken belief that a small sample drawn from a population should be representative of that population.

We study the portfolio choice and asset pricing implications of belief heterogeneity when a subset of investors believe in the LSN by adding three features to the standard Lucas economy. First, we allow for belief heterogeneity. A subset of investors believes in the LSN and this belief affects their forecast of future dividend growth. We refer to these investors as Freddy, a terminology we adopt from Rabin (2002). The
remaining investors forecast future dividend growth using Bayes’ rule. In equilibrium of an endowment economy, shocks to beliefs impact both expected future cash-flows and expected future discount rates and cash-flow news affects discount rates. We disentangle the two through the second future feature we add to the standard Lucas economy: We consider an economy with multiple trees, that is, a Lucas Orchard. The final feature is the assumption of incomplete information: Future dividend growths are unobservable.

The belief in the LSN has implications when estimating and forecasting quantities. For illustration, suppose that Freddy observes sample $S_n$ of $n$ independent coin tosses from a *fair* coin. Moreover, assume that the number of heads is higher than the number of tails in $S_n$. Then, Freddy believes that the next coin toss has a distribution with the property that tail occurs with probability higher than 50%. This follows from Freddy’s expectation that the sample $S_{n+1}$ will be representative of the population of fair coins. This belief about the distribution of the next coin toss affects Freddy’s estimations and forecasts related to that toss.

We develop a continuous time model of belief in the LSN. We assume that Freddy’s forecast of future dividend growth differs from his estimate of dividend growth. We model the difference as a weighted sum of past shocks to dividend growth, where recent shocks have higher weights. The intuition is that shocks to dividend growth cause the realized dividend growth to be different from its expected value. These deviations from the expected dividend growth result in the observed sample of dividend growth to be different from its population. The weighted sum, the aggregation of the deviations, is used by Freddy in his forecast since he ex-
pects realizations of future dividend growth to be such that future samples are more representative of the population.

This model of belief in the LSN yields the belief in (short-term) reversal, that is, the *Gambler’s fallacy*: Freddy’s subjective expected dividend growth is higher (lower) after observing a recent history of shocks to dividend growth whose aggregate value is negative (positive). The model also yields the belief in (long-term) extrapolation, that is, the *Hot-Hand fallacy* in the presence of incomplete information: Freddy forecasts that future dividend growth will be higher (lower) following long sequences of positive (negative) shocks to dividend growth.

We embed the belief in the LSN in a Lucas Orchard with incomplete information and heterogeneous beliefs. The model generates the Disposition Effect. The growth in Freddy’s holding of the risky asset is lower following positive asset returns relative to negative returns. The mechanism generating this effect is that Freddy’s portfolio choices exhibit asymmetric *V-shaped* trading patterns. Both increases and decreases in returns are associated with non-zero probabilities that Freddy will sell the risky asset. However, the likelihood of selling the risky asset is higher for an increase in returns relative to a decrease in returns. The opposite pattern holds for buys: Freddy buys the risky asset more frequently following a decrease in returns relative to an increase in returns. These asymmetric V-shaped patterns are consistent with the empirical findings of Ben-David and Hirshleifer (2012).

An asset pricing implication of the LSN is a higher equity premium relative to Odean (1998a)’s definition of the Disposition Effect is not applicable in our case since there is continuous rebalancing in our continuous time economy. We propose a new definition for the Disposition Effect that adapts Odean’s definition to our setting.
the benchmark case without Freddy. Freddy’s belief induces additional risk for the rational investor because there is a non-zero probability that future dividend growth will be more in line with Freddy’s expectations. This risk reduces the overall demand of the risky asset and results in lower prices and higher returns. The equity premium is increasing in the magnitude of disagreement between the two agents about the mean dividend growth.

The time-varying beliefs also generate high trading volume and excess volatility. Agents trade because they disagree about the expected dividend growth. Freddy’s beliefs are endogenous and continuously varying which cause greater fluctuations in prices. We find that excess volatility tends to be higher (lower) when Freddy is moderately optimistic (pessimistic). Volatility is lower when Freddy is either very pessimistic or very optimistic because one of the two agents is reluctant to hold the risky asset in such a case. As a result, the risk-free rate is higher.

The momentum strategy yields positive returns in our model. That is, a portfolio of past winners outperforms a portfolio of past losers in the short run. (Recent) Past winners are associated with high realized dividends. These high realized dividends lead to lower prices, and thus high future returns, because Freddy reduces his demand for the risky asset since he expects low future dividends. The momentum effect is stronger when Freddy holds a higher share of consumption, consistent with the effect being driven by the belief in the LSN. The same mechanism leads to positive autocorrelation in returns for individual stocks, consistent with time-series momentum (see Moskowitz et al. (2012)), and to price underreacting to dividend news.

The key to our model is the belief in the LSN. Tversky and Kahneman (1971)
document this belief within a group of mathematical psychologists. Rapoport and Budescu (1997) present experimental evidence confirming this belief in a setting where agents are assigned production tasks. They find that agents tend to produce sequences with too many reversals and too few long runs when asked to produce a sequence of fair coin tosses. Bar-Hillel and Wagenaar (1991) and Burns and Corpus (2004) provide corresponding evidence for judgment tasks (for example, agents are asked if a given sequence of fair coin tosses) and prediction tasks (for example, agents are asked to assign probabilities to the next coin toss given a sequence of fair coin tosses).

The relevance of our model hinges on the presence of the belief in the LSN when agents make decision in high stakes settings such as investment decisions. There is empirical evidence consistent with it. Chen et al. (2015) provide evidence that belief in the LSN affects the (professional) decisions of judges, loans officers, and Major League Baseball umpires. Haigh and List (2005), Coval and Shumway (2005), and Locke and Mann (2005) find that cognitive biases can affect trading by professional traders.

Rabin (2002) is the first to propose a model of belief in the LSN. In a discrete time setting, he assumes agents believe that pairs of coin tosses are drawn from the

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3There is disagreement about the importance of cognitive biases in financial markets. Friedman (1953) argues that cognitive biases are irrelevant because investors holding incorrect beliefs will not survive in the long-run. Arrow (1982) points to limits to arbitrage to counter Friedman’s argument (see Barberis and Thaler (2003) and Gromb and Vayanos (2010) for more on limits to arbitrage). Kogan et al. (2006) and Cvitanic and Malamud (2011) show that irrational beliefs can affect long-run effects even when irrational agents do not survive in the long-run. Borovička (2015) shows that investors with irrational beliefs can survive in the long-run when investors have recursive preferences. Freddy does not survive in the long-run in our model. However, her presence affects price for over 40 years within our simulations.
same urn without replacement. This belief generates the Gambler’s fallacy. Rabin and Vayanos (2010) extend this approach to more general sequences. They are the first to make the connection between belief in the LSN and the Hot-Hand fallacy. They show that belief in the LSN can explain both the active-fund and the fund-flow puzzles. Our model of the LSN is a continuous time analogue of the model in Rabin and Vayanos (2010) and we derive our results in a general equilibrium.

Our paper is related to the literature concerned with non-rational beliefs in financial markets. Papers here often assume that a subset of investors are either always optimistic or always pessimistic. Scheinkman and Xiong (2003), David (2008), Dumas et al. (2009), Xiong and Yan (2010), and Han et al. (2014) depart from this characteristic and consider time-varying beliefs due to overconfidence. Scheinkman and Xiong (2003) and Dumas et al. (2009) focus on speculative trading and excess volatility. David (2008) addresses the equity premium, and Xiong and Yan (2010) are concerned with the failure of the expectation hypothesis. Han et al. (2014) study the spillover effect of disagreement of the overconfidence bias within a two-trees Lucas economy. More recently, Ehling et al. (2014), Collin-Dufresne et al. (2016) and Mal-mendier et al. (2016) study an Overlapping Generation economy where all agents “learn from experience”, which leads to time-varying irrational beliefs. All these papers resort to irrational beliefs to explain various empirical irregularities.

There is a vast literature using belief based behavioral models to explain important empirical irregularities. Abel (1989) was the first to point out that heterogeneous beliefs can generate a high risk-premium. Barberis et al. (1998) and Daniel et al. (1998) model a representative agent with two cognitives biases to explain momentum
and reversal (along with other irregularities) in a discrete time economy. [Hong and Stein (1999)] model the interaction of two boundedly rational agents in a discrete time economy and show their model generates both momentum and reversal. [Barberis et al. (2015)] is closely related to our paper. They also consider a two-investors economy where one of the investors has endogenous subjective belief. They assume that the irrational agent forms expectations about future prices by *extrapolating* past prices in their model, which is justified by survey evidence.

A different line of research uses preference based behavioral models with the same objective of explaining empirical irregularities. [Barberis and Xiong (2009), Ingersoll and Jin (2012), and Li and Yang (2013)] show that prospect theory preferences can explain the disposition effect. [Barberis and Xiong (2009)] note that the choice of the reference point is crucial. [Ingersoll and Jin (2012)] embed prospect theory preferences in a dynamic partial equilibrium model that generates V-shaped trading patterns along with the disposition effect. [Li and Yang (2013)] show that prospect theory preferences can simultaneously generate a high equity premium, momentum, and a disposition effect.
2.1 Model

We consider a continuous time infinite horizon pure exchange economy. The economy is endowed with $N$ trees, each of which produces a flow of non-storable goods. We denote the endowment processes by $D_{jt}$ and assume that they are Geometric Brownian processes on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P_t)$:

$$\frac{dD_{jt}}{D_{jt}} = \mu_{jt}dt + \sigma_j dB_{jt}$$

where $D_{j0} > 0, \mu_{jt} > 0, \sigma_j > 0$, and $B_{jt}$ is a Brownian Motion of the probability space. Agents observe the dividend growth $dD_{jt}/D_{jt}$ but do not observe the mean dividend growth $\mu_j$.

Let

$$D_t = \sum_{j=1}^{N} D_{jt}; \quad s_j = \frac{D_{jt}}{D_t}; \quad y_j = \ln D_j \quad u_j = \ln \frac{s_j}{s_1} = y_j - y_1.$$

Moreover, let

$$\Sigma = \text{diag}(\sigma_1, \cdots, \sigma_N); \quad B = (B_1, \cdots, B_N)'; \quad \mu = (\mu_1, \cdots, \mu_N)';$$

$$s = (s_1, \cdots, s_N)'; \quad y = (y_1, \cdots, y_N)'; \quad u = (u_1, \cdots, u_N)'.$$

It follows that

$$\frac{dD_t}{D_t} = \mu_t dt + \sigma_t dB_t.$$

where
\[
\mu_t = s'\mu; \quad \sigma_t = \sqrt{s'\Sigma s}; \quad \text{and} \quad dB_t = \frac{1}{\sigma} s'\Sigma dB.
\]

There is a market where shares of the trees are traded. There is also a locally risk-free asset in zero-net supply that can be traded.

We assume that there is a unit mass of agents in the market. All agents have identical risk preferences (constant relative risk-aversion with coefficient \(\gamma\)) and time discounting (exponential at rate \(\beta\)). We assume that agents do not observe the mean dividend growths \(\{\mu_j\}_{j=1}^N\).

Agents choose consumption-allocation policies to maximize their expected lifetime utility:
\[
U_i = E_i \left[ \int_0^\infty e^{-\beta t} \frac{C_{i,t}^{1-\gamma}}{1-\gamma} dt \right], \quad (2.1)
\]

where \(\beta, \gamma > 0\).

The expectation in Equation (2.1) is taken with respect to each agent’s subjective probability distribution of future dividend growth. The agents differ with respect to their belief about the dividend growth. We assume that agents can be put in one of two groups, 1 or 2, according to their beliefs. Members of the same group hold the same belief about each dividend growth. Beliefs of agents in Group I are captured by the filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t^j, P_{t,t}^j)\). A sufficient statistic for these beliefs
is the Radon-Nikodym derivative of $P_{I,t}^j$ with respect to $P_{t}^j$:

$$\eta_{I,t}^j = \frac{dP_{I,t}^j}{dP_t^j}; \quad \eta_{I,0}^j = 1.$$ 

**Assumptions**

1. $\eta_{I,t}^j$ is a strictly positive martingale with respect to $P_t^j$.

2. 

$$\frac{d\eta_{I,t}^j}{\eta_{I,t}^j} = \theta_{I,j}(t)dB_{jt} \quad \text{and} \quad \int_0^T \theta_{I,j}(t)dt < \infty \quad \text{almost surely,} \quad \forall \ T > 0.$$ 

3. The variable $\theta_{I,j}$ is a stochastic process adapted to the filtration $\mathcal{F}^j$:

$$d\theta_{I,j} = \mu_{\theta_{I,j}}dt + \sigma_{\theta_{I,j}}dB_{jt}.$$ 

Let

$$\theta_I = (\theta_{I,1}, \cdots, \theta_{I,N})'.$$

Define

$$\frac{d\eta_{I,t}}{\eta_{I,t}} = \theta_{I,t}'dB_t \quad \eta_{I,0} = 1.$$ 

Then, $\eta_{I,t}$ is the Radon-Nikodym derivative of Agent $I$’s subjective probability distribution with respect to the true probability distribution. We have

$$\eta_{I,t} = \prod_{j=1}^N \eta_{I,t}^j.$$
Proposition 2 Survival and long-term price impact are equivalent within our model. Moreover, Agent $I$ does not survive if and only if

$$\lim_{t \to \infty} \int_0^t \left[ \| \theta_I(s) \|^2 - \| \theta_{-I}(s) \|^2 \right] ds = \infty$$

where

$$\|(x_1, \cdots, x_n)\| = \sqrt{x_1^2 + \cdots + x_n^2}.$$

We can interpret $\theta_I$ as the error made in estimating the expected dividend growth. The proposition above says that an agent does not survive if her mean-squared error grows too fast relative to the other agent.

2.2 Beliefs in the Law of Small Numbers in Continuous Time

The dynamics of $\theta_{I,j}$ determine the belief of Agent $I$. We motivate our choice for these dynamics for a subset of agents in our model using the belief in the Law of Small Numbers.

The belief in the LSN is the mistaken belief that a small sample drawn from a population should be representative of that population. Experimental evidence shows that individuals hold this belief (see Tversky and Kahneman (1971), Rapoport and Budescu (1997), Bar-Hillel and Wagenaar (1991), Burns and Corpus (2004), and references therein). Moreover, belief in the LSN affects individuals, decision-making process in high stakes settings (see Clotfelter and Cook (1993), Coval and Shumway...
The following example illustrates how belief in the LSN influences the way agents update their beliefs. Suppose that Freddy knows that a coin is fair. Then the belief in the LSN leads Freddy to think that any sequence of independent coin tosses using this fair coin is representative of the population of fair coins. That is, Freddy believes that any sequence of coin tosses will have a ratio of number of Heads to number of Tails that is close to 50%. Assume that Freddy observes four independent coin tosses and is asked about his beliefs concerning the fifth coin toss. If the first four tosses resulted in four Heads, then the belief in the LSN leads Freddy to assign a probability lower that 50% to the event \{The next coin toss is a Head.\}. This belief about the fifth coin toss rationalizes the belief that the random sample of the five coin tosses will be representative of the fair coins’ population. Similarly, Freddy assigns a higher probability to the event \{The next coin toss is a Head.\} following three Heads than she does following four Heads. This monotonicity results from the fact that the three Heads sample is more representative of the population relative to the four Heads sample.

We present a continuous-time model of the belief in the LSN within a Lucas economy with a single tree and complete information. We then derive the model’s implications within a Lucas Orchard with incomplete information.\(^4\)

2.2.1 Beliefs in the Law of Small Numbers with complete information

We assume in this subsection that the mean dividend is observable. Freddy knows that the expected instantaneous dividend growth of the \( j \)th tree is \( \mu_j \, dt \). Freddy’s belief about the tree’s dividend growth at time \( t \) is a function of the history up to time \( t \) of deviations from \( \mu_j \). We use the following state variable as a proxy for the history:

\[
\theta_{jt} \equiv -b \int_0^t e^{-\kappa(t-s)} \frac{1}{\sigma_j} \left[ \frac{dD_{js}}{D_{js}} - \mu_{jt} dt \right] = -b \int_0^t e^{-\kappa(t-s)} dB_{js}, \quad \theta_0 = 0, \tag{2.2}
\]

where \( \kappa, b \geq 0 \). The integral in Equation (2.2) is the aggregate of the realized deviations from the expected growth rate \( \mu \). The term \( \kappa \) captures the fact that recent deviations are more salient. The term \( b \) measures the degree of the belief in the LSN. The dynamics of \( \theta_j \) are

\[
d\theta_{jt} = -\kappa \theta_{jt} dt - bdB_{jt}.
\]

Thus, Freddy believes that the dividend growth process of the \( j \)th tree is

\[
\frac{dD_{jt}}{D_{jt}} = (\mu_j + \sigma_j \theta_{jt}) dt + \sigma_j dB_{jt},
\]
where \( \hat{B}_{jt} \) is the Brownian motion with respect to Freddy’s subjective probability and

\[
dB_{jt} = d\hat{B}_{jt} + \theta_{jt}.
\]

That is, Freddy believes that the expected dividend growth at time \( t \) is

\[
\mu_j + \sigma_j \theta_{jt}.
\]

Let \( \eta^j_F \) be the Radon-Nikodym derivative of Freddy’s subjective probability measure with respect to the true probability measure governing random dividend growth of the \( j \)th tree. Then,

\[
\begin{cases}
\frac{d\eta^j_F(t)}{\eta^j_F(t)} = \theta_{jt} dB_{jt} \\
\eta^j_F(0) = 1
\end{cases}
\] (2.3)

We state an immediate property of the belief in the LSN, one that follows by construction: The Gambler’s fallacy.

**Proposition 3** Let \( h_t \) be an arbitrary history of dividend growth up to time \( t \) and \( \Pr^F \) denote Freddy’s subjective probability. Then,

\[
\Pr^F \left[ \frac{dD_{ju}}{D_{ju}} \geq \mu_j dt | B_{ju} - B_{jt} > 0, h_t \right] < \Pr^F \left[ \frac{dD_{jt}}{D_{jt}} \geq \mu_j dt | h_t \right] < \Pr^F \left[ \frac{dD_{ju}}{D_{ju}} \geq \mu_j dt | B_{ju} - B_{jt} < 0, h_t \right]
\]

for \( u > t \) sufficiently small.
The proposition implies that Freddy believes in reversal in the short run: Freddy assigns a lower probability to a positive dividend growth shocks occurring in the (near) future if she observes a positive dividend growth today. Thus, Freddy has the Gambler’s fallacy. In the absence of the recency bias (that is, when $\kappa = 0$), $\theta_{jt}$ satisfies

$$\theta_{ju} - \theta_{jt} = -b(B_{ju} - B_{jt}).$$

Thus, in this case, the Gambler’s fallacy holds at all horizons.

### 2.2.2 Learning with Beliefs in the Law of Small Numbers

In the complete information case, the rational agent believes correctly that the assets dynamics are

$$\frac{dD_{jt}}{D_{jt}} = \mu_{jt}dt + \sigma_j dB_{jt}$$

$$d\mu_{jt} = -\alpha(\mu_{jt} - \bar{\mu}_j)dt + \sigma_{1j} dB_{1jt}$$

while Freddy believes that

$$\frac{dD_{jt}}{D_{jt}} = \left(\mu_{jt} + \sigma_j \theta_{jt}\right)dt + \sigma_j d\hat{B}_{jt}$$

$$d\mu_{jt} = -\alpha_j(\mu_{jt} - \bar{\mu}_j)dt + \sigma_{1j} dB_{1jt}$$

$$d\theta_{jt} = -\kappa_j \theta_{jt}dt - b_j dB_{jt}$$

$$d\hat{B}_{jt} = dB_{jt} - \theta_{jt} dt.$$
In the incomplete information case, the agents use these dynamics and the observations of \( [dD_{1t}/D_{1t}, \ldots, dD_{Nt}/D_{Nt}]^T \) to estimate \( [\mu_{1t}, \ldots, \mu_{Nt}] \). We solve their filtering problem, starting with that of the rational agent:

**Lemma 2.1**

The rational agent estimates \( \mu_{jt} \) as follows:

\[
\begin{align*}
\frac{dD_{jt}}{D_{jt}} &= \hat{\mu}_{jt}dt + \sigma_j dZ_{jt} \\
\frac{d\hat{\mu}_{jt}}{\sigma_j} &= -\alpha_j(\hat{\mu}_{jt} - \bar{\mu}_j)dt + \sigma_{1jt} dZ_{jt} \\
dZ_{jt} &= \frac{1}{\sigma_j} \left[ \frac{dD_{jt}}{D_{jt}} - \hat{\mu}_{jt}dt \right]
\end{align*}
\]

where

\[
\sigma_{1jt} = c_{j0} \frac{e^{2\xi_j t} - 1}{e^{2\xi_j t} - \frac{c_{j0}^2}{\sigma_{j1}^2}}; \quad c_{j0} = \sigma_j^2 (\xi_j - \alpha_j); \quad \text{and} \quad \xi_j = \sqrt{\alpha_j^2 + \frac{\sigma_{11}^2}{\sigma_j^2}}.
\]

The long-run value of \( \sigma_{1jt} \) is

\[
\hat{\sigma}_{1j} \equiv \lim_{t \to \infty} \hat{\sigma}_{1jt} = c_{j0}.
\]

Freddy’s belief in short-term reversal affects both her estimate of the mean dividend growth and her estimate of the reversal. The lemma below provides these
estimates:

**Lemma 2.2**

Freddy estimates $\mu_t$ as follows:

\[
\frac{dD_{jt}}{D_{jt}} = \left( \hat{\mu}_{jt}^F + \sigma_j \hat{\theta}_{jt} \right) dt + \sigma_j dZ_{jt}^F
\]

\[
d\hat{\mu}_{jt}^F = -\alpha_j (\hat{\mu}_{jt}^F - \bar{\mu}_j) dt + \frac{1}{\sigma_j} (\sigma_{jt11} + \sigma_j \sigma_{jt12}) dZ_{jt}^F
\]

\[
d\hat{\theta}_{jt} = -(\kappa_j + b) \hat{\theta}_{jt} dt + \frac{1}{\sigma_j} (\sigma_{jt21} + \sigma_j \sigma_{jt22} - b \sigma_j) dZ_{jt}^F
\]

\[
dZ_{jt}^F = \frac{1}{\sigma_j} \left[ \frac{dD_{jt}}{D_{jt}} - \left( \hat{\mu}_{jt}^F + \sigma_j \hat{\theta}_{jt} \right) dt \right]
\]

\[= dZ_{jt} - \hat{\theta}_{jt} dt + \frac{1}{\sigma_j} (\hat{\mu}_{jt} - \hat{\mu}_{jt}^F) dt
\]

where

\[
\sigma_{jt}^F = \begin{bmatrix} \sigma_{jt11}^F & \sigma_{jt12}^F \\ \sigma_{jt21}^F & \sigma_{jt22}^F \end{bmatrix}
\]

is the error covariance matrix. We provide $\sigma_{jt}^F$ in closed-form in the Appendix. The
long-run values of $\sigma_{jt}$’s components are

\[
\begin{align*}
\sigma_{j12}^F &= \sigma_{j21}^F \equiv \lim_{t \to \infty} \sigma_{jt11}^F = \sqrt{\sigma_{1j}^2 - 2\alpha_j \sigma_{jt11}^F - \frac{\sigma_{jt11}^F}{\sigma_j}}, \\
\sigma_{j22}^F &= \lim_{t \to \infty} \sigma_{jt11}^F = \frac{(b\sigma_j - c)\sigma_{jt11}^F - \sigma_j (\sigma_{jt12}^F + \alpha_j \sigma_j + \kappa \sigma_j)\sigma_{jt12}^F}{\sigma_j (\sigma_{jt11}^F + \sigma_j \sigma_{jt12}^F)}, \\
\sigma_{j11}^F &= \lim_{t \to \infty} \sigma_{jt11}^F = \frac{\sigma_{1j}^2 - y_j^2}{2\alpha_j},
\end{align*}
\]

where $y_j$ is the positive root of a fourth degree polynomial both of which are provided in the Appendix.

We assume that of both the agent’s beliefs are in their steady-states for the remainder of the paper. That is, we assume that both $\hat{\sigma}_{1jt}$ and $\sigma_{jt}^F$ are at their long-run values.

The rational agent’s estimates of $\mu_t$ coincides with her expected dividend growth at time $t$. This is not the case for Freddy. Freddy’s expectation for the dividend growth is

\[
E^F \left[ \frac{dD_{jt}}{D_{jt}} \right] = \hat{\mu}_{jt}^F + \sigma_j \hat{\theta}_{jt},
\]

which differs from her estimate $\hat{\mu}_{jt}^F$ of $\mu$. We use this difference to define beliefs reversal and extrapolation. We first define a benchmark rational agent:

*Definition 2.1* Given a history up to time $t$ of the economy, Freddy’s benchmark
rational agent at time $t$ is a rational agent assumed to have the same estimate of $\mu$ as Freddy.

**Definition 2.2** Freddy believes in reversal if her expectation of future dividend growth is lower (higher) than that of her corresponding benchmark rational agent following a positive (negative) shock to dividend growth.

**Definition 2.3** Freddy believes in extrapolation if her expectation of future dividend growth is higher (lower) than that of her corresponding benchmark rational agent following a flow of positive (negative) shock to dividend growth.

The length of time associated with the flow of shocks in Definition 2.3 is not specified. It will depend on the model’s parameters. It follows from the two definitions above that studying belief in reversal/extrapolation is equivalent to examining impulse responses of dividend growth expectation to shocks to dividend growth. We use Malliavin derivatives to compute these impulse responses. Let $D_t\hat{\mu}_{js}^F$ (resp. $D_t\hat{\theta}_{js}$) denotes the Malliavin derivative of $\mu_{js}^F$ (resp. $\theta_{js}$) to a positive shock to dividend growth at time $t$ ($t < s$). This derivative is also the impulse-response of $\mu_{js}^F$ (resp. $\theta_{js}$) to a positive shock to dividend growth at time $t$. 
Freddy exhibits both the *Gambler’s fallacy* (belief in short-term reversal) and the *Hot-Hand fallacy* (belief in long-term extrapolation).

