RICE UNIVERSITY

TWO APPLICATIONS OF RUNGE'S TECHNIQUES ON APPROXIMATION

by

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SECTION I

Given $\mu(r)$ for $r \geq 0$ with $0 < \mu(r) \uparrow \infty$, G. R. MacLane [1, Thm. 3, Coroll.] has constructed a function $\phi(z)$, meromorphic in $|z| < \infty$, with the properties

(1) <u>the image under</u> $w = \phi(z)$ <u>of an unbounded curve</u>: $z = \gamma(t)$ ($0 \leq t < \infty$, γ <u>continuous</u>, $\limsup_{t \to \infty} |\gamma(t)| = \infty$) <u>is dense on</u> $|w| < \infty$, <u>and</u>

(2) <u>the Nevanlinna characteristic of</u> ϕ <u>satisfies</u>

$$T(r) \leq \mu(r) \overset{+}{\log} r.$$

The function ϕ was constructed geometrically by specifying the Riemann surface of its inverse as a covering of the sphere. For a given sequence $\{a_n\}_1^{\infty}$ of points on $|w| < \infty$ such that

(3) $$\{a_n\}_1^{\infty} \text{ is dense on } |w| < \infty,$$

the construction defines an expanding sequence $\{C_n\}_1^{\infty}$ of analytic curves with the property that

(4) $$|\phi(z) - a_n| < \frac{1}{n} \quad \text{for } z \text{ on } C_n \quad (n \geq 1).$$

It is clear that (1) follows from (3) and (4).

It should be noted that (2) is as strong as possible. For a rational function clearly cannot satisfy (1), and the characteristic of a function which is meromorphic in $|z| < \infty$ and non-rational satisfies [3,p.218]

$$\lim_{r \to \infty} \frac{T(r)}{\log r} = \infty$$

The object of this section is to construct, by giving an explicit formula, a function possessing the essential properties of the function ϕ. More specifically, let $\{a_n\}_1^\infty$ be a sequence satisfying (3) and $a_n \neq 0$ $(n \geq 1)$, and choose $\{\varepsilon_n\}_1^\infty$ with $\varepsilon_1 < 1$ and $\varepsilon_n \downarrow 0$. We will define a double sequence $\{(\rho_n, r_n)\}_1^\infty$ satisfying

(5) $$0 < \rho_1 < r_1 < \rho_2 < r_2 < \ldots \uparrow \infty$$

and such that if we let

(6) $$\phi(z) = \sum_1^\infty a_k \left(\frac{\rho_{k+1}}{\rho_{k+1} - z} - \frac{\rho_k}{\rho_k - z} \right) ,$$

then $\phi(z)$ is meromorphic in $|z| < \infty$ and

(7) $$|\phi(z) - a_n| < \varepsilon_n \quad \text{for } |z| = r_n \quad (n \geq 1) .$$

Then since $\phi(z)$ has poles only on the $\{|z| = \rho_n\}$ and has only one pole on each $\{|z| = \rho_n\}$, (2) will be satisfied if $\rho_n \uparrow \infty$ sufficiently fast.

The idea of the proof of (7) is as follows. For a fixed n and $|z|$ sufficiently large

$$\frac{\rho_n}{\rho_n - z} \approx 1$$

where we are using "$\approx$" to mean "approximately equal to"; and for a fixed $|z|$ and n sufficiently large

$$\frac{\rho_n}{\rho_n - z} \approx 1$$

Thus if $\left\{(\rho_n, r_n)\right\}_1^\infty$ is properly chosen,

$$\frac{\rho_{n+1}}{\rho_{n+1} - z} - \frac{\rho_n}{\rho_n - z} \approx \begin{cases} 1 & \text{for } |z| = r_n \\ 0 & \text{for } \frac{|z|}{\rho_n} \text{ small or } \frac{|z|}{\rho_{n+1}} \text{ large.} \end{cases}$$

Thus in particular

$$\frac{\rho_{n+1}}{\rho_{n+1} - z} - \frac{\rho_n}{\rho_n - z} \approx \begin{cases} 1 & \text{for } |z| = r_n \\ 0 & \text{for } |z| = r_k, \quad k \neq n \end{cases}$$

Now let $g(r) = \frac{1}{\pi} \int_0^{2\pi} \overset{+}{\log} \left| \frac{1}{1 - re^{i\theta}} \right| d\theta$. We would like to let

$$\text{(8)} \qquad B = \max_{0 \leq r \leq 1} g(r)$$

and to establish that

$$\text{(9)} \qquad g(r) \leq g(1) \text{ for } r \geq 1.$$

It is elementary that $\left| \frac{1}{1 - re^{i\theta}} \right|$ is a decreasing function of r for $r \geq 1$. Thus to establish (8) and (9), it is sufficient to prove the continuity of $g(r)$ at $r = 1$. To this end we note that

$$|1 - re^{i\theta}|^2 = 1 - 2r \cos \theta + r^2 = (1 - r \cos \theta)^2 + r^2 \sin^2 \theta$$

