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DICHOTOMY FOR ARITHMETIC PROGRESSIONS IN SUBSETS OF REALS

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Abstract. Let $\mathcal{H}$ stand for the set of homeomorphisms $\phi\colon [0,1] \to [0,1]$. We prove the following dichotomy for Borel subsets $A \subset [0,1]$:

- either there exists a homeomorphism $\phi \in \mathcal{H}$ such that the image $\phi(A)$ contains no 3-term arithmetic progressions;
- or, for every $\phi \in \mathcal{H}$, the image $\phi(A)$ contains arithmetic progressions of arbitrary finite length.

In fact, we show that the first alternative holds if and only if the set A is meager (a countable union of nowhere dense sets).

1. Definitions

Let $\mathbb{R}$, $\mathbb{Q}$ denote the sets of real and rational numbers, respectively. By an AP (arithmetic progression) we mean a finite strictly increasing sequence in $\mathbb{R}$ of the form $\boldsymbol{x} = (x + kd)_{k=0}^{n-1}$, with $d > 0$ and $n \geq 3$. The convention is sometimes abused by identifying the sequence $\boldsymbol{x}$ with the set of its elements. An AP is completely determined by its first term $x = \min \boldsymbol{x}$, its length $n = |\boldsymbol{x}|$ and its step (difference) $d > 0$.

Denote by $\mathcal{H}$ the set of homeomorphisms $\phi\colon [0,1] \to [0,1]$ of the unit interval. The result presented in the abstract can be restated as follows.

Theorem 1. *Let $S \subset [0,1]$ be a Borel subset. Then exactly one of the following two assertions holds:*

(1) *(either) there exists $\phi \in \mathcal{H}$ such that $\phi(S)$ does not contain 3-term APs;*
(2) *(or) for every $\phi \in \mathcal{H}$, $\phi(S)$ contains APs of arbitrarily large finite length.*

Moreover, (1) holds if and only if S is meager.

Recall some basic relevant definitions. Let $S \subset \mathbb{R}$. A set S is called *nowhere dense* if its closure $\bar{S} \subset \mathbb{R}$ has empty interior. S is called *meager* (or a *set of first category*), if it is a countable union of nowhere dense sets. S is called *residual*, or *co-meager*, if $\mathbb{R} \setminus S$ is meager; S is called *residual* in a subinterval $X \subseteq \mathbb{R}$ if the complement $X \setminus S$ is meager. Finally, S is called *a set of second category* if it is not meager.

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The following proposition lists some "largeness" properties of a set $A \subset \mathbb{R}$ which force it to contains APs of arbitrary large finite length. Denote by λ the Lebesgue measure on $\mathbb{R}$.

Proposition 1. *The sets in each of the following five classes* $\mathcal{E}_i, 1 \le i \le 5$, *contain arbitrarily long finite arithmetic progressions:*

$$\mathcal{E}_1 = \{S \subset \mathbb{R} \mid S \textit{ is Lebesgue measurable with } 0 < \lambda(S) \le \infty\},$$
$$\mathcal{E}_2 = \{S \subset \mathbb{R} \mid S \textit{ is residual in some interval } U \subset \mathbb{R} \textit{ of positive length}\},$$
$$\mathcal{E}_3 = \{S \subset \mathbb{R} \mid S \textit{ is winning in Schmidt's game}\},$$
$$\textit{(several versions of Schmidt's games are possible; see } [5],[8]),$$
$$\mathcal{E}_4 = \{S \subset \mathbb{R} \mid S \textit{ is Borel and not meager}\},$$
$$\mathcal{E}_5 = \{S \subset \mathbb{R} \mid S \textit{ has Baire property and is not meager}\}.$$

Recall that a set $S \subset \mathbb{R}$ has the Baire property if it can be represented as the symmetric difference $S = F \triangle P = (F \setminus P) \cup (P \setminus F)$ where $F \subset \mathbb{R}$ is open and $P \subset \mathbb{R}$ is meager.

The family **BP** of subsets of $\mathbb{R}$ which have Baire property forms a σ-algebra containing the σ-algebra **B** of Borel subsets of $\mathbb{R}$. (We refer to [6, §4, page 19] for the short review of relevant standard material.)