Specifically, $D_t \hat{\mu}_{j_s}$ and $D_t \hat{\theta}_{j_s}$ are

$$D_t \hat{\mu}_{j_s} = \dot{\sigma}_1 e^{-\alpha_j(s-t)} \tag{2.4}$$

$$D_t \hat{\mu}_{j_s}^F = \Sigma_{1j} e^{-(\eta_j + \Delta_j)(s-t)} \left[ (\eta_j - \kappa - \Sigma_{2j}) (1 - e^{2\Delta_j(s-t)}) + \Delta_j (1 + e^{2\Delta_j(s-t)}) \right] \tag{2.5}$$

$$D_t \hat{\theta}_{j_s} = (\Sigma_{2j} - b) e^{-(\eta_j + \Delta_j)(s-t)} \left[ (\eta_j - \alpha_j) (1 - e^{2\Delta_j(s-t)}) + \Delta_j (1 + e^{2\Delta_j(s-t)}) \right] \tag{2.6}$$

where

$$\Sigma_{\ell j} = \frac{\sigma_{j \ell 1} + \sigma_j \sigma_{j \ell 2}}{\sigma_j}, \quad \ell = 1, 2$$

$$\Delta_j = \sqrt{\eta_j^2 - \frac{1}{\sigma_j} \left[ (b + \kappa) \Sigma_{1j} + \alpha_j \sigma_j (\kappa + \Sigma_{2j}) \right]}$$

$$\eta_j = \frac{\sigma_j \Sigma_{2j} + \kappa \sigma_j + \Sigma_{1j} + \alpha_j \sigma_j}{2 \sigma_j}.$$
from Lemmas 2.1 and 2.2 that

\[
\hat{\mu}_{jt} = \hat{\mu}_j + (\hat{\mu}_{j0} - \mu_j)e^{-\hat{\alpha}_j t} + \int_0^t e^{-\hat{\alpha}_j(t-s)}\nu_j(s)ds + c^2_{j0} \int_0^t e^{-\hat{\alpha}_j(t-u)}dB_{ju}
\]  

(2.7)

\[
\begin{bmatrix}
\hat{\mu}_{jt}^F \\
\hat{\theta}_{jt}
\end{bmatrix} = \begin{bmatrix}
\hat{\mu}_j \\
0
\end{bmatrix} + B(t) \left( \begin{bmatrix}
\hat{\mu}_{j0}^F \\
\hat{\theta}_{j0}
\end{bmatrix} - \begin{bmatrix}
\bar{\mu}_j \\
0
\end{bmatrix} \right) + \int_0^t B(t-s) \begin{bmatrix}
\nu_j(s) \\
0
\end{bmatrix} ds + \int_0^t \begin{bmatrix}
\mathcal{D}_u \hat{\mu}_{jt}^F \\
\mathcal{D}_u \hat{\theta}_{jt}
\end{bmatrix} dB_{ju}
\]  

(2.8)

where

\[
\hat{\alpha}_j = \alpha_j + \frac{c^2_{j0}}{\sigma_j}; \quad \nu_j(s) = \sigma_{1j} \int_0^s e^{-\alpha_j(s-u)}dB_{ju};
\]

\[
A_0 = \begin{bmatrix}
\alpha_j \bar{\mu}_j + \Sigma_{1j} \mu_j \\
\Sigma_{2j} - b \bar{\mu}_j
\end{bmatrix}; \quad A = \begin{bmatrix}
\alpha_j + \frac{\Sigma_{1j} \mu_j}{\sigma_j} & \Sigma_{1j} \\
\Sigma_{2j} - b \bar{\mu}_j & \kappa + \Sigma_{2j}
\end{bmatrix};
\]

and

\[
B(t) = \frac{e^{-(\eta_j + \Delta_j)t}}{2\Delta_j} \left[ (\eta_j - \kappa - \Sigma_{2j}) (1 - e^{2t\Delta_j}) + \Delta_j (1 + e^{2t\Delta_j}) \right] 2\sigma_j \Sigma_{1j} \left( 1 - e^{2t\Delta_j} \right)
\]

\[
\frac{1}{2(\Sigma_{2j} - b)(1 - e^{2t\Delta_j})} \left( (\eta_j - \kappa - \Sigma_{2j}) (1 - e^{2t\Delta_j}) + \Delta_j (1 + e^{2t\Delta_j}) \right)
\]

We plot these dynamics for two distinct stocks. Figure 2.1 confirms the results of Theorem 2.1. Both stocks experience positive dividend shocks at the start of the economy, with these shocks continuing to be positive for the first stock but reverting to negative for the second (see Figure 2.1 (a)). The bias caused by the belief in the LSN, \( \theta_{it} \), tends to have the opposite sign of the stock’s cumulative dividend shocks (see Figure 2.1 (b)). This bias leads Freddy to have an estimate of the mean dividend growth that is different from that of the rational agent (see Figure 2.1 (c)).
of positive shocks for both stocks at the start of the economy leads Freddy to have higher estimates relative to the rational agent. The second stock experiences a long sequence of negative shocks toward the end of the economy, which results in Freddy adjusting her estimate to the point that it becomes lower than that of the rational agent. Figure 2.1 (d) shows that Freddy believes in reversal after the short history of positive shocks at the start of the economy. As the positive shocks persist for both stocks, Freddy starts to extrapolate positive shocks in that he forecasts higher dividend growth relative to the rational agent. A long sequence of negative shocks to the second asset’s dividend growth toward the end of the economy leads Freddy to extrapolate that more negative shocks will occur.
Figure 2.1: Assets, Estimates, and Beliefs Dynamics. We plot the paths of shocks to dividend growth for two assets (a). For each path, we plot the corresponding, $\theta_t$, the bias caused by the belief in the Law of Small Number (b), both the rational agent and Freddy’s estimates of the unobservable expected dividend growth (c), and the difference between the forecast of future dividend growth by both agents (d). The expected dividend growth is unobservable and agents need to estimate it. The rational agent’s forecast is her estimate of the expected dividend growth while Freddy’s forecast is his estimate plus the volatility multiplied by the bias. Freddy believes in the LSN.
2.3 Asset Pricing with Belief in the Law of Small Numbers

We study an \( N \)-trees Lucas economy with two agents. The trees’ dividend dynamics are as described in Section 1.1. Agent 1 and Agent 2 have beliefs about each tree’s dividend growth, and these beliefs are captured by the Radon-Nikodym derivative of their subjective probabilities with respect to the true probability. This economy with incomplete information is equivalent to a complete information continuous \( N \)-trees Lucas economy with two agents where the dividend growth dynamics are those given in Lemma 2.1:

\[
\frac{dD_{jt}}{D_{jt}} = \hat{\mu}_{jt} dt + \sigma_j dZ_{jt}
\]

\[
d\hat{\mu}_{jt} = -\alpha_j (\hat{\mu}_{jt} - \bar{\mu}_j) + \hat{\sigma}_j dZ_{jt}
\]

In this equivalent economy, Freddy believes that the dividend growth dynamics are those given in Lemma 2.2:

\[
\frac{dD_{jt}^F}{D_{jt}} = \left( \hat{\mu}_{jt}^F + \sigma_j \hat{\theta}_{jt} \right) dt + \sigma_j dZ_{jt}^F
\]

\[
d\hat{\mu}_{jt} = -\alpha_j (\hat{\mu}_{jt} - \bar{\mu}_j) dt + \Sigma_1 j dZ_{jt}^F
\]

\[
d\hat{\theta}_{jt} = -(\kappa + b) \hat{\theta}_{jt} dt + (\Sigma_2 j - b) dZ_{jt}^F
\]

\[
dZ_{jt}^F = \frac{1}{\sigma_j} \left[ \frac{dD_{jt}^F}{D_{jt}} - \left( \hat{\mu}_{jt}^F + \sigma_j \hat{\theta}_{jt} \right) dt \right]
\]

\[= dZ_{jt} - \hat{\theta}_{jt} dt + \frac{1}{\sigma_j} (\hat{\mu}_{jt} - \hat{\mu}_{jt}^F) dt.
\]
The difference between the Brownian motions with respect to each agent filtration is
\[ \hat{\theta}_j dt + \frac{1}{\sigma_j} (\hat{\mu}_j - \hat{\mu}_j^F) dt. \]
This expression is the sum of two Ornstein-Uhlenbeck processes mean-reverting around zero.

We derive the equilibrium and asset pricing implications in a more general setting. We will then specialize to the case where Agent 1 is rational and Agent 2 is Freddy. Specifically, we consider a complete information $N$-trees Lucas economy where dividend growth dynamics are of the form:
\[ \frac{dD_{jt}}{D_{jt}} = \mu_{jt} dt + \sigma_j dZ_{jt}, \]
\[ d\mu_{jt} = -\alpha_j (\mu_{jt} - \bar{\mu}_j) dt + \sigma_{1j} dZ_{jt} \]

There are two sets of agents in this economy and they have heterogeneous beliefs. Let $\eta_1^j$ and $\eta_2^j$ denote the Radon-Nikodym derivative of the first and second sets of agents respectively. We assume that
\[ \frac{d\eta_\ell^j(t)}{\eta_\ell^j(t)} = \theta_{\ell,j}(t) dZ_{jt}; \quad \eta_\ell^j(0) = 1; \quad \ell = 1, 2, \]
where $\theta_{\ell,j}(t) = c_{1j} \theta_{1,j}^1(t) + c_{2j} \theta_{2,j}^2(t)$ and
\[ d\theta_{\ell,j}^k = -\kappa_{\ell kj} \theta_{\ell,j}^k dt - b_{kj} dZ_{jt}. \]
for arbitrary constant $c_{1j}, c_{2j}, \kappa_{kj}$, and arbitrary function $b_{kj}$.

### 2.3.1 Equilibrium

The market is dynamically complete. Therefore it is enough to consider the social planner’s problem. The social planner’s problem is

$$\sup_{C_{1,t} + C_{2,t} = D_t} E^1 \left\{ e^{-\beta t} \left[ \lambda C_{1,t}^{1-\gamma} + (1 - \lambda) \xi_t C_{2,t}^{1-\gamma} \right] \right\}$$

where with take the expectation with respect to Agent 1’s subjective probability distribution and

$$\xi_t \equiv \frac{\eta_{2,t}}{\eta_{1,t}} = \prod_{j=1}^N \xi_{j,t}; \quad \text{where} \quad \xi_{j,t} \equiv \frac{\eta_{2,t}^j}{\eta_{1,t}^j}.$$

The first order conditions yield

$$\lambda C_{1,t}^{1-\gamma} = (1 - \lambda) \xi_t C_{2,t}^{1-\gamma}.$$

Let

$$\nu_{I,t} = \frac{C_{I,t}}{D_t}$$

be Agent $I$’s consumption share. The investors’ individual optimization problems imply that the following is a state price density:

$$\pi_t = \lambda e^{-\beta t} \nu_{1,t}^{1-\gamma} D_t^{1-\gamma} = (1 - \lambda) \xi_t e^{-\beta t} \nu_{2,t}^{1-\gamma} D_t^{1-\gamma}.$$
Let

\[ \hat{\pi}_{1,t} = \lambda e^{-\beta t} D_t^{-\gamma}; \quad \hat{\pi}_{2,t} = (1 - \lambda) e^{-\beta t} \xi_t D_t^{-\gamma}; \quad \text{and} \quad A_t = \left( \frac{\hat{\pi}_{1,t}}{\hat{\pi}_{2,t}} \right)^{-1/\gamma}. \]

The optimal consumption shares are

\[ \nu_{1,t} = \frac{A_t}{1 + A_t} = \frac{1}{1 + (\alpha \xi_t)^\gamma} \quad \text{and} \quad \nu_{2,t} = 1 - \nu_{1,t}, \]

where \( \alpha = (1 - \lambda)/\lambda. \)

The state price density is

\[ \pi_t = \lambda e^{-\beta t} D_t^{-\gamma} \left[ 1 + (\alpha \xi_t)^\gamma \right]^\gamma. \]

Applications of Ito’s Lemma yield the following Proposition:

**Proposition 4** The instantaneous risk-free rate is

\[ r_{f,t} = r_{f,t}^b + \sum_{j=1}^{N} r_{f,t}^{h_j} \quad (2.10) \]
where

\[
    r^{b}_{f,t} = \gamma \mu_t + \beta - \frac{\gamma(1 + \gamma)}{2} \sum_{j=1}^{N} \sigma_{jt}^2 s_{jt}^2.
\]

_Benchmark: Only rational beliefs_

\[
    r^{bj}_{f,t} = \nu_{2,t} s_{jt}(\theta_{2,j}(t) - \theta_{1,j}(t)) \left[ \gamma \sigma_j + \frac{\gamma - 1}{2\gamma} \nu_{1,t}(\theta_{2,j}(t) - \theta_{1,j}(t)) + \theta_{1,j}(t) \right].
\]

The market price of risk of the \( j \)th tree (as perceived by Agent 1) is

\[
    \sigma_j^j = \gamma \sum_{j=1}^{N} s_{jt} \left[ \sigma_{jt} - \frac{\nu_{2,t}}{\gamma}(\theta_{2,j}(t) - \theta_{1,j}(t)) \right]. \tag{2.11}
\]

The market price of risk the \( j \)th tree as perceived by Agent 1 has two components. The first component is the standard compensation for bearing aggregate risk. The second component is decreasing (and linear) in both the consumption share of Agent 2 and the difference in beliefs \( \theta_{2,j} - \theta_{1,j} \). To understand this relation, note that Agent 1’s consumption share satisfies

\[
    \frac{d\nu_{1,t}}{\nu_{1,t}} dB_{jt} = -\frac{\nu_{2,t}}{\gamma}(\theta_{2,j}(t) - \theta_{1,j}(t)) dt.
\]

A positive shock to the \( j \)th tree’s dividends results in lower consumption share for Agent 1 when \( (\theta_{2,j} - \theta_{1,j}) \) is positive. Intuitively, Agent 2 is more _optimistic_ about the \( j \)th tree’s future dividends relatively to Agent 1 when \( (\theta_{2,j} - \theta_{1,j}) > 0 \). Thus, Agent 1 becomes relatively wealthier following positive shock to the \( j \)th tree’s dividends.
because these are consistent with the optimism, which implies lower consumption share for Agent 2. Therefore, when \((\theta_{2,j} - \theta_{1,j})\) is positive, a positive shock to the \(j\)th tree’s dividends is correlated with lower consumption share for Agent 2 and as a result risk associated with the \(j\) tree carries negative price.

The instantaneous risk-free rate is the sum of the instantaneous risk-free rate in the corresponding Lucas economy without Freddy and linear-quadratic terms in \((\theta_{2,j}(t) - \theta_{1,j}(t))\). The variable \((\theta_{2,j}(t) - \theta_{1,j}(t))\) has two effects on the risk-free rate. First, Freddy expects higher (lower) dividends from the \(j\)th tree in the future and thus has lower (higher) future marginal utility when \((\theta_{2,j}(t) - \theta_{1,j}(t))\) is positive (negative). Thus, the risk-free rate is higher (lower) when \((\theta_{2,j}(t) - \theta_{1,j}(t))\) is positive (negative) because Freddy consumes more (less) and saves less (more) today. Second, the two agents use the credit market (borrowing/lending the risk-free asset) to share risk when \(\gamma > 1\).\(^5\) Divergence in beliefs, that is higher values of \(|\theta_{2,j}(t) - \theta_{1,j}(t)|\), increases the need for risk-sharing through the credit market. Thus, higher values of \(|\theta_{2,j}(t) - \theta_{1,j}(t)|\) yield higher interest rates. The two effects combine when \((\theta_{2,j}(t) - \theta_{1,j}(t))\) is positive. The first effect dominates for small negative values of \((\theta_{2,j}(t) - \theta_{1,j}(t))\) while the second effect dominates for large negative values of \((\theta_{2,j}(t) - \theta_{1,j}(t))\). Figure 2.2 illustrates these relations between the instantaneous risk-free rate and \((\theta_{2,j}(t) - \theta_{1,j}(t))\) in a single tree economy where Agent 1 is rational (that is, when \(\theta_1 \equiv 0\)), for different values of \(\xi_t\).

\(^5\)See Longstaff and Wang (2012) for a discussion of the credit market’s role for risk-sharing purposes.
Figure 2.2: **Instantaneous Risk-free Rate.** We plot the instantaneous risk-free rate as a function of $\theta_t$ for different values of $\xi_t$. The believer in the Law of Small Number (Freddy) expects dividend growth to be the true dividend growth plus $\sigma \theta_t$. $\xi_t$ is the Radon-Nikodym derivative of Freddy’s subjective probability density function with respect to the true density function. The baseline case is the interest rate in the corresponding economy where Freddy is absent.

### 2.3.2 Asset Prices and Portfolio Allocation

Let $P_\alpha$ be the price of an asset paying dividends

$$D^\alpha \equiv \prod D_j^{\alpha_j}, \quad \text{where} \quad \alpha = (\alpha_1, \ldots, \alpha_N).$$

Then,

$$\frac{P_\alpha}{D^\alpha} = E_t \left\{ \int_0^\infty e^{-\beta(u-t)} \left( \frac{1 + \alpha^{1/\gamma} \xi_u^{1/\gamma}}{1 + \alpha^{1/\gamma} \xi_t^{1/\gamma}} \right)^\gamma \left( \frac{D_u}{D_t} \right)^{-\gamma} \frac{D_u^\alpha}{D_t^\alpha} \, du \right\}. $$
The following integrals are useful when pricing assets:

\[
f(t, D_t, \xi_t, y_t, \theta_t, \mu_t; \alpha, n, m) = \int_0^\infty e^{-\beta \tau} E_t \left\{ \left( \frac{1+\alpha^{1/\gamma} \xi_u^{1/\gamma}}{1+\alpha^{1/\gamma} \xi_t^{1/\gamma}} \right)^m \left( \frac{D_u}{D_t} \right)^n \prod \left( \frac{D_j,u}{D_j,t} \right)^{\alpha_j} \right\} d\tau
\]

(2.12)

for \(m, n, \alpha_1, \ldots, \alpha_N \in \mathbb{R}\),

\[
\theta_t = (\theta_{21t}-\theta_{11t}, \ldots, \theta_{2Nt}-\theta_{1Nt}); \quad y_t = (y_{1t}, \ldots, y_{Nt}) \quad \text{and} \quad \mu_t = (\mu_{1t}, \ldots, \mu_{Nt}).
\]

We estimate these integrals by computing expectations of the form

\[
H(\tau, D_t, y_t, \theta_t, \mu_t; \alpha, -n, m) = E_t \left\{ \left( \frac{\xi_u}{\xi_t} \right)^m \left( \frac{D_u}{D_t} \right)^{-n} \prod \left( \frac{D_j,u}{D_j,t} \right)^{\alpha_j} \right\}.
\]

We follow [Dumas et al. (2009) and Martin (2013)] and show in the Appendix that \(H\) is

\[
H(\tau, D_t, y_t, \theta_t, \mu_t; \alpha, -n, m) = e^{-n' y_t/N} D_t^n e^{K_N(\theta, \tau)} \int e^{iz'(Qy_t)} e^{\tau e((\alpha-n/N+iQ'z')y_t)} F_N(z) dz
\]

where

\[
K_N(\theta, \tau) = \sum_{j=1}^N \left[ A_{j,0}(\tau) + \theta_j A_{j,1}(\tau) + \theta_j^2 A_{j,2}(\tau) \right]
\]

and the functions \(Q, F_N, A_{j,0}, A_{j,1},\) and \(A_{j,2}\) are defined in the Appendix. We assume
for the remainder of the paper that $m \in \mathbb{N}$. Then

$$f(t, D_t, \xi_t, y_t, \theta_t, \mu_t; \alpha, n, m) = \int_0^\infty e^{-\beta \tau} \left(1 + \alpha^{1/\gamma} \xi_t^{1/\gamma}\right)^{-m} \sum_{j=0}^m \left(\frac{m}{j}\right) (\xi_t)^{j/\gamma} H(\tau, D_t, y_t, \alpha, \theta_t, \mu_t; -n, j/\gamma) d\tau.$$

The dynamics of $f$ can be obtained by applying Ito’s lemma. We use the following notation

$$f_\eta(t, D_t, \xi_t, y_t, \theta_t, \mu_t, \alpha) \equiv f(t, D_t, \xi_t, y_t, \theta_t, \mu_t; \alpha, -\eta, \eta)$$

$$f_{i,\eta}(t, D_t, \xi_t, y_t, \theta_t, \mu_t, \alpha) \equiv f(t, D_t, \xi_t, y_t, \theta_t, \mu_t; \alpha, -\eta, \eta)$$

$$df_\eta = \mu_{f_\eta} dt + \sum_{j=1}^N \sigma_{f_\eta}^j dB_j = \mu_{f_\eta} dt + \sigma_{f_\eta} dB$$

$$df_{i,\eta} = \mu_{f_{i,\eta}} dt + \sum_{j=1}^N \sigma_{i,f_\eta}^j dB_j = \mu_{f_{i,\eta}} dt + \sigma_{i,f_\eta} dB.$$

The following proposition gives the price to dividend ratio of a claim on a portfolio of trees and each agent’s equilibrium wealth and optimal portfolio:

**Proposition 5** Consider an asset with dividend stream

$$D^\alpha \equiv \prod D_j^{\alpha_j}, \quad \text{where} \quad \alpha = (\alpha_1, \cdots, \alpha_N)$$
and denote its price by $P_\alpha$. Then its price to dividend ratio is

$$\frac{P_\alpha}{D_\alpha} = f_\gamma(t, D_t, \xi_t, y_t, \theta_t, \mu_t, \alpha).$$

Let $P_M$ be the price of the market portfolio. Then,

$$\frac{P_{M,t}}{D_t} \equiv f_{M,t} = f(t, D_t, \xi_t, y_t, \theta_t, \mu_t; 0, -\gamma, \gamma).$$

The rational agent’s wealth to consumption ratio is

$$\frac{W_{1,t}}{C_{1,t}} = f_{\gamma-1}(t, D_t, \xi_t, y_t, \theta_t, \mu_t, 0).$$

Freddy’s wealth is

$$W_{2t} = P_t - W_{1t}.$$

We plot the price to dividend ratio of tree 1 as a function of the bias $\theta_{2,1}$ when tree 1 is small in a two trees Lucas economy in Figure 2.3. The figure shows two salient properties: First, the price to dividend ratio is increasing in a neighborhood of zero. In general, the price to dividend ratio exhibits the following asymmetry: It is higher for positive values of $\theta_{2,1}$ relative to the corresponding negative values. Second, the price to dividend ratio has a reverse U-shape. These results are intuitive. Variations in $\theta_{2,1}$ have a negligible effect on the Freddy’s discount rate when the tree is small so all the action comes from cash-flow news. Cash flow news have two direct
effects on valuation in our model. Freddy expects high (low) dividend growth when \( \theta_{2,1} \) is positive (negative) which leads to high (low) valuations when \( \theta_{2,1} \) is positive (negative). This effect explains the first salient property. The second effect is that the correlation between cash-flow and consumption is higher for larger values of \(|\theta_{2,1}|\), which leads to lower valuation and explains the fact that the price to dividend ratio is reversed U-shaped.

Figure 2.3 also shows that the price to dividend ratio in our model approaches that in the benchmark model where Freddy is absent as \( \xi_t \) goes to 0. This result follows because Freddy’s consumption share converges to zero as \( \xi_t \) goes to 0.

Figure 2.3 suggests that our model with belief in the LSN generates short-term momentum and long-term reversal. We know that high price to dividend ratios predict low future returns. The figure implies that the price to dividend ratio is higher for positive values of \( \theta_{2,1} - \theta_{1,1} \) relative to the symmetric negative values of \( \theta_{2,1} - \theta_{1,1} \). This difference is positive following a short sequence of negative dividend growths when the two agents are a rational agent and Freddy. Moreover, we expect that such sequence of negative dividend growths is also associated with low contemporaneous returns. Thus, we expect that our model of belief in the LSN will generate short-term momentum. The presence of long-term reversal follows a similar argument since the difference \( \theta_{2,1} - \theta_{1,1} \) is negative following a long sequence of negative dividend growths in our model with belief in the LSN. We confirm the presence of both short-term momentum and long-term reversal in our model with belief in the LSN using numerical simulations.

We use Proposition 5 to derive the returns dynamics. Let \( R_{i,t} \) be asset \( i \)’s cumu-
Figure 2.3: **Price to Dividend Ratio.** We plot the price to dividend ratio of tree 1 as a function of $\theta_{2,1}$ when tree 1 is a small tree within a two-trees Lucas economy, for different values of $\xi$. The believer in the Law of Small Number (Freddy) expects dividend growth to be the true dividend growth plus $\sigma \theta_{2,1}$. $\xi$ is the aggregate Radon-Nikodym derivative of Freddy’s subjective probability density function with respect to the true density function. The baseline case is the price to dividend ratio in the corresponding economy where Freddy is absent.

\[
\begin{align*}
dR_{i,t} & \equiv \frac{dP_{i,t} + D_{i,t} dt}{P_{i,t}} = \left[ \mu_i + \frac{1 + \mu_{f_{i,\gamma}} + \sigma_i \sigma_{f_{i,\gamma}}}{f_{i,\gamma}} \right] dt + \sigma_i dB_{i,t} + \frac{1}{f_{i,\gamma}} \sum_{j=1}^{N} \sigma_{f_{i,\gamma}}^j dB_{j,t}. \\
& \quad (2.13)
\end{align*}
\]
The market return is

\[
    dR_{M,t} \equiv \frac{dP_{M,t} + D_t dt}{P_{M,t}} = \left[ \mu + \frac{1 + \mu_{f_{M,t}} + \sigma_M \sigma_{f_{M,t}}}{f_{M,t}} \right] dt + \left[ \sigma_M + \frac{1}{f_{M,t}} \sigma_{f_{M,t}} \right] dB_t
\]

(2.14)

where

\[
    \sigma_M = s^\prime \Sigma \quad \text{and} \quad df_{M,t} = \mu_{f_{M,t}} dt + \sigma_{f_{M,t}} dB_t.
\]

Both returns and volatilities are stochastic and different from the corresponding quantities in the benchmark model. Whether they are higher or lower relative to their benchmark counterparts depends in part on how the price to dividend ratios correlate with dividend growth. The expected return has the form the standard form dividend drift + dividend yield

plus a term accounting for the variability of the price to dividend ratio. Similarly, the return volatility is equal to the fundamental volatility plus a term accounting for the volatility of the price to dividend ratio. The latter generates excess volatility in our model and is

\[
    \frac{1}{f_{i,\gamma}} \sqrt{\sum_{j=1}^{N} \left( \sigma_{f_{i,\gamma}}^j \right)^2}
\]

for a single asset and

\[
    \frac{1}{f_{M,t}} \sigma_{f_{M,t}}
\]

for the aggregate market.
We plot both the instantaneous equity premium and the excess volatility in our
general economy when the first agent is rational in Figure 2.4. Figure 2.4 (a) shows
that the instantaneous equity premium is higher in our model relative to the cor-
responding economy where both agents are rational. Figure 2.4 (b) shows there is po-
itive excess volatility in our model for small and intermediate levels of disagreement
between the two agents. Excess volatility is negative for large levels of disagreement.
In additional, excess volatility is increasing when there is little disagreement between
the two agents.

The implications of Figure 2.4 (b) for case where the second agent believes in the
LSN are as follows:

1. Idiosyncratic volatility is higher (lower) following short sequences of negative
   (positive) dividend growth shocks.

2. Idiosyncratic volatility is lower (higher) following long sequences of negative
   (positive) dividend growth shocks.

We now consider the agents’ portfolio choices. Let \((M_{i,t}, N_{i,j,t})\) the Player’s \(i\)
portfolio, where \(M_{i,t}\) is the amount of money invested in the risk-free asset and \(N_{i,j,t}\)
is the number of shares of the \(j\)th risky asset she holds. The intertemporal budget
constraint implies that

\[
\frac{dW_{i,t} - \mathbb{E}[dW_{i,t}]}{W_{i,t}} = \sum_{j=1}^{N} \pi_{i,j} \frac{dP_j - \mathbb{E}[dP_j]}{P_j} \quad \implies \quad W_{i,t} - \mathbb{E}[dW_{i,t}] = \sum_{j=1}^{N} N_{i,j,t} (dP_j - \mathbb{E}[dP_j]).
\]

We obtain both the wealth and price volatilities from the previous proposition. It
thus follows from the independence of Brownian motions that the portfolio choices
Figure 2.4: **Instantaneous Equity Premium and Excess Volatility.** We plot the instantaneous equity premium (a) and excess volatility (b) as functions of $\theta_t$ for different values of $\xi_t$. We consider a two-agents economy where one agent is rational and the other expects dividend growth to be the true dividend growth plus $\sigma \theta_t$. $\xi_t$ is the Radon-Nikodym derivative of Freddy’s subjective probability density function with respect to the true density function. The baseline cases are the respective quantities in the corresponding economy with rational representative agent.

can be derived by solving a system of equation:

**Proposition 6** Let $(M_{i,t}, N_{i,j,t})$ the Player’s $i$ portfolio, where $M_{i,t}$ is the amount of money invested in the risk-free asset and $N_{i,j,t}$ is the number of shares of the $j$th risky asset she holds. The equilibrium number of shares of the risky asset held by the agents at time $t$ are

\[
N_{2,j,t} = 1 - N_{1,j,t}; \quad M_{2,t} = -M_{1,t}; \quad M_{1,t} = W_{1,t} - \sum_{j=1}^{N} N_{1,j,t} P_{j,t}
\]
and

$$
\begin{bmatrix}
N_{1,j,t} \\
N_{1,2,t} \\
\vdots \\
N_{1,N,t}
\end{bmatrix} = \Sigma_{p}^{-1}
\begin{bmatrix}
\sigma_{W_1}^1 \\
\sigma_{W_1}^2 \\
\vdots \\
\sigma_{W_1}^N
\end{bmatrix}
$$

where $\Sigma_{p} = [(dP_k)(dB_i)]$ and $\sigma_{W_1}^j = (dW_1)(dB_j)$. 
2.4 Numerical Analysis

We simulate our model under the assumption that one agent is rational and the other is Freddy, who believes in the LSN. We simulate 100 economies at the one-week frequency, each for a 50 years period and compute relevant quantities for each economy. We then average across economies.

Table 2.1: Model Values. We present the parameter values used in our simulations. The main source of asset parameter values is Dumas et al. (2009).

<table>
<thead>
<tr>
<th></th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Endowment economy</strong></td>
<td></td>
</tr>
<tr>
<td>Dividend volatility</td>
<td>0.13</td>
</tr>
<tr>
<td>Dividend growth long-run mean</td>
<td>0.015</td>
</tr>
<tr>
<td>Dividend growth volatility</td>
<td>0.03</td>
</tr>
<tr>
<td>Dividend growth mean reversion</td>
<td>0.2</td>
</tr>
<tr>
<td><strong>Agents</strong></td>
<td></td>
</tr>
<tr>
<td>Risk aversion</td>
<td>4</td>
</tr>
<tr>
<td>Time discounting</td>
<td>0.1</td>
</tr>
<tr>
<td>Initial ratio of consumption shares</td>
<td>1</td>
</tr>
<tr>
<td><strong>Freddy</strong></td>
<td></td>
</tr>
<tr>
<td>LSN parameter</td>
<td>0.4</td>
</tr>
<tr>
<td>Recency bias</td>
<td>2</td>
</tr>
</tbody>
</table>
2.4.1 State Variables

We first show the simulated values for the two state variables related to the belief in the LSN: $\theta_t$ and $\xi_t$. Figure 2.5 (b) shows that Freddy’s influence vanishes in the long-run. However, Freddy’s impact in the economy disappears slowly.