Thus

$$|1 - re^{i\theta}| \leq r|\sin \theta| \leq \tfrac{1}{2}|\sin \theta| \text{ for } r \geq \tfrac{1}{2}$$

and

$$\overset{+}{\log}\left|\frac{1}{1-re^{i\theta}}\right| \leq \overset{+}{\log}(2|\csc\theta|) = \log(2|\csc\theta|).$$

Therefore, if we choose $r_n \to 1$, we have from Lebesgue's Dominated Convergence Thoerem

$$g(r_n) \to g(1) .$$

We define the sequence $\{c_n\}_0^\infty$ by

$$(10)\begin{cases} c_o = \frac{1}{2}B + \overset{+}{\log}|a_1| + \log 6 \\ c_1 = B + \frac{1}{2}g(1) + \sum_1^2 \overset{+}{\log}|a_k| + \log 18 \\ c_n = B + g(1) + \sum_{n-1}^{n+1} \overset{+}{\log}|a_k| + \log 30 \quad (n \geq 2) . \end{cases}$$

For notational convenience we will assume that $\mu(r)$ is continuous and strictly increasing. This is no restriction since for any $\mu(r)$ given for $r \geq 0$ with $0 < \mu(r) \uparrow \infty$, we may choose a continuous, strictly increasing $\mu'(r)$ with $0 < \mu'(r) < \mu(r)$ $(r \geq 0)$ and such that $\mu' \uparrow \infty$. Then

$$\mu'(r)\overset{+}{\log} r < \mu(r)\overset{+}{\log} r .$$

We may also assume without restriction that $\mu(0) < 2$.

Now define r_o and the sequence $\{(\rho_n, r_n)\}_1^\infty$ by

$$(11)\begin{cases} r_o = \max\left\{\mu^{-1}(2c_o), e\right\} \\ \rho_1 = \max\left\{r_o\left(1 + \frac{2|a_1|}{\varepsilon_1}\right), \mu^{-1}(2c_1)\right\} \\ r_1 = \rho_1\left(1 + \max\left\{\frac{2^3|a_1|}{\varepsilon_1}, 2\right\}\right) \\ \rho_n = \max\left\{r_{n-1}\left(1 + \max\left[\frac{2^{n+1}|a_{n-1}|}{\varepsilon_{n-1}}, \frac{2^n|a_n|}{\varepsilon_n}, 2\right]\right), \mu^{-1}(2c_n), \mu^{-1}(2n)\right\} \\ r_n = \rho_n\left(1 + \max\left\{\frac{(n-1)2^{n+2}\sum_1^n |a_k|}{\varepsilon_n}, 2\right\}\right) \quad (n \geq 2). \end{cases}$$

Note that $\{(\rho_n, r_n)\}_1^{\infty}$ satisfies (5).

Upon taking (6) as our definition of the function ϕ we can prove

Theorem 1. (6) converges subuniformly to a meromorphic function ϕ and satisfies (7) and (2).

Then we have

Corollary. The function $\phi(z)$ is meromorphic in $|z| < \infty$ and has the properties (1) and (2).

Now for $|z| \leq r_{k-1}$ $(k \geq 1)$ we have from (5)

$$\left|\frac{\rho_{k+1}}{\rho_{k+1}-z} - \frac{\rho_k}{\rho_k-z}\right| = \left|\frac{z(\rho_k - \rho_{k+1})}{(\rho_{k+1}-z)(\rho_k-z)}\right|$$

$$\leq \frac{r_{k-1}(\rho_{k+1}-\rho_k)}{(\rho_{k+1}-r_{k-1})(\rho_k-r_{k-1})}$$

$$= \frac{\rho_k}{\rho_k-r_{k-1}} - \frac{\rho_{k+1}}{\rho_{k+1}-r_{k-1}}$$

$$< \frac{\rho_k}{\rho_k-r_{k-1}} - 1$$

$$= \frac{1}{\frac{\rho_k}{r_{k-1}} - 1} .$$

So from (11)

$$(12) \qquad \left|\frac{\rho_{k+1}}{\rho_{k+1}-z} - \frac{\rho_k}{\rho_k-z}\right| \leq \frac{\varepsilon_k}{2^k|a_k|} \text{ for } |z| \leq r_{k-1} \ (k \geq 1).$$

We may now prove that (6) converges subuniformly in $|z| < \infty$. Let K be a compact subset of $|z| < \infty$, and choose r_n such that $K \subset |z| < r_n$. Let

$$R_n(z) = \sum_{1}^{n} a_k \left(\frac{\rho_{k+1}}{\rho_{k+1}-z} - \frac{\rho_k}{\rho_k - z} \right) .$$

