Short communication

On the time for Brownian motion to visit every point on a circle

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A B S T R A C T

Consider a Wiener process $W$ on a circle of circumference $L$. We prove the rather surprising result that the Laplace transform of the distribution of the first time, $\theta_L$, when the Wiener process has visited every point of the circle can be solved in closed form using a continuous recurrence approach.

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1. Introduction

Consider a Wiener process on a circle of circumference $L$. The distribution of the first time, $\theta_L$, when the Wiener process has visited every point of the circle is equivalent, via the natural bijection between and interval of the form $(b, b+L)$ on the real line and a circle of circumference $L$, to the distribution of the first time when the range of the Wiener process on the real line is of length $L$. This distribution is well-known and it has the following Laplace transform: [see, for example, (Borodin and Salminen, 2002), p. 242]

$$E \left[ e^{-s \theta_L} \right] = \frac{1}{\cosh \left( \frac{L}{2} \sqrt{\frac{s}{2}} \right)}, \quad s \geq 0. \quad (1)$$

Feller (1951), in writing about the range of a Wiener process, did so using explicit probability density calculations. Imhof (1986) discovered Laplace transform for the first time, $\theta_L$, when the Wiener process has visited every point of the circle, again via explicit probability density calculations. Further computations employing the Laplace transform for $\theta_L$ were presented in Vallois (1993). However, in departure from these previous works, we prove the result in Eq. (1) using a continuous recurrence setup. We do so by calculating the left hand side in terms of random variables representing how long it takes to cover a range of length $L$, given that one is already at an endpoint of a range of length $a$ (which counts as being covered already). This is the idea behind the definition of $\theta_{a,L}$, which is defined in Section 2.

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Key to our recurrence will be the concept of a switchback. Imagine we pick some \( a \in \mathbb{R}^+ \) that is less than \( L \). Consider the maximum, \( M_a \), of \( W \) until the first visit to the point, \( -a \), on the negative half-axis. (Here, \( M_a > 0 \); otherwise, the process would have moved directly from 0 to \( -a \), which occurs w.p. 0.) We call the time of this first visit \( \tau_{-a} \). We say that a “switchback” occurs when \( W \) hits \( -a \) before the length of the range, \( a + M_a \), is \( L \). Formally, let \( I_{a,L} \) be the indicator random variable for the event of a switchback, defined as follows:

\[
I_{a,L} = \begin{cases} 1 & \text{if } \inf\{ t : 0 \leq t < \infty | W_t = -a \} \leq \inf\{ t : 0 \leq t < \infty | W_t = L - a \} \\ 0 & \text{otherwise.} \end{cases}
\]

After a switchback, the process continues from \( -a \) with a starting range of \( M_a + a \) (i.e., the interval \( [-a, M_a] \) has been covered). By translation and reflection invariance, as well as the symmetry of Brownian motion, we may just as well assume that we are at the point \( 0 \) and have covered the interval \( [-a + M_a, 0] \). Then we repeat the process and say that a second switchback occurs if we reach \( -(a + M_a) \) before covering a range of length \( L \). To summarize:

Step 1: We start our process at the right hand end of \( [-a, 0] \) and consider this interval as already being covered. \( M_a \) is the maximal positive value attained before the first hit \( -a \). The total range is \( a + M_a \). If \( M_a \geq L - a \), then we have covered an interval of length \( L \) before reaching \( -a \), and no switchback occurs. If not, a switchback occurs and we continue to Step 2.

Step 2: We have covered a range of length \( a + M_a \). Without loss of generality, we consider the interval \( [-a + M_a, 0] \) to have been covered. Let \( -(a + M_a) := -a' \), and start the process on the right hand end of \( [-a', 0] \). If \( M_{a'} \geq L - a' \), no switchback occurs. Otherwise, another switchback occurs and we continue to Step 3.