Figure 2.5: State Variables.
2.4.2 Risk-Free rate and Market Price of Risk

Figure 2.6 shows that the risk-free rate is higher in our model relative to the baseline without Freddy. Both the risk-free rate and the market price of risk vary over time. Freddy has long-term impact on the economy despite his long-run extinction: The risk-free in our model remains significantly different from that in the benchmark economy for long horizons.

![Figure 2.6: Risk-Free Rate and Market Price of Risk.](image)
2.4.3 Disposition Effect

There is continuous rebalancing in our model due to the assumption of continuous
time trading and the absence of frictions (such as trading costs). Thus, if we follow
Odean (1998a)’s definition of the disposition effect, there is no disposition effect
in our model. We propose a definition of the disposition effect applicable in our
setting:

Definition 2.4 Freddy exhibits the disposition effect if, on average,

\[ \frac{N_{2,t}|R_t-R_s>a}{N_{2,s}} < \frac{N_{2,t}|R_t-R_s<a}{N_{2,s}} \]

where \( a > 0 \) and \( t > s \).

This definition is consistent with that of Odean (1998a) when applied to a discrete
time setting. Following Odean (1998a), we use the following notations:

\[ \text{PRG} = \frac{N_{2,t}|R_t-R_s>a}{N_{2,s}} \quad \text{and} \quad \text{PRL} = \frac{N_{2,t}|R_t-R_s<a}{N_{2,s}}. \]

Therefore, Freddy exhibits the disposition effect if

\[ \text{PRG} < \text{PRL}. \]

\[ ^6 \text{A similar issue arises when applying the empirical definition of volume to continuous time trading models.} \]
We consider Freddy’s portfolio holding following increases/decreases in returns. Every month, we rank stocks into quintiles based on the difference between the present month and the previous month returns and compute the average growth in holding of the risky asset relative to the previous month within each group. We summarize the results in Table 2.2. Table 2.2 shows that Freddy exhibits the disposition effect in our model.

Table 2.2 : **Disposition Effect.** The economy is simulated 100 times at the weekly frequency, each over a 50 years period, and the portfolio of each agent is computed. We define PRG and RL as follows:

\[
PRG = \frac{N_{2,t} | R_t - R_{t-1} > 0}{N_{2,t}} \quad \text{and} \quad PRL = \frac{N_{2,t} | R_t - R_{t-1} < 0}{N_{2,t}}.
\]

Each month, we compute the difference between PRG and PRL and report the summary statistics of this series.

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Std Dev</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>PRG - PRL</strong></td>
<td>0.41</td>
<td>0.1</td>
<td>0.055</td>
<td>0.70</td>
</tr>
</tbody>
</table>

Next, we examine Freddy’s trading patterns at a more granular level. We compute the likelihood of Freddy selling (buying) the risky asset conditional on changes in returns. Figure 2.7 shows that Freddy is more likely to sell the risky asset following an increase in returns relative to a decrease in returns, with both probabilities been positive. The opposite holds for buys. Thus, Freddy exhibits a \( V \) - shaped trading pattern with respect to returns. These patterns are consistent with the empirical
findings of Ben-David and Hirshleifer (2012).

Figure 2.7: Probability of Selling/Buying Additional Shares.
Every months, we rank stocks into quintiles based on the difference between the present month and the previous month returns. We compute the number of sells (buys) within each group scaled by the total number of sells (buys), where a sell (buy) is an instance of Freddy reducing (increasing) her holding of the risky asset. We report the average across all months.
2.4.4 Momentum

Trading by Freddy causes prices to deviate from the rational benchmark. We study the asset pricing implications of the LSN. An obvious starting point is momentum, given the V-shaped patterns associated with Freddy’s trading.

Table 2.3 shows that our model replicates the return momentum documented empirically. The momentum is stronger in the first half of our sample, when Freddy’s consumption is a large percentage of the dividend flow, relative to the second half. Therefore, the momentum is driven by the belief in the LSN.

Although we do not attempt to match empirical moment, we note that the momentum in our simulated data is low relative to the empirical momentum. Thus, our model does not explain the majority of the empirical momentum. This result is consistent with the findings of Birru (2015). Birru shows that momentum is present in the market even when the disposition effect is absent among retail investors.

2.5 Conclusion

Beliefs in the Law of Small Numbers has non-trivial asset pricing and portfolio choices implications. Agents with such beliefs exhibit V-shaped trading patterns, that is, they have a higher (lower) likelihood to sell (buy) winners relative to losers, which implies the disposition effect. These trading patterns result in returns having “momentum”, a high average equity premium, and excess volatility. We obtain these results by
Table 2.3: **Momentum.** The market is simulated 100 times, each over a 50 years period and monthly returns are computed. Each month, we group stocks into deciles based on the cumulative returns over the previous twelve months. The table shows descriptive statistics of the differences of average returns between the high decile and the low decile for the month following the formation of portfolios.

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Std Dev</th>
<th>T-Stat</th>
<th>Min</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Full Sample</strong></td>
<td>0.0018</td>
<td>0.007</td>
<td>6.24</td>
<td>−0.019</td>
<td>0.025</td>
</tr>
<tr>
<td><strong>First Half</strong></td>
<td>0.003</td>
<td>0.007</td>
<td>6.67</td>
<td>−0.017</td>
<td>0.025</td>
</tr>
<tr>
<td><strong>Second Half</strong></td>
<td>0.0007</td>
<td>0.006</td>
<td>1.9</td>
<td>−0.019</td>
<td>0.018</td>
</tr>
</tbody>
</table>

developing a continuous time model of the belief in the LSN and embedding these beliefs in a heterogeneous Lucas Orchard.
Chapter 3

On The Volatility and Volume Relationship: Evidence From a Natural Experiment

Does trading activity in itself affect volatility above what can be justified by fundamentals? We investigate this question in the context of US equity markets. Trading on US stock exchanges has significantly increased over the last 50 years (see Figure 3.1). Both researchers and market participants often use observed volatility as a proxy for the firm’s riskiness (see for example [Campbell et al. (2001) and Ang et al. (2006)]). As a result, observed volatility affects the firm’s cost of capital, its managers’ compensation, and the type of investors willing to hold equity in the firm. Therefore examining the effects of volume on observed volatility and the link through which trading activity affects volatility is of first order importance in economics and has important implications for policy, market design, and empirical studies on volatility.

The main issue faced by researchers studying this question empirically is that volume and volatility are endogenously determined (see Schwert (1989)). In addition, theoretical models do not agree on whether the sign of the correlation between volume and volatility is positive or negative.\footnote{Higher trading volume can yield better price discovery and thus lower observed volatility. It is also possible that higher volume harms price discovery if it is uninformed and can thus move prices away from fundamentals and increase volatility. Examples of uninformed trades include noise trades, speculative trades, trades for hedging purposes, and liquidity trades.} We use a natural experiment to identify the
causal effect of an exogenous shock to volume on volatility in a setting where firms’ fundamentals do not vary during the experiment.

Our main finding is that trading activity on the New York Stock Exchange (NYSE) generates excess volatility above what can be justified by firms’ fundamentals. A 10% exogenous reduction in turnover of NYSE-listed firms leads to a 2.1% decrease in firms’ observed volatility even though firms’ fundamentals remain unchanged. This result is economically significant. A 2.1% permanent increase in a firm’s volatility should be accompanied by a 17% increase in the firm’s cost of capital for an investor with CARA utility preference to keep her holding in the firm unchanged. In addition to the clean identification, an advantage of our empirical setting is its external validity. A large portion of US publicly traded firms are listed on the NYSE. Although NYSE-listed firms are older and larger on average relative to non NYSE-listed firms, we believe that our results are applicable to non NYSE-listed firms. Our results should also remain valid in other markets such as commodity markets and developed international exchanges.

We identify the effects of volume on volatility using the exogenous shock to volume caused by blizzards in Manhattan, New York, NY. We classify a trading day as a blizzard day if there is at least 10 inches of snowfall on that day (or on the previous if the previous day is a non-trading day). This classification results in 21 blizzard events in Manhattan since 1941. We collect information about the real effects of these events from newspapers and find that several of our blizzard events lead to disruptions such

\[2\text{We assume that the risk-free rate is } 1.75\% \text{ and that the firm’s original cost of capital is } 6.97\%\]
Figure 3.1: Value-Weighted Stock Trading Turnover for NYSE-Listed Firms. We plot the value-weighted yearly turnover of all NYSE-listed firms from 1926 to 2013. Turnover is defined as the volume dividend by the total number of share outstanding. We compute daily market value-weighted turnover as the sum(turnover*lagmktcap)/sum(lagmktcap) where the sum is over all listed firms. lagmktcap is the previous market capitalization. We average the daily market value-weighted turnover within each year to obtain the yearly measure of market value-weighted turnover.
as cancellation of classes and transportation in and out of New York City and power outages. Predicting these events and their exact location in advance is difficult. An atmospheric scientist noted in the New York Times that “a small error in predicting the path of the storm can cause a much larger error in impact. The bigger the event, the bigger the bust potential”\(^3\) Thus, cancellations caused by the blizzard are often announced after they have occurred or the day before. In some cases, blizzards do not occur when expected. For example, New York City was essentially shut down on January 27th 2015 because authorities expected the (in)famous “Blizzard That Wasn’t”. Blizzards are thus a natural experiment, clearly exogenous to both trading volume and volatility.

Blizzards in Manhattan are associated with a statistically significant 10% average reduction in turnover for NYSE-listed firms. We do not identify the exact channel yielding this result. However, blizzards disrupt economic activities, cause flights, trains, and school cancellations, and result in power outages. Thus, blizzards can reduce trading volume directly through power outages or indirectly by preventing traders from going to work. We only consider NYSE-listed firms whose headquarters are not in the state of New York or its neighboring states to isolate the effect on trading volume orthogonal to firm’s fundamentals. In addition, Loughran and Schultz (2004) show that there is little correlation between occurrence of inclement weather across major US cities. Therefore, the 10% average reduction in turnover is an exogenous shock to the trading volume of the firms in our sample, which allows for

a clean identification of the effects of a change in volume on volatility.

We find indirect evidence suggesting that the channel through which trading generates excess volatility is uninformed trading by institutional investors. First, the exogenous shock to volume we identify disproportionately affects institutional investors. Retail investors trade mainly stocks of companies located near them (Coval and Moskowitz (1999) and Coval and Moskowitz (2001)) thus changes to their trading volume due to blizzards in New York City are negligible since we eliminate their local firms from our sample. Thus the effects we document is likely to be the result of a reducing of trading volume by institutional investors. Second, the effect of the shock to volume on volatility is greater after 2003. Between January 29th, 2003 and May 27th, 2003 the NYSE implemented “autoquote” \cite{Hendershott2011} find that this change led to a significant increase in trading by algorithmic traders. In addition, several algorithmic trading strategies, such as momentum trading and front-running do not have firms’ fundamentals as first order inputs. Taken together this evidence indicates that trading by institutional investors, specifically uninformed trading, increases firm’s volatility

The effects of volume in financial markets in general and on volatility in particular have been studied in the extant literature. The research closest to ours was done by Elyasiani et al. (2000), Zhang (2010a), and Dichev et al. (2014). Elyasiani et al. (2000) find that an increase in volume due to switches between ex- changes lead to lower volatility. Dichev et al. (2014) arrive at a different conclusion using natural

experiments that include switches between exchanges. Dichev et al. (2014) look at differences in volume when stocks switch exchanges or are added or deleted from the S&P500 and the difference in volume between dual-class stocks to identify shocks to volume. They use these shocks to volume to study the effect of trading activity of volatility and find it to be positive. The decision to move from one exchange to another one is made by the firm and could be related to fundamentals. Different classes of stocks have different voting-rights and those rights have different values (see Zingales (1995)). Zhang (2010a) shows that there is a positive correlation between high frequency trading and volatility. However Zhang does not establish the causal direction between volume and volatility. We contribute to this literature by providing an identification strategy that is clean and has strong external validity to show that trading activity contributes to volume.

Understanding the effects of volume on volatility is part of the general effort in the finance and economic literature to understanding the sources of the observed volatility in markets. Both Campbell et al. (2001) and Irvine and Pontiff (2009) document a sharp increase in idiosyncratic volatility over the recent years. They attribute their findings to institutional ownership and cash-flow volatility, respectively. French and Roll (1986), Schwert (1989), and Koudijs (2015) find that private information in the principal driver of volatility relative to public information. Fleming et al. (2006) present the complementary evidence that public information plays an important role in determining volatility in weather-sensitive markets. Our findings contribute to this literature by showing that trading, specifically trading by institutional investors, contributes to volatility.
Theoretical models concerned with the determinants of volume and volatility often emphasize the links between the two. These links include speculative trades (see Foster and Viswanathan (1995), Campbell et al. (1993), De Long et al. (1990)), behavioral biases such as overconfidence (see Odean (1998b) Scheinkman and Xiong (2003), and Goetzmann and Massa (2008)), and trading by institutional investors (see Martinez and Rosu and Malamud (2015)). In particular, Martinez and Rosu model high frequency traders as news-watchers who are ambiguous about firms’ fundamentals and show that higher high frequency trading increases both volume and volatility. Malamud (2015) shows that trading by exchange traded funds does not increase volatility in general. Our empirical findings support the thesis that uninformed trading by institutional investors can increase volatility.

Our work is also related to the literature concerned with factors affecting trading activities and returns. Hirshleifer and Shumway (2003) show that mood, proxied by the degree of cloudiness, affects stock returns. Coval and Moskowitz (1999, 2001) document the “home bias” in investing. Loughran and Schultz (2004) show that blizzards reduce trading by local investors which in turn reduces the trading volume of local firms and that trading volume is lower on holidays such as the Yom Kippur. Chordia et al. (2011) show that the degree of institutional holding is an important determinant in the recent increase in volume. We contribute to this literature by showing that inclement weather in Manhattan, New York, NY can reduce the trading volume of firms located far from New York, NY.

The remainder of the paper is organized as follows. We describe our data selection and proxies in Section II. We present our empirical result on trading volume during
blizzards in Manhattan, New York, NY in Section II. Section III documents the relation between volume and volatility. We discuss possible explanations of our results in Section IV and Section V concludes.

3.1 Data

Exchanges have idiosyncratic microstructure designs that affect trading volume which makes it difficult to jointly study volume across several exchanges. We circumvent this potential issue by focusing our analysis on firms listed on the New York Stock Exchange (NYSE). NYSE-listed firms tend to be larger and older relative to firms listed on the NASDAQ. As a result, the “home bias” documented by Coval and Moskowitz (1999, 2001) is weaker for NYSE-listed relative to NASDAQ-listed firms. Thus, we expect that stockholders of NYSE-listed firms are located further away from the firm’s headquarter relative to NASDAQ-listed firms. Therefore, the effects of investors biases (local bias, mood bias) should be lower for NYSE-listed firms. Moreover, NASDAQ is an electronic exchange while a significant portion of trading in NYSE-listed firms takes place at the NYSE through a specialist. Thus Manhattan’s weather should have a higher effect on trading volume for NYSE-listed firms.

We obtain data about snow precipitations in New York, New York from the International Surface Weather Observations dataset provided by the National Oceanic and Atmospheric Administration. Both Hirshleifer and Shumway (2003) and Loughran and Schultz (2004) use this dataset in their studies examining the effects of weather on stock returns and trading volume. We use data from the Central Park Observa-
tory at Belvedere Tower, the observatory closest to Wall Street. We classify a trading day as a blizzard day if there is at least 10 inches of snowfall on that day or on the previous day, if that the previous day is a non-trading day. For example, if there is 4 inches of snowfall in Manhattan on a Monday when the NYSE is open and there was 12 inches of snowfall on the Sunday before, we consider the Monday a blizzard day. In comparison, we find that the average snowfall when it snows at least 0.1 inches in Manhattan is about 2 inches. The amount of snowfall is about three standard deviations away from that yearly average. Blizzards are natural events that affect trading volume for the entire universe of firms traded on an exchange located in New York City.

There have been 21 blizzard events in Manhattan since 1941. 11 of these snow events led “snow day” (cancellation of classes) in New York City public schools. The blizzard event with the largest snowfall in recent year is the “Presidents Day Blizzard” from the 16th to the 18th of February 2003, which brought more than 16 inches of snowfall in Manhattan. This snowfall also affected states around New York, from North Carolina to Massachusetts. This example shows that blizzard events in Manhattan could directly impact firms in states adjacent to New York.

We collect firm’s headquarter location information (state and zip code) from Compustat. We exclude firms headquartered in the following states from our analysis: New York, New Jersey, Pennsylvania, Massachusetts, Maryland, Delaware, and Connecticut. This restriction is done to ensure that the blizzard events we consider have no direct effects on the firms’ fundamental. We obtain firms’ accounting information from Compustat.
Daily returns, trading volume, price (daily high, daily low, and close), share outstanding, and S&P500 returns information comes from the Center for Research in Security Prices (CRSP). We require each firm to trade at or above $4 for 120 days prior a blizzard event to eliminate the impact of penny stocks. This restriction has little effect on our sample since we are working with NYSE-listed firms.

Finally, we obtain the Market Volatility Index (VIX) from the Chicago Board Options Exchange (CBOE), firms’ implied volatility from Ivy DB OptionMetrics, and firms’ daily trades from the Trade and Quote (TAQ) database. We select the implied volatility of at-the-money call options with date to maturity closest to 14 days.

Table 3.1 presents the summary statistics of our events. The average snowfall on a blizzard event is 15.25 inches. We observe that volume is lower on blizzard days relative to pre-blizzard days on average. The drop in the order of magnitude of 28%, which represents an economically important decline. We caution however that there are well-known patterns in trading volume that are not accounted for by these averages.

Figure 3.1 illustrates the growth of volume in our sample. It shows the time series of yearly value-weighted turnover of NYSE-listed firms from 1926 to 2013. Following a period of decline at the start of our sample, volume has been increasing since 1942. There appears to be a structural break in the time series around 2004. The beginning of the 2007 financial crisis corresponds to the start a sharp drop in turnover.
We collect snow precipitations in Manhattan New York, New York as measure by at Central Park Observatory at Belvedere Tower from the International Surface Weather Observations dataset provided by the National Oceanic and Atmospheric Administration. We classify a trading day as a blizzard day if there is at least 10 inches of snowfall on that day or on the previous, given that the previous day is a non-trading day. We obtain volume information from the CRSP database. We exclude firms with headquarters in the following states: New York, New Jersey, Pennsylvania, Massachusetts, Maryland, Delaware, and Connecticut. Firms’ headquarters location information (state and zip code) come from Compustat. In the Table N is the number of (non-excluded) firms traded on the NYSE on the day of the event. Snow \( t \) is the measure of snowfall on the day of the event, in mm. Snow Year is the daily measure of snowfall in the year of the event, conditional on having a minimum of 2.5 mm of snowfall. Volume \( v \) is the firm’s volume on the day of the event. Volume \( [t-11; t-2] \) is the firm’s volume from 11 to 2 trading days prior to the event.

<table>
<thead>
<tr>
<th>Event</th>
<th>N</th>
<th>Snow (_{t=0})</th>
<th>Snow Year</th>
<th>Volume (_{t=0})</th>
<th>Volume ([t-11; t-2])</th>
</tr>
</thead>
<tbody>
<tr>
<td>08Jan96</td>
<td>1900</td>
<td>345</td>
<td>77.85</td>
<td>59865.66</td>
<td>169790.1</td>
</tr>
<tr>
<td>18Feb03</td>
<td>1706</td>
<td>414</td>
<td>43.8</td>
<td>527337.13</td>
<td>605500.49</td>
</tr>
<tr>
<td>13Feb06</td>
<td>1863</td>
<td>612</td>
<td>78.45</td>
<td>713070.91</td>
<td>962298.96</td>
</tr>
<tr>
<td>10Feb10</td>
<td>1635</td>
<td>254</td>
<td>125.08</td>
<td>1720648.05</td>
<td>2132950.45</td>
</tr>
<tr>
<td>27Dec10</td>
<td>1581</td>
<td>310</td>
<td>127.7</td>
<td>756581.41</td>
<td>1660084.67</td>
</tr>
</tbody>
</table>
3.2 Blizzards and Trading Volume

In this section we examine the impact of blizzards on trading volume. Blizzards can directly affect trading volume by causing (temporary) power outages (see for example Shive (2012)). They can also affect trading indirectly through their impact on economic activities in the cities where they occur. Transportation by air, train, or road is usually reduced or canceled during blizzards. Blizzards can also cause closure of schools and daycare establishments. As a result a subset of parents who are stock markets participants may be unable to trade on a blizzard days as easily as they can on other days. Both Loughran and Schultz (2004) and Shive (2012) find that blizzards affect the trading volume of local firms. Blizzards in New York, NY could affect the trading volume of firms located “far” from New York, NY because several institutional traders such as algorithmic traders, proprietary trading firms, and hedge funds are located in or near New York, NY. The home bias is less prevalent among those investors. Thus, if blizzards affect these investors’ ability to trade, then it is likely that it would reflected in trading volume of the firms we consider.

Following Lo and Wang (2000), we use turnover as our proxy for daily volume. A stock’s daily turnover is the total number of shares traded that day over the total number of shares outstanding. We start by comparing abnormal turnover on blizzards days. We adapt the market model for expected stocks returns to compute expected turnover (see Chae (2005)). We regress each stocks turnover on a constant
and the turnover of the value weighted portfolio of the firms in our sample:

\[ \ln \text{Turn}_{it} = \alpha_i + \beta_i \ln \text{PortTurn}_t + \epsilon_{it}. \]

The estimation period is trading days \(-100\) to \(-11\). Next we compute each firm expected turnover on trading days \(-4\) to \(+3\) using the estimated coefficient and the corresponding portfolio’s daily turnover:

\[ \mathbb{E}[\ln \text{Turn}_{it}] = \hat{\alpha}_i + \hat{\beta}_i \ln \text{PortTurn}_t. \]

The abnormal turnover is the difference between the expected turnover and the realized turnover:

\[ \text{AbTurnover}_{it} = \ln \text{Turn}_{it} - \mathbb{E}[\ln \text{Turn}_{it}]. \]

Figure 3.2 shows the abnormal turnover around a blizzard day. The figure shows that there is volume is abnormally low on blizzards days. In addition, this abnormally low volume last for only a day. Volume returns to normal levels a day after the blizzards.

We test whether blizzards affect trading volume using two related multivariate approaches. First, we examine the difference in trading turnover between blizzard days and pre-blizzards days. Our pre-blizzard window consists of trading days \(-11\) to \(-2\), where trading day 0 is the blizzard event day. We regress firm’s daily turnover on a dummy equal one if the trading day is a blizzard day. The coefficient on the dummy variable is the mean difference of turnover between blizzard days and non-blizzards
days. We repeat this exercise using logged values of turnover because firms’ turnover are not normally distributed (see [Ajinkya and Jain (1989)]). The results are reported in Table 3.2. We find that turnover is lower on blizzards days relative to pre-blizzards days. In the regression with logged values, the coefficient on the dummy variable Snow is $-0.216$ and statistically significant at the 99.99% level. This coefficient implies that the ratio of turnover between non-blizzard days and blizzard days is approximately

$$e^{-0.216} \approx 0.806.$$  

Thus, turnover is about 19.4% lower on blizzard days relative to non-blizzard days.

Next, we test for the difference in trading volume between blizzards days and non-blizzards days following the approach in Loughran and Schultz (2004). We perform a regression of firm’s daily turnover on both the blizzard dummy and the mean turnover on trading days $-11$ to $-2$. Table 3.2 shows that turnover is lower on blizzard days relative to non-blizzards days, which is consistent with our previous findings.

These results show that blizzard events in Manhattan, New York, NY have a significant and economically important effect on the trading volume of firms located far from New York, NY. Our results complement the work of Loughran and Schultz (2004) and Shive (2012).

Each firm’s abnormal turnover is defined as the difference between its realized turnover and its expected turnover. Figure 2 shows the time series of the mean abnormal turnover around the blizzards events. We see that there is a significant
negative abnormal turnover on blizzard days, consistent with our multivariate results.

The empirical evidence in this section shows that blizzards have a significant impact on trading volume. Firms’ turnover is lower on blizzard days. We study how this exogenous change in trading volume affects firm’s volatility in the next section.

3.3 Volume and Volatility

We identified blizzards as exogenous shocks to firms’ volume. If volume affects volatility then this shock to volume should result in change in volatility. To test for the effects of the shock on volume on volatility we need both a proxy for volatility and a measure of abnormal volatility. We use two proxies for volatility. The first is the option implied volatility. The second is the difference between the daily high and low prices, scaled by the previous day close price. We use a market model to compute abnormal volatility. Similar to the market model for volume we estimate the relation between each firm’s daily turnover and the daily market volatility using the following regression:

\[
\text{Volatility}_{it} = \alpha_i + \beta_i \text{Portf} - \text{Volatility} + \epsilon,
\]

where Portf-Volatility is the reference portfolio volatility. We use VIX, the Chicago Board Options Exchange Market Volatility Index, as the reference portfolio volatility when working with option implied volatilities. The estimation period is trading days -100 to -11. Next we compute each firm expected volatility on trading days -4 to +3
Table 3.2: **Blizzards and Trading Volume.**

We examine the effects of blizzards on trading turnover. We regress firm’s turnover on the dummy Snow which is equal to one on blizzard days. We classify a trading day as a blizzard day if there is at least 10 inches of snowfall on that day or on the previous, given that the previous day is a non-trading day in Manhattan, New York, NY. Turnover is the ratio of firm’s volume to firm’s number of share outstanding. Turn is the firm’s turnover and LTurn is the firm’s log turnover. PreMeanTurn is the mean turnover for days -11 to -2, where day 0 is the event day. LPreMeanTurn is the mean log turnover for days -11 to -2, where day 0 is the event day. Dummy 2001 is a dummy equal to one if the year is 2001 or later. Dummy 2004 is a dummy equal to one if the year is 2004 or later.

<table>
<thead>
<tr>
<th></th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
<th>(6)</th>
<th>(7)</th>
<th>(8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turn</td>
<td>-0.00217***</td>
<td>-0.216***</td>
<td>-0.0013***</td>
<td>-0.00276***</td>
<td>-0.00152***</td>
<td>-0.132***</td>
<td>-0.402***</td>
<td>-0.201***</td>
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<tr>
<td></td>
<td>(-19.75)</td>
<td>(-29.09)</td>
<td>(-8.86)</td>
<td>(-12.96)</td>
<td>(-8.58)</td>
<td>(-22.83)</td>
<td>(-32.61)</td>
<td>(-22.18)</td>
</tr>
<tr>
<td>LTurn</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>PreMeanTurn</td>
<td>0.602***</td>
<td>0.595***</td>
<td>0.596***</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>(17.83)</td>
<td>(17.61)</td>
<td>(17.24)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dummy2001</td>
<td></td>
<td>0.000294</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>-0.0609***</td>
<td></td>
</tr>
<tr>
<td></td>
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<td>(1.07)</td>
<td></td>
<td></td>
<td></td>
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<td>(-5.52)</td>
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<tr>
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<td></td>
<td>0.00187***</td>
<td></td>
<td></td>
<td></td>
<td>0.346***</td>
<td></td>
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<tr>
<td></td>
<td></td>
<td></td>
<td>(6.75)</td>
<td></td>
<td></td>
<td></td>
<td>(24.97)</td>
<td></td>
</tr>
<tr>
<td>Dummy2004</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.000472***</td>
<td></td>
<td></td>
<td>-0.0310***</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(1.63)</td>
<td></td>
<td></td>
<td>(-3.41)</td>
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<td>Snow*Dummy2004</td>
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<td></td>
<td>0.000379</td>
<td></td>
<td></td>
<td>-0.117***</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(1.35)</td>
<td></td>
<td></td>
<td>(-10.00)</td>
<td></td>
</tr>
<tr>
<td>PreLMeanTurn</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.850***</td>
<td>0.826***</td>
<td>0.841***</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(162.34)</td>
<td>(146.15)</td>
<td>(140.32)</td>
<td></td>
</tr>
<tr>
<td>Constant</td>
<td>0.00741***</td>
<td>-4.906***</td>
<td>0.00199***</td>
<td>0.00182***</td>
<td>0.00176***</td>
<td>-0.822***</td>
<td>-0.893***</td>
<td>-0.846***</td>
</tr>
<tr>
<td></td>
<td>(148.28)</td>
<td>(-175.11)</td>
<td>(8.24)</td>
<td>(8.05)</td>
<td>(9.14)</td>
<td>(-30.34)</td>
<td>(-27.26)</td>
<td>(-25.55)</td>
</tr>
<tr>
<td>N</td>
<td>68623</td>
<td>68623</td>
<td>17195</td>
<td>17195</td>
<td>17195</td>
<td>17195</td>
<td>17195</td>
<td>17195</td>
</tr>
<tr>
<td>Adjusted R²</td>
<td>0.003</td>
<td>0.011</td>
<td>0.332</td>
<td>0.335</td>
<td>0.333</td>
<td>0.656</td>
<td>0.673</td>
<td>0.658</td>
</tr>
</tbody>
</table>
as:

\[ E[\text{Volatility}_{it}] = \hat{\alpha}_i + \hat{\beta}_i \text{Portf} - \text{Volatility}_t. \]

Each firm’s abnormal volatility is defined as the difference between its realized volatility and its expected volatility:

\[ \text{AbVolatility}_{it} = \text{Volatility}_{it} - E[\ln \text{Turn}_{it}]. \]

We formally test for the effects of volume on volatility by sorting firms into quintile of abnormal turnover on blizzard days and computing mean abnormal volatility of stocks within each quintile. We present the results in Table 3.3.