Observe that from (12) we have

$$\sum_{m}^{\infty} \left| a_k \left(\frac{\rho_{k+1}}{\rho_{k+1}-z} - \frac{\rho_k}{\rho_k - z} \right) \right| < \varepsilon_m \quad z \in K \quad (m > n) .$$

Thus since $\varepsilon_n \downarrow 0$ we see that

$$\phi(z) = R_n(z) + \sum_{n+1}^{\infty} a_k \left(\frac{\rho_{k+1}}{\rho_{k+1}-z} - \frac{\rho_k}{\rho_k - z} \right)$$

is the sum of the rational function $R_n(z)$ and the infinite sum which is uniformly convergent on K. Thus $\phi(z)$ is meromorphic in $|z| < \infty$.

Now for $|z| \geq r_n$ $(n \geq 2)$ we have for $k \leq n$

$$\left| \frac{\rho_k}{\rho_k - z} \right| \leq \frac{\rho_k}{r_n - \rho_k} = \frac{1}{\frac{r_n}{\rho_k} - 1} - \frac{1}{\frac{r_n}{\rho_n} - 1} .$$

Thus from (11)

$$\left| \frac{\rho_k}{\rho_k - z} \right| \leq \frac{\varepsilon_n}{(n-1)2^{n+2} \sum_{1}^{n} |a_n|} \leq \frac{\varepsilon_n}{(n-1)2^{n+2}(|a_k| + |a_{k-1}|)} \quad (n \geq 2)$$

and for $|z| \geq r_n$ $(n \geq 2)$ and $k < n$ we have

$$(13)\quad \left|\frac{\rho_{k+1}}{\rho_{k+1}-z}-\frac{\rho_k}{\rho_k-z}\right| \le \frac{\varepsilon_n}{(n-1)2^{n+2}(|a_{k+1}|+|a_k|)} + \frac{\varepsilon_n}{(n-1)2^{n+2}(|a_k|+|a_{k-1}|)} < \frac{\varepsilon_n}{(n-1)2^{n+1}|a_k|}$$

Now for $|z| = r_n$, $(n \geq 1)$ we have from (11)

$$\left|\frac{\rho_{n+1}}{\rho_{n+1}-z} - 1\right| = \left|\frac{1}{\frac{\rho_{n+1}}{z}-1}\right| \le \frac{1}{\frac{\rho_{n+1}}{r_n}-1} \le \frac{\varepsilon_n}{2^{n+2}|a_n|} .$$

Also from (11) for $|z| = r_1$

$$\left|\frac{\rho_1}{\rho_1-z}\right| \le \frac{\rho_1}{r_1-\rho_1} = \frac{1}{\frac{r_1}{\rho_1}-1} \le \frac{\varepsilon_1}{2^3|a_1|}$$

and for $|z| = r_n$ $(n \geq 2)$

$$\left|\frac{\rho_n}{\rho_n-z}\right| \le \frac{\rho_n}{r_n-\rho_n} = \frac{1}{\frac{r_n}{\rho_n}-1} \le \frac{\varepsilon_n}{2^{n+2}|a_n|} .$$

Thus for $|z| = r_n$ $(n \geq 1)$ we have

$$(14)\quad \left|\frac{\rho_{n+1}}{\rho_{n+1}-z} - \frac{\rho_n}{\rho_n-z} - 1\right| \le \left|\frac{\rho_{n+1}}{\rho_{n+1}-z} - 1\right| + \left|\frac{\rho_n}{\rho_n-z}\right| \le \frac{\varepsilon_n}{2^{n+1}|a_n|} .$$

Combining (12), (13), and (14) we have for $|z| = r_1$

$$|\phi(z) - a_1| \le |a_1| \left| \frac{\rho_2}{\rho_2 - z} - \frac{\rho_1}{\rho_1 - z} - 1 \right| + \sum_{2}^{\infty} |a_k| \left| \frac{\rho_{k+1}}{\rho_{k+1} - z} - \frac{\rho_k}{\rho_k - z} \right|$$

$$\le \frac{\varepsilon_1}{2^2} + \sum_{2}^{\infty} \frac{\varepsilon_k}{2^k}$$

$$< \varepsilon_1$$

and for $|z| = r_n \quad (n \ge 2)$

$$|\phi(z) - a_n| \le \sum_{1}^{n-1} |a_k| \left| \frac{\rho_{k+1}}{\rho_{k+1} - z} - \frac{\rho_k}{\rho_k - z} \right| + |a_n| \left| \frac{\rho_{n+1}}{\rho_{n+1} - z} - \frac{\rho_n}{\rho_n - z} - 1 \right|$$

$$+ \sum_{n+1}^{\infty} |a_k| \left| \frac{\rho_{k+1}}{\rho_{k+1} - z} - \frac{\rho_k}{\rho_k - z} \right|$$

$$\le \sum_{1}^{n-1} \frac{\varepsilon_n}{(n-1)2^{n+1}} + \frac{\varepsilon_n}{2^{n+1}} + \sum_{n+1}^{\infty} \frac{\varepsilon_k}{2^k}$$

$$= \sum_{n}^{\infty} \frac{\varepsilon_k}{2^k}$$

$$< \varepsilon_n$$

Thus (7) has been proved.