Proof of Proposition 1. For a set $S \in \mathcal{E}_1$, one easily produces APs in S near any of its Lebesgue density points. The argument for the classes E_2 and E_3 is even easier because the class E_3 and the class of residual subsets of a fixed subinterval are closed under finite (and even countable) intersections.

Since $\mathcal{E}_4 \subset \mathcal{E}_5$ (because $\mathbf{B} \subset \mathbf{BP}$), the proof of Proposition 1 is completed by showing that $\mathcal{E}_5 \subset \mathcal{E}_2$. Given $S = F \triangle P \in \mathcal{E}_5$, the open set F cannot be empty (otherwise $S = P$ would be meager, contradicting $S \in \mathcal{E}_5$). Let $U \neq \emptyset$ be a subinterval of F; then S is residual in U, so that $S \in \mathcal{E}_2$. $\square$

For more references on Borel sets and Baire property we refer to [1] and [9] (in particular, see Proposition 3.5.6 and Corollary 3.5.2 in [9, page 108]).

Note that the problems of finding finite or countable configurations F in sets $S \subset \mathbb{R}$, under various "largeness" metric assumptions on S, have been considered by several mathematicians.

Following Kolountzakis [3], a set F is called *universal* for a class $\mathcal{E}$ of subsets of reals if $F \ll S$ for all $S \in \mathcal{E}$. Henceforth $F \ll S$ means that S contains an affine image of F, i.e., that $aF + b \subset S$, for some $a, b \in \mathbb{R}$, $a > 0$.

Every finite subset of reals is universal for all the classes $\mathcal{E}_k$, $1 \le k \le 5$. Every bounded countable subset is universal for the classes $\mathcal{E}_k$, $2 \le k \le 5$.

An old question of Erdős is whether there is a universal infinite set $F \subset \mathbb{R}$ for the class $\mathcal{E}_1$ (of sets of positive measure). The question is still open even though some families of countable sets F are shown not to contain universal sets; see Kolountzakis [3], Humke and Laczkovich [7] and the references there. In [7] an elegant combinatorial characterization of universal sets F (for the class $\mathcal{E}_1$) is given which reproduces earlier results in the subject.

Keleti [2] constructed a compact set $A \subset [0, 1]$ of Hausdorff dimension 1 which does not contain 3-term APs; on the other hand, Laba and Pramanik in [4] showed that under certain assumptions (on the Fourier transform of supported measure) compact sets of fractional dimension close to 1 must contain 3-term APs. We refer to [4] for a survey of related questions.

The central result of the paper, Theorem 1, completely characterizes the topological (rather than metric) properties of a Borel set $S \subset \mathbb{R}$ which guarantee that it contains arbitrarily long APs. This theorem is an immediate consequence of the following proposition and the fact that the sets $S \in \mathcal{E}_4$ contain arbitrarily long APs (Proposition 1).

Proposition 2. *For every meager subset $C \subset [0,1]$, there is a map $\phi \in \mathcal{H}$, $\phi\colon [0,1] \to [0,1]$, such that $\phi(C)$ does not contain 3-term APs.*

A stronger version of Proposition 2 (Proposition 3) is presented and proved in the next section.

2. Proofs of Propositions 2 and 3

Denote by $\mathcal{C}$ the Banach space of continuous maps $f\colon [0,1] \to \mathbb{R}$ equipped with the norm

$$\|f\| = \|f\|_\infty = \max_{x \in [0,1]} |f(x)| . \tag{2.1}$$

Denote by $\mathcal{F}$ and $\mathcal{H}^+$ the following subsets of $\mathcal{C}$:

$$\mathcal{F} = \{f \in \mathcal{C} \mid f \text{ is non-decreasing with } f(0) = 0;\ f(1) = 1\}, \tag{2.2}$$

$$\mathcal{H}^+ = \{f \in \mathcal{F} \mid f \text{ is injective}\} = \{f \in \mathcal{H} \mid f \text{ is increasing on } [0,1]\}. \tag{2.3}$$

The set $\mathcal{F}$ is a closed subset of $\mathcal{C}$, while $\mathcal{H}^+$ is residual in $\mathcal{F}$. (Indeed,

$$\mathcal{H}^+ = \bigcap_{\substack{0<a<b<1 \\ a,b \in \mathbb{Q}}} F_{a,b}; \qquad F_{a,b} = \big\{ f \in \mathcal{F} \mid f(a) < f(b) \big\},$$

where $\mathbb{Q}$ stands for the set of rationals, and $F_{a,b}$ are open dense subsets of $\mathcal{F}$.)