There is a monotonic relationship between abnormal volume and abnormal volatility. The decrease in volume caused by blizzards leads to a decrease in volatility. The differences between the first and fifth quintiles imply that a decrease of 0.00162 in level of turnover leads to a decrease of 0.0034 in level of volatility. Relating these results to the mean levels of turnover and volatility, we conclude that a 10% decrease in turnover implies a 2.1% decrease in volatility.

The component of volume generated by trading activity that we document is economically important. A 2% increase in an asset’s volatility should be accompanied with a 2% increase in the asset’s excess return for the asset’s shape ratio to remain constant. Assuming that the a firm current gross return is 6.97% and that the risk-free rate is 1.75%, a 2% increase in a firm’s observed volatility will lead to a 16%

\[^5\text{These values come from the equity premium and risk-free rate puzzles literature, see for example Weil (1989).}\]
increase in its cost of capital.

We repeat the quintile test we just performed, this time controlling for the abnormal volatility five trading days before blizzards events. For each firm-event pairing we subtract from the firm’s abnormal volatility its abnormal volatility five trading days prior to the event. The choice of five trading days ensure that the “control” abnormal volatility is computed on the same day of the week as the day of the event since volume and volatility tend to have trading day patterns (for example volume is higher on Wednesdays relative to Mondays). Table 3.3 shows that the results remain qualitatively consistent to the case without controls.

Table 3.3: Causal Effects of Volume on Volatility.

We sort firms into quintiles of abnormal turnover on blizzard days and compute mean abnormal volatility of stocks within each quintile. We use market models to compute both abnormal volume and abnormal volatility.

<table>
<thead>
<tr>
<th>Quantiles AbTurn_{t=0}</th>
<th>AbTurn_{t=0} Mean</th>
<th>Tstat.</th>
<th>AbVolatility_{t=0} Mean</th>
<th>Tstat.</th>
<th>Diff Turn_{-5,0} Mean</th>
<th>Tstat.</th>
<th>Diff Volatility_{-5,0} Mean</th>
<th>Tstat.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q1</td>
<td>0.00081</td>
<td>0.002</td>
<td>0.00055</td>
<td>0.0013</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q2</td>
<td>0.0005</td>
<td>0.0006</td>
<td>0.00003</td>
<td>0.0003</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Q3</td>
<td>-0.00059</td>
<td>-0.00024</td>
<td>-0.00009</td>
<td>0.0002</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>Q4</td>
<td>-0.00017</td>
<td>-0.0010</td>
<td>-0.00015</td>
<td>-0.0003</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>Q5</td>
<td>-0.00084</td>
<td>-0.0014</td>
<td>-0.00046</td>
<td>-0.0009</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q5 - Q1</td>
<td>0.00162</td>
<td>2.7</td>
<td>0.0034</td>
<td>3.1</td>
<td>0.00102</td>
<td>2.4</td>
<td>0.00227</td>
<td>1.83</td>
</tr>
</tbody>
</table>

The univariate results we presented show how volume would affect volatility for a randomly selected NYSE-listed firm. To further our understanding of the volume-
volatility relationship, we perform multivariate tests that account for both firms and market characteristics such as idiosyncratic liquidity, market volatility, fixed effects, and firm size. Our identification strategy is to perform regressions of abnormal volatility on abnormal volume, controlling both firms and market characteristics:

$$\text{AbVolatility}_{it} = \alpha_i + \beta_i \text{AbVolume}_{it} + \gamma_i \mathbf{X} + \epsilon$$

where $\mathbf{X}$ is a vector of controls.

Table 3.4 presents the regression results. The table shows there is significant negative abnormal volatility on blizzard days. That is, volatility is lower on blizzard days relative to pre-blizzard days. This difference is partially explained by the abnormal volume on blizzard days.

The lower volatility on blizzard days is explained by the lower volume on blizzard days. The coefficient estimates on both abnormal turnover and abnormal turnover interacted with the event day dummy are positive. Next we identify a possible channel through which volume affects volatility.

### 3.4 Channel

We have shown that an exogenous negative shock to volume leads to a reduction in volatility. Firms in our sample, that is NYSE-listed firms (located far from New York, NY), tend to be larger and older than other publicly traded firms. We thus expect that institutional investors form a significant percentage of equity-holders of firms in our sample. Moreover, the volume of firms with a higher percentage of equity-
Table 3.4: Causal Effects of Volume on Volatility: Multivariate Tests.

We examine the effects of an exogenous shock to volume on volatility. We regress firm’s abnormal volatility on its abnormal volume. Abnormal volume and abnormal volatility are computed use market models. The event is a trading with at least 10 inches of snowfall on that day or on the previous, given that the previous day is a non-trading day in Manhattan, New York, NY. Turnover is the ratio of firm’s volume to firm’s number of share outstanding. Turn is the firm’s turnover and LTurn is the firm’s log turnover. Snow is a dummy equal to one on the event day. Dummy 2001 is a dummy equal to one if the year is 2001 or later. Dummy 2004 is a dummy equal to one if the year is 2004 or later.

<table>
<thead>
<tr>
<th></th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>AbVola</td>
<td>AbVola</td>
<td>AbVola</td>
<td>AbVola</td>
</tr>
<tr>
<td>Snow</td>
<td>-0.00508*</td>
<td>-0.0021*</td>
<td>-0.00530*</td>
<td>-0.00656***</td>
</tr>
<tr>
<td></td>
<td>(-1.99)</td>
<td>(-1.89)</td>
<td>(-1.97)</td>
<td>(-3.38)</td>
</tr>
<tr>
<td>Ab_Turn</td>
<td>0.174*</td>
<td>0.0191*</td>
<td>(2.02)</td>
<td>(1.90)</td>
</tr>
<tr>
<td>Snow*Ab_Turn</td>
<td>0.676**</td>
<td>0.246*</td>
<td>(2.61)</td>
<td>(1.97)</td>
</tr>
<tr>
<td>Vix</td>
<td></td>
<td></td>
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<td>-0.0125***</td>
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<td></td>
<td></td>
<td>(-6.46)</td>
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<tr>
<td>MktCap</td>
<td>-2.3e-11*</td>
<td>-1.99e-1***</td>
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<td>(-10.39)</td>
<td>(-30.20)</td>
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<td>Amihud</td>
<td>11.71*</td>
<td>13.64</td>
<td>(1.49)</td>
<td>(1.71)</td>
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</tr>
<tr>
<td>N</td>
<td>37237</td>
<td>44723</td>
<td>44723</td>
<td>43265</td>
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<tr>
<td>Adjusted R²</td>
<td>0.000</td>
<td>0.000</td>
<td>0.401</td>
<td>0.495</td>
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</table>
holders that are institutional investors is more likely to be affected by blizzards in New York, NY. The reason is that, among investors affected by blizzards in New York, NY, institutional investors are more likely to buy/sell shares of firms located far from New York on a regular trading day. A possible channel for the link between trading volume and volatility is the trading activity of some institutional investors. We examine whether the lower volatility on blizzard days we documented is caused by the trading activity of institutional investors.

Institutional investors do not report their trades daily. However we have access to a proprietary database of institutional investors’ daily transactions from 1999 to 2009 provided to us by Abel Noser Solutions (formerly ANcerno Ltd. and the Abel/Noser Corporation). Approximately 10% of the universe of institutional investors report their trades to Abel Noser Solutions. This database is a representative sample of the universe of institutional investors and has been used in the extant finance literature (see Anand, Irvine, Puckett, and Venkataraman (2012), Anand, Irvine, Puckett, and Venkataraman (2013), Puckett and Yan (2011) and references therein).

Our empirical strategy is to repeat the multivariate tests we performed in Section 3.3 adding a proxy for trading volume due to institutional investors. For each firm-day we compute the percentage of the firm’s volume due to trading by institutional investors in our subset of institutional investors which we refer to as institutional volume. We then estimate the abnormal institutional volume around each blizzard events occurring between 1999 to 2009 using a market model. Finally we create a dummy variable equal to one for firms whose abnormal institutional volume is in the lowest quintile of institutional volume for a blizzard event (note that we
include year fixed effects in our analysis).

The coefficient of the interaction between the event dummy and the institutional volume dummy gives a difference-in-difference result where the first difference is between abnormal volume pre-blizzard event and during blizzard events and the second difference is between firms in the lowest quintile of abnormal institutional volume and those outside that quintile. This coefficient gives the proportion of the drop in volatility of blizzard days that is explained by the drop in trading activities by institutional investors. Table 3.5 contains the results of our analysis. We find that trading by institutional investors from Ancerno does not lead to higher decrease in volatility. It is possible that the Ancerno data base does not contain enough trades from institutional investors driving our results.

We repeat the multivariate tests of Section IV for sub-periods characterized by changed trading microstructure on the NYSE and increased trading activities by institutional investors. The first NYSE rule change that we consider is the gradual shift from a fractional pricing system to a decimal pricing between August 28th, 2000 and January 29th, 2001 in anticipation of a rule change by the Securities and Exchange Commission (SEC).\(^6\)\(^\text{Chakravarty et al. (2001)}\) find that decimalization did not affect volume.\(^\text{Vuorenmaa (2010)}\) finds that observed volatility is lower following decimalization, and \(^\text{Bessembinder (2003)}\) finds that decimalization improves the liquidity of NYSE-listed firms. The second NYSE rule change affecting trading that we consider is the implementation of autoquote between January 29th, 2003 and May

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\(^6\)See Release No. 34-42360/January 28, 2000 File No. 4-430.
27th, 2003\cite{Hendershott et al. 2011} find that this change increased trading by algorithmic traders. We observe from Figure 3.1 that there is an exponential growth in trading volume on the NYSE after 2003, which coincides with this rule changed.

We regress abnormal volatility on abnormal volume controlling both both firms and market characteristics. In addition we add two dummy variables, the first equal to one for firm-blizzards occurring in 2001 or later and the second equal to one for firm-blizzards occurring in 2004 or later. Interacting these dummy variables with the abnormal turnover variable will give us an indication of whether or not the relation between volume and volatility is stronger following the rule changes the NYSE. The result would give indirect evidence of the role that different market participants play linking trading volume to volatility for NYSE-listed firms.

Table 3.5 contains the regression results. We find that the excess volatility created by trading activity is higher post 2004. This evidence indirectly suggests that trading by algorithmic traders is the source of the excess volatility we document.

### 3.5 Robustness and Alternative Explanations

The change in volatility we documented could be caused by factors other than the change in volume. Such factors will have to occur on the blizzard days (or possibly a few days prior to a blizzard day) and affect all NYSE-listed firms. Potential candidates include macroeconomic news such as announcements of the Consumer Price Index (CPI), the Producer Price Index (PPI), the unemployment rate, Federal

---

Table 3.5: Causal Effects of Volume on Volatility: Channel.

We examine the effects of an exogenous shock to volume on volatility. We repeat our multivariate tests above with an additional dummy variable: DummyInst is a dummy equal to one for firms whose abnormal institutional volume is in the lowest quintile of institutional volume for a blizzard event. Institutional volume is the percentage of firm’s volume coming from trading by institutional investors in the Ancerno data base.

<table>
<thead>
<tr>
<th></th>
<th>(1) AbVola</th>
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<th>(3) AbVola</th>
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<th>(5) AbVola</th>
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<tr>
<td>Ab_Turn</td>
<td>0.298**</td>
<td>0.298*</td>
<td>0.910*</td>
<td>0.910*</td>
<td>0.00191*</td>
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<td></td>
<td>(2.75)</td>
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<tr>
<td></td>
<td>(0.57)</td>
<td>(1.09)</td>
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<tr>
<td>Dummy2001*Ab_Turn</td>
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<td></td>
<td>-1.031</td>
<td></td>
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<tr>
<td></td>
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<td></td>
<td>(-1.20)</td>
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<td>0.114***</td>
<td>0.0638***</td>
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<td>(33.57)</td>
<td>(19.10)</td>
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<tr>
<td>Dummy2003*Ab_Turn</td>
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<td>0.0128***</td>
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<td>(1.71)</td>
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<tr>
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<tr>
<td>Adjusted $R^2$</td>
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<td>0.014</td>
<td>0.498</td>
<td>0.498</td>
<td>0.495</td>
</tr>
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</table>
Open Market Committee (FOMC) decisions, or results of a presidential election. Both Flannery and Protopapadakis (2002) and Savor and Wilson (2013) document that macroeconomic announcements affect firms’ return. We do not have a control group of firms to measure the counterfactual change in volatility in our setting. We hand checked the blizzard days in our sample and they do not systematically occur on macroeconomic news days.

It is possible that our results are driven by shocks to order imbalance. If shocks to volume affect order imbalance, then changes in implied volatility will be driven by changes in prices due to changes in order imbalance. We look at order imbalance on blizzard days. Table VII shows that order imbalance is not affected by blizzards. This evidence support the interpretation that our results are driven by traders who get in and out of positions within trading days. Examples of such investors include algorithmic traders and high frequency traders.

Our results could also be driven by changes in mood or discount rates. Changes in discount rates are unlikely to explain our result since our event happens in a trading day. Changes in mood are also unlikely to explain our results because the investors susceptible to mood changes due to weather tend to be retail investors. Those investors also tend to invest close to home.

3.6 Conclusion

We document that trading activity in itself generates excess volatility. We use an exogenous shock to volume to identify this relation. Firms located far from New York
state experience a significant decline in their trading volume on days when there is a blizzard in Manhattan, New York, NY. This exogenous shock to volume reduces firms’ observed volatility on the day of the event. We present indirect evidence that the link through which volume affects volatility in our setting is the trading activity of a subset of institutional investors. The excess volatility generated by trading activity is higher post 2004, when a change in trading rule led to a significant increase in trading volume by algorithmic traders.
We plot the abnormal turnover around blizzard events of NYSE-listed firms located outside of the following states: New York, New Jersey, Pennsylvania, Massachusetts, Maryland, Delaware, and Connecticut. A blizzard event is a trading day with at least 10 inches of snowfall on that day or on the previous, given that the previous day is a non-trading day. Turnover is defined as the volume dividend by the total number of share outstanding. We use a market model for volume to compute expected turnover and the abnormal turnover is defined as the realized turnover minus the expected turnover. The estimation period is trading days -100 to -11.

Figure 3.2: Abnormal Turnover Around Blizzards.
Appendix A

Asymmetric Information and Liquidity Provision: Proofs, Numerical Analysis, and Open-Loop Game

A.1 Equilibrium

We prove Theorem 1.1 by characterizing the set of best-response strategies for each player and then the equilibrium strategies.

A.1.1 Potential predator best-response

We first assume that the distressed trader follows a linear strategy of the form:

\[ Y^d(t) = a_1(t) + a_2(t)\Delta x \]  

(A.1)

where \(a_1\) and \(a_2\) are continuously differentiable. Let

\[ \bar{a}_1(t) = \gamma \int_0^t a_1(s)ds + \lambda a_1(t); \quad \bar{a}_2(t) = \gamma \int_0^t a_2(s)ds + \lambda a_2(t). \]

The state variables relevant to the potential predator’s optimization problem are the price \(P\), her asset holding \(X^t\), and her estimate of \(\widetilde{\Delta}x\) which we denote \(\hat{X}\). The price component providing additional information to the potential predator is the
variable $Z$ defined as

$$Z_t \equiv \gamma X_t^d + \lambda Y_t^d + F_t.$$  \hspace{1cm} (A.2)

The informative component of price (to the potential predator) generates a filtration $\{\mathcal{F}(t), 0 \leq t < T\}$. The potential predator learns about $\Delta x$ as follows:

**Lemma A.1**

Suppose that the distressed trader follows a strategy of the form given in Equation (A.1). Then the time $t$ estimate of $\tilde{\Delta}x$, denoted

$$\hat{X}_t = E \left[ \tilde{\Delta}x | \mathcal{F}(t); \tilde{S}_1 \right],$$

is

$$\hat{X}_t = \hat{X}_0 + \int_0^t \sigma_t(u) dW(u)$$

where

$$\hat{X}_0 = \mu + (1 - \kappa)(\tilde{S} - \mu); \hspace{1cm} \Omega(t) = \left[ \int_0^t (\tilde{a}_2(u))^2 du + \frac{1}{\kappa \sigma^2} \right]^{-1}; \hspace{1cm} (A.3)$$

$$\sigma_t(t) = \tilde{a}_2(t) \Omega(t); \hspace{1cm} dW = dF + \tilde{a}_2(\Delta x - \hat{X}) dt.$$ \hspace{1cm} (A.4)

**Proof A.1**

The proof follows from applying the Kalman Bucy filter and basic conditional ex-
pectation formulas for multivariate normal random variables.

We now study the dynamics of $\hat{X}_t$. It follows from (A.3) and (A.4) that

$$\Omega(t)' = -(\bar{a}_2^2(t))^2\Omega(t)^2$$

$$= -\sigma_\ell(t)\bar{a}_2(t)\Omega(t)$$

$$\Rightarrow \int_0^t \sigma_\ell(s)\bar{a}_2'(s)ds = -\int_0^t \frac{\Omega(t)'}{\Omega(t)}ds$$

$$= -\ln \frac{\Omega(t)}{\Omega(0)}. \quad (A.5)$$

Let

$$\delta(t) = \frac{\Omega(t)}{\Omega(0)} \Rightarrow \left(\frac{1}{\delta(t)}\right)' = \Omega(0)(\bar{a}_2'(t))^2. \quad (A.6)$$

$\delta(t)$ is the percentage of the initial variance remaining at time $t$. We shall refer to $\delta(t)$ as the percentage of uncertainty left at time $t$. Lemma A.1 implies that the variable $\hat{X}_t$ satisfies

$$d\hat{X}_t = \sigma_\ell(t)\bar{a}_2'(t)(\Delta x - \hat{X}_t)dt + \sigma_\ell(t)dF.$$  

This implies that

$$d\left(\exp\left[\int_0^t \sigma_\ell(s)\bar{a}_2'(s)ds\right] \hat{X}_t\right) = \left[\sigma_\ell(t)\bar{a}_2(t)\Delta x dt + \sigma_\ell(t)d\hat{X}_t\right] \exp\left[\int_0^t \sigma_\ell(s)\bar{a}_2'(s)ds\right]$$

$$= \left[\sigma_\ell(t)\bar{a}_2(t)\Delta x dt + \sigma_\ell(t)dF\right] \exp\left[\int_0^t \sigma_\ell(s)\bar{a}_2'(s)ds\right].$$
Therefore,
\[
\frac{1}{\delta(t)} \dot{X}_t - \dot{X}_0 = \int_0^t \left\{ [\sigma_{\ell}(u)\tilde{a}'_2(u)\Delta xdu + \sigma_{\ell}(u)dF_u] \frac{1}{\delta(u)} \right\}
\]
\[
= \Delta x \int_0^t \Omega(0)(\tilde{a}'_2(u))^2du + \Omega(0) \int_0^t \tilde{a}'_2(u)dF_u.
\]
Hence,
\[
\dot{X}_t = \Delta x + \left[ \dot{X}_0 - \Delta x \right] \delta(t) + \Omega(t) \int_0^t \tilde{a}'_2(u)dF(u). \tag{A.7}
\]

Next we turn our attention to the potential predator’s optimization problem. Let \(J\) denote the potential predator’s value. \(J\) is a function of \((Z,X_{\ell t},\hat{X}_t,t)\). Given the state variables dynamics, the HJB equation associated with the potential predator’s optimization problem is

\[
\max_Y \{[J_X - Z - \gamma X]Y - \lambda Y^2 \}
\]
\[
+ J_t + \left( \tilde{a}_1' + \tilde{a}_2' \hat{X} \right) J_Z + \frac{1}{2} J_{Z Z} + \frac{1}{2} \sigma_{\tilde{X}}^2 J_{\tilde{X} \hat{X}} + \sigma_{\tilde{X}} J_{\hat{X} \hat{X}} \tag{A.8}
\]

The optimal strategy is then
\[
Y^* = \frac{1}{2\lambda} [J_X - Z - \gamma X]. \tag{A.9}
\]

Equation \(A.8\) becomes
\[
0 = J_t + \left( \tilde{a}_1' + \tilde{a}_2' \hat{X} \right) J_Z + \frac{1}{2} J_{Z Z} + \frac{1}{2} \sigma_{\tilde{X}}^2 J_{\tilde{X} \hat{X}} + \sigma_{\tilde{X}} J_{\hat{X} \hat{X}} + \frac{1}{4\lambda} [J_X - Z - \gamma X]^2 \tag{A.10}
\]
We conjecture a solution of the form:

\[
J(t, Z, X, \hat{X}) = b_1(t)Z^2 + b_2(t)X^2 + b_3(t)\hat{X}^2 + b_4(t)ZX + b_5(t)Z\hat{X} + b_6(t)X\hat{X} + b_7(t)Z
+ b_8(t)X + b_9(t)\hat{X} + b_{10}(t). \tag{A.11}
\]

The terminal value of the optimization problem implies the following terminal values for \(b_i, i = 1, \ldots, 10:\)

\[
\begin{align*}
b_1(T) &= 0; & b_2(T) &= -\frac{C}{2}\gamma; & b_3(T) &= 0; & b_4(T) &= 1; & b_5(T) &= 0; \\
b_6(T) &= -\lambda a_2(T); & b_7(T) &= 0; & b_8(T) &= -\lambda a_1(T); & b_9(T) &= 0; & b_{10}(T) &= 0.
\end{align*}
\]

Define the liquidity ratio as

\[
\rho = \frac{\gamma}{\lambda}.
\]
Plugging (A.11) into Equation (A.10) we obtain the following system of equations:

\[
\begin{align*}
    b_1' + \frac{1}{4\lambda} (b_4 - 1)^2 &= 0. \quad (A.12) \\
    b_2' + \frac{1}{4\lambda} (2b_2 - \gamma)^2 &= 0. \quad (A.13) \\
    b_3' + \bar{a}_2 b_5 + \frac{1}{4\lambda} b_6^2 &= 0. \quad (A.14) \\
    b_4' + \frac{1}{2\lambda} (2b_2 - \gamma)(b_4 - 1) &= 0. \quad (A.15) \\
    b_5' + 2\bar{a}_2' b_1 + \frac{1}{2\lambda} b_6(b_4 - 1) &= 0. \quad (A.16) \\
    b_6' + \bar{a}_2 b_4 + \frac{1}{2\lambda} b_6(2b_2 - \gamma) &= 0. \quad (A.17) \\
    b_7' + 2\bar{a}_1' b_1 + \frac{1}{2\lambda} b_8(b_4 - 1) &= 0. \quad (A.18) \\
    b_8' + \bar{a}_1' b_4 + \frac{1}{2\lambda} b_8(2b_2 - \gamma) &= 0. \quad (A.19) \\
    b_9' + \bar{a}_1' b_5 + \bar{a}_2' b_7 + \frac{1}{2\lambda} b_8 b_6 &= 0. \quad (A.20) \\
    b_{10}' + \bar{a}_1' b_7 + b_1 + \sigma_X^2 b_3 + \sigma_X b_5 + \frac{1}{4\lambda} b_8^2 &= 0. \quad (A.21)
\end{align*}
\]

The general solutions to equations (A.12), (A.13), and (A.15) are

\[
\begin{align*}
    b_2(t) &= \frac{1}{2} \gamma \left[ 1 - \frac{2(C + 1)}{2 + \rho(C + 1)(T - t)} \right] \quad (A.22) \\
    b_4(t) &= 1 \quad (A.23) \\
    b_1(t) &= 0. \quad (A.24)
\end{align*}
\]
Substituting these into the previous system we get that

\[
\begin{align*}
    b_1(t) &= 0 \\
    b'_3 + \frac{1}{4\lambda} b_6^2 &= 0. \\
    b_5(t) &= 0. \\
    b_7(t) &= 0. \\
    b'_9 + \frac{1}{2\lambda} b_8 b_6 &= 0. \\
    b'_10 + \sigma_X^2 b_3 + \frac{1}{4\lambda} b_8^2 &= 0.
\end{align*}
\]

Therefore we obtain the optimal strategy \( Y^* \) once we solve for \( b_6 \) and \( b_8 \). Equations (A.17) and (A.19) have the same homogeneous solution:

\[
-\frac{(C + 1)}{1 + \rho(C + 1)(T - t)}.
\]

It is straightforward to obtain the homogeneous solutions to the remaining equations. The existence and uniqueness results for the equations follow from the assumption that \( a_1 \) and \( a_2 \) are continuously differentiable.

The existence of a solution to the HJB equation implies the existence of a unique best response strategy. It follows from Equation (A.9) that the unique best response
strategy is the linear strategy

\[ Y^* = \frac{1}{2\lambda} \left[ (2a_2(t) - \gamma)X + a_6(t)\dot{X} + a_8(t) \right] \]
\[ = -\frac{(C + 1)\rho}{2 + \rho(C + 1)(T - t)} X + \frac{1}{2\lambda} \left[ a_6(t)\dot{X} + a_8(t) \right]. \quad (A.25) \]

A.1.2 Distressed trader best-response

Assume that the potential predator follows a linear strategy

\[ Y^\ell(t, Z, X^\ell, \dot{X}) = c_1(t)X^\ell + c_2(t)\dot{X} + c_3(t) \]

where \( c_1, c_2, \) and \( c_3 \) are continuously differentiable. Then \( X^\ell \) evolves as

\[ dX^\ell = Y^\ell dt = \left[ (c_2\dot{X} + c_3) + c_1 X^\ell \right] dt. \]

Therefore,

\[ X^\ell_t = A(t) \int_0^t A(-s) \left[ c_2(s)\dot{X}_s + c_3(s) \right] ds \]
\[ \Rightarrow E^d[X^\ell_t] = A(t) \int_0^t A(-s) \left[ c_2(s)B(s) + c_3(s) \right] ds, \]

where

\[ A(t) = \exp \left[ \text{Sign}(t) \int_0^{[t]} c_1(s)ds \right] \quad \text{and} \quad B(t) = E^d[\dot{X}_1]. \]
Equation (A.7) and standard Normal-Normal updating results imply that

\[ B(t) = [1 - \kappa \delta(t)] \Delta x + \mu \kappa \delta(t). \]  

(A.26)

We now consider the distressed trader’s optimization problem. Recall that

\[ P(t) = U + \gamma (X^d_t + X^f_t) + \lambda (Y^d_t + Y^f_t) \]

\[ = U + \gamma X^d_t + \lambda Y^d_t + (\gamma + \lambda c_1(t)) X^f_t + \lambda c_2(t) \hat{X}_t + \lambda c_3(t). \]

We can rewrite the optimization problem as

\[
\max_{Y \in \mathcal{Y}} \left[ \int_0^T \mathcal{L} (t, X^d, Y^d) \, dt \right] \\
\text{subject to} \\
X^d_0 = 0 \\
X^d_T = \Delta x \\
dX^d = Y^d dt \tag{A.27}
\]

where

\[
\mathcal{L} (t, X^d, Y^d) = -Y^d_t \left\{ u + \gamma X^d_t + \lambda Y^d_t + \lambda c_2(t) B(t) + \lambda c_3(t) + h(t) \right\}. \\
h(t) = (\gamma + \lambda c_1(t)) A(t) \int_0^t A(-s) (c_2(s) B(s) + c_3(s)) \, ds. \tag{A.28}
\]

Using standard techniques, that is the Pontryagin Maximization Principle (PMP), we obtain that the optimal \( Y \), if it exists, satisfies the following Euler-Lagrange equa-
tion:

\[
\frac{d}{dt}Y(t) = -\frac{1}{2\lambda} \frac{d}{dt} \left[ h(t) + \lambda (B(t)c_2(t) + c_3(t)) \right].
\]

We deduce that \(Y_t\) is of the form

\[
Y_t = \text{cst} - \frac{1}{2\lambda} \left[ h(t) + \lambda (B(t)c_2(t) + c_3(t)) \right]; \quad \text{cst} = Y_0 + \frac{1}{2} (B(0)c_2(0) + c_3(0)).
\]

The boundary conditions in Equation (A.27) imply that

\[
\Delta x = \int_0^T Y_t dt \Rightarrow \Delta x - \text{cst} \times T = -\int_0^T \frac{1}{2\lambda} \left[ h(s) + \lambda (B(s)c_2(s) + c_3(s)) \right] ds.
\]

Therefore,

\[
Y_0 = -\frac{1}{2} (B(0)c_2(0) + c_3(0)) + \frac{1}{T} \left[ \Delta x + \int_0^T \frac{1}{2\lambda} \left[ h(s) + \lambda (B(s)c_2(s) + c_3(s)) \right] ds \right].
\]

Hence, the distressed trader’s best-response, if it exists, is

\[
Y_t^d = a_{11}(t) + a_{21}(t) \Delta x \quad \text{(A.29)}
\]
where

\begin{align}
a_{12}(t) &= \frac{1}{T} - \frac{1}{2\lambda} [h_0(t) + \lambda B_0(t)c_2(t)] + \frac{1}{2\lambda T} \int_0^T [h_0(s) + \lambda B_0(s)c_2(s)] \, ds \\ a_{11}(t) &= -\frac{1}{2\lambda} [h_1(t) + \lambda (B_1(t)c_2(t) + c_3(t))] + \frac{1}{2T\lambda} \int_0^T [h_1(s) + \lambda (B_1(s)c_2(s) + c_3(s))] \, ds,
\end{align}

(A.30) \quad (A.31)

and

\begin{align*}
B_0(t) &= 1 - \kappa \delta(t); \quad h_0(t) = (\gamma + \lambda c_1(t))A(t) \int_0^t A(-s)c_2(s)B_0(s) \, ds; \\
B_1(t) &= \mu \kappa \delta(t); \quad h_1(t) = (\gamma + \lambda c_1(t))A(t) \int_0^t A(-s) [c_2(s)B_1(s) + c_3(s)] \, ds.
\end{align*}

The differentiability of \(c_1, c_2\) and \(c_3\) implies that \(a_{11}\) and \(a_{12}\) are well-defined and differentiable.