Now for $|z| \le \rho_n \ (n \ge 1)$ we have from (11)

$$\left| \frac{\rho_{n+1}}{\rho_{n+1} - z} - 1 \right| = \left| \frac{1}{\frac{\rho_{n+1}}{z} - 1} \right| \le \frac{1}{\frac{\rho_{n+1}}{\rho_n} - 1}$$

$$< \frac{1}{\frac{\rho_{n+1}}{r_n} - 1} \le \frac{1}{2} .$$

Thus

$$(15)\qquad \left|\frac{\rho_{n+1}}{\rho_{n+1}-z}\right| < \frac{3}{2} \text{ for } |z| \le \rho_n \ (n \ge 1) .$$

Also from (11) for $|z| \ge \rho_n \quad (n \ge 2)$

$$(16)\qquad \left|\frac{\rho_{n-1}}{\rho_{n-1}-z}\right| \le \frac{\rho_{n-1}}{\rho_n - \rho_{n-1}} = \frac{1}{\frac{\rho_n}{\rho_{n-1}} - 1} < \frac{1}{\frac{r_{n-1}}{\rho_{n-1}} - 1} \le \frac{1}{2}$$

We are now in a position to prove

$$(17)\qquad m(r,\infty) = \frac{1}{2\pi}\int_0^{2\pi} \overset{+}{\log}|\phi(re^{i\theta})|\, d\theta < \frac{1}{2}\mu(r) \overset{+}{\log} r \quad (r \ge 0) .$$

In the following computation we will use the facts that

$$\overset{+}{\log} \prod_1^p \alpha_i \le \sum_1^p \overset{+}{\log}\, \alpha_i \quad \text{and}$$

$$\overset{+}{\log} \sum_1^p \alpha_i \le \sum_1^p \overset{+}{\log}\, \alpha_i + \log p .$$

For $0 \le r \le r_0$ we have from (12)

$$|\phi(re^{i\theta})| \le \sum_1^\infty |a_k| \left| \frac{\rho_{k+1}}{\rho_{k+1} - re^{i\theta}} - \frac{\rho_k}{\rho_k - re^{i\theta}} \right|$$

$$< \sum_1^\infty \frac{\varepsilon_k}{2^k}$$

$$< 1$$

Thus

(18) $$m(r,\infty) = 0 \text{ for } 0 \leq r \leq r_o$$

For $r_o < r \leq \rho_1$ we have again from (12)

$$\sum_{2}^{\infty} |a_k| \left| \frac{\rho_{k+1}}{\rho_{k+1}-re^{i\theta}} - \frac{\rho_k}{\rho_k - re^{i\theta}} \right| < \sum_{2}^{\infty} \frac{\varepsilon_k}{2^k} < 1$$

Thus for $r_o < r \leq \rho_1$ we have from (15)

$$\overset{+}{\log}|\phi(re^{i\theta})| \leq \overset{+}{\log}\left(|a_1| \left| \frac{\rho_2}{\rho_2-re^{i\theta}} - \frac{\rho_1}{\rho_1-re^{i\theta}} \right| \right)$$

$$+ \overset{+}{\log} \sum_{2}^{\infty} |a_k| \left| \frac{\rho_{k+1}}{\rho_{k+1}-re^{i\theta}} - \frac{\rho_k}{\rho_k-re^{i\theta}} \right| + \log 2$$

$$\leq \overset{+}{\log}\left(\left| \frac{\rho_1}{\rho_1-re^{i\theta}} \right| + \frac{3}{2} \right) + \overset{+}{\log}|a_1| + \log 2$$

$$\leq \overset{+}{\log} \left| \frac{1}{1-\frac{r}{\rho_1}e^{i\theta}} \right| + \overset{+}{\log}|a_1| + \log 6$$