Step 3: We have covered a range of length \( a' + M_{a'} \). Without loss of generality, we consider the interval \( [-a' + M_{a'}, 0] \) to have been covered. Let \( -(a' + M_{a'}) \) be called \( -a'' \), and start the process on the right hand end of \( [-a'', 0] \). If \( M_{a''} \geq L - a'' \), a switchback occurs. Otherwise, continue Step 3 recursively until a range of length \( L \) has been covered.

Steps 1–3 are illustrated in Fig. 1.

In Section 3 we prove that the recurrence can be solved in closed form. In Section 4 we prove that the number \( v = v_{a,L} \) of switchbacks before covering an interval of length \( L \) has a Poisson distribution with parameter \( \lambda = \log \frac{L}{a} \). Thus, as \( a \downarrow 0 \), the number of switchbacks goes to infinity at a logarithmic rate.

2. Solving the recurrence

We proceed to solve for the recurrence. First, consider a Wiener process \( W(t), \ t \geq 0 \). For each fixed, \( a > 0 \), let \( M_a \) denote the maximum positive value of \( W(t) \) before the first hitting time of \( -a \). Assuming that \( L - a \) is positive, we have

\[
P(M_a \leq y) = P(\tau_{-a} < \tau_y) = \frac{y}{a+y},
\]

by the logic of the gambler’s ruin.

Let \( I(t) \) be the range of the Wiener process up to time \( t \). Define \( \theta_{a,L} \) to be the random variable representing the time until \( I(t) \cup [-a, 0] \) has length \( L \). We proceed by defining

\[
f(s, a, L) := \mathbb{E}\left[ \exp(-s\theta_{a,L}) \right],
\]

where \( f(s, a, L) \) is considered a function of \( a \) with \( s \) and \( L \) being held constant. By abuse of notation, we label \( f(s, a, L) \) as \( f(a) \).

Let us define the following functions

\[
F(s, y) = \mathbb{E}\left[ \exp(-s\tau_{-a}) \mathbf{1}_{\tau_{-a} < \tau_y} \right] \quad \text{and} \quad G(s, y) = \mathbb{E}\left[ \exp(-s\tau_y) \mathbf{1}_{\tau_y < \tau_{-a}} \right].
\]

We now employ the well-known fact (see Borodin and Salminen, 2002, amongst other sources), that for any \( c \),

\[
\exp\left( cW(t) - \frac{c^2}{2} t \right) \quad t \geq 0
\]

is a martingale. If \( s = \frac{c^2}{2} \), we easily obtain the following standard and well known forms of \( F(s, y) \) and \( G(s, y) \) (see Borodin and Salminen, 2002, amongst other sources),

\[
F(s, y) = \frac{\sinh cy}{\sinh (c(a+y))} \quad \text{and} \quad G(s, y) = \frac{\sinh ca}{\sinh (c(a+y))}.
\]

Continuing from above, our goal is to write a recurrence for \( f(a) \) in terms of \( f(a+y) \) for \( 0 < y \leq L - a \). To do so, we define \( f(a) \) using indicator functions. With the process starting at 0, let the first indicator function represent the case of a switchback, in which \( -a \) is hit before the length of the range is \( L \). Let the second indicator function denote the case of no switchback. We may then write

\[
f(a) = \mathbb{E}\left[ \exp(-s\theta_{a,L}) \mathbf{1}_{\tau_{-a} < \tau_y} \right] + \mathbb{E}\left[ \exp(-s\theta_{a,L}) \mathbf{1}_{\tau_y < \tau_{-a}} \right].
\]
Letting $y = L - a$, and using the expression for $G(s, y)$ in equation (5), we have:

\[
G(s, L - a) = \frac{\sinh ca}{\sinh cL},
\]

which is exactly the "no switchback" term. To calculate the switchback term, we integrate over all possible values of $M_a$, from 0 to $L - a$, using $y$ as a dummy variable. $f(a)$ becomes:

\[
f(a) = \frac{\sinh ca}{\sinh cL} + \int_0^{L-a} f(a + y) \frac{d}{dy} F(s, y) \, dy, \quad 0 < a \leq L.
\]  

We note that it is possible for $a$ to be $L$ since the original definition of $f(s, a, L)$ gives $f(L) = 1$.