Equation \((A.29)\) gives the form the distressed trader’s best-response \textit{necessarily} takes if it exists. The following lemma proves the existence of the distressed trader’s best-response:

\textit{Lemma A.2}

Suppose the potential predator’s strategy is linear with continuous coefficients. Then the distressed trader’s best-response strategy is

\[ Y_t^d = a_{11}(t) + a_{21}(t) \Delta x \]

where \(a_{11}\) and \(a_{12}\) are given by Equations \((A.30)\) and \((A.31)\).
Proof A.2

The integrand in Equation (A.27) is concave. Theorem 3 in Rockafellar (1974) then implies that the integral functional we are optimizing is concave. Therefore the necessary conditions are also sufficient.

A.1.3 Equilibrium

Solving for the equilibrium is done by combining the results from the previous two sections. Linear equilibrium strategies are of the form

\[ Y^\ell = c_1(t)X^\ell + c_2(t)\hat{X} + c_3(t). \]
\[ Y^d = a_1(t) + a_2(t)\Delta x. \]

The distressed trader’s strategy satisfies

\[ \int_0^T Y^d(t)dt = \Delta x \quad \forall \Delta x. \]

For a linear strategy, this implies that

\[ \int_0^T a_2(t)dt = 1. \quad (A.32) \]
\[ \int_0^T a_1(t)dt = 0. \quad (A.33) \]

The coefficients \( c_1, c_2, c_3, a_1, \) and \( a_2 \) are related through Equations (A.25) and (A.29). Using the results in the previous two sections, we have the following relations between
the coefficients:

\[
\begin{align*}
    a_1(t) &= -\frac{1}{2} \left[ \rho \left( 1 - C + \rho(C + 1)(T - t) \right) H_1(t) + B_1(t) c_2(t) + c_3(t) \right] + \mu_{a_1}.
    \\
    a_2(t) &= -\frac{1}{2} \left[ \rho \left( 1 - C + \rho(C + 1)(T - t) \right) H_0(t) + B_0(t) c_2(t) \right] + \mu_{a_2}.
    \\
    \mu_{a_1} &= \frac{1}{2T} \int_0^T \left[ \rho \left( 1 - C + \rho(C + 1)(T - s) \right) H_1(s) + B_1(s) c_2(s) + c_3(s) \right] ds.
    \\
    \mu_{a_2} &= \frac{1}{T} + \frac{1}{2T} \int_0^T \left[ \rho \left( 1 - C + \rho(C + 1)(T - s) \right) H_0(s) + B_0(s) c_2(s) \right] ds.
    \\
    B_0(t) &= 1 - \kappa \delta(t).
    \\
    B_1(t) &= \mu \kappa \delta(t).
    \\
    H_0(t) &= \int_0^t \frac{c_2(s) B_0(s)}{2 + \rho(C + 1)(T - s)} ds.
    \\
    H_1(t) &= \int_0^t \frac{c_2(s) B_1(s) + c_3(s)}{2 + \rho(C + 1)(T - s)} ds.
    \\
    c_1(t) &= -\frac{\rho(C + 1)}{2 + \rho(C + 1)(T - t)}.
    \\
    0 &= c_2' - \frac{\rho(C + 1)}{2 + \rho(C + 1)(T - t)} c_2 + \frac{1}{2\lambda} \ddot{a}_2.
    \\
    0 &= c_3' - \frac{\rho(C + 1)}{2 + \rho(C + 1)(T - t)} c_3 + \frac{1}{2\lambda} \ddot{a}_1.
    \\
    c_2(T) &= \frac{1}{2} a_2(T).
    \\
    c_3(T) &= \frac{1}{2} a_1(T).
\end{align*}
\]

Therefore, solving for the equilibrium is equivalent to solving for a fixed-point problem in \((a_1, a_2, c_2, c_3)\). This fixed-point problem can be broken into two fixed-point problems, the first involving only \(a_2\) and \(c_2\). We do not have existence and
uniqueness results regarding this fixed-point problem, and standard techniques do not apply here. We shall transform this fixed-point problem into a system of differential equations that we will solve numerically. Using some algebra, we obtain from the relations above the following system of equations

\[0 = \delta'(t) + \lambda^2 \kappa \sigma^2 [\rho a_2(t) + a_2'(t)]^2 \delta^2(t)\]  
(A.34)

\[0 = H'_0(t) - \frac{[1 - \kappa \delta(t)] c_2(t)}{2 + \rho(C + 1)(T - t)}\]  
(A.35)

\[0 = H'_1(t) - \frac{\mu \kappa \delta(t) c_2(t) + c_3(t)}{2 + \rho(C + 1)(T - t)}\]  
(A.36)

\[0 = c_2'(t) + c_1(t) c_2(t) + \frac{1}{2} [\rho a_2(t) + a_2'(t)]\]  
(A.37)

\[0 = c_3'(t) + c_1(t) c_3(t) + \frac{1}{2} [\rho a_1(t) + a_1'(t)]\]  
(A.38)

\[0 = a_2'(t) + \frac{1}{2} \left[ -\rho^2(C + 1) H_0(t) + \rho[1 - \kappa \delta(t)] c_2(t) + \lambda^2 \kappa^2 \sigma^2 [\rho a_2(t) + a_2'(t)]^2 \delta^2(t) c_2(t) - \frac{1}{2} [1 - \kappa \delta(t)] [\rho a_2(t) + a_2'(t)] \right]\]  
(A.39)

\[0 = a_1'(t) - \frac{1}{3} \rho a_1(t) - \frac{2}{3} \left[ \rho^2(C + 1) H_1(t) - \rho (\mu \kappa \delta(t) c_2(t) + c_3(t)) + \lambda^2 \mu \kappa^2 \sigma^2 [\rho a_2(t) + a_2'(t)]^2 \delta^2(t) c_2(t) + \frac{\mu \kappa \delta(t)}{2} [\rho a_2(t) + a_2'(t)] \right]\]  
(A.40)

with boundary conditions

\[H_0(0) = 0; \quad a_2(0) = \mu a_2 - \frac{1}{2} (1 - \kappa) c_2(0); \quad c_2(T) = -\frac{1}{2} a_2(T);\]

\[H_1(0) = 0; \quad a_1(0) = \mu a_1 - \frac{1}{2} [\mu \kappa c_2(0) + c_3(0)]; \quad c_3(T) = -\frac{1}{2} a_1(T); \quad \delta(0) = 1.\]
The existence and uniqueness results from the HJB theory and both the PMP and Lemma A.2 imply that a linear equilibrium exists if and only if the system of equations (A.34)–(A.40) has a solution. This result completes the proof of Theorem 1.1.

The system of equations (A.34)–(A.40) has a unique solution on a subset of $(0, T)$ for any given set of initial values since $[2 + \rho(C + 1)(T - t)]^{-1}$ is smooth on $(0, T)$. The existence and uniqueness problem we face is more complicated because our problem is a boundary value problem.

A.1.4 Proof of the Corollary

Suppose that

$$\gamma = 0.$$ 

This implies that

$$c_1 \equiv 0.$$ 

The system of equations (A.34)–(A.40) then reduces to
0 = \delta'(t) + \lambda^2 \kappa \sigma^2 [a'_2(t)]^2 \delta^2(t) \quad \text{(A.41)}

0 = H'_0(t) - \frac{[1 - \kappa \delta(t)] c_2(t)}{2} \quad \text{(A.42)}

0 = H'_1(t) - \frac{\mu \kappa \delta(t) c_2(t) + c_3(t)}{2} \quad \text{(A.43)}

0 = c'_2(t) + \frac{1}{2} a'_2(t) \quad \text{(A.44)}

0 = c'_3(t) + \frac{1}{2} a'_1(t) \quad \text{(A.45)}

0 = a'_2(t) + \frac{1}{2} \left[ \lambda^2 \kappa^2 \sigma^2 [a'_2(t)]^2 \delta^2(t) c_2(t) - \frac{1}{2} [1 - \kappa \delta(t)] a'_2(t) \right] \quad \text{(A.46)}

0 = a'_1(t) - \frac{2}{3} \left[ \lambda^2 \mu \kappa^2 \sigma^2 [a'_2(t)]^2 \delta^2(t) c_2(t) + \frac{\mu \kappa \delta(t)}{2} a'_2(t) \right] \quad \text{(A.47)}

with boundary conditions

\[ H_0(0) = 0; \quad a_2(0) = \mu a_2 - \frac{1}{2} (1 - \kappa) c_2(0); \quad c_2(T) = -\frac{1}{2} a_2(T); \]

\[ H_1(0) = 0; \quad a_1(0) = \mu a_1 - \frac{1}{2} [\mu \kappa c_2(0) + c_3(0)]; \quad c_3(T) = -\frac{1}{2} a_1(T); \quad \delta(0) = 1. \]

Equations \text{(A.44)} and \text{(A.45)}, together with the terminal boundary conditions for \( c_2 \) and \( c_4 \), imply that

\[ c_2(t) = -\frac{1}{2} a_2(t) \quad \text{and} \quad c_3(t) = -\frac{1}{2} a_1(t). \]
Plugging the first equality above into Equation (A.46) leads to

\[0 = (3 + \kappa \delta(t))a_2'(t) - \lambda^2 \kappa^2 \sigma^2 [a_2'(t)]^2 \delta^2(t)a_2(t)
= (3 + \kappa \delta(t))a_2'(t) + \kappa \delta'(t)a_2(t)
\Rightarrow a_2(t) = a_2(0) \frac{3 + \kappa}{3 + \kappa \delta(t)}.\]

We used Equation (A.41) to obtain the second equality. Taking the derivative of \(a_2\) with respect to \(t\) and plugging the result in Equation (A.41) yields

\[0 = \delta'(t) \left( [3 + \kappa \delta(t)]^4 + D \delta^2(t) \delta'(t) \right) \quad \text{where} \quad D = \lambda^2 \kappa^3 \sigma^2 a_2^2(0)[3 + \kappa]^2.\]

The solution \(\delta\) thus satisfies either

\[0 = \delta'(t) \quad \forall \, t \in [0, T] \quad \text{or} \quad 0 = [3 + \kappa \delta(t)]^4 + D \delta^2(t) \delta'(t) \quad \forall \, t \in [0, T]\]

because we require smooth solutions. We shall show that the unique solution is

\[\delta'(t) = 0 \quad \forall \, t \in [0, T].\]

To do so, we show that the solution to the ODE

\[\delta'(t) = -\frac{1}{D} \frac{[3 + \kappa \delta(t)]^4}{\delta^2(t)}.\]
cannot be smooth and satisfy the requirement that

$$\delta(t) \geq 0 \quad \forall t,$$

that is, the requirement that the percentage of uncertainty remaining in the game is non-negative. Suppose that $\delta$ is smooth,

$$\delta'(t) = -\frac{1}{D} \frac{[3 + \kappa \delta(t)]^4}{\delta^2(t)}, \quad \text{and} \quad \delta(t) \geq 0 \quad \forall t.$$

The expression for $a_2(t)$ yields that

$$a_2(0) \leq a_2(t) \leq a_2(0) \frac{3 + \kappa}{3}$$

since $0 \leq \delta(t) \leq 1$ for all $t \geq 0$. It thus follows from Equation (A.32) that

$$a_2(0)T \leq \int_0^T a_2(t) dt = 1 \leq a_2(0) \frac{3 + \kappa}{3} T \quad \Rightarrow \quad \frac{3}{3 + \kappa} \frac{1}{T} \leq a_2(0) \leq \frac{1}{T}$$

Moreover, for $t > 0,$

$$\delta'(t) < -\frac{[3 + \kappa]^2}{\lambda^2 \kappa^3 \sigma^2 a_2^2(0)} \quad \Rightarrow \quad \delta(t) < 1 - \frac{[3 + \kappa]^2}{\lambda^2 \kappa^3 \sigma^2 a_2^2(0)} t$$
since the function $-([3 + \kappa x]^4)/x^2$ is an increasing function for $x \in (0, 1]$ and $\delta(t)$ is bounded above by 1. It thus follows that $\delta(t) < 0$ for

$$t > \frac{\lambda^2 \kappa^3 \sigma^2 a_2^2(0)}{[3 + \kappa]^2} > \frac{9 \lambda^2 \kappa^3 \sigma^2}{[3 + \kappa]^4} \frac{1}{T^2}.$$  

This result contradicts both the assumption that $\delta(t) \geq 0$ and that $\delta(t)$ is smooth since $\delta(0) = 1$ and $\delta'(t)$ is not defined for $\delta(t) = 0$ (the contradiction holds for $T$ sufficiently large). The contradiction implies that the only possible solution is

$$0 = \delta'(t) \ \forall \ t \in [0, T] \ \Rightarrow \ 1 = \delta(t) \ \forall \ t \in [0, T].$$

For this solution, we have

$$a_2(t) = a_2(0) \ \forall \ t \in [0, T] \ \Rightarrow \ -2c_2(t) = a_2(t) = \frac{1}{T} \ \forall \ t \in [0, T].$$

It thus follows from Equations (A.47) and (A.33) that

$$a_1(t) = a_1(0) \ \forall \ t \in [0, T] \ \Rightarrow \ a_1(t) = c_3(t) = 0 \ \forall \ t \in [0, T].$$

The assumption $\gamma = 0$ and the fact that $a_2$ is constant imply that

$$\ddot{a}_2(t) = 0.$$
Thus, Equations (A.3) and (A.7) imply that

$$\hat{X}_t = \hat{X}_0 = \mu + (1 - \kappa)(\tilde{S} - \mu).$$

This completes the derivation of the equilibrium strategies.

We now derive the distressed trader’s equilibrium expected value and the probability of the potential predator providing liquidity in equilibrium.

$$V^d = E^d \left\{ \int_0^T - [\mathcal{F}_t + \lambda(Y^d + Y^\ell)] Y^d dt \right\}$$

$$= V^{d,0} - \lambda T E^d [Y^\ell Y^d]$$

$$= V^{d,0} + \frac{\lambda}{2T} E^d [\tilde{\Delta} x E^d [\hat{X}_t | \tilde{\Delta} x]]$$

$$= V^{d,0} + \frac{\lambda}{2T} E^d [\tilde{\Delta} x \left( \mu + (1 - \kappa)(\tilde{\Delta} x - \mu) \right)]$$

$$= V^{d,0} + \frac{\lambda}{2T} \left[ \mu^2 + (1 - \kappa)\sigma^2 \right].$$

Here, $V^{d,0}$ is the distressed trader’s equilibrium expected value in the absence of the potential predator. We rewrite the signal $\tilde{S}$ as

$$\tilde{S} = \tilde{\Delta} x + \sigma \sqrt{\frac{\kappa}{1 - \kappa}} \tilde{\epsilon}_0 \quad \text{where} \quad \tilde{\epsilon}_0 \sim N(0, 1) \quad \text{and} \quad \kappa \neq 1.$$
For a given pair \((\Delta x; \epsilon_0)\), the distressed trader’s equilibrium expected value is

\[
V^d(\Delta x; \epsilon_0) = \mathbb{E}^B \left\{ \int_0^T - \left[ F_t + \lambda (Y^d + Y^\ell) \right] Y^d dt \right\} \\
= V^{d,0}(\Delta x) - \lambda T Y^\ell Y^d \\
= V^{d,0}(\Delta x) + \frac{\lambda}{2T} \left[ \kappa \mu \Delta x + (1 - \kappa)(\Delta x)^2 + \sigma \sqrt{\kappa(1 - \kappa)} \epsilon_0 \Delta x \right]
\]

where the expectation is taken with respect to the Brownian motion \(B_t\) and \(V^{d,0}(\Delta x)\) is the distressed trader’s equilibrium expected value in the absence of the potential predator. Define \(\bar{Y}\) as

\[
\bar{Y} = \bar{Y}_0 \Delta x \quad \text{where} \quad \bar{Y}_0 \equiv \kappa \mu + (1 - \kappa) \Delta x + \sigma \sqrt{\kappa(1 - \kappa)} \epsilon_0.
\]

The probability of liquidity provision occurring is the same as \(P[\bar{Y} > 0]\). Clearly,

\[
P[\bar{Y} > 0] = 1 \quad \text{if} \quad \kappa = 0.
\]
Assume that $\kappa \neq 0$.

$$P[\tilde{Y} > 0] = P[\tilde{Y}_0 > 0, \Delta x > 0] + P[\tilde{Y}_0 < 0, \Delta x < 0]$$

$$P[\tilde{Y}_0 > 0, \Delta x > 0] = P[\tilde{Y}_0 > 0|\Delta x > 0]P[\Delta x > 0]$$

$$= P \left\{ \varepsilon_0 > -\frac{1}{\sigma} \left[ \sqrt{\frac{\kappa}{1-\kappa}} \mu + \sqrt{\frac{1-\kappa}{\kappa}} \Delta x \right] | \Delta x > 0 \right\} \Phi \left( \frac{\mu}{\sigma} \right)$$

$$= \Phi \left( \frac{\mu}{\sigma} \right) \int_0^\infty \Phi \left( \frac{1}{\sigma} \left[ \sqrt{\frac{\kappa}{1-\kappa}} \mu + \sqrt{\frac{1-\kappa}{\kappa}} x \right] \right) \phi \left( \frac{x-\mu}{\sigma} \right) dx$$

$$= \int_0^\infty \Phi \left( \frac{1}{\sigma} \left[ \sqrt{\frac{\kappa}{1-\kappa}} \mu + \sqrt{\frac{1-\kappa}{\kappa}} x \right] \right) \phi \left( \frac{x-\mu}{\sigma} \right) dx$$

$$P[\tilde{Y}_0 < 0, \Delta x < 0] = \int_{-\infty}^0 \Phi \left( -\frac{1}{\sigma} \left[ \sqrt{\frac{\kappa}{1-\kappa}} \mu + \sqrt{\frac{1-\kappa}{\kappa}} x \right] \right) \phi \left( \frac{x-\mu}{\sigma} \right) dx.$$

### A.2 Numerical methods

#### A.2.1 Numerical solutions to differential equations

**Arbitrary $C$**

We first consider an arbitrary constant $C$. We shall set

$$C = 1$$

when solving the system numerically. Let

$$H_2 = \frac{1}{2\lambda} \tilde{a}_2'(t) = \frac{1}{2} [\rho a_2(t) + a_2'(t)].$$
We can use Equation (A.39) to derive a differential equation satisfied by $H_2$. For numerical simplicity, we transform the system of equations (A.34) — (A.40) into the following system of ordinary first-order differential equations:

\[\begin{align*}
0 &= \delta'(t) + 4\lambda^2\kappa^2\sigma^2 H_2^2(t)\delta^2(t) \\
0 &= H_0'(t) - \frac{[1 - \kappa\delta(t)] c_2(t)}{2 + \rho(C + 1)\rho(T - t)} \\
0 &= H_1'(t) - \frac{\mu\kappa\delta(t)c_2(t) + c_3(t)}{2 + \rho(C + 1)(T - t)} \\
0 &= c'_2(t) + c_1(t)c_2(t) + H_2(t) \\
0 &= c'_3(t) + c_1(t)c_3(t) + 2\rho a_1(t) + \frac{1}{3} \left[ \rho^2(C + 1)H_1(t) - \rho (\mu\kappa\delta(t)c_2(t) + c_3(t)) \\
&\quad + 4\lambda^2\mu\kappa^2\sigma^2 H_2^2(t)\delta^2(t)c_2(t) + \mu\kappa\delta(t)H_2(t) \right] \\
0 &= a'_2(t) + \frac{1}{2} \left[ -\rho^2(C + 1)H_0(t) + \rho[1 - \kappa\delta(t)]c_2(t) + 4\lambda^2\kappa^2\sigma^2 H_2^2(t)\delta^2(t)c_2(t) \\
&\quad - [1 - \kappa\delta(t)]H_2(t) \right] \\
0 &= a'_1(t) - \frac{1}{3} \rho a_1(t) - \frac{2}{3} \left[ \rho^2(C + 1)H_1(t) - \rho (\mu\kappa\delta(t)c_2(t) + c_3(t)) \\
&\quad + 4\lambda^2\mu\kappa^2\sigma^2 H_2^2(t)\delta^2(t)c_2(t) + \mu\kappa\delta(t)H_2(t) \right] \\
0 &= H_2'(t) + \frac{d_2(t) + d'_2(t)}{2d_2(t)H_2(t) + 4 - d_1(t)} H_2^2(t) - \frac{d_1(t) + d'_1(t)}{2d_2(t)H_2(t) + 4 - d_1(t)} H_2(t) + \frac{d_0(t) + d'_0(t)}{2d_2(t)H_2(t) + 4 - d_1(t)} \\
\end{align*}\]

with boundary conditions

\[\begin{align*}
H_0(0) &= 0; \quad a_2(0) = \mu_{a_2} - \frac{1}{2}(1 - \kappa)c_2(0); \quad c_2(T) = \frac{1}{2}a_2(T); \\
H_1(0) &= 0; \quad a_1(0) = \mu_{a_1} - \frac{1}{2}[\mu\kappa c_2(0) + c_3(0)]; \quad c_3(T) = \frac{1}{2}a_1(T); \quad \delta(0) = 1.
\]
The functions $d_0$, $d_1$, and $d_2$ are

\[
d_0(t) = -\rho^2(C + 1)H_0(t) + \rho[1 - \kappa\delta(t)]c_2(t);
\]

\[
d_2(t) = 4\lambda^2\kappa^2\sigma^2\delta^2(t)c_2(t);
\]

\[
d_1(t) = 1 - \kappa\delta(t).
\]

The function \textit{odeint} from the Scipy library for Python can be used to solve the system of first order equations. However a difficulty arises because \textit{odeint} handles Initial Value Problems (IVP) and we have a Boundary Value Problem (BVP). We use the standard shooting method to solve this issue.

Given initial values of $H_2$ and $c_2$, we can solve the IVP consisting of Equations (A.48), (A.49), (A.51), (A.53), and (A.55). Note that

\[
a_2(0) = \frac{1}{2\rho} \left[ 4\lambda^2\kappa^2\sigma^2c_2(0)H_2^2(0) + (3 + \kappa)H_2(0) + \rho(1 - \kappa)c_2(0) \right].
\]

\[
a_2'(0) = 2H_2(0) - \rho a_2(0).
\]

We use the shooting method to find initial values of $H_2$ and $c_2$ for which the boundary conditions for $a_2$ and $c_2$ are satisfied. We then repeat the exercise, this time selecting initial values of $a_1$ and $c_3$ and solving the entire system of equations.

\textbf{No time to liquidate excess holding}

We consider the equilibrium in the limit

\[
C \rightarrow \infty \quad \iff \quad \Delta T \rightarrow 0.
\]
In this limit, the system of equations determining the equilibrium is:

\[ a_1(t) = -\frac{1}{2} \left[ \left( -1 + \rho(T - t) \right) H_1(t) + B_1(t)c_2(t) + c_3(t) \right] + \mu_{a_1}, \]

\[ a_2(t) = -\frac{1}{2} \left[ \left( -1 + \rho(T - t) \right) H_0(t) + B_0(t)c_2(t) \right] + \mu_{a_2}, \]

\[ \mu_{a_1} = \frac{1}{2T} \int_0^T \left[ \left( -1 + \rho(T - s) \right) H_1(s) + B_1(s)c_2(s) + c_3(s) \right] ds. \]

\[ \mu_{a_2} = \frac{1}{T} + \frac{1}{2T} \int_0^T \left[ \left( -1 + \rho(T - s) \right) H_0(s) + B_0(s)c_2(s) \right] ds. \]

\[ B_0(t) = 1 - \kappa \delta(t). \]

\[ B_1(t) = \mu \kappa \delta(t). \]

\[ H_0(t) = \int_0^t \frac{c_2(s)B_0(s)}{T - s} ds. \]

\[ H_1(t) = \int_0^t \frac{c_2(s)B_1(s) + c_3(s)}{T - s} ds. \]

\[ c_1(t) = -\frac{1}{T - t}. \]

\[ 0 = c_2' - \frac{1}{T - t} c_2 + \frac{1}{2\lambda} \bar{a}_2'. \]

\[ 0 = c_3' - \frac{1}{T - t} c_3 + \frac{1}{2\lambda} \bar{a}_1'. \]

\[ c_2(T) = -\frac{1}{2} a_2(T). \]

\[ c_3(T) = -\frac{1}{2} a_1(T). \]
The system of equations corresponding to the differential equations (A.34)—(A.40) is:

\[0 = \delta'(t) + \lambda^2 \kappa \sigma^2 \left[ \rho a_2(t) + a'_2(t) \right]^2 \delta^2(t)\]  
(A.56)

\[0 = H'_0(t) - \frac{[1 - \kappa \delta(t)] c_2(t)}{T - t}\]  
(A.57)

\[0 = H'_1(t) - \frac{\mu \kappa \delta(t) c_2(t) + c_3(t)}{T - t}\]  
(A.58)

\[0 = c'_2(t) + c_1(t)c_2(t) + H_2(t)\]  
(A.59)

\[0 = c'_3(t) + c_1(t)c_3(t) + \frac{1}{2} \left[ \rho a_1(t) + a'_1(t) \right]\]  
(A.60)

\[0 = a'_2(t) + \frac{1}{2} \left[ -\rho H_0(t) + \rho [1 - \kappa \delta(t)] c_2(t) + 4 \lambda^2 \kappa^2 \sigma^2 H_2(t)^2 \delta^2(t)c_2(t) - [1 - \kappa \delta(t)] H_2(t) \right]\]  
(A.61)

\[0 = a'_1(t) - \frac{1}{3} \rho a_1(t) - \frac{2}{3} \left[ \rho H_1(t) - \rho (\mu \kappa \delta(t) c_2(t) + c_3(t)) + 4 \lambda^2 \mu \kappa^2 \sigma^2 H_2(t)^2 \delta^2(t)c_2(t) + \mu \kappa \delta(t) H_2(t) \right]\]  
(A.62)

\[0 = H'_2(t) + \frac{d_2(t) + d'_2(t)}{2d_2(t)H_2(t) + 4 - d_1(t)} H_2^2(t) - \frac{d_1(t) + d'_1(t)}{2d_2(t)H_2(t) + 4 - d_1(t)} H_2(t) + \frac{d_0(t) + d'_0(t)}{2d_2(t)H_2(t) + 4 - d_1(t)}\]  
(A.63)

with boundary conditions

\[H_0(0) = 0; \quad a_2(0) = \mu_{a_2} - \frac{1}{2} (1 - \kappa) c_2(0); \quad c_2(T) = -\frac{1}{2} a_2(T);\]

\[H_1(0) = 0; \quad a_1(0) = \mu_{a_1} - \frac{1}{2} [\mu \kappa c_2(0) + c_3(0)]; \quad c_3(T) = -\frac{1}{2} a_1(T); \quad \delta(0) = 1.\]
The functions $d_0, d_1,$ and $d_2$ are

$$d_0(t) = -\rho H_0(t) + \rho [1 - \kappa \delta(t)]c_2(t);$$
$$d_2(t) = 4\lambda^2 \kappa^2 \sigma^2 \delta^2(t)c_2(t);$$
$$d_1(t) = 1 - \kappa \delta(t).$$

We solve for the equilibrium in this case numerically and compute the probability of predatory trading occurring in Table A.1.

