DICHOTOMY FOR ARITHMETIC PROGRESSIONS
IN SUBSETS OF REALS

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Abstract. Let $\mathcal{H}$ stand for the set of homeomorphisms $\phi: [0, 1] \to [0, 1]$. We prove the following dichotomy for Borel subsets $A \subset [0, 1]$: 

• either there exists a homeomorphism $\phi \in \mathcal{H}$ such that the image $\phi(A)$ contains no 3-term arithmetic progressions;
• or, for every $\phi \in \mathcal{H}$, the image $\phi(A)$ contains arithmetic progressions of arbitrary finite length.

In fact, we show that the first alternative holds if and only if the set $A$ is meager (a countable union of nowhere dense sets).

1. Definitions

Let $\mathbb{R}, \mathbb{Q}$ denote the sets of real and rational numbers, respectively. By an AP (arithmetic progression) we mean a finite strictly increasing sequence in $\mathbb{R}$ of the form $x = (x + kd)^{n-1}_{k=0}$, with $d > 0$ and $n \geq 3$. The convention is sometimes abused by identifying the sequence $x$ with the set of its elements. An AP is completely determined by its first term $x = \min x$, its length $n = |x|$ and its step (difference) $d > 0$.

Denote by $\mathcal{H}$ the set of homeomorphisms $\phi: [0, 1] \to [0, 1]$ of the unit interval. The result presented in the abstract can be restated as follows.

Theorem 1. Let $S \subset [0, 1]$ be a Borel subset. Then exactly one of the following two assertions holds:

1. (either) there exists $\phi \in \mathcal{H}$ such that $\phi(S)$ does not contain 3-term APs;
2. (or) for every $\phi \in \mathcal{H}$, $\phi(S)$ contains APs of arbitrarily large finite length.

Moreover, (1) holds if and only if $S$ is meager.

Recall some basic relevant definitions. Let $S \subset \mathbb{R}$. A set $S$ is called nowhere dense if its closure $\bar{S} \subset \mathbb{R}$ has empty interior. $S$ is called meager (or a set of first category), if it is a countable union of nowhere dense sets. $S$ is called residual, or co-meager, if $\mathbb{R} \setminus S$ is meager; $S$ is called residual in a subinterval $X \subset \mathbb{R}$ if the complement $X \setminus S$ is meager. Finally, $S$ is called a set of second category if it is not meager.

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The following proposition lists some “largeness” properties of a set $A \subset \mathbb{R}$ which force it to contains APs of arbitrary large finite length. Denote by $\lambda$ the Lebesgue measure on $\mathbb{R}$.

**Proposition 1.** The sets in each of the following five classes $\mathcal{E}_i$, $1 \leq i \leq 5$, contain arbitrarily long finite arithmetic progressions:

- $\mathcal{E}_1 = \{ S \subset \mathbb{R} \mid S$ is Lebesgue measurable with $0 < \lambda(S) \leq \infty \}$,
- $\mathcal{E}_2 = \{ S \subset \mathbb{R} \mid S$ is residual in some interval $U \subset \mathbb{R}$ of positive length $\}$,
- $\mathcal{E}_3 = \{ S \subset \mathbb{R} \mid S$ is winning in Schmidt’s game $\}$,
  
  (several versions of Schmidt’s games are possible; see [5, 8]),
- $\mathcal{E}_4 = \{ S \subset \mathbb{R} \mid S$ is Borel and not meager $\}$,
- $\mathcal{E}_5 = \{ S \subset \mathbb{R} \mid S$ has Baire property and is not meager $\}$.

Recall that a set $S \subset \mathbb{R}$ has the Baire property if it can be represented as the symmetric difference $S = F \triangle P = (F \setminus P) \cup (P \setminus F)$ where $F \subset \mathbb{R}$ is open and $P \subset \mathbb{R}$ is meager.

The family $\mathcal{BP}$ of subsets of $\mathbb{R}$ which have Baire property forms a $\sigma$-algebra containing the $\sigma$-algebra $\mathcal{B}$ of Borel subsets of $\mathbb{R}$. (We refer to [6, §4, page 19] for the short review of relevant standard material.)

**Proof of Proposition 1.** For a set $S \in \mathcal{E}_1$, one easily produces APs in $S$ near any of its Lebesgue density points. The argument for the classes $\mathcal{E}_2$ and $\mathcal{E}_3$ is even easier because the class $\mathcal{E}_3$ and the class of residual subsets of a fixed subinterval are closed under finite (and even countable) intersections.