The following proposition is a stronger version of Proposition 2.

Proposition 3. *Let $C \subset [0,1]$ be a meager subset. Then, for a residual subset of $\phi \in \mathcal{H}^+$, the image $\phi(C)$ does not contain 3-term APs.*

Since a meager set is a countable union of nowhere dense sets, it is enough to prove the above proposition under the weaker assumption that C is nowhere dense. Indeed, a meager set C has a representation in the form $C = \bigcup_{k=1}^{\infty} C_k$ where C_k are nowhere dense.

Then the unions $U_k = \bigcup_{i=1}^{k} C_i$ form a non-decreasing sequence of nowhere dense sets, and $\phi(C)$ may contain a 3-term AP only if some $\phi(U_k)$ does.

Let

$$\mathcal{H}_\varepsilon(C) = \{\phi \in \mathcal{H}^+ \mid \phi(C) \text{ has no 3-term APs of step } d \geq \varepsilon\}. \tag{2.4}$$

In the proof of Proposition 3 we need the following lemma. Its proof is provided at the end of the next section.

Lemma 1. *Let $C \subset [0,1]$ be a nowhere dense subset and $\varepsilon > 0$. Then $\mathcal{H}_\varepsilon(C)$ contains a dense open subset of $\mathcal{H}^+$. In particular, $\mathcal{H}_\varepsilon(C)$ is residual in $\mathcal{H}^+$.*

Proof of Proposition 3. We may assume that C is nowhere dense (see the sentence following Proposition 3). We may also assume that C is compact (otherwise replacing C by its closure $\bar{C}$).

By Lemma 1, each of the sets $\mathcal{H}_\varepsilon(C)$, $\varepsilon > 0$, is residual in $\mathcal{H}^+$. It follows that the set $\mathcal{H}_0(C) = \bigcap_{k=1}^\infty \mathcal{H}_{1/k}(C)$ is residual. It is also clear that, for $\phi \in \mathcal{H}_0(C)$, the images $\phi(C)$ do not contain 3-term APs.

This completes the proof of Proposition 3. □

3. Proof of Lemma 1

First we prepare some auxiliary results.

Lemma 2. *Let $C \subset [0,1]$ be a nowhere dense set, let $f \in \mathcal{H}^+$ and let $\varepsilon > 0$ be given. Then there exists $g \in \mathcal{H}^+$ such that $\|g - f\| < \varepsilon$ and the set $g(C)$ has no 3-term APs with step $d \geq \varepsilon$.*

Proof. Without loss of generality, we assume that $\varepsilon < 1/2$. Pick an integer $r \geq 3$ such that $r\varepsilon > 1$.

Since C is nowhere dense, so is $f(C)$, and one can select $r-1$ points $x_1, x_2, \ldots, x_{r-1} \in (0,1) \setminus f(\bar{C})$,

$$0 = x_0 < x_1 < x_2 < \ldots < x_{r-1} < x_r = 1,$$

partitioning the unit interval into r subintervals $X_k = (x_{k-1}, x_k)$, each shorter than ε:

$$0 < |X_k| = x_k - x_{k-1} < \varepsilon \quad (1 \leq k \leq r).$$

Then one selects non-empty open subintervals $Y_k = (y_k^-, y_k^+) \subset X_k$, $1 \leq k \leq r$, in such a way that the following four conditions are met:

$$\begin{aligned}
(3.1) \qquad & (c1)\ f(C) \subset \bigcup_{k=1}^r \bar{Y}_k, \\
& (c2)\ x_{k-1} < y_k^- < y_k^+ < x_k \text{ (i.e., } \bar{Y}_k \subset X_k), \quad \text{for } 2 \leq k \leq r-1, \\
& (c3)\ 0 = x_0 = y_1^- < y_1^+ < x_1, \quad \text{and} \\
& (c4)\ x_{r-1} < y_r^- < y_r^+ = x_r = 1.
\end{aligned}$$

Set $p_1 = 0$, $p_r = 1$ and then select the $r-2$ points $p_k \in Y_k$, $2 \leq k \leq r-1$, so that the set $P = \{p_k\}_{k=1}^r$ contain no 3-term APs. Then the sequence $(p_k)_1^r$ is strictly increasing, and

$$\delta = \min_{1 \leq m < n < k \leq r} |p_m + p_k - 2p_n| > 0.$$

Next, for $1 \leq k \leq r$, we select open subintervals $Z_k \subset Y_k$, each shorter than $\frac{\delta}{4}$, with $p_k \subset \bar{Z}_k$.