Table A.1: Probability of Predatory Trading Occurring in the Presence of Information Asymmetry: No Time to Liquidate.

We run $100 \times 100$ simulations of the game assuming that each player follows her linear equilibrium strategy and estimate the probability that predatory trading occurs when the potential predator has no time to liquidate her excess holding at the end of the game. In effect, she has zero excess holding at the end of the game. See A.2.3 for more details on the simulations. Parameters: $\tilde{\Delta}x \sim N(-10, \sqrt{0.5})$, $\lambda = 1$ and $T = 1.0$.

<table>
<thead>
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A.2.2 Performance of the Numerical Solutions

We evaluate the performance of our numerical solutions. Recall that the distressed trader strategy is of the form

\[ Y^d_t = a_1(t) + a_2(t)\Delta x; \quad \forall \Delta x \in \mathbb{R}. \]

Moreover, \( Y^d \) satisfies

\[ \int_0^T Y^d_t dt = \Delta x; \quad \forall \Delta x \in \mathbb{R}. \]

Therefore,

\[ \int_0^T a_1(t)dt = 0 \quad \text{(A.64)} \]
\[ \int_0^T a_2(t)dt = 1. \quad \text{(A.65)} \]

We compute

\[ \left| \int_0^T a_1(t)dt \right| \quad \text{and} \quad \left| 1 - \int_0^T a_2(t)dt \right| \]

for our numerical solutions presented in the body of the paper and present the results in Table A.2. Table A.2 shows that our numerical solutions perform well, at least as far as conditions (A.64) and (A.65) are concerned.
We numerically solve for each player’s linear equilibrium strategy and estimate both $A_0 = \left| \int_0^T a_1(t) \, dt \right|$ and $A_1 = \left| 1 - \int_0^T a_2(t) \, dt \right|$. See A.2.1 for details of the numerical solutions. Parameters: $\tilde{\Delta}x \sim N(-10, 0.5)$ and $T = 1$.

(a) Fixed $\lambda = 1$; $A_0$.

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(b) Fixed $\lambda = 1$; $A_1$.

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(c) Fixed $\gamma = 2.5$; $A_0$.

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(b) Fixed $\gamma = 2.5$; $A_1$.

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A.2.3 Simulations

We run $100 \times 100$ simulations of the game assuming that each player follows her linear equilibrium strategy. Below is the algorithm describing the simulations

1. Randomly pick a realization $\Delta x$ using the distribution of $\tilde{\Delta}x$. Calculate

$$p_{0i} = \Pr \left[ \left| \frac{\tilde{\Delta}x - \Delta x_i}{\sigma} \right| < 0.017 \right]$$

2. Simulate 100 paths of the potential predator’s equilibrium strategy. Compute $p_{1i}$, the percentage of paths for which the potential predator engages in predatory trading. Also compute the realized value of both players for each path. Denote the potential predator’s (distressed trader’s) mean value for the 100 paths $v^\ell (v^d)$.

3. Repeat steps one and two 100 times.

We use the ratio

$$\sum_{i=1}^{100} (p_{0i} \times p_{1i}) / \sum_{i=1}^{100} p_{0i}$$

as our proxy for the probability that predatory trading will occur. Similar proxies are made for each player’s expected value.

We use the Euler-Maruyama method to solve the system of stochastic differential equations for the state variables numerically in Step 2 above.

\footnote{We choose 0.017 in the definition of $p_{0i}$ to ensure that $\sum_{i=1}^{100} p_{0i} \approx 1$ when we randomly select 100 realizations of $\tilde{\Delta}x \sim N (-10, \sqrt{0.5})$.}
A.3 Open-Loop Equilibrium

We assume that both traders follow time-dependent strategies. We start by stating the following useful lemmas and definition:

Lemma A.3

Suppose that the distressed trader follows a linear strategy of the form

\[ Y^d_t = a_1(t) + a_2(t) \Delta x \]

for any realization $\Delta x$ of the random variable $\Delta x$. Then,

\[ \int_0^T a_1(t) dt = 0 \quad \text{and} \quad \int_0^T a_2(t) dt = 1. \]

Proof A.3

The result follows from the requirement that

\[ \int_0^T Y^d_t(t) dt = \Delta x. \]

Lemma A.4

The potential predator’s estimate of the random liquidation size $\Delta x$ at time $t = 0$, denoted

\[ \hat{X}_0 = E^f[\Delta x] = E^f[\Delta x | S_1], \]
\[ \hat{X}_0 = \mu + (1 - \kappa)(\tilde{S} - \mu). \] (A.66)

The distressed trader’s estimate of the random variable \( \hat{X}_0 \) is

\[ E^d[\hat{X}_0] = E^d[\hat{X}_0|\Delta x] = (1 - \kappa)\Delta x + \mu \kappa. \] (A.67)

**Proof A.4**

The proof follows from applying basic conditional expectation formulas for multivariate normal random variables.

**Definition A.1** Let \( g : [0, \infty] \to \mathbb{R} \) be an arbitrary integrable function. We define \( \bar{g} \) as the function:

\[ \bar{g}(t) = \int_{t}^{\infty} g(s)ds. \]

**A.3.1 Best-Response: Potential Predator**

Suppose that the distressed trader follows a strategy of the form

\[ Y^d_t = a_1(t) + a_2(t)\Delta x \]
where both $a_1$ and $a_2$ are smooth. The potential predator’s best response strategy solves the optimization problem:

$$\max_{Y^\ell \in Y} \mathbb{E}^\ell \left\{ \int_0^T \left[ F_t + \gamma \left( X_t^\ell + X_d^\ell \right) + \lambda \left( Y_t^\ell + Y_d^\ell \right) \right] Y_t^\ell \, dt + X_T^\ell \left( F_T + \gamma X_T^d \right) - \frac{C}{2} \gamma (X_T^\ell)^2 \right\}$$

subject to

$$\begin{cases} dX_t^\ell = Y_t^\ell \, dt \\ X_0^\ell = 0. \end{cases}$$

The Euler-Lagrange equation associated with this problem is

$$0 = \gamma \left( \mathbb{E}^\ell[Y_d^\ell] + Y_t^\ell \right) \, dt + \lambda \left( \mathbb{E}^\ell[dY_d^\ell] + dY_t^\ell \right) + \lambda dY_t^\ell - \gamma Y_t^\ell \, dt$$

$$= 2\lambda dY_t^\ell + \gamma \left( a_1(t) + a_2(t) \tilde{X}_0 \right) \, dt + \lambda \left( a'_1(t) + a'_2(t) \tilde{X}_0 \right) \, dt \quad (A.68)$$

with transversality condition

$$2\lambda Y_T^\ell + \gamma (1 + C)X_T^\ell = -\gamma \mathbb{E}^\ell[X_T^d] - \lambda \mathbb{E}^\ell[Y_T^d] + \gamma \mathbb{E}^\ell[\Delta x]$$

$$= -\lambda (a_1(T) + a_2(T) \tilde{X}_0). \quad (A.69)$$

It follows from the Euler-Lagrange equation that

$$dY_t^\ell = -\frac{1}{2} \left[ (\rho a_1(t) + a'_1(t)) + (\rho a_2(t) + a'_2(t)) \tilde{X}_0 \right] \, dt \quad (A.70)$$

$$\Rightarrow Y_t^\ell = Y_0^\ell - \frac{1}{2} \left[ f_1(t) + f_2(t) \tilde{X}_0 \right]$$

$$\Rightarrow X_t^\ell = tY_0^\ell - \frac{1}{2} \left[ \tilde{f}_1(t) + \tilde{f}_2(t) \tilde{X}_0 \right]$$
where
\[ f_i(t) = \rho \bar{a}_i(t) + a_i(t) - a_i(0). \]

We can derive both terminal values \( Y^t_\ell \) and \( X^t_\ell \) and combine them with the transversality Equation (A.69) to solve for \( Y^t_0 \):

\[
\begin{align*}
Y^t_\ell &= Y^t_0 - \frac{1}{2} \left[ f_1(T) + f_2(T) \hat{X}_0 \right] \\
X^t_T &= T Y^t_0 - \frac{1}{2} \left[ \bar{f}_1(T) + \bar{f}_2(T) \hat{X}_0 \right] \\
Y^t_0 &= \frac{2 f_1(T) + \rho (1 + C) \bar{f}_1(T) - 2a_1(T)}{2[2 + \rho T (1 + C)]} + \frac{2 f_2(T) + \rho (1 + C) \bar{f}_2(T) - 2a_2(T)}{2[2 + \rho T (1 + C)]} \hat{X}_0.
\end{align*}
\]  

(A.71)

Therefore, the potential predator’s best-response is

\[
Y^t_\ell = \frac{2 f_1(T) + \rho (1 + C) \bar{f}_1(T) - 2a_1(T)}{2[2 + \rho T (1 + C)]} - \frac{1}{2} f_1(t) + \frac{2 f_2(T) + \rho (1 + C) \bar{f}_2(T) - 2a_2(T)}{2[2 + \rho T (1 + C)]} - \frac{1}{2} f_2(t) \hat{X}_0.
\]  

We can simplify the expression for \( Y^d_0 \) using Lemma A.3

\[
f_1(T) = a_1(T) - a_1(0); \quad \bar{f}_1(T) = \rho \bar{a}_1(T) - Ta_1(0); \\
f_2(T) = \rho + a_2(T) - a_2(0); \quad \bar{f}_2(T) = \rho \bar{a}_2(T) + 1 - Ta_2(0).
\]
A.3.2 Best-Response: Distressed Trader

Suppose that the potential predator follows a strategy of the form

\[ Y^t = c_1(t) + c_2(t) \hat{X}_0 \]

where both \( c_1 \) and \( c_2 \) are smooth. The distress trader Euler-Lagrange equation yields

\[
\begin{align*}
\frac{dY^d}{dt} &= -\frac{1}{2} \left[ \rho \left( c_1(t) + c_2(t) \left( (1 - \kappa) \Delta x + \mu \kappa \right) \right) + \left( c_1'(t) + c_2'(t) \left( (1 - \kappa) \Delta x + \mu \kappa \right) \right) \right] \\
\Rightarrow Y^d_t &= Y^d_0 - \frac{1}{2} \left[ e_1(t) + \mu \kappa e_2(t) + (1 - \kappa) e_2(t) \Delta x \right]
\end{align*}
\]

where

\[
e_i(t) = \rho \tilde{e}_i(t) + c_i(t) - c_i(0); \quad i = 1, 2
\]

and we made use of Equation (A.67). The boundary condition

\[
\int_0^T Y^d_t dt = \Delta x
\]

together with Equation (A.74) yield \( Y^d_0 \):

\[
\begin{align*}
\Delta x &= TY^d_0 - \frac{1}{2} \left[ \tilde{e}_1(T) + \mu \kappa \tilde{e}_2(T) + (1 - \kappa) \tilde{e}_2 \Delta x \right] \\
y^d_0 &= \frac{\tilde{e}_1(T) + \mu \kappa \tilde{e}_2(T)}{2T} + \frac{1}{T} \left[ 1 + \frac{(1 - \kappa) \tilde{e}_2(T)}{2} \right] \Delta x.
\end{align*}
\]
Hence,

\[ Y_t = \frac{\bar{e}_1(T) + \mu \bar{e}_2(T)}{2T} - \frac{e_1(t) + \mu \epsilon_2(t)}{2} + \left[ \frac{1}{T} + \frac{(1 - \kappa) \bar{e}_2(T)}{2T} - \frac{(1 - \kappa) \epsilon_2(t)}{2} \right] \Delta Y_t \]
A.3.3 Equilibrium

We showed that each trader’s best-response strategy to a linear strategy by her opponent is also a linear strategy. We now solve for the linear equilibrium. Equations (A.73) and (A.70) imply that

\[
\begin{align*}
\frac{dY_t^d}{dt} &= -\frac{1}{2} \left[ \rho c_1(t) + c_1'(t) + \mu \kappa (\rho c_2(t) + c_2'(t)) + (1 - \kappa)(\rho c_2(t) + c_2'(t)) \Delta x \right] dt \\
\frac{dY_t^l}{dt} &= -\frac{1}{2} \left[ (\rho a_1(t) + a_1'(t)) + (\rho a_2(t) + a_2'(t)) \hat{X}_0 \right] dt \\
2a_1'(t) &= -\rho c_1(t) - c_1'(t) - \mu \kappa (\rho c_2(t) + c_2'(t)) \\
2a_2'(t) &= -(1 - \kappa)(\rho c_2(t) + c_2'(t)) \\
2c_1'(t) &= -(\rho a_1(t) + a_1'(t)) \\
2c_2'(t) &= -(\rho a_2(t) + a_2'(t))
\end{align*}
\]

We use Equations (A.71) and (A.75) to get the boundary conditions of the system above:

\[
\begin{align*}
a_2(0) &= \frac{1}{T} \left[ 1 + \frac{1-\kappa}{2} \left( \rho \tilde{c}_2(T) + \tilde{c}_2(T) - T c_2(0) \right) \right] \\
c_2(0) &= \frac{1}{2[2+\rho T(1+C)]} \left[ 2\left( \rho - a_2(0) \right) + \rho(1+C) \left( \rho a_2 + 1 - Ta_2(0) \right) \right] \\
a_1(0) &= \frac{1}{2T} \left[ \rho \tilde{c}_1(T) + \tilde{c}_1(T) - T c_1(0) + \mu \kappa \left( \rho \tilde{c}_2(T) + \tilde{c}_2(T) - T c_2(0) \right) \right] \\
c_1(0) &= \frac{1}{2[2+\rho T(1+C)]} \left[ -2a_1(0) + \rho(1+C) \left( \rho a_1 - Ta_1(0) \right) \right]
\end{align*}
\]

We rewrite the system of first-order differential equations above as two matrix
equations:

\[
\begin{bmatrix}
2 & 1 - \kappa \\
1 & 2
\end{bmatrix}
\begin{bmatrix}
a'_2(t) \\
c'_2(t)
\end{bmatrix}
= -\rho
\begin{bmatrix}
0 & 1 - \kappa \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
a_2(t) \\
c_2(t)
\end{bmatrix}
\]

(A.79)

\[
\begin{bmatrix}
2 & 1 \\
1 & 2
\end{bmatrix}
\begin{bmatrix}
a'_1(t) \\
c'_1(t)
\end{bmatrix}
= -\rho
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
a_1(t) \\
c_1(t)
\end{bmatrix}
- \mu \kappa
\begin{bmatrix}
\rho c_2(t) + c'_2(t) \\
0
\end{bmatrix}.
\]

(A.80)

The matrices on the LHS of Equations (A.79) and (A.80) are both invertible. Therefore, we have

\[
\begin{bmatrix}
a'_2(t) \\
c'_2(t)
\end{bmatrix}
= -\frac{\rho}{3 + \kappa}
\begin{bmatrix}
-1 + \kappa & 2(1 - \kappa) \\
2 & -1 + \kappa
\end{bmatrix}
\begin{bmatrix}
a_2(t) \\
c_2(t)
\end{bmatrix}
\]

(A.81)

\[
\begin{bmatrix}
a'_1(t) \\
c'_1(t)
\end{bmatrix}
= -\frac{\rho}{3}
\begin{bmatrix}
-1 & 2 \\
2 & -1
\end{bmatrix}
\begin{bmatrix}
a_1(t) \\
c_1(t)
\end{bmatrix}
- \frac{\mu \kappa}{3}
\begin{bmatrix}
2(\rho c_2(t) + c'_2(t)) \\
-(\rho c_2(t) + c'_2(t))
\end{bmatrix}.
\]

(A.82)

The matrix in Equation (A.81) satisfies

\[
\begin{bmatrix}
-1 + \kappa & 2(1 - \kappa) \\
2 & -1 + \kappa
\end{bmatrix}
= \begin{bmatrix}
\sqrt{1 - \kappa} & \sqrt{1 - \kappa} \\
1 & -1
\end{bmatrix}
\begin{bmatrix}
\lambda_1(\kappa) & 0 \\
0 & \lambda_2(\kappa)
\end{bmatrix}
\begin{bmatrix}
\sqrt{1 - \kappa} \\
1 - \kappa
\end{bmatrix}
\]

where

\[
\lambda_1(\kappa) = -(1 - \kappa) + 2\sqrt{1 - \kappa} \quad \text{and} \quad \lambda_2(\kappa) = -(1 - \kappa) - 2\sqrt{1 - \kappa}.
\]
The solution to Equation (A.81) is thus

\[
\begin{bmatrix}
a_2(t) \\
c_2(t)
\end{bmatrix} = \exp\left\{-\frac{t\rho}{3+\kappa}\begin{bmatrix}
-1 + \kappa & 2(1 - \kappa) \\
2 & -1 + \kappa
\end{bmatrix}\right\} \begin{bmatrix}
a_2(0) \\
c_2(0)
\end{bmatrix}
= \begin{bmatrix}
\sqrt{1-\kappa} & \sqrt{1-\kappa} \\
1 & -1
\end{bmatrix} \begin{bmatrix}
\frac{1}{2\sqrt{1-\kappa}} & \frac{1}{2} \\
\frac{1}{2\sqrt{1-\kappa}} & \frac{-1}{2}
\end{bmatrix} \begin{bmatrix}
a_2(0) \\
c_2(0)
\end{bmatrix}.
\]

(A.83)

We integrate Equation (A.83) to obtain \(\bar{a}_2(t), \bar{c}_2(t), \bar{a}_2(t),\) and \(\bar{c}_2(t).\) All four functions are linear in \(a_2(0)\) and \(c_2(0).\) We then plug \(a_2(T), c_2(T), \bar{a}_2(T), \bar{c}_2(T), \bar{a}_2(T),\) and \(\bar{c}_2(T)\) into Equations (A.75) and (A.71) to solve for both \(a_2(0)\) and \(c_2(0).\)

We use the same approach to solve Equation (A.82) and then for both \(a_1(0)\) and \(c_1(0).\)

\[
\begin{bmatrix}
a_1(t) \\
c_1(t)
\end{bmatrix} = \exp\left\{-\frac{t\rho}{3}\begin{bmatrix}
-1 & 2 \\
2 & -1
\end{bmatrix}\right\} \begin{bmatrix}
a_1(0) \\
c_1(0)
\end{bmatrix} - \frac{\mu\kappa}{3} \int_0^t \exp\left\{-\frac{s\rho}{3}\begin{bmatrix}
-1 & 2 \\
2 & -1
\end{bmatrix}\right\} \begin{bmatrix}
2(\rho c_2(s) + c_2'(s)) \\
-2(\rho c_2(s) + c_2'(s))
\end{bmatrix} ds
= \begin{bmatrix}
e^{-\frac{tp}{2} + e^{tp}} & e^{-\frac{tp}{2} - e^{tp}} \\
e^{-\frac{tp}{2} - e^{tp}} & e^{-\frac{tp}{2} + e^{tp}}
\end{bmatrix} \begin{bmatrix}
a_1(0) \\
c_1(0)
\end{bmatrix} - \frac{\mu\kappa}{3} \int_0^t \begin{bmatrix}
e^{\frac{sp}{2} + e^{-sp}} & e^{\frac{sp}{2} - e^{-sp}} \\
e^{\frac{sp}{2} - e^{-sp}} & e^{\frac{sp}{2} + e^{-sp}}
\end{bmatrix} \begin{bmatrix}
2(\rho c_2(s) + c_2'(s)) \\
-2(\rho c_2(s) + c_2'(s))
\end{bmatrix} ds.
\]

(A.84)

The distressed trader’s equilibrium surplus returns for a given path of the Brownian motion \(B,\) a realization \(\Delta x\) of \(\Delta \bar{x}\), and a realization \(S\) of \(\tilde{S}\) relative to the case
when the potential predator is absent is

\[ \Delta V^d = - \int_0^T \left[ F_t + \gamma \left( \bar{a}_1(t) + \bar{c}_1(t) + \bar{a}_2(t) \Delta x + \bar{c}_2(t) \hat{X}_0 \right) \right. \]

\[ + \lambda \left( a_1(t) + c_1(t) + a_2(t) \Delta x + c_2(t) \hat{X}_0 \right) \left( a_1 + a_2 \Delta x \right) dt + \frac{\Delta x}{T} \int_0^T F_t dt + \left( \gamma + \frac{\lambda}{T} \right) \Delta x^2. \]

(A.85)

The potential predator’s value is

\[ V^t = - \int_0^T \left[ F_t + \gamma \left( \bar{a}_1(t) + \bar{c}_1(t) + \bar{a}_2(t) \Delta x + \bar{c}_2(t) \hat{X}_0 \right) \right. \]

\[ + \lambda \left( a_1(t) + c_1(t) + a_2(t) \Delta x + c_2(t) \hat{X}_0 \right) \left( c_1(t) + c_2(t) \hat{X}_0 \right) dt \]

\[ + \left( \bar{c}_1(T) + \bar{c}_2(T) \hat{X}_0 \right) \left( F_T + \gamma \Delta x \right) - \frac{C}{2} \gamma \left( \bar{c}_1(T) + \bar{c}_2(T) \hat{X}_0 \right)^2. \]

(A.86)

A.3.4 Numerical Analysis

The constant \( C \) affects the equilibrium only through its effects on the initial values \( a_1(0), c_1(0), c_2(0) \) and \( a_2(0) \). We proceed by presenting, for each limit, the system of equations corresponding to systems (A.77) and (A.78).

In the case \( C = 1 \),
the systems (A.77) and (A.78) become:

\[
\begin{align*}
  a_2(0) &= \frac{1}{T} \left[ 1 + \frac{1-\kappa}{2} \left( \rho \bar{c}_2(T) + \bar{c}_2(T) - Tc_2(0) \right) \right] \\
  c_2(0) &= \frac{1}{2[1+\rho T]} \left[ \left( \rho - a_2(0) \right) + \rho \left( \rho \bar{a}_2 + 1 - Ta_2(0) \right) \right] \\
  a_1(0) &= \frac{1}{2T} \left[ \rho \bar{c}_1(T) + \bar{c}_1(T) - Tc_1(0) + \mu \kappa \left( \rho \bar{c}_2(T) + \bar{c}_2(T) - Tc_2(0) \right) \right] \\
  c_1(0) &= \frac{1}{2[1+\rho T]} \left[ -a_1(0) + \rho \left( \rho \bar{a}_1 - Ta_1(0) \right) \right].
\end{align*}
\]

In the limit

\[ C \to \infty \iff \Delta T \to 0, \]

the systems (A.77) and (A.78) become:

\[
\begin{align*}
  a_2(0) &= \frac{1}{T} \left[ 1 + \frac{1-\kappa}{2} \left( \rho \bar{c}_2(T) + \bar{c}_2(T) - Tc_2(0) \right) \right] \\
  c_2(0) &= \frac{1}{2T} \left[ \rho \bar{a}_2 + 1 - Ta_2(0) \right] \\
  a_1(0) &= \frac{1}{2T} \left[ \rho \bar{c}_1(T) + \bar{c}_1(T) - Tc_1(0) + \mu \kappa \left( \rho \bar{c}_2(T) + \bar{c}_2(T) - Tc_2(0) \right) \right] \\
  c_1(0) &= \frac{\bar{a}_1 - Ta_1(0)}{2T}.
\end{align*}
\]

We compute the probability of predatory trading occurring in the case

\[ C = 1 \]

and present the results in Table A.3.

We run $100 \times 100$ simulations of the game assuming that each player follows her open loop equilibrium strategy and estimate the probability that predatory trading occurs. See A.2.3 for more details on the simulations. Parameters: $\Delta x \sim N(-10, \sqrt{10})$, $\lambda = 1$ and $T = 1.0$. 

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Appendix B

Lucas Orchard: Proofs

B.1 Proofs: Properties of beliefs in the Law of Small Numbers

Proof B.1 (Proof of Proposition 3)

The state variable $\theta_{jt}$ satisfies

$$\theta_{ju} - \theta_{jt} = -b(B_{ju} - B_{jt}) - b\kappa \int_{t}^{u} \theta_{js} ds, \quad u > t.$$ 

$\theta_{js}$ is continuous in $s$ and thus bounded in the interval $[t, u]$ for any $u > t$. The result then follows. In the special case $\kappa = 0$, we have

$$\theta_{ut} - \theta_{jt} = -b(B_{ut} - B_{jt}).$$

Thus, in this case, Freddy believes in reversal for any $u > t$.

Proof B.2 (Proof of Lemma 2.1)

Standard filtering theory results implies that
\[ d\hat{\mu}_{jt} = -\alpha_j(\hat{\mu}_{jt} - \bar{\mu}_j) + \hat{\sigma}_{1jt} dZ_{jt} \]
\[ dZ_{jt} = \frac{1}{\sigma_j} \left[ \frac{dD_{jt}}{D_{jt}} - \hat{\mu}_{jt} dt \right] \]

where \( \hat{\sigma}_{1jt} \) satisfies

\[ \hat{\sigma}'_{1jt} = -2\alpha_j \hat{\sigma}_{1jt} - \frac{1}{\sigma_j^2} \hat{\sigma}_{1jt}^2 + \sigma_1j; \quad \hat{\sigma}_{1j0} = 0. \]

Suppose that \( \hat{\sigma}_{1jt} \) takes the form

\[ \hat{\sigma}_{1jt} = c_{j0} \frac{e^{tc_{1j}} - 1}{e^{tc_{1j}} - c_{2j}}. \]

Plugging this guess into the differential equation characterizing \( \hat{\sigma}_{1jt} \), then matching coefficients and solving the resulting system of equations yields the solution of \( \hat{\sigma}_{1jt} \).