Since $\mathcal{E}_4 \subset \mathcal{E}_5$ (because $\mathcal{B} \subset \mathcal{BP}$), the proof of Proposition 1 is completed by showing that $\mathcal{E}_5 \subset \mathcal{E}_2$. Given $S = F \triangle P \in \mathcal{E}_5$, the open set $F$ cannot be empty (otherwise $S = P$ would be meager, contradicting $S \in \mathcal{E}_5$). Let $U \neq \emptyset$ be a subinterval of $F$; then $S$ is residual in $U$, so that $S \in \mathcal{E}_2$. \[\square\]

For more references on Borel sets and Baire property we refer to [1] and [9] (in particular, see Proposition 3.5.6 and Corollary 3.5.2 in [9, page 108]).

Note that the problems of finding finite or countable configurations $F$ in sets $S \subset \mathbb{R}$, under various “largeness” metric assumptions on $S$, have been considered by several mathematicians.

Following Kolountzakis [8], a set $F$ is called universal for a class $\mathcal{E}$ of subsets of reals if $F \ll S$ for all $S \in \mathcal{E}$. Henceforth $F \ll S$ means that $S$ contains an affine image of $F$, i.e., that $aF + b \subset S$, for some $a, b \in \mathbb{R}$, $a > 0$.

Every bounded countable subset is universal for the classes $\mathcal{E}_k$, $1 \leq k \leq 5$. Every finite subset of reals is universal for all the classes $\mathcal{E}_k$, $1 \leq k \leq 5$.

An old question of Erdős is whether there is a universal infinite set $F \subset \mathbb{R}$ for the class $\mathcal{E}_1$ (of sets of positive measure). The question is still open even though some families of countable sets $F$ are shown not to contain universal sets; see Kolountzakis [8], Humke and Laczkovich [7] and the references there. In [7] an elegant combinatorial characterization of universal sets $F$ (for the class $\mathcal{E}_1$) is given which reproduces earlier results in the subject.

Keleti [2] constructed a compact set $A \subset [0,1]$ of Hausdorff dimension 1 which does not contain 3-term APs; on the other hand, Laba and Pramanik in [4] showed that under certain assumptions (on the Fourier transform of supported measure)
compact sets of fractional dimension close to 1 must contain 3-term APs. We refer to [4] for a survey of related questions.

The central result of the paper, Theorem 1, completely characterizes the topological (rather than metric) properties of a Borel set $S \subset \mathbb{R}$ which guarantee that it contains arbitrarily long APs. This theorem is an immediate consequence of the following proposition and the fact that the sets $S \in \mathcal{E}_4$ contain arbitrarily long APs (Proposition 1).

**Proposition 2.** For every meager subset $C \subset [0,1]$, there is a map $\phi \in \mathcal{H}$, $\phi: [0,1] \to [0,1]$, such that $\phi(C)$ does not contain 3-term APs.

A stronger version of Proposition 2 (Proposition 3) is presented and proved in the next section.

2. Proofs of Propositions 2 and 3

Denote by $C$ the Banach space of continuous maps $f: [0,1] \to \mathbb{R}$ equipped with the norm

$$\|f\| = \|f\|_\infty = \max_{x \in [0,1]} |f(x)|.$$  

Denote by $\mathcal{F}$ and $\mathcal{H}^+$ the following subsets of $C$:

$$\mathcal{F} = \{ f \in C \mid f \text{ is non-decreasing with } f(0) = 0; f(1) = 1 \},$$  

$$\mathcal{H}^+ = \{ f \in \mathcal{F} \mid f \text{ is injective} \} = \{ f \in \mathcal{H} \mid f \text{ is increasing on } [0,1] \}.$$  

The set $\mathcal{F}$ is a closed subset of $C$, while $\mathcal{H}^+$ is residual in $\mathcal{F}$. (Indeed,

$$\mathcal{H}^+ = \bigcap_{0 < a < b < 1 \text{ or } a,b \in \mathbb{Q}} F_{a,b}; \quad F_{a,b} = \{ f \in \mathcal{F} \mid f(a) < f(b) \},$$

where $\mathbb{Q}$ stands for the set of rationals, and $F_{a,b}$ are open dense subsets of $\mathcal{F}$.)