This recurrence structure is an integral equation. The key idea is that Eq. (8) shows that the expected time it takes to get from point $a$ to point $b$, conditional on starting at a left most point of an interval, can be found by integrating over all possible left most points of the subsequent path.

**Remark 2.1.** This continuous recurrence approach presents an enormously valuable alternative to a direct density calculation. This approach should be helpful in many applied statistical settings in which such a calculation is intractable!

### 3. A closed form solution for the recurrence

**Theorem 3.1.** The recurrence structure in Eq. (8) can be solved in closed form. Letting $a \downarrow 0$, we obtain $f(0) = \frac{1}{\cosh^2(\frac{1}{2})}$.

**Proof.** Using the expression for $F(s, y)$ in equation (5) we have

\[
f(a) = \frac{\sinh ca}{\sinh cL} + \int_0^{L-a} f(a + y) \frac{d}{dy} \left[ \frac{\sinh cy}{\sinh c(a + y)} \right] dy, \quad 0 < a \leq L.
\]  

Differentiating, we obtain:

\[
f(a) = \frac{\sinh ca}{\sinh cL} + \int_0^{L-a} \frac{c \sinh ca}{\sinh^2 c(a + y)} f(a + y) dy, \quad 0 < a \leq L.
\]
Finally, substituting \( x = a + y \) gives the integral equation
\[
f(a) = \frac{\sinh ca}{\sinh cl} + \int_a^L \frac{c \sinh ca}{\sinh^2 cx} f(x) \, dx, \quad 0 < a \leq L.
\]
Fortunately, this is easy to solve: we divide by \( \sinh ca \) and let
\[
g(x) = \frac{f(x)}{\sinh cx}
\]
to arrive at:
\[
g(a) = \frac{1}{\sinh cl} + \int_a^L \frac{c}{\sinh cx} g(x) \, dx, \quad 0 < a < L. \tag{10}
\]
Differentiating with respect to \( a \), we obtain the following differential equation for \( g \)
\[
g'(a) = -\frac{c}{\sinh ca} g(a). \tag{11}
\]
Noting that \( f(L) = 1 \),
\[
g(L) = \frac{1}{\sinh cl}. \tag{12}
\]
We now have that
\[
g(a) = \frac{1}{\sinh cl} \exp \left( \int_a^L \frac{c}{\sinh cu} \, du \right),
\]
which is a unique solution to Eq. (11) with (12) as its initial condition.
Further,
\[
f(a) = \frac{\sinh ca}{\sinh cl} \exp \left( \int_a^L \frac{c}{\sinh cu} \, cu \, du \right).
\]
We now let \( a \downarrow 0 \). The limit is
\[
f(0) = \mathbb{E} [\exp (-s\theta_1)] = \lim_{a \to 0} \exp \left( \int_a^L \left( \frac{c}{\sinh cu} - \frac{c \cosh cu}{\sinh cu} \right) \, du \right). \tag{13}
\]
Combining the fractions, integrating, and letting \( c = \sqrt{2} s \), we obtain
\[
f(0) = \mathbb{E} [\exp (-s\theta_1)] = \frac{2}{1 + \cosh cl} = \frac{1}{\cosh^2 \left( L \sqrt{\frac{c}{2}} \right)}. \tag{14}
\]
Since by Oberhettinger and Badii (1973)
\[
\int_0^\infty \exp (-st) \exp \left( \frac{-a^2}{2t} \right) \frac{a}{\sqrt{2\pi t}} \, t^{-\frac{3}{2}} \, dt = \exp \left( -a\sqrt{2s} \right), \tag{15}
\]
we can expand \( \left( \cosh^2 \sqrt{\frac{c}{2}} \right)^{-1} \) in powers of \( e^{-\sqrt{2s}} \) to obtain an infinite series representation of the density of \( \theta_1 \). Namely, for \( L = 1 \), we can write:
\[
\int_0^\infty e^{-st} p_{\theta_1}(t) \, dt = 4 \exp \left( -\sqrt{2s} \right) \left( 1 + \exp \left( -\sqrt{2s} \right) \right)^{-2}
\]
\[
= \sum_{n=0}^\infty 4(-1)^n(n+1)\exp \left( -(n+1)\sqrt{2s} \right). \tag{16}
\]
Since the Laplace transform is invertible, it suffices to find a formula for \( p_{\theta_1}(t) \) that makes the above equation true. If we take
\[
p_{\theta_1}(t) = \sum_{n=0}^\infty \frac{4(-1)^n(n+1)^2}{\sqrt{2\pi t^\frac{3}{2}}} \exp \left( -\frac{(n+1)^2}{2t} \right)
\]
and plug this into the left hand side of (16), then using (15) gives us the equality (16). Since the amount of time to cover a range of length $L$ is equal to $L^2$ multiplied by the amount of time to cover a range of length 1, we write: $\theta_L \sim L^2 \theta$. Thus, the density of $\theta_L$ is