Thus from (9) and (10), for $r_o < r \leq \rho_1$

$$\frac{1}{2\pi}\int_0^{2\pi} \overset{+}{\log}|\phi(re^{i\theta})|\, d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} \overset{+}{\log}\left| \frac{1}{1-\frac{r}{\rho_1}e^{i\theta}} \right| d\theta$$

$$+ \overset{+}{\log}|a_1| + \log 6$$

$$\leq \frac{1}{2}B + \overset{+}{\log}|a_1| + \log 6$$

$$= c_o ,$$

and we have

(19) $$m(r,\infty) \le c_o \text{ for } r_o < r \le \rho_1 .$$

For $\rho_1 < r \le \rho_2$ we have as before from (12)

$$\sum_3^\infty |a_k| \left| \frac{\rho_{k+1}}{\rho_{k+1}-re^{i\theta}} - \frac{\rho_k}{\rho_k-re^{i\theta}} \right| < \sum_3^\infty \frac{\varepsilon_k}{2^k} < 1 .$$

Thus for $\rho_1 < r \le \rho_2$ it follows from (15) and (16) that

$$\overset{+}{\log}|\phi(re^{i\theta})| \le \overset{+}{\log}\left(|a_1| \left|\frac{\rho_2}{\rho_2-re^{i\theta}} - \frac{\rho_1}{\rho_1-re^{i\theta}}\right|\right)$$

$$+ \overset{+}{\log}\left(|a_2| \left|\frac{\rho_3}{\rho_3-re^{i\theta}} - \frac{\rho_2}{\rho_2-re^{i\theta}}\right|\right)$$

$$+ \overset{+}{\log} \sum_3^\infty |a_k| \left|\frac{\rho_{k+1}}{\rho_{k+1}-re^{i\theta}} - \frac{\rho_k}{\rho_k-re^{i\theta}}\right| + \log 3$$

$$\le \overset{+}{\log}\left(\left|\frac{\rho_2}{\rho_2-re^{i\theta}}\right| + \left|\frac{\rho_1}{\rho_1-re^{i\theta}}\right|\right)$$

$$+ \overset{+}{\log}\left(\left|\frac{\rho_2}{\rho_2-re^{i\theta}}\right| + \frac{3}{2}\right) + \sum_1^2 \overset{+}{\log}|a_k| + \log 3$$

$$\le 2\overset{+}{\log}\left|\frac{1}{1-\frac{r}{\rho_2}e^{i\theta}}\right| + \overset{+}{\log}\left|\frac{1}{1-\frac{r}{\rho_1}e^{i\theta}}\right| + \sum_1^2 \overset{+}{\log}|a_k|$$

$$+ \log 18 ,$$

and from (8), (9), and (10) we have for $\rho_1 < r \le \rho_2$

$$\frac{1}{2\pi}\int_0^{2\pi} \overset{+}{\log}|\phi(re^{i\theta})|\, d\theta \le \frac{1}{\pi}\int_0^{2\pi} \overset{+}{\log}\left|\frac{1}{1-\frac{r}{\rho_2}e^{i\theta}}\right| d\theta$$

$$+ \frac{1}{2\pi}\int_0^{2\pi} \overset{+}{\log}\left|\frac{1}{1-\frac{r}{\rho_1}e^{i\theta}}\right| d\theta + \sum_1^2 \overset{+}{\log}|a_k|$$

$$+ \overset{+}{\log}\, 18$$

$$\le B + \frac{1}{2\pi}\int_0^{2\pi} \overset{+}{\log}\left|\frac{1}{1-e^{i\theta}}\right| d\theta + \sum_1^2 \overset{+}{\log}|a_k|$$

$$+ \log 18$$

$$= c_1 .$$

Thus

(20) $$M(r,\infty) \le c_1 \text{ for } \rho_1 < r \le \rho_2 .$$

Now for $\rho_n < r \le \rho_{n+1}$ $(n \ge 2)$ we have from (12) and (13), if we interpert $\sum_1^0 (\ ) = 0$,

$$\sum_1^{n-2} |a_k| \left|\frac{\rho_{k+1}}{\rho_{k+1}-re^{i\theta}} - \frac{\rho_k}{\rho_k - re^{i\theta}}\right| \le \sum_1^{n-2} \frac{\varepsilon_{n-1}}{(n-2)2^n} < 1 \quad \text{and}$$

$$\sum_{n+2}^{\infty} |a_k| \left|\frac{\rho_{k+1}}{\rho_{k+1}-re^{i\theta}} - \frac{\rho_k}{\rho_k - re^{i\theta}}\right| \le \sum_{n+2}^{\infty} \frac{\varepsilon_k}{2^k} < 1 .$$