The following proposition is a stronger version of Proposition 2.

**Proposition 3.** Let $C \subset [0,1]$ be a meager subset. Then, for a residual subset of $\phi \in \mathcal{H}^+$, the image $\phi(C)$ does not contain 3-term APs.

Since a meager set is a countable union of nowhere dense sets, it is enough to prove the above proposition under the weaker assumption that $C$ is nowhere dense. Indeed, a meager set $C$ has a representation in the form $C = \bigcup_{k=1}^{\infty} C_k$ where $C_k$ are nowhere dense.

Then the unions $U_k = \bigcup_{i=1}^{k} C_i$ form a non-decreasing sequence of nowhere dense sets, and $\phi(C)$ may contain a 3-term AP only if some $\phi(U_k)$ does.

Let

$$\mathcal{H}_\varepsilon(C) = \{ \phi \in \mathcal{H}^+ \mid \phi(C) \text{ has no 3-term APs of step } d \geq \varepsilon \}.$$  

In the proof of Proposition 3 we need the following lemma. Its proof is provided at the end of the next section.

**Lemma 1.** Let $C \subset [0,1]$ be a nowhere dense subset and $\varepsilon > 0$. Then $\mathcal{H}_\varepsilon(C)$ contains a dense open subset of $\mathcal{H}^+$. In particular, $\mathcal{H}_\varepsilon(C)$ is residual in $\mathcal{H}^+$. 


Proof of Proposition 3. We may assume that \( C \) is nowhere dense (see the sentence following Proposition 3). We may also assume that \( C \) is compact (otherwise replacing \( C \) by its closure \( \overline{C} \)).

By Lemma 1, each of the sets \( \mathcal{H}_\varepsilon(C) \), \( \varepsilon > 0 \), is residual in \( \mathcal{H}^+ \). It follows that the set \( \mathcal{H}_0(C) = \bigcap_{k=1}^{\infty} \mathcal{H}_{1/k}(C) \) is residual. It is also clear that, for \( \phi \in \mathcal{H}_0(C) \), the images \( \phi(C) \) do not contain 3-term APs.

This completes the proof of Proposition 3. \( \square \)

3. Proof of Lemma 1

First we prepare some auxiliary results.

Lemma 2. Let \( C \subset [0,1] \) be a nowhere dense set, let \( f \in \mathcal{H}^+ \) and let \( \varepsilon > 0 \) be given. Then there exists \( g \in \mathcal{H}^+ \) such that \( \|g - f\| < \varepsilon \) and the set \( g(C) \) has no 3-term APs with step \( d \geq \varepsilon \).

Proof. Without loss of generality, we assume that \( \varepsilon < 1/2 \). Pick an integer \( r \geq 3 \) such that \( \varepsilon r > 1 \).

Since \( C \) is nowhere dense, so is \( f(C) \), and one can select \( r - 1 \) points \( x_1, x_2, \ldots, x_{r-1} \in (0,1) \setminus f(\overline{C}) \),

\[
0 = x_0 < x_1 < x_2 < \ldots < x_{r-1} < x_r = 1,
\]

partitioning the unit interval into \( r \) subintervals \( X_k = (x_{k-1}, x_k) \), each shorter than \( \varepsilon \):

\[
0 < |X_k| = x_k - x_{k-1} < \varepsilon \quad (1 \leq k \leq r).
\]

Then one selects non-empty open subintervals \( Y_k = (y_k^-, y_k^+) \subset X_k, 1 \leq k \leq r \), in such a way that the following four conditions are met:

\[
\begin{align*}
(3.1) \quad & f(C) \subset \bigcup_{k=1}^{r} \bar{Y}_k, \\
(3.2) \quad & x_{k-1} < y_k^- < y_k^+ < x_k \text{ (i.e., } \bar{Y}_k \subset X_k) \text{, for } 2 \leq k \leq r - 1, \\
(3.3) \quad & 0 = x_0 = y_1^- < y_1^+ < x_1, \text{ and} \\
(3.4) \quad & x_{r-1} < y_r^- < y_r^+ = x_r = 1.
\end{align*}
\]

Set \( p_1 = 0, p_r = 1 \) and then select the \( r - 2 \) points \( p_k \in Y_k, 2 \leq k \leq r - 1 \), so that the set \( P = \{p_k\}_{k=1}^{r} \) contains no 3-term APs. Then the sequence \( \{p_k\}_1^r \) is strictly increasing, and

\[
\delta = \min_{1 \leq m < n < k \leq r} |p_m + p_k - 2p_n| > 0.
\]

Next, for \( 1 \leq k \leq r \), we select open subintervals \( Z_k \subset Y_k \), each shorter than \( \frac{\delta}{4} \), with \( p_k \subset \bar{Z}_k \).