$$p_{\theta_L}(t) = \frac{p_{\theta}(\frac{t}{L^2})}{L^2}.$$  \hspace{1cm} (17)

4. The number of switchbacks

The result that the number of switchbacks is distributed as a Poisson random variable comes naturally when one accounts for the Markov and scaling properties of Brownian motion. The formal proof of the result follows.

**Theorem 4.1.** The number of switchbacks has a Poisson distribution with parameter $\lambda = \log \left( \frac{L}{a} \right)$.

**Proof.** The argument that the number of switchbacks has a Poisson distribution with parameter $\lambda = \log \left( \frac{L}{a} \right)$ is similar to the argument in Section 3 used to obtain the distribution of $\theta_L$. We begin by defining

$$f(a) = f(a, L, t) = \mathbb{E}[t^{\nu_{L,a}}],$$  \hspace{1cm} (18)

where $t$ is a dummy variable and $\nu_{L,a}$ is the number of switchbacks starting from an endpoint of an interval of length $a > 0$ before the interval grows to length $L$. $\nu_{L,a} = 0$ with probability $\frac{a}{L}$ and

$$\mathbb{P}(M_a \leq y) = \frac{y}{a+y}.$$ 

We rewrite $f(a)$ as we did in Section 3, splitting it up by indicator random variables which account for whether or not a switchback has occurred. If no switchbacks have occurred, $\nu_{L,a} = 0$, and thus:

$$\mathbb{E}[t^{\nu_{L,a}} 1_{M_a > L-a}] = \mathbb{E}[1_{M_a > L-a}] = \frac{a}{L}.$$  \hspace{1cm} (19)

Employing Eq. (19) and recalling that a switchback occurs when $M_a < L - a$, we write:

$$f(a) = \frac{a}{L} + \int_0^{L-a} t^{\nu_L+y} \mathbb{P}(M_a \in dy) \, dy.$$ 

which simplifies to

$$f(a) = \frac{a}{L} + \int_0^{L-a} tf(a+y) \frac{d}{dy} \left[ \frac{y}{a+y} \right] \, dy.$$ 

Thus,

$$f(a) = \frac{a}{L} + t \int_0^{L-a} f(a+y) \frac{a}{(a+y)^2} \, dy = \frac{a}{L} + at \int_0^L f(x) \, dx.$$  \hspace{1cm} (20)

It is straightforward to verify directly, or to use the differential equation argument used above, to find the Laplace transform of $\theta_L$, so that

$$f(a) = \exp \left( \ln \left( \frac{L}{a} \right) (t - 1) \right).$$

Since the above is the form of the Laplace transform of the Poisson distribution, we have the desired result that the number of switchbacks is distributed Poisson with mean $\lambda = \ln \left( \frac{L}{a} \right)$. \hspace{1cm} $\Box$

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