The limit follows easily given that \( c_{1j} \) is positive.
Proof B.3 (Proof of Lemma 2.2)

Within this proof, define

\[
H = \begin{bmatrix} 1 \\ \sigma_j \end{bmatrix}^T; \quad X = \begin{bmatrix} \mu_{jt} \\ \theta_{jt} \end{bmatrix}; \quad A_0 = \begin{bmatrix} \alpha_{jt} \hat{\mu}_j \\ 0 \end{bmatrix}; \quad A = \begin{bmatrix} -\alpha_{jt} & 0 \\ 0 & -(b + \kappa) \end{bmatrix}; \quad C_X = \begin{bmatrix} \sigma_{ij} & 0 \\ 0 & -b \end{bmatrix};
\]

\[
C_Y = \begin{bmatrix} 0 \\ \sigma_j \end{bmatrix}^T; \quad M = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]

Then, Freddy’s estimates of \( \mu_t \) satisfies

\[
\frac{dD_{jt}}{D_{jt}} = H\dot{X}dt + (C_Y M)dZ_{jt}^F
\]

\[
d\dot{X} = (A_0 + A\hat{X})dt + \frac{1}{\sigma_j}(\sigma_{jt}^F H^T + C_X C_Y^T)dZ_{jt}^F
\]

\[
dZ_{jt}^F = \frac{1}{\sigma_j} \left[ \frac{dD_{jt}}{D_{jt}} - H\dot{X}dt \right]
\]

where \( \sigma_{jt}^F \) satisfies

\[
(\sigma_{jt}^F)' = A\sigma_{jt}^F + A^T \sigma_{jt}^F - \frac{1}{\sigma_j^2}(\sigma_{jt}^F H^T + C_X C_Y^T) \left( H\sigma_{jt}^F + C_Y C_X^T \right) + C_X C_X^T
\]

\[
= G_0\sigma_{jt}^F + G_0^T \sigma_{jt}^F - \sigma_{jt}^F G_1 \sigma_{jt}^F + C_X C_X^T
\]

where

\[
G_0 = A - \frac{1}{\sigma_j^2} C_X C_Y^T H \quad \text{and} \quad G_1 = \frac{1}{\sigma_j^2} H^T H.
\]
We start by solving for the long-run limit $\sigma^F_j$ of $\sigma^F_{jt}$. This limit satisfies

$$0 = G_0\sigma^F_{jt} + G_0^T\sigma^F_{jt} - \sigma^F_{jt}G_1\sigma^F_{jt} + C_XC_T^T.$$ 

Suppose that

$$\sigma^F_j = \begin{bmatrix} \sigma^F_{j11} & \sigma^F_{j12} \\ \sigma^F_{j12} & \sigma^F_{j22} \end{bmatrix}.$$ 

Then,

$$0 = \begin{bmatrix} -\frac{(\sigma^F_{j11}+\sigma^F_{j12})^2}{\sigma^2_j} + \sigma^2_{j1} - 2\sigma^F_{j11}\alpha_j \sigma^F_{j12} \sigma^F_{j1} & -\frac{\sigma^F_{j11}(\sigma^F_{j12}+\sigma^F_j(\sigma^F_{j22}-b))+\sigma^F_j(\sigma^F_{j12}+\sigma^F_j(\sigma^F_{j22}+\alpha_j+\kappa))}{\sigma^2_j} \\ -\frac{\sigma^F_{j12}(\sigma^F_{j12}+\sigma^F_j(\sigma^F_{j22}-b))+\sigma^F_j(\sigma^F_{j12}+\sigma^F_j(\sigma^F_{j22}+\alpha_j+\kappa))}{\sigma^2_j} & -\frac{(\sigma^F_{j12})^2+2\sigma^F_j(\sigma^F_{j22}-b)\sigma^F_{j12}+\sigma^2_j(\sigma^F_{j22}+2\kappa)\sigma^F_{j12}}{\sigma^2_j} \end{bmatrix}.$$ 

$\sigma^F_{j11}$ given in the Lemma satisfies this equation with $y$ a real solution of

$$a_0 + a_1y + a_2y^2 + a_3y^3 + a_4y^4.$$
where

\[ a_0 = \frac{\sigma_{ij}^4 (\alpha_j + b + \kappa)^2}{4\alpha_j^2} > 0 \]
\[ a_1 = -\sigma_j \sigma_{ij}^2 (\alpha_j + b + \kappa) < 0 \]
\[ a_2 = \frac{2\alpha_j^2 (\alpha_j^2 - \kappa^2) \sigma_j^2 - (\alpha_j^2 + (b + \kappa)^2) \sigma_{ij}^2}{2\alpha_j^2} \]
\[ a_3 = -\sigma_j (b + \kappa - \alpha_j) \]
\[ a_4 = \frac{(b + \kappa - \alpha_j)^2}{4\alpha_j^2} > 0. \]

We now return to the time dependent equation:

\[(\sigma_{jt}^F)' = G_0 \sigma_{jt}^F + G_0^T \sigma_{jt}^F - \sigma_{jt}^F G_1 \sigma_{jt}^F + C_X C_X^T \]

The equation above is a matrix differential Riccati equation. To solve this equation, we need to find the exponential of the matrix

\[ \Psi = \begin{bmatrix} G_0 & C_X C_X^T \\ \frac{1}{\sigma_j} H^T H & -G_0^T \end{bmatrix} = \begin{bmatrix} -\alpha_j & 0 & \sigma_{ij}^2 & 0 \\ \frac{b}{\sigma_j} & -\kappa & 0 & b^2 \\ \frac{1}{\sigma_j} & \frac{1}{\sigma_j} & \sigma_j & -\frac{b}{\sigma_j} \\ \frac{1}{\sigma_j} & 1 & 0 & \kappa \end{bmatrix}. \]

In fact,

\[ \Sigma_{jt} = N_\Sigma D_\Sigma^{-1} \]
where $N_{\Sigma}$ and $D_{\Sigma}$ are $2 \times 2$ matrices such that

$$
\begin{bmatrix}
N_{\Sigma} \\
D_{\Sigma}
\end{bmatrix} = e^{\Psi t} \begin{bmatrix}
\sigma_{0j} & 0 \\
0 & 0 \\
1 & 0 \\
0 & 1
\end{bmatrix}.
$$

We compute Jordan form of $\Psi t$, use it to compute the matrix exponential and derive $\sigma^F_{jt}$.

**Proof B.4 (Proof: Theorem 2.1)**

It is enough to derive the impulse-response results given Definitions 2.2 and 2.3. It follows from Lemma 2.2 that

$$
d \begin{bmatrix}
\hat{\mu}^F_{jt} \\
\hat{\theta}_{jt}
\end{bmatrix} = \left( A_0 + A \begin{bmatrix}
\hat{\mu}^F_{jt} \\
\hat{\theta}_{jt}
\end{bmatrix} \right) dt + \begin{bmatrix}
\Sigma_{1j} \\
\Sigma_{2j} - b
\end{bmatrix} dB_{jt}
$$

where we define $A_0$ and $A$ in this proof as

$$
A_0 = \begin{bmatrix}
\alpha_j \bar{\mu}_j + \frac{\Sigma_{1j} \bar{\mu}_j}{\sigma_j} \\
\frac{\Sigma_{2j} - b}{\sigma_j} \bar{\mu}_j
\end{bmatrix}
$$

and

$$
A = \begin{bmatrix}
-\left( \alpha_j + \frac{\Sigma_{1j}}{\sigma_j} \right) & -\Sigma_{1j} \\
-\frac{\Sigma_{2j} - b}{\sigma_j} & -(\kappa + \Sigma_{2j})
\end{bmatrix}.
$$

Thus, it follows from [Detemple et al., 2005] that the solution to the impulse-response
problem is
\[ e^{A(s-t)} \begin{bmatrix} \Sigma_{1j} \\ \Sigma_{2j} - b \end{bmatrix}. \]

This solution is
\[ \begin{bmatrix} D_t \hat{\mu}^F_{js} \\ D_t \hat{\theta}_{js} \end{bmatrix} \]
where \( D_t \hat{\mu}^F_{js} \) and \( D_t \hat{\theta}_{js} \) are defined in the Proposition. \( D_t \hat{\theta}_{js} \) has a unique critical value and satisfies
\[
\lim_{s \to \infty} D_t \hat{\theta}_{js} > 0 \quad \text{and} \quad \lim_{s \to \infty} D_t \hat{\theta}_{js} < 0.
\]

This completes the proof.

B.2 Conditional Expectations

Each month \( t \), we will classify each economy as H (L) if
\[ R_t - R_{t-h} > 0 \quad (R_t - R_{t-h} < 0) \]
where \( h > 0 \) is an integer multiple of one month.

For a given stochastic process \( X_t \), we define the conditional expectation
\[ E[X_{t+h'} - X_t | R_t - R_{t-h} > 0] \]
as the average of $X_{t+h'} - X_t$ among the economies that are in the H group at time $t$. Other conditional expectations are defined along the same lines.
B.3 Ito's Results

\[
\frac{dD^{-\gamma}}{D^{-\gamma}} = \left[ -\gamma \mu + \frac{1}{2} \gamma (\gamma + 1) \sigma^2 \right] dt - \gamma \sigma dB
\]

\[
d\frac{(\alpha \xi)^{1/\gamma}}{(\alpha \xi)^{1/\gamma}} = \frac{1}{2 \gamma \nu} (\theta_2 - \theta_1) \left[ (\theta_2 - \theta_1) - \gamma (\theta_2 + \theta_1) \right] dt + \frac{1}{\gamma} (\theta_2 - \theta_1) dB
\]

\[
\frac{d\nu_1}{\nu_1} = \frac{d \left[ 1 + (\alpha \xi)^{1/\gamma} \right]^{-1}}{[1 + (\alpha \xi)^{1/\gamma}]^{-1}} = -\nu_1 d(\alpha \xi)^{1/\gamma} + \nu_1^2 (d(\alpha \xi)^{1/\gamma})^2
\]

\[
= -\nu_1 \left\{ \frac{1}{2 \gamma^2} (\theta_2 - \theta_1) \left[ (\theta_2 - \theta_1) - \gamma (\theta_2 + \theta_1) \right] dt + \frac{1}{\gamma} (\theta_2 - \theta_1) dB \right\} + \nu_1^2 \frac{1}{2 \gamma^2} (\theta_2 - \theta_1)^2 dt
\]

\[
= \left\{ -\nu_1 \frac{1}{2 \gamma^2} (\theta_2 - \theta_1) \left[ (\theta_2 - \theta_1) - \gamma (\theta_2 + \theta_1) \right] + \nu_1^2 \frac{1}{2 \gamma^2} (\theta_2 - \theta_1)^2 \right\} dt - \nu_2 \frac{1}{\gamma} (\theta_2 - \theta_1) dB
\]

\[
= \frac{\nu_2}{2 \gamma^2} (\theta_2 - \theta_1) \left\{ -\left[ (\theta_2 - \theta_1) - \gamma (\theta_2 + \theta_1) \right] + 2 \nu_2 (\theta_2 - \theta_1) \right\} dt - \nu_2 \frac{1}{\gamma} (\theta_2 - \theta_1) dB
\]

\[
\frac{d\nu_2}{\nu_1^\gamma} = \left\{ -\frac{\nu_2}{2 \gamma} (\theta_2 - \theta_1) \left[ (\theta_2 + \theta_1) + (\nu_2 - \nu_1)(\theta_2 - \theta_1) \right] + \frac{\gamma + 1}{2 \gamma} \nu_2^2 (\theta_2 - \theta_1)^2 \right\} dt + \nu_2 (\theta_2 - \theta_1) dB
\]

\[
= \frac{\nu_2}{2 \gamma} (\theta_2 - \theta_1) \left[ -\left[ (\theta_2 + \theta_1) + (\nu_2 - \nu_1)(\theta_2 - \theta_1) \right] + (\gamma + 1) \nu_2 (\theta_2 - \theta_1) \right] dt + \nu_2 (\theta_2 - \theta_1) dB
\]

\[
= \frac{\nu_2}{2 \gamma} (\theta_2 - \theta_1) \left[ -\gamma (\theta_2 + \theta_1) + (\gamma \nu_2 + \nu_1)(\theta_2 - \theta_1) \right] dt + \nu_2 (\theta_2 - \theta_1) dB
\]

\[
\frac{d\pi}{\pi} = -\left\{ \beta + \gamma \mu - \frac{1}{2} \gamma (\gamma + 1) \sigma^2 - \frac{\nu_2}{2 \gamma} (\theta_2 - \theta_1) \left[ (1 - \gamma)(\theta_2 - \theta_1) - 2 \gamma \theta_1 \right] + \gamma \sigma \nu_2 (\theta_2 - \theta_1) \right\} dt
\]

\[
- [\gamma \sigma - \nu_2 (\theta_2 - \theta_1)] dB
\]

\[
= -\left\{ \beta + \gamma \mu - \frac{1}{2} \gamma (\gamma + 1) \sigma^2 + \nu_2 (\theta_2 - \theta_1) \left[ \gamma \sigma + \frac{\gamma - 1}{2 \gamma} (\theta_2 - \theta_1) + \theta_1 \right] \right\} dt
\]

\[
- [\gamma \sigma - \nu_2 (\theta_2 - \theta_1)] dB
\]
### B.4 Pricing Assets

Let $P_\alpha$ be the price of an asset paying dividends

$$D_\alpha^\alpha \equiv \prod D_j^{\alpha_j}, \quad \text{where} \quad \alpha = (\alpha_1, \ldots, \alpha_N).$$

Let

$$D^\alpha \equiv D^{\alpha}, \quad \text{where} \quad \alpha = \sum \alpha_j.$$

Then,

$$\frac{P_\alpha}{D^\alpha} = E_t \left\{ \int_0^\infty e^{-\beta(u-t)} \left( \frac{1 + \alpha^{1/\gamma} \xi^{1/\gamma}_u}{1 + \alpha^{1/\gamma} \xi^{1/\gamma}_t} \right)^\gamma \left( \frac{D_u}{D_t} \right)^{-\gamma} \frac{D^{\alpha,u}}{D^{\alpha,t}} du \right\}.$$

Computing this ratio will require computing expectation of the form

$$H() = E \left\{ \left( \frac{1 + \alpha^{1/\gamma} \xi^{1/\gamma}_u}{1 + \alpha^{1/\gamma} \xi^{1/\gamma}_t} \right)^m \left( \frac{D_u}{D_t} \right)^n \prod \left( \frac{D^{\alpha,u}}{D^{\alpha,t}} \right)^{\alpha_j} \right\}. \quad (B.1)$$

We shall evaluate this expectation under several assumptions about whether or not some/all agents have incorrect beliefs and the number of trees in the economy, gradually arriving to the case treated in the paper. a—a

The simplest case is that where all agents have correct beliefs and there is a single tree in the economy. This case is that of the standard Lucas economy. However, the dividend growth is not a Levy Process in our economy because the mean dividend growth is an Ornstein—Uhlenbeck (OU) process. If the mean dividend growth were
The expression inside the bracket is a normal random variable. To compute its moments, first note that

\[
\sigma_1 \int_t^u \int_s^r e^{-\alpha(s-v)} dZ_v ds = \int_t^u \sigma_1 \int_s^r e^{-\alpha(s-v)} dZ_v ds + \sigma_1 \int_t^u \int_s^r e^{-\alpha(s-v)} dZ_v ds
\]

\[
= \int_t^u e^{-\alpha(s-t)} \sigma_1 \int_0^t e^{-\alpha(t-v)} dZ_v ds + \int_t^u \int_t^s e^{-\alpha(s-v)} dZ_v ds
\]

\[
= \int_t^u e^{-\alpha(s-t)} (\mu_t - \bar{\mu}) ds + \int_t^u \int_t^s e^{-\alpha(s-v)} dZ_v ds
\]

\[
= \frac{1}{\alpha} (\mu_t - \bar{\mu}) \left[ 1 - e^{-\alpha(u-t)} \right] + \int_t^u \int_t^s e^{-\alpha(s-v)} dZ_v ds
\]

\[
= \frac{1}{\alpha} (\mu_t - \bar{\mu}) \left[ 1 - e^{-\alpha(u-t)} \right] + \int_t^u \int_v^s e^{-\alpha(s-v)} dsdZ_v.
\]

Its mean is

\[
n \left( \mu - \frac{1}{2} \sigma^2 \right) (u - t) + \frac{n}{\alpha} (\mu_t - \bar{\mu}) \left[ 1 - e^{-\alpha(u-t)} \right]
\]
For the variance, we need
\[
\text{Cov} \left( Z_u - Z_t; \int_t^u \int_0^s e^{-\alpha(s-v)} dZ_v ds \right) = \text{Cov} \left( \int_t^u dZ_s; \int_t^u \int_v^u e^{-\alpha(s-v)} dsdZ_v \right) \\
= \int_t^u \int_v^u e^{-\alpha(s-v)} dsdZ_v \\
= \frac{\alpha(u-t) - 1 + e^{-\alpha(u-t)}}{\alpha^2}.
\]

Moreover,
\[
\text{Var} \left( \int_t^u \int_0^s e^{-\alpha(s-v)} dZ_v ds \right) = \int_t^u \int_t^u \mathbb{E} \left[ \int_0^{s_1} e^{-\alpha(s_1-v)} dZ_v \int_0^{s_2} e^{-\alpha(s_2-v)} dZ_v \right] ds_1 ds_2 \\
= \int_t^u \int_t^u \int_0^{\min\{s_1, s_2\}} e^{-\alpha(s_1+s_2)} dv ds_1 ds_2 \\
= \int_t^u \int_t^u \min\{s_1, s_2\} e^{-\alpha(s_1+s_2)} ds_1 ds_2 \\
= \frac{e^{-2\alpha(t+u)} \left( (3 + 2\alpha u) e^{2\alpha t} + (2\alpha t + 1) e^{2\alpha u} - 4(\alpha t + 1) e^{\alpha(t+u)} \right)}{2\alpha^3}.
\]

It thus follows that
\[
\ln H() = n \left[ (\bar{\mu} - \frac{1}{2}\sigma^2)(u - t) + \frac{1}{\alpha}(\mu_t - \bar{\mu}) (1 - e^{-\alpha(u-t)}) \right] \\
+ \frac{n^2}{2} \sigma^2 \left[ u - t + \frac{\sigma^2}{\sigma^2} \alpha(u-t) - 1 + e^{-\alpha(u-t)} \right] \\
+ 2 \frac{\sigma_1}{\sigma} \frac{\left( (3 + 2\alpha u) e^{-2\alpha u} + (2\alpha t + 1) e^{-2\alpha t} - 4(\alpha t + 1) e^{-\alpha(t+u)} \right)}{2\alpha^3}. 
\]
Following Martin (2013), we define

\[ c(x; t, u) \equiv x \left[ (\bar{\mu} - \frac{1}{2} \sigma^2)(u - t) + \frac{1}{\alpha}(\mu - \bar{\mu})(1 - e^{-\alpha(u-t)}) \right] 
+ \frac{x^2\sigma^2}{2} \left[ u - t + \frac{\sigma^2}{\alpha^2} \frac{\alpha(u - t) - 1 + e^{-\alpha(u-t)}}{\alpha^2} \right. 
+ \left. 2 \frac{\sigma_1}{\sigma} \left[ (3 + 2\alpha u)e^{-2\alpha u} + (2\alpha t + 1)e^{-2\alpha t} - 4(\alpha t + 1)e^{-\alpha(t+u)} \right] \right]. \]

This is Martin (2013)'s *cumulant-generating function* that is appropriate to our setting.
B.4.1 No Bias, Two Trees

We consider a Two-Trees Lucas economy where all agents have rational beliefs. We follow the technique developed by Martin (2013). The dividend growth is not a Levy Process in our economy because the mean dividend growth is an Ornstein-Uhlenbeck process. We shall use the *cumulant-generating function* defined in the previous subsection.

In this case, we need expectations of the form:

\[
D_t^n E \left\{ D_u^{n} \left( \frac{D_{1,u}}{D_{1,t}} \right)^{\alpha} \left( \frac{D_{2,u}}{D_{2,t}} \right)^{\beta} \right\} = D_t^n E \left\{ \frac{e^{\alpha \tilde{y}_{1,u} + \beta \tilde{y}_{2,u}}}{(e^{\tilde{y}_{1,u} + y_{1,t}} + e^{\tilde{y}_{2,u} + y_{2,t}})^n} \right\}
\]

where

\[
\tilde{y}_{i,u} \equiv y_{i,u} - y_{i,t} = \ln D_{i,u} - \ln D_{i,t}.
\]
We have

\[ H() = E \left\{ \frac{e^{\alpha \tilde{y}_1,u + \beta \tilde{y}_2,u}}{\left( e^{\tilde{y}_1,u + y_1,t} + e^{\tilde{y}_2,u + y_2,t} \right)^n} \right\} \]

\[ = E \left\{ e^{-\frac{1}{2} \tilde{y}_1,u + y_1,t + \frac{1}{2} \tilde{y}_2,u} e^{\alpha \tilde{y}_1,u + \beta \tilde{y}_2,u} \left( e^{\tilde{y}_1,u + y_1,t - \frac{1}{2}(\tilde{y}_2,u + y_2,t)} + e^{-\frac{1}{2}(\tilde{y}_2,u + y_2,t)} \right)^n \right\} \]

\[ = E \left\{ e^{-\frac{1}{2} y_1,t + y_2,t} e^{(\alpha - n/2) \tilde{y}_1,u + (\beta - n/2) \tilde{y}_2,u} \left( \frac{e^{\tilde{y}_1,u + y_1,t - \frac{1}{2}(\tilde{y}_2,u + y_2,t)}}{2 \cosh \left( \frac{1}{2}(\tilde{y}_2,u + y_2,t) \right)} \right)^n \right\} \]

\[ = E \left\{ e^{-\frac{1}{2} y_1,t + y_2,t} e^{(\alpha - n/2) \tilde{y}_1,u + (\beta - n/2) \tilde{y}_2,u} \left( \frac{2 \cosh \left( \frac{1}{2}(\tilde{y}_2,u + y_2,t) \right)}{\tilde{y}_1,u + y_1,t - (\tilde{y}_2,u + y_2,t)} \right)^n \right\} \]

\[ = e^{-\frac{1}{2} y_1,t + y_2,t} E \left\{ e^{(\alpha - n/2) \tilde{y}_1,u + (\beta - n/2) \tilde{y}_2,u} \int_{-\infty}^{\infty} e^{i[y_2 + y_1,t - (\tilde{y}_1,u + y_1,t)]z} F_n(z) dz \right\} \]

\[ = e^{-\frac{1}{2} y_1,t + y_2,t} \int_{-\infty}^{\infty} e^{i[y_2,t - y_1,t]z} E \left\{ e^{(\alpha - n/2 - iz) \tilde{y}_1,u + (\beta - n/2 + iz) \tilde{y}_2,u} \right\} F_n(z) dz \]

\[ = e^{-\frac{1}{2} y_1,t + y_2,t} \int_{-\infty}^{\infty} e^{i[y_2,t - y_1,t]z} e^{(\alpha - n/2 - iz,\beta - n/2 + iz; t,u)} F_n(z) dz \]

where

\[ F_n(z) = \frac{1}{2\pi} \frac{\Gamma\left(\frac{z}{2} + iz\right) \Gamma\left(\frac{z}{2} - iz\right)}{\Gamma(n)} \]

is the Fourier transform of \(1/(2 \cosh(x/2))^n\) and

\[ c(x, y; t, u) = c(x; t, u) + c(y; t, u) \]
B.4.2 No Bias, \( N \) Trees

We consider a Two-Trees Lucas economy where all agents have rational beliefs. The approach is a generalization of that in the previous Subsection. We have

\[
H() = E \left\{ (D_u)^{-n} \prod \left( \frac{D_{j,u}}{D_{j,t}} \right)^{\alpha_j} \right\} \\
= E \left\{ e^{\sum \alpha_j \bar{y}_{j,u}} \right\} \left( \sum e^{\bar{y}_{j,u} + y_{j,t}} \right)^n \\
= E \left\{ e^{\alpha^t \bar{y}_u} \right\} \left( \sum e^{\bar{y}_{j,u} + y_{j,t}} \right)^n \\
= E \left\{ e^{\alpha^t \bar{y}_u - \sum (\bar{y}_u + y_t)}/N \right\} \left( \sum e^{\bar{y}_{j,u} + y_{j,t} - 1' (\bar{y}_u + y_t)/N} \right)^n \\
= E \left\{ e^{\alpha^t \bar{y}_u - \sum (\bar{y}_u + y_t)/N} \right\} \left( \sum e^{x_j/N} \right)^n \\
= E \left\{ e^{\alpha^t \bar{y}_u - \sum (\bar{y}_u + y_t)/N} \right\} \int e^{iz^t x} F_n^N(z) dz \\
= e^{-\sum (\bar{y}_u + y_t)/N} \int E \left\{ e^{(\alpha - n/N) \bar{y}_u e^{iz^t x}} \right\} F_n^N(z) dz \\
= e^{-\sum (\bar{y}_u + y_t)/N} \int E \left\{ e^{(\alpha - n/N) \bar{y}_u e^{iz^t (Q(\bar{y}_u + y_t))}} \right\} F_n^N(z) dz \\
= e^{-\sum (\bar{y}_u + y_t)/N} \int e^{iz^t (Q y_t)} E \left\{ e^{(\alpha - n/N + iQ^t z) \bar{y}_u} \right\} F_n^N(z) dz \\
= e^{-\sum (\bar{y}_u + y_t)/N} \int e^{iz^t (Q y_t)} e^{c(\alpha - n/N + iQ^t z; t, u)} F_n^N(z) dz \\
\]

where

\[
Q \equiv (NI - 1 \cdot 1')_{N \times (N-1)}; \quad x \equiv Q(\bar{y}_u + y_t); 
\]
and
\[ F_n^N(z) = \left( \frac{n}{N} + i z_1 + \cdots + i z_{N-1} \right) \left( \frac{n}{N} - i z_k \right) \prod_{k=1}^{N-1} \Gamma \left( \frac{n}{N} - i z_k \right). \]

**B.4.3 One Freddy, One Tree**

We consider a single-tree Lucas economy with one rational agent and one believer in the LSN. This case is similar to the model considered by Dumas et al. (2009). We are interested in the expectation

\[ H(\tau) = \mathbb{E} \left\{ D^{-n} \xi_t^m \right\}. \]

I obtained this expectation following the approach of Dumas et al. (2009). By the Feynman-Kac formula, \( H \) satisfies the PDE

\[
0 = \frac{\partial H}{\partial t} + \mu D \frac{\partial H}{\partial D} - \alpha (\mu - \bar{\mu}) \frac{\partial H}{\partial \mu} - \kappa \theta \frac{\partial H}{\partial \theta} + \frac{1}{2} (\sigma D)^2 \frac{\partial^2 H}{\partial D^2} + \frac{1}{2} (\xi \theta)^2 \frac{\partial^2 H}{\partial \xi^2} + \frac{1}{2} b^2 \frac{\partial^2 H}{\partial \theta^2} + \sigma \theta D \frac{\partial^2 H}{\partial \xi \partial D} - b \theta \xi \frac{\partial^2 H}{\partial \xi \partial \theta} - b \sigma D \frac{\partial^2 H}{\partial \theta \partial D} \tag{B.2}
\]

with boundary condition \( H(\tau) = D_t^n \xi_t^m \). We conjecture that the solution is of the form

\[ H(\tau) = D_t^n \xi_t^m e^{c(n; t, u)} e^{A_0(\tau) + \theta A_1(\tau) + \theta^2 A_2(\tau)}. \tag{B.3} \]

Plugging Equation (B.3) into Equation (B.2) we find that \( (A_0, A_1, A_2) \) is a solu-
tion to the system

\[ A_2' = a_2A_2^2 - 2a_1A_2 + a_0 \]  
(B.4)

\[ A_1' = (a_2A_2 - a_1)A_1 + nm\sigma - 2nb\sigma A_2 \]  
(B.5)

\[ A_0' = \frac{1}{4}a_2(A_1^2 + 2A_2) - nb\sigma A_1 \]  
(B.6)

with boundary conditions

\[ A_2(0) = 0; \quad A_1(0) = 0; \quad A_0(0) = 0, \]

where

\[ a_0 = \frac{1}{2}m(m - 1); \quad a_1 = \kappa + mb; \quad a_2 = 2b^2. \]

Let

\[ q = \sqrt{a_1^2 - a_2a_0}. \]

The solution to Equation (B.19) is

\[ A_2(\tau) = \frac{a_0 (1 - e^{-2q\tau})}{q + a_1 + (q - a_1)e^{-2q\tau}}. \]  
(B.7)

We now turn our attention to Equation (B.20). One can verify that

\[ \int A_2(\tau)d\tau = \frac{a_1 - q}{a_2} - \frac{1}{a_2} \ln [q + a_1 + (q - a_1)e^{-2q\tau}] \]

\[ \Rightarrow - \int (a_2A_2 - a_1) \, d\tau = q\tau + \ln [q + a_1 + (q - a_1)e^{-2q\tau}]. \]
Therefore,

\[ u(\tau) \equiv e^{-\int a_2[A_2(\tau) + a_1]d\tau} = \left[q + a_1 + (q - a_1)e^{-2q\tau}\right] e^{q\tau} \]

\[ \int u(\tau)d\tau = \frac{1}{q} \left[q + a_1 - (q - a_1)e^{-2q\tau}\right] e^{q\tau}. \]

Moreover,

\[ u' = -(a_2A_2 - a_1)u \implies \int u(\tau)A_2(\tau) = \frac{1}{a_2} \int u(\tau)d\tau - u(\tau). \]

Define

\[ H(\tau) \equiv \left[mn\sigma \int u(\tau)d\tau - 2nb\sigma \int u(\tau)A_2(\tau)d\tau\right] = \frac{n\sigma}{a_2} \left[\left(ma_2 - 2ba_1\right) \int u(\tau)d\tau + 2bu(\tau)\right] = \frac{n\sigma}{a_2} \left\{a_3 \left[q + a_1 - (q - a_1)e^{-2q\tau}\right] + 2bq \left[q + a_1 + (q - a_1)e^{-2q\tau}\right]\right\} e^{q\tau} \]

Equation (B.20) is a First-Order ODE with \( u \) as integrating factor. Its solution is

\[ A_1(\tau) = \frac{1}{u(\tau)} [H(\tau) - H(0)] = \frac{n\sigma}{a_2q} a_3 \left[q + a_1 - (q - a_1)e^{-2q\tau} - 2a_1e^{-q\tau}\right] + 2bq \left[q + a_1 + (q - a_1)e^{-2q\tau} - 2qe^{-q\tau}\right]. \]

(B.8)
where we used the boundary condition for $A_1$ in the first equality.