Define $u \in \mathcal{H}$ to be the homeomorphism $[0,1] \to [0,1]$ which affinely contracts $\bar{Y}_k$ to $\bar{Z}_k$ and affinely expands the gaps between the intervals $\bar{Y}_k$ to fill it in. Note that

$$(3.2) \qquad |u(x) - x| < \varepsilon, \qquad \text{for } x \in \bigcup_{k=1}^r \bar{Y}_k,$$

because $x \in \bar{Y}_k$ implies $u(x) \in \bar{Y}_k$ and hence $|u(x) - x| \leq |Y_k| < |X_k| < \varepsilon$.

Since $u(x) - x$ is linear on each of the $(r-1)$ gaps between the intervals $\bar{Y}_k$, the inequality (3.2) extends to the whole unit interval: $\|u(x) - x\| < \varepsilon$.

Define $g \in \mathcal{H}$ as the composition $g(x) = (u \circ f)x = u(f(x))$. Then

$$\|g - f\| = \|u \circ f - f\| = \|u(x) - x\| < \varepsilon.$$

It remains to show that $g(C)$ has no 3-term APs with step $d \geq \varepsilon$. In view of (3.1),

$$\bigcup_{k=1}^{r} \bar{Z}_k = u(\bigcup_{k=1}^{r} \bar{Y}_k) \supset u(f(C)) = g(C),$$

so it would suffice to prove that $\bigcup_{k=1}^{r} \bar{Z}_k$ has no 3-term APs with step $d \geq \varepsilon$.

Assume to the contrary that such an AP exists, say a_1, a_2, a_3, with $d = a_2 - a_1 = a_3 - a_2 \geq \varepsilon$. Let $a_i \in \bar{Z}_{k_i}$, for $i = 1, 2, 3$. These k_i are uniquely determined, and since $|Z_{k_i}| < |X_{k_i}| < \varepsilon \leq d$, we have $k_1 < k_2 < k_3$. Taking into account that $|a_i - p_{k_i}| \leq |Z_{k_i}| < \delta/4$, we obtain

$$\begin{aligned}|a_1 + a_3 - 2a_2| \geq |p_{k_1} + p_{k_3} - 2p_{k_2}| \\ - (|a_1 - p_{k_1}| + |a_3 - p_{k_3}| + 2|a_2 - p_{k_2}|) > \delta - 4 \cdot \tfrac{\delta}{4} = 0,\end{aligned}$$

a contradiction with the assumption that a_1, a_2, a_3 forms an AP. □

Corollary 1. *Let $C \subset [0, 1]$ be a nowhere dense set. Then for all $\varepsilon > 0$, the sets $\mathcal{H}_\varepsilon(C)$ (defined by (2.4)) are dense in $\mathcal{H}^+$.*

Proof. Note that the sets $\mathcal{H}_\varepsilon(C)$ are monotone in $\varepsilon > 0$: $\mathcal{H}_{\varepsilon_2}(C) \subset \mathcal{H}_{\varepsilon_1}(C)$ if $0 < \varepsilon_2 < \varepsilon_1$.

By the previous lemma (Lemma 2), all sets $\mathcal{H}_\varepsilon(C)$ are ε-dense. Then, for a given $\varepsilon > 0$, the set $\mathcal{H}_\varepsilon(C)$ is δ-dense for every positive $\delta < \varepsilon$ (because even the smaller set $\mathcal{H}_\delta(C) \subset \mathcal{H}_\varepsilon(C)$ is δ-dense). This argument completes the proof of Corollary 1. □

Lemma 3. *Let $C \subset [0, 1]$ be a compact nowhere dense set, let $g \in \mathcal{H}$ and let $\varepsilon > 0$ be given. Assume that the set $g(C)$ has no 3-term APs with step $d \geq \varepsilon$. Then there exists a $\delta > 0$ such that for all $h \in \mathcal{H}$ such that $\|h - g\| < \delta$ the sets $h(C)$ have no 3-term APs with step exceeding 2ε.*