Thus

$$\overset{+}{\log}\ |\phi(re^{i\theta})| \leq \overset{+}{\log} \sum_{1}^{n-2} |a_k| \left| \frac{\rho_{k+1}}{\rho_{k+1}-re^{i\theta}} - \frac{\rho_k}{\rho_k-re^{i\theta}} \right|$$

$$+ \sum_{n-1}^{n+1} \overset{+}{\log} \left(|a_k| \left| \frac{\rho_{k+1}}{\rho_{k+1}-re^{i\theta}} - \frac{\rho_k}{\rho_k-re^{i\theta}} \right| \right)$$

$$+ \overset{+}{\log} \sum_{n+2}^{\infty} |a_k| \left| \frac{\rho_{k+1}}{\rho_{k+1}-re^{i\theta}} - \frac{\rho_k}{\rho_k-re^{i\theta}} \right| + \log 5$$

$$\leq \sum_{n-1}^{n+1} \overset{+}{\log} \left| \frac{\rho_{k+1}}{\rho_{k+1}-re^{i\theta}} - \frac{\rho_k}{\rho_k-re^{i\theta}} \right| + \sum_{n-1}^{n+1} \overset{+}{\log}\ |a_k|$$

$$+ \log 5 .$$

So from (15) and (16) we have for $\rho_n < r \leq \rho_{n+1}$ $(n \geq 2)$

$$\overset{+}{\log}|\phi(re^{i\theta})| \leq \overset{+}{\log}\left(\left|\frac{\rho_n}{\rho_n-re^{i\theta}}\right| + \frac{1}{2}\right) + \overset{+}{\log}\left(\left|\frac{\rho_{n+1}}{\rho_{n+1}-re^{i\theta}}\right| + \left|\frac{\rho_n}{\rho_n-re^{i\theta}}\right|\right)$$

$$+ \overset{+}{\log}\left(\left|\frac{\rho_{n+1}}{\rho_{n+1}-re^{i\theta}}\right| + \frac{3}{2}\right) + \sum_{n-1}^{n+1} \overset{+}{\log}\ |a_k| + \log 5$$

$$\leq 2\ \overset{+}{\log} \left| \frac{1}{1-\frac{r}{\rho_{n+1}}e^{i\theta}} \right| + 2\ \overset{+}{\log} \left| \frac{1}{1-\frac{r}{\rho_n}e^{i\theta}} \right|$$

$$+ \sum_{n-1}^{n+1} \overset{+}{\log}\ |a_k| + \log 30$$

Thus for $\rho_n < r \leq \rho_{n+1}$ $(n \geq 2)$ we have from (8), (9), and (10)

$$\frac{1}{2\pi}\int_0^{2\pi} \overset{+}{\log}|\phi(re^{i\theta})|\, d\theta \leq \frac{1}{\pi}\int_0^{2\pi} \overset{+}{\log}\left|\frac{1}{1-\frac{r}{\rho_{n+1}}e^{i\theta}}\right| d\theta$$

$$+ \frac{1}{\pi}\int_0^{2\pi} \overset{+}{\log}\left|\frac{1}{1-\frac{r}{\rho_n}e^{i\theta}}\right| d\theta$$

$$+ \sum_{n-1}^{n+1} \overset{+}{\log}|a_k| + \log 30$$

$$\leq B + \frac{1}{\pi}\int_0^{2\pi} \overset{+}{\log}\left|\frac{1}{1-e^{i\theta}}\right| d\theta$$

$$+ \sum_{n-1}^{n+1} \overset{+}{\log}|a_k| + \log 30$$

$$= c_n ,$$

and

(21) $$m(r,\infty) \leq c_n \text{ for } \rho_n < r \leq \rho_{n+1} \quad (n \geq 2).$$

Thus from (18) through (21) and (11)

$$m(r,\infty) = 0 \quad , \quad 0 \leq r \leq r_o$$

and

$$m(r,\infty) \leq \begin{cases} \frac{1}{2}\mu(r_o) < \frac{1}{2}\mu(r)\log r \quad , & r_o < r \leq \rho_1 \\ \frac{1}{2}\mu(\rho_n) < \frac{1}{2}\mu(r)\log r \quad , & \rho_n < r \leq \rho_{n+1} \ (n \geq 1), \end{cases}$$

and we see that (17) has been proved.

Now since $\rho_1 > 1$, we have for $\rho_n < r \leq \rho_{n+1}$ $(n \geq 1)$

$$N(r,\infty) = \int_0^r \frac{n(t,\infty) - n(0,\infty)}{t}\, dt + n(0,\infty) \log r$$

$$\leq n \int_1^r \frac{dt}{t}$$

$$= n \log r$$

Thus for $\rho_n < r \leq \rho_{n+1}$ $(n \geq 1)$ we have from (11)

$$N(r,\infty) \leq \tfrac{1}{2}\mu(\rho_n) \log r < \tfrac{1}{2}\mu(r) \log r$$

Then since it is clear that $N(r,\infty) = 0$, for $r \leq \rho_1$, we have

$$N(r,\infty) < \tfrac{1}{2}\mu(r) \overset{+}{\log} r \quad (r \geq 0)$$

This result combined with (17) yields (2).