Define \( u \in \mathcal{H} \) to be the homeomorphism \([0,1] \to [0,1]\) which affinely contracts \( \bar{Y}_k \) to \( \bar{Z}_k \) and affinely expands the gaps between the intervals \( \bar{Y}_k \) to fill it in. Note that

\[
(3.2) \quad |u(x) - x| < \varepsilon, \quad \text{for } x \in \bigcup_{k=1}^{r} \bar{Y}_k,
\]

because \( x \in \bar{Y}_k \) implies \( u(x) \in \bar{Y}_k \) and hence \( |u(x) - x| \leq |Y_k| < |X_k| < \varepsilon \).

Since \( u(x) - x \) is linear on each of the \( (r-1) \) gaps between the intervals \( \bar{Y}_k \), the inequality \((3.2)\) extends to the whole unit interval: \( \|u(x) - x\| < \varepsilon \).
Let $\delta > 0$ exist a $M$.

**Corollary 1.** Let $g$ which is positive because $|\epsilon| > |\alpha|$. Note that the sets $H$ assumes its minimum $\gamma > 0$. Assume to the contrary that such an AP exists, say $A = \{a_1, a_2, a_3\}$. Then, for both $\epsilon > 0$, the set $H$ contains an AP with step $d = 2\epsilon$. It remains to show that $\gamma > 0$. Define $g = 2\epsilon$. Then there exists a $\gamma > 0$ such that for all $h \in H$ such that $\|h - g\| < \delta$ the sets $h(C)$ have no 3-term APs with step exceeding $2\delta$.

**Proof.** Let $M = \{(x_1, x_2, x_3) \in g(C)^3 \mid x_2 - x_1 \geq \epsilon$ and $x_3 - x_2 \geq \epsilon\}$. Then $M$ is compact, and $F: M \to \mathbb{R}$ defined by $F(x_1, x_2, x_3) = |x_1 + x_3 - 2x_2|$ assumes its minimum $\gamma = \min_{x \in M} F(x) > 0$ which is positive because $g(C)$ has no 3-term APs with step $d \geq \epsilon$. Take $\delta = \min(\epsilon/2, \gamma/5)$.

Assume to the contrary that for some $h \in H$ with $\|h - g\| < \delta$, the set $h(C)$ contains an AP with step $d' > 2\epsilon$, i.e., there are $c_1, c_2, c_3 \in C$ such that $h(c_3) - h(c_2) = h(c_2) - h(c_1) > 2\epsilon$.

Then, for both $i = 1, 2$, we have $g(c_{i+1}) - g(c_i) > h(c_{i+1}) - h(c_i) - 2\delta > 2\epsilon - 2\delta \geq \epsilon$, as claimed.

Define $g \in H$ as the composition $g(x) = (u \circ f)x = u(f(x))$. Then

$$\|g - f\| = \|u \circ f - f\| = \|u(x) - x\| < \epsilon.$$
whence \((g(c_1), g(c_2), g(c_3)) \in M\) and hence
\[
\gamma \leq F(g(c_1), g(c_2), g(c_3)) = |g(c_1) + g(c_3) - 2g(c_2)| \\
\leq |h(c_1) + h(c_3) - 2h(c_2)| + 4\delta = 0 + 4\delta \leq \frac{4\delta}{5} < \gamma,
\]
a contradiction. \(\square\)

**Proof of Lemma 1.** It follows from Lemma 3 that there is an (intermediate) open subset \(U \subset H^+\) such that
\[
H_\varepsilon(C) \subset U \subset H_{2\varepsilon}(C) \subset H^+.
\]
This set \(U\) is dense in \(H^+\) because its subset \(H_\varepsilon(C)\) is (by Corollary 1). Thus the set \(H_{2\varepsilon}(C)\) contains an open dense subset \(U \subset H^+\). Since \(\varepsilon > 0\) is arbitrary, the proof is complete. \(\square\)

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