Finally, the solution to Equation (B.21) is

$$A_0(\tau) = \sum_{i=0}^{2} d_i D_1(i, \tau) + \sum_{i=0}^{4} e_i D_2(i, \tau) \tag{B.9}$$

where

$$c_1 = c_0(a_3 + 2bq)(q + a_1); \quad c_2 = -2c_0(a_1a_3 + 2bq^2); \quad c_3 = c_0(q - a_1)(2bq - a_3); \quad c_0 = \frac{n\sigma}{qa_2};$$

$$d_0 = b^2a_0 - nb\sigma c_1; \quad d_1 = -nb\sigma c_2; \quad d_2 = -(b^2a_0 + nb\sigma c_3);$$

$$e_0 = \frac{b^2c_1^2}{2}; \quad e_1 = b^2c_1c_2; \quad e_2 = \frac{b^2(2c_1c_3 + c_2^2)}{2};$$

$$e_3 = b^2c_2c_3; \quad e_4 = \frac{b^2c_3^2}{2}; \quad a_3 = ma_2 - 2ba_1$$

The functions $D_i(j, \tau)$ are

$$D_1(j, \tau) = \int_0^{\tau} \frac{e^{-js}}{q + a_1 + (q - a_1)e^{-2js}} ds = \begin{cases} \frac{\tau}{q+a_1} + \frac{1}{2q(q+a_1)} \ln \frac{q+a_1+(q-a_1)e^{-2q\tau}}{2q} \quad & \text{if } j = 0 \\ \frac{1}{q(q+a_1)} \left[ 2F_1 \left(1, \frac{j}{2}; \frac{j}{2} + 1; z \right) - e^{-jq\tau} 2F_1 \left(1, \frac{j}{2}; \frac{j}{2} + 1; \bar{z} \right) \right] \quad & \text{if } j > 0. \end{cases}$$

$$D_2(j, \tau) = \int_0^{\tau} \frac{e^{-js}}{[q + a_1 + (q - a_1)e^{-2js}]^2} ds = \frac{1}{2q(q - a_1)} \left[ \frac{e^{(2-j)q\tau}}{q + a_1 + (q - a_1)e^{-2q\tau}} - \frac{1}{2q} + (j - 2)qD_1(j - 2, \tau) \right].$$
Here,
\[ z = -\frac{q - a_1}{q + a_1} \quad \text{and} \quad \bar{z} = e^{-2q\tau} z. \]

The functions are well defined since
\[ |\bar{z}| \leq z < 1 \]
for the parameters we consider.

Suppose that \( j > 0 \).

\[
D_1(j, \tau) = \int_0^\tau \frac{e^{-jsq}}{q + a_1 + (q - a_1)e^{-2qs}} ds
\]

\[
= \frac{1}{q + a_1} \int_0^\tau e^{-jsq} \left( 1 - ze^{-2qs} \right)^{-1} ds
\]

\[
= \frac{1}{q + a_1} \int_0^\tau \left( e^{-2qs} \right)^{\frac{j}{2} - 1} \left( 1 - ze^{-2qs} \right)^{-1} e^{-2qs} ds
\]

\[
= \frac{1}{-2q(q + a_1)} \int_1^e e^{-2q\tau} x^{\frac{j}{2} - 1} (1 - zx)^{-1} dx
\]

\[
= \frac{1}{2q(q + a_1)} \int_{e^{-2q\tau}}^1 x^{\frac{j}{2} - 1} (1 - zx)^{-1} dx
\]

\[
= \frac{1}{2q(q + a_1)} \left[ \int_0^1 x^{\frac{j}{2} - 1} (1 - zx)^{-1} dx - \int_0^{e^{-2q\tau}} x^{\frac{j}{2} - 1} (1 - zx)^{-1} dx \right]
\]

\[
= \frac{1}{2q(q + a_1)} \left[ \int_0^1 x^{\frac{j}{2} - 1} (1 - zx)^{-1} dx - e^{-jq\tau} \int_0^1 x^{\frac{j}{2} - 1} (1 - \bar{z}x)^{-1} dx \right]
\]

\[
= \frac{1}{2q(q + a_1)} \frac{2}{j} \left[ {}_2F_1 \left( 1, \frac{j}{2}; \frac{j}{2} + 1; z \right) - e^{-jq\tau} {}_2F_1 \left( 1, \frac{j}{2}; \frac{j}{2} + 1; \bar{z} \right) \right]
\]
B.4.4 One Freddy, Two Trees

Let \( P_{\alpha} \) be the price of an asset paying dividends \( D_{\alpha,j}^u \equiv D_{1,j}^u D_{2,j}^u \), where \( \alpha = (\alpha_1, \alpha_2) \).

Let

\[
D_{u}^{\alpha} \equiv D_{u}^{\alpha}, \quad \text{where} \quad \alpha = \alpha_1 + \alpha_2.
\]

Then,

\[
\left[ (1 + \alpha^{1/\gamma} \xi_t^{1/\gamma})^\gamma D_t^{-\gamma} \right] P_{\alpha} = \int_0^\infty e^{-\beta(u-t)} E_t \left\{ (1 + \alpha^{1/\gamma} \xi_t^{1/\gamma})^\gamma D_u^{-\gamma} D_{j,u}^\alpha \right\} du.
\]

Computing this ratio will require computing expectations of the form

\[
H() = E \left\{ \xi^{m} D_{u}^{-n} D_{1,u}^{\alpha} D_{2,u}^{\beta} \right\}.
\]

We proceed as before:

\[
H() = E \left[ \frac{e^{\alpha \tilde{y}_{1,u} + \beta \tilde{y}_{2,u}}}{e^{\tilde{y}_{1,u} + y_{1,t} + e^{\tilde{y}_{2,u} + y_{2,t}}}^{n}} \right]
\]

\[
= e^{-n \frac{y_{1,t} + y_{2,t}}{2}} \int_{-\infty}^{\infty} e^{i[y_{2,t} - y_{1,t}]z} E \left[ \xi^{m} e^{(\alpha-n/2-iz) \tilde{y}_{1,u} + (\beta-n/2+iz) \tilde{y}_{2,u}} \right] F_n(z) dz
\]

\[
= e^{-n \frac{y_{1,t} + y_{2,t}}{2}} \int_{-\infty}^{\infty} e^{i[y_{2,t} - y_{1,t}]z} E \left[ e^{m \log \xi_u + (\alpha-n/2-iz) \tilde{y}_{1,u} + (\beta-n/2+iz) \tilde{y}_{2,u}} \right] F_n(z) dz.
\]

(B.10)
Consider the following expectation:

\[
G() = E \left[ e^m \log \xi_{u} + p_1 y_{1,u} + p_2 y_{2,u} \right] \\
= E \left[ e^m \log \xi_{1,u} + \log \xi_{2,u} + p_1 y_{1,u} + p_2 y_{2,u} \right] \\
= E \left[ e^m \log \xi_{1,u} + p_1 y_{1,u} \right] E \left[ e^m \log \xi_{2,u} + p_2 y_{2,u} \right] 
\]

where the last equality follows from the fact that the Brownian motions are uncorrelated:

\[
dB_2 dB_2 = 0.
\]

We use the result from Equation (B.3) to compute each of the two expectations above:

\[
E \left[ e^m \log \xi_{j,u} + p_j y_{j,u} \right] = e^m \log \xi_{j,t} + p_j y_{j,t} e^{c_j(p_j)\tau + A_j,0(\tau) + \theta_j A_j,1(\tau) + \theta_j^2 A_j,2(\tau)}
\]

where

\[
c_j(x; t, u) \equiv x \left[ (\bar{\mu}_j - \frac{1}{2} \sigma_j^2)(u - t) + \frac{1}{\alpha_j}(\mu_{jt} - \bar{\mu}_j) \left( 1 - e^{-\alpha_j(u-t)} \right) \right] \\
+ \frac{x^2 \sigma_j^2}{2} \left[ u - t + \frac{\sigma_j^2}{\alpha_j^2} \frac{\alpha_j(u - t) - 1 + e^{-\alpha_j(u-t)}}{\alpha_j^2} \right] \\
+ 2 \frac{\sigma_j^2}{\sigma_j} \left[ (3 + 2\alpha_j u)e^{-2\alpha_j u} + (2\alpha_j t + 1)e^{-2\alpha_j t} - 4(\alpha_j t + 1)e^{-\alpha_j(t+u)} \right]
\]

and \((A_{j,0}, A_{j,1}, A_{j,2})\) is obtained from Equations (B.7)–(B.9).
We now use the solution for the function $G$ in the Equation \(B.10\):

\[
H() = e^{-\frac{y_{1,t} + y_{2,t}}{2}} \int_{-\infty}^{\infty} e^{i[y_{2,t} - y_{1,t}]z} e^{m \log \xi_t + \tau c(a - n/2 - iz; \beta - n/2 + iz)} e^{K(\theta_1, \theta_2, \tau)} F_n(z) dz
\]

(B.11)

where

\[
K(\theta_1, \theta_2, \tau) = \sum_{j=1}^{2} [A_{j,0}(\tau) + \theta_j A_{j,1}(\tau) + \theta_j^2 A_{j,2}(\tau)].
\]

**B.4.5 One Freddy, $N$ Trees**

Let $P_\alpha$ be the price of an asset paying dividends

\[
D^\alpha_j \equiv \prod D^{\alpha_j}_j, \quad \text{where} \quad \alpha = (\alpha_1, \cdots, \alpha_N).
\]

Let

\[
D^\alpha \equiv D^\alpha, \quad \text{where} \quad \alpha = \sum \alpha_j.
\]

Then,

\[
\frac{P_\alpha}{D^\alpha_j} = E_t \left\{ \int_{0}^{\infty} e^{-\beta(u-t)} \left( \frac{1 + \alpha^{1/\gamma} \xi_u^{1/\gamma}}{1 + \alpha^{1/\gamma} \xi_t^{1/\gamma}} \right) \gamma \left( \frac{D_u}{D_t} \right)^{-\gamma} \frac{D^\alpha_{j,u}}{D^\alpha_{j,t}} du \right\}.
\]

Computing this ratio will require computing expectations of the form

\[
E \left\{ \left( \frac{1 + \alpha^{1/\gamma} \xi_u^{1/\gamma}}{1 + \alpha^{1/\gamma} \xi_t^{1/\gamma}} \right)^m \left( \frac{D_u}{D_t} \right)^n \prod \left( \frac{D_{j,u}}{D_{j,t}} \right)^{\alpha_j} \right\}
\]

(B.12)
Using Taylor series expansion, we can derive the expectation above by computing

\[ H() = E \left\{ \xi^m D_u^{-n} \prod \left( \frac{D_{j,u}}{D_{j,t}} \right)^{\alpha_j} \right\}. \]

We proceed as before:

\[ H() = E \left\{ \xi^m e^{\alpha \hat{y}_u} \left( \sum e^\theta_j \right)^n \right\} \]

\[ = e^{-n'y_N/N} \int e^{iz'(Qy_t)} \left\{ \xi^m e^{(\alpha-n/N+iQ'z)\hat{y}_u} \right\} F_n^N(z) dz \]

\[ = e^{-n'y_N/N} \int e^{iz'(Qy_t)} \left\{ e^{m \log \xi + (\alpha-n/N+iQ'z)\hat{y}_u} \right\} F_n^N(z) dz \]

\[ = e^{-n'y_N/N} \int e^{iz'(Qy_t)} e^{m \log \xi + \tau e((\alpha-n/N+iQ'z)\hat{y}_u)} e^{K_N(\theta_1, \theta_2, \tau)} F_n^N(z) dz \]

where

\[ K_N(\theta, \tau) = \sum_{j=1}^N \left[ A_{j,0}(\tau) + \theta_j A_{j,1}(\tau) + \theta_j^2 A_{j,2}(\tau) \right]. \]
B.4.6 Two Freddy, One Tree

Let $H$ be the expectation

$$H(\tau, D_t, \xi_t, x, y; n, m) = E[\eta_{1,u} D_u^n \xi_u^m], \quad (B.13)$$

where

$$x = \theta_{1,t} \quad \text{and} \quad y = \theta_{2,t}.$$ 

By the Feynman-Kac formula, $H$ satisfies the PDE

$$0 = -\frac{\partial H}{\partial \tau} + \mu D \frac{\partial H}{\partial D} + \mu_x \frac{\partial H}{\partial x} + \mu_y \frac{\partial H}{\partial y} - x(y - x)\xi \frac{\partial H}{\partial \xi}$$

$$+ \frac{1}{2}(\sigma D)^2 \frac{\partial^2 H}{\partial D^2} + \frac{1}{2}\sigma_x^2 \frac{\partial^2 H}{\partial x^2} + \frac{1}{2}\sigma_y^2 \frac{\partial^2 H}{\partial y^2} + \frac{1}{2}(y - x)^2 \xi^2 \frac{\partial^2 H}{\partial \xi^2} + \frac{1}{2}x^2 \eta_1^2 \frac{\partial^2 H}{\partial \eta_1^2}$$

$$+ (y - x)\xi \left[ (\sigma D) \frac{\partial H^2}{\partial D \partial \xi} + \sigma_x \frac{\partial H^2}{\partial x \partial \xi} + \sigma_y \frac{\partial H^2}{\partial y \partial \xi} + x\eta_1^1 \frac{\partial^2 H}{\partial \eta_1^1 \partial \xi} \right]$$

$$+ \sigma D \left[ \sigma_x \frac{\partial H^2}{\partial x \partial D} + \sigma_y \frac{\partial H^2}{\partial y \partial D} + x\eta_1 \frac{\partial^2 H}{\partial \eta_1 \partial D} \right]$$

$$+ \theta_1 \eta_1 \left[ \sigma_x \frac{\partial H^2}{\partial x \partial \eta_1} + \sigma_y \frac{\partial H^2}{\partial y \partial \eta_1} \right] + \sigma_x \sigma \frac{\partial^2 H}{\partial x \partial y} \quad (B.14)$$

with boundary condition

$$H(t, D_t, \xi_t, x, y; n, m) = \eta_{1,t} D_t^n \xi_t^m.$$
We conjecture that the solution is of the form

\[ H() = \eta_{t,l} D_l^\alpha e^{\int_i^t \eta(t,x,y,z)\,dz} e^{h(x,y,\tau)} \]  

(B.15)

where

\[ h(x, y, \tau) = A_0(\tau) + B_1(\tau)x + B_2(\tau)y + C_0(\tau)x^2 + 2C_1 xy + B_2(\tau)y^2. \]

Plugging Equation (B.15) into Equation (B.14) we obtain

\[
\frac{\partial h}{\partial \tau} = \mu_x \frac{\partial h}{\partial x} + \mu_y \frac{\partial h}{\partial y} - mx(y-x) + 1 \frac{\sigma_x^2}{2} \left( \frac{\partial h}{\partial x} \right)^2 + \frac{\sigma_y^2}{2} \left( \frac{\partial h}{\partial y} \right)^2 \\
+ \frac{m(m-1)}{2} (y-x)^2 + m(y-x) \left[ \sigma_n + \sigma_x \frac{\partial h}{\partial x} + \sigma_y \frac{\partial h}{\partial y} + x \right] \\
+ n \sigma \left[ \sigma_x \frac{\partial h}{\partial x} + \sigma_y \frac{\partial h}{\partial y} + x \right] + x \left[ \sigma_x \frac{\partial h}{\partial x} + \sigma_y \frac{\partial h}{\partial y} \right] + \sigma_x \sigma_y \left[ \frac{\partial h}{\partial x} \frac{\partial h}{\partial y} + \frac{\partial^2 h}{\partial x \partial y} \right]
\]

\[
\frac{\partial h}{\partial \tau} = \frac{m(m-1)}{2} (y-x)^2 + mn \sigma (y-x) + n \sigma x \\
+ \mu_x \frac{\partial h}{\partial x} + \mu_y \frac{\partial h}{\partial y} + m(y-x) \left[ \sigma_x \frac{\partial h}{\partial x} + \sigma_y \frac{\partial h}{\partial y} \right] + n \sigma \left[ \sigma_x \frac{\partial h}{\partial x} + \sigma_y \frac{\partial h}{\partial y} \right] + x \left[ \sigma_x \frac{\partial h}{\partial x} + \sigma_y \frac{\partial h}{\partial y} \right] \\
+ \frac{1}{2} \sigma_x^2 \left( \frac{\partial h}{\partial x} \right)^2 + \frac{\partial^2 h}{\partial x^2} \right] + \frac{1}{2} \sigma_y^2 \left( \frac{\partial h}{\partial y} \right)^2 + \sigma_x \sigma_y \left[ \frac{\partial h}{\partial x} \frac{\partial h}{\partial y} + \frac{\partial^2 h}{\partial x \partial y} \right]
\]
We assume that
\[ \mu_x = \kappa_x x; \quad \mu_y = \kappa_y y \]
and that \( \kappa_x, \kappa_y, \sigma_x, \sigma_y \) are all constant. We define
\[
    X = \begin{bmatrix} x \\ y \end{bmatrix}; \quad S = \begin{bmatrix} \sigma_x \\ \sigma_y \end{bmatrix}; \quad \frac{\partial h}{\partial X} = \begin{bmatrix} \frac{\partial h}{\partial x} \\ \frac{\partial h}{\partial y} \end{bmatrix}; \quad \frac{\partial h^2}{\partial X \partial X'} = \begin{bmatrix} \frac{\partial^2 h}{\partial x^2} & \frac{\partial^2 h}{\partial x \partial y} \\ \frac{\partial^2 h}{\partial y \partial x} & \frac{\partial^2 h}{\partial y^2} \end{bmatrix}
\]
\[
    D = \begin{bmatrix} \kappa_x + (1 - m)\sigma_x & m\sigma_x \\ (1 - m)\sigma_y & \kappa_y + m\sigma_y \end{bmatrix}; \quad E = \frac{m(m - 1)}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; \quad V = n\sigma \begin{bmatrix} 1 - m \\ m \end{bmatrix};
\]
\[
    \mu_X = n\sigma S + DX
\]
\[
    h = A + B'X + X'CX.
\]

We can rewrite the Feynman-Kac equation as
\[
    \frac{\partial h}{\partial \tau} = \frac{m(m - 1)}{2} (y - x)^2 + mn\sigma(y - x) + n\sigma x + \mu_x' \frac{\partial h}{\partial X} + \frac{1}{2} \left[ S' \frac{\partial^2 h}{\partial X \partial X'} \right ] S + \left( S' \frac{\partial h}{\partial X} \right)^2
\]
\[
    = X'EX + V'X + \mu_x' [B + 2CX] + \frac{1}{2} \left[ 2S'CS + (S'B + 2S'C)^2 \right ]
\]
\[
    = S'CS + n\sigma S'B + \frac{1}{2} (S'B)^2 + [V' + B'D' + 2n\sigma S'C + 2B'SS'C] X
\]
\[
    + X' \left[ CD + D'C + 2C'SS'C + E \right ] X.
\]
We used the following relation in the last equality:

\[
2X'D'CX = X'D'CX + X'D'CX \\
= X'D'CX + (X'D'CX)' \\
= X'D'CX + X'C'DX \\
= X'D'CX + X'CDX \\
= X'[D'C + CD]X
\]

where the second equality follows from the fact that \( X'D'CX \) is a real number and the fourth equality follows from the fact that \( C \) is symmetric. We perform this transformation to obtain a symmetric term \( (D'C + CD) \) instead of working with a non-symmetric term \( D'C \) because \( C \) is symmetric. The importance of this transformation will become clear later on.

We deduce that \((A, B, C)\) is a solution to the following system:

\[
\begin{align*}
\frac{\partial C}{\partial \tau} &= 2C'SS'C + CD + D'C + E \quad \text{(B.16)} \\
\frac{\partial B}{\partial \tau} &= [D + 2C'SS']B + V + 2n\sigma C'S \\
\frac{\partial A}{\partial \tau} &= S'CS + n\sigma S'B + \frac{1}{2} (S'B)^2. \quad \text{(B.18)}
\end{align*}
\]

A necessary condition for Equation (B.16) to have a solution is that its RHS is symmetric because its LHS is symmetric since it is the case for \( C \). This condition holds because both \( E \) and \( D'C + CD \) are symmetric.
Remark: We recover the “One Freddy” case by setting

\[
x = (\kappa_x = ) \sigma_x = 0.
\]

\[
A'_2 = a_2A_2^2 - 2a_1A_2 + a_0 \quad (B.19)
\]

\[
A'_1 = (a_2A_2 - a_1)A_1 + nm\sigma - 2nb\sigma A_2 \quad (B.20)
\]

\[
A'_0 = \frac{1}{4}a_2(A_1^2 + 2A_2) - nb\sigma A_1 \quad (B.21)
\]

Equation (B.16) is a matrix Riccati equation. Its solution is

\[
C(\tau) = C_{22}^{-1}(\tau)C_{21}(\tau) \quad (B.22)
\]

where

\[
\begin{bmatrix}
C_{11}(\tau) & C_{21}(\tau) \\
C_{12}(\tau) & C_{22}(\tau)
\end{bmatrix} = \exp\left\{ \tau \begin{bmatrix}
D & -2SS' \\
E & -D
\end{bmatrix} \right\}.
\]

Consider Equation (B.17). The integrating factor of this equation is

\[
H(\tau) = e^{\int(D + 2C'SS')d\tau}.
\]

The solution to Equation (B.17) is thus

\[
B(\tau) = H^{-1}(\tau) \int_0^\tau H(s) [V + 2n\sigma C'S] ds. \quad (B.23)
\]
The solution to Equation (B.18) is obtained through an integral once we have both $C$ and $B$:

$$A(\tau) = \int_{0}^{\tau} \left[ S' C(s) S + n \sigma S' B(s) + \frac{1}{2} (S'B(s))^2 \right] ds. \quad (B.24)$$

Note that the integrand in the RHS of Equation (B.24) is a real-value function. Thus, the integral is simple to evaluate numerically.

The matrix exponential are difficult to compute for arbitrary parameter values. For a given set of values, we shall use the Jordan decomposition to compute the matrix exponentials.

**Lemma B.1**

Suppose that

$$\kappa_x = \kappa_y = \kappa \quad \text{and} \quad \sigma_x = -\sigma_y.$$ 

Then, the $4 \times 4$ matrix

$$\begin{bmatrix}
D & -2SS' \\
E & -D
\end{bmatrix}$$

has at least two real eigenvalues, one of which is $\kappa$.

We can use the lemma above when finding a simple form for $C$. 
B.4.7 Two Freddy, Two Trees

\[ H(\cdot) = \mathbb{E} \left\{ \eta_{1,u} \xi_m D_{-n}^{-\alpha} \left( \frac{D_{1,u}}{D_{1,t}} \right)^\alpha \left( \frac{D_{2,u}}{D_{2,t}} \right)^\beta \right\} \]

We proceed as before:

\[
H(\cdot) = \mathbb{E} \left[ \eta_{1,u} \xi_u e^{\alpha \tilde{y}_{1,u} + \beta \tilde{y}_{2,u}} \right] \\
= e^{-\frac{y_{1,t} + y_{2,t}}{2}} \int_{-\infty}^{\infty} e^{i[y_{2,t}-y_{1,t}]z} \mathbb{E} \left[ \eta_{1,u} \xi_u e^{(\alpha-n/2-iz)\tilde{y}_{1,u} + (\beta-n/2+iz)\tilde{y}_{2,u}} \right] F_n(z) \, dz \\
= e^{-\frac{y_{1,t} + y_{2,t}}{2}} \int_{-\infty}^{\infty} e^{i[y_{2,t}-y_{1,t}]z} \eta_{1,t} \xi_t e^{\alpha \tilde{y}_{1,u} + \beta \tilde{y}_{2,u}} e^{K(\theta_1, \theta_2, \tau)} F_n(z) \, dz
\]

where

\[
K(\theta_1, \theta_2, \tau) = \sum_{j=1}^{2} \left[ A_{j,0}(\tau) + \theta_j' B_j(\tau) + \theta_j' C_{j,2}(\tau) \theta_j \right]
\]

where

\[
\theta_j = (\theta_{1,j}, \theta_{2,j})'.
\]
B.4.8 Two Freddy, $N$ Trees

$$H() = E \left\{ \xi^m D_u^{-n} \prod \left( \frac{D_{j,u}}{D_{j,t}} \right)^{\alpha_j} \right\} .$$

We proceed as before:

$$H() = E \left\{ \frac{\xi^m e^{\alpha' \tilde{y}_u}}{(\sum e^{\tilde{y}_{j,u} + y_{j,t}})^n} \right\}$$

$$= e^{-n'y_t/N} \int e^{iz'(Qy_t)} E \left\{ \xi^m e^{(\alpha - n/N + iQ'z')\tilde{y}_u} \right\} F_N^n(z) dz$$

$$= e^{-n'y_t/N} \int e^{iz'(Qy_t)} E \left\{ e^{m \log \xi + (\alpha - n/N + iQ'z')\tilde{y}_u} \right\} F_N^n(z) dz$$

$$= e^{-n'y_t/N} \int e^{iz'(Qy_t)} e^{m \log \xi + \tau c((\alpha - n/N + iQ'z')\tilde{y}_u)} e^{K_N(\theta_1, \theta_2, \tau)} F_N^n(z) dz$$

where

$$K_N(\theta, \tau) = \sum_{j=1}^N \left[ A_{j,0}(\tau) + \theta_j A_{j,1}(\tau) + \theta_j^2 A_{j,2}(\tau) \right] .$$
\[
\frac{\partial h}{\partial \tau} = -mxy + \frac{1}{2}m(m-1)y^2 + y[m\sigma + mx] + n\sigma x
\]

\[
+ \mu_x h_x + \mu_y h_y + \frac{1}{2}\sigma_x^2(h_x^2 + h_{xx}) + \frac{1}{2}\sigma_y^2(h_y^2 + h_{yy})
\]

\[
+ my[\sigma_x h_x + \sigma_y h_y] + n\sigma [\sigma_x h_x + \sigma_y h_y]
\]

\[
+ x[\sigma_x h_x + \sigma_y h_y] + \sigma_x \sigma_y (h_x h_y + h_{xy})
\]

where

\[
h(\tau, x, y) \equiv A_0(\tau) + A_1(\tau)x + B_0(\tau) + B_1(\tau)y + B_2(\tau)y^2.
\]

\[
\frac{\partial h}{\partial \tau} = \frac{1}{2}m(m-1)y^2 + m\sigma y + n\sigma x + \sigma_x \sigma_y (h_x h_y + h_{xy})
\]

\[
+ \mu_x h_x + \mu_y h_y + \frac{1}{2}\sigma_x^2(h_x^2 + h_{xx}) + \frac{1}{2}\sigma_y^2(h_y^2 + h_{yy})
\]

\[
+ (my + x)[\sigma_x h_x + \sigma_y h_y] + n\sigma [\sigma_x h_x + \sigma_y h_y]
\]

A necessary condition for this approach to work is that the “second” part of the RHS above is polynomial. One way to achieve it is to model \(\mu_{\theta_1}, \mu_{\theta_2}\), and \(h\) as polynomial functions.

**Assumptions:**

\[
\mu_{\theta_1} = a_1 + b_1 \theta_1 \quad \text{and} \quad \sigma_{\theta_1} \text{ is constant.}
\]
Assumptions:

\[ \mu_{\theta_i} = a_I + b_I \theta_I \quad \text{and} \quad \sigma_{\theta_i} \text{ is constant.} \]

\[
\frac{\partial h}{\partial \tau} = -m\theta_1(\theta_2 - \theta_1) + \frac{1}{2}m(m - 1)(\theta_2 - \theta_1)^2 + (\theta_2 - \theta_1)[m n \sigma + m \theta_1] + n \sigma \theta_1
\]

\[+ \mu_{\theta_1} h_1 + \mu_{\theta_2} h_2 + \frac{1}{2} \sigma_{\theta_1}^2 (h_1^2 + h_{11}) + \frac{1}{2} \sigma_{\theta_2}^2 (h_2^2 + h_{22})\]

\[+ (\theta_2 - \theta_1)[m \sigma_{\theta_1} h_1 + m \sigma_{\theta_2} h_2]\]

\[+ \sigma [n \sigma_{\theta_1} h_1 + n \sigma_{\theta_2} h_2]\]

\[+ \theta_1[\sigma_{\theta_1} h_1 + \sigma_{\theta_2} h_2] + \sigma_{\theta_1} \sigma_{\theta_2} (h_1 h_2 + h_{12})\]

Plugging Equation (B.15) into Equation (B.14) we find that \((A_0, A_1, A_2)\) is a solution to the system

\[ A'_2 = a_2 A_2^2 - 2a_1 A_2 + a_0 \]  \hfill (B.25)

\[ A'_1 = (a_2 A_2 - a_1) A_1 + n m \sigma \]  \hfill (B.26)

\[ A'_0 = \frac{1}{4} a_2 (A_1^2 + 2A_2) - \frac{na_2}{2} \sigma (A_1 + 2A_2) \]  \hfill (B.27)

with boundary conditions

\[ A_2(0) = 0; \quad A_1(0) = 0; \quad A_0(0) = 0, \]
where

\[ a_0 = \frac{1}{2}m(m - 1); \quad a_1 = \kappa + mb; \quad a_2 = 2b^2. \]

Let

\[ q = \sqrt{a_1^2 - a_2a_0}. \]
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