SECTION II

G.R. MacLane has proved

Theorem 2. Let $f(z)$ be holomorphic in $|z| < 1$ and suppose there exists a dense set Θ on $[0,2\pi]$ such that

$$(22) \qquad \int_0^1 (1 - r) \overset{+}{\log}|f(re^{i\theta})| \, dr < \infty \qquad \theta \in \Theta .$$

Then any component of $\{z \,|\, |f(z)| = c\}$ which is not compact must tend, at each end, to a definite point of $\{|z| = 1\}$.

The purpose of this section is to construct a meromorphic function with "wobbly" level curves, which indicates that some condition, such as (22), is necessary to conclude that level curves end at points . More specifically, we will construct a function, holomorphic in $|z| < 1$, which has a level curve one component of which tends, at one end, to an arc on $\{|z| = 1\}$.

Let S denote the partially open square $\{ z = x + iy \,|\, 0 < x < 1, \ 0 \leq y \leq 1\}$. Let $1 = a_1 > a_2 > \cdots > a_n \downarrow 0$ be a given sequence. Define

$$\Gamma = \left(\bigcup_1^\infty \left\{ z \,\middle|\, x = a_n, \ 0 < y \leq \tfrac{3}{4} \right\} \right) \cup \{ z \,|\, 0 < x \leq 1 , \ y = 0 \}$$

$$\Gamma^* = \left(\bigcup_1^\infty \left\{ z \,\middle|\, x = \frac{a_{n+1}+a_n}{2} , \ \tfrac{1}{4} \leq y < 1 \right\} \right) \cup \left\{ z \,\middle|\, 0 < x \leq \frac{a_1 + a_2}{2} , \ y = 1 \right\}$$

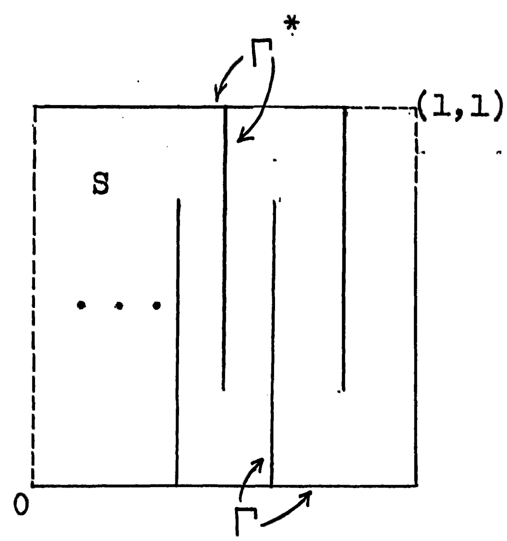

Figure 1

We can prove

<u>Theorem 3.</u> <u>Given</u> $\varepsilon > 0$. <u>There exists a function</u> $f(z)$, <u>holomorphic in</u> S, <u>such that</u>

$$(23) \qquad \begin{cases} |f(z) - 1| < \varepsilon & , z \in \Gamma \\ |f(z)| < \varepsilon & , z \in \Gamma^* \end{cases}$$

From Theorem 3 the desired result easily follows. We choose $0 < \varepsilon < \frac{1}{2}$ and let $T = \left\{ z \,\middle|\, |f(z)| < \frac{1}{2} \right\}$. Then Γ^* is contained in some component, say T', of T. We let A denote the boundary of T' and note that A is a component of the level curve $\left\{ z \,\middle|\, |f(z)| = \frac{1}{2} \right\}$. Note also that $\Gamma \cap (T' \cup A) = \emptyset$. We now let ψ' be a conformal map of $|z| < 1$ onto the interior of S. Then ψ' has a continuous extension ψ to $|z| \leq 1$ and ψ gives a one to one correspondence between $\{|z| = 1\}$ and the boundary of S. Thus letting $\Psi(z) = f(\psi(z))$ we have

<u>Corollary</u>. <u>The function</u> $\Psi(z)$ <u>is holomorphic in</u> $|z| < 1$ <u>and the component</u> $\psi^{-1}(A)$ <u>of the level curve</u> $\left\{ z \,\middle|\, |\Psi(z)| = \frac{1}{2} \right\}$ <u>tends, at one end, to an arc on</u> $\{|z| = 1\}$.

The proof of Theorem 3 is an extension of a familiar argument. If there existed a rectifiable Jordan curve C which contained Γ in its interior and Γ^* in its exterior, then we would have

$$(24) \qquad F(z) = \frac{1}{2\pi i} \int_C \frac{d\zeta}{\zeta - z} = \begin{cases} 0 & z \in \Gamma^* \\ 1 & z \in \Gamma \end{cases} .$$

We could then find a rational function $R(z)$, with poles only on C, approximating $F(z)$ on Γ and Γ^* and, using Runge's pole-pushing process, obtain a function $f(z)$, holomorphic in S, approximating $R(z)$ on Γ and Γ^*. Then Theorem 3 would be proved.