Proof. Let

$$M = \{(x_1, x_2, x_3) \in g(C)^3 \mid x_2 - x_1 \geq \varepsilon \text{ and } x_3 - x_2 \geq \varepsilon\}.$$

Then M is compact, and $F \colon M \to \mathbb{R}$ defined by $F(x_1, x_2, x_3) = |x_1 + x_3 - 2x_2|$ assumes its minimum

$$\gamma = \min_{\boldsymbol{x} \in M} F(\boldsymbol{x}) > 0$$

which is positive because $g(C)$ has no 3-term APs with step $d \geq \varepsilon$. Take $\delta = \min(\varepsilon/2, \gamma/5)$.

Assume to the contrary that for some $h \in \mathcal{H}$ with $\|h - g\| < \delta$, the set $h(C)$ contains an AP with step $d' > 2\varepsilon$, i.e., that there are $c_1, c_2, c_3 \in C$ such that

$$h(c_3) - h(c_2) = h(c_2) - h(c_1) > 2\varepsilon.$$

Then, for both $i = 1, 2$, we have

$$g(c_{i+1}) - g(c_i) > h(c_{i+1}) - h(c_i) - 2\delta > 2\varepsilon - 2\delta \geq \varepsilon,$$

whence $(g(c_1), g(c_2), g(c_3)) \in M$ and hence

$$\gamma \leq F(g(c_1), g(c_2), g(c_3)) = |g(c_1) + g(c_3) - 2g(c_2)|$$
$$\leq |h(c_1) + h(c_3) - 2h(c_2)| + 4\delta = 0 + 4\delta \leq \tfrac{4\gamma}{5} < \gamma,$$

a contradiction. □

Proof of Lemma 1. It follows from Lemma 3 that there is an (intermediate) open subset $U \subset \mathcal{H}^+$ such that

$$\mathcal{H}_\varepsilon(C) \subset U \subset \mathcal{H}_{2\varepsilon}(C) \subset \mathcal{H}^+.$$

This set U is dense in $\mathcal{H}^+$ because its subset $\mathcal{H}_\varepsilon(C)$ is (by Corollary 1). Thus the set $\mathcal{H}_{2\varepsilon}(C)$ contains an open dense subset $U \subset \mathcal{H}^+$. Since $\varepsilon > 0$ is arbitrary, the proof is complete. □

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References

[1] Alexander S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995. MR1321597

[2] Tamás Keleti, *A 1-dimensional subset of the reals that intersects each of its translates in at most a single point*, Real Anal. Exchange **24** (1998/99), no. 2, 843–844. MR1704757

[3] Mihail N. Kolountzakis, *Infinite patterns that can be avoided by measure*, Bull. London Math. Soc. **29** (1997), no. 4, 415–424, DOI 10.1112/S0024609397003056. MR1446560

[4] Izabella Łaba and Malabika Pramanik, *Arithmetic progressions in sets of fractional dimension*, Geom. Funct. Anal. **19** (2009), no. 2, 429–456, DOI 10.1007/s00039-009-0003-9. MR2545245

[5] Curtis T. McMullen, *Winning sets, quasiconformal maps and Diophantine approximation*, Geom. Funct. Anal. **20** (2010), no. 3, 726–740, DOI 10.1007/s00039-010-0078-3. MR2720230

[6] John C. Oxtoby, *Measure and category*, 2nd ed., Graduate Texts in Mathematics, vol. 2, Springer-Verlag, New York-Berlin, 1980. A survey of the analogies between topological and measure spaces. MR584443

[7] Paul D. Humke and Miklós Laczkovich, *A visit to the Erdős problem*, Proc. Amer. Math. Soc. **126** (1998), no. 3, 819–822, DOI 10.1090/S0002-9939-98-04167-7. MR1425126

[8] Wolfgang M. Schmidt, *On badly approximable numbers and certain games*, Trans. Amer. Math. Soc. **123** (1966), 178–199. MR0195595

[9] S. M. Srivastava, *A course on Borel sets*, Graduate Texts in Mathematics, vol. 180, Springer-Verlag, New York, 1998. MR1619545

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