But clearly no such rectifiable curve can exist. We will define a curve C which contains Γ in its interior and Γ^* in its exterior and prove that the improper integral $\int_C \frac{d\zeta}{\zeta - z}$ "converges" (in a suitable sense) and satisfies (24). It will then be a simple matter to approximate $\int_C \frac{d\zeta}{\zeta - z}$ on Γ and Γ^* by a function holomorphic in S.

Now define

$$(25) \qquad \delta_n = \begin{cases} \min\left\{\frac{a_1 - a_2}{4}, \frac{1}{8}\right\}, & n = 1 \\ \min\left\{\frac{a_{n-1} - a_n}{4}, \frac{a_n - a_{n+1}}{4}, \frac{1}{8}\right\}, & n \geq 2 \end{cases}$$

and let

$$C_n = \left\{z \middle| a_{n+1} + \delta_{n+1} \leq x \leq a_n + \delta_n,\ y = -\frac{1}{8}\right\}$$
$$\cup \left\{z \middle| a_{n+1} + \delta_{n+1} \leq x \leq a_n - \delta_n,\ y = \frac{1}{8}\right\}$$
$$\cup \left\{z \middle| x = a_n + \delta_n,\ \frac{1}{8} \leq y \leq \frac{3}{4}\right\} \cup \left\{z \middle| x = a_n - \delta_n,\ \frac{1}{8} < y \leq \frac{3}{4}\right\}$$
$$\cup \left\{z \middle| |z - a_n - \tfrac{3}{4} i| = \delta_n,\ y > \frac{3}{4}\right\} \qquad (n \geq 1),$$

$$\gamma_n = \left\{z \middle| x = a_n + \delta_n,\ -\frac{1}{8} < y < \frac{1}{8}\right\} \qquad (n \geq 1), \text{ and}$$

$$\gamma = \left\{z \middle| x = 0,\ -\frac{1}{8} < y < \frac{1}{8}\right\}.$$

The curve γ will be considered as oriented with the negative direction of the imaginary axis. We will consider γ_n and C_n oriented as shown in the following diagram.

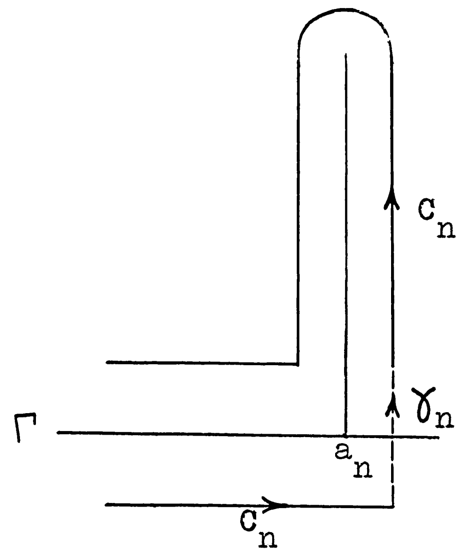

Figure 2

We also define

$$C = \bigcup_1^\infty C_n \cup \gamma_1 \cup \gamma \quad ,$$

$$D_n = \left\{ z \,\middle|\, \text{dist}\,(z, C_n) < \frac{\delta_n}{2} \right\} \qquad (n \geq 1) \ , \quad \text{and}$$

$$D = \bigcup_1^\infty D_n \quad .$$

Note from (25) it is clear that Γ is interior and Γ^* exterior to C. Note also that $C \subset D$ and $(\Gamma \cup \Gamma^*) \cap D = \emptyset$.

We now write formally

$$(26) \quad F(z) = \frac{1}{2\pi i} \int_C \frac{d\zeta}{\zeta - z} = \sum_1^\infty \frac{1}{2\pi i} \int_{C_n} \frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i} \int_{\gamma_1} \frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i} \int_{\gamma} \frac{d\zeta}{\zeta - z}$$

and prove

<u>Lemma 1</u>. (26) <u>converges subuniformly in</u> $S - C$ <u>to a function</u> $F(z)$ <u>which satisfies</u>

$$(27) \qquad F(z) = \begin{cases} 0, & z \text{ exterior to } C \\ 1, & z \text{ interior to } C \ . \end{cases}$$

Write $\zeta = \xi + i\eta$. Now choose $z_o \in S - C$ and let N be a disc about z_o and k_o an integer such that $x - a_k - \delta_k \geq \rho > 0$ for $z = x + iy \in N$ and $k \geq k_o$. Then

$$\left| \frac{1}{2\pi i} \int_{\gamma_k} \frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i} \int_{-\gamma} \frac{d\zeta}{\zeta - z} \right| = \frac{1}{2\pi} \left| \int_{-\frac{1}{8}}^{\frac{1}{8}} \frac{d\eta}{a_k + \delta_k + i\eta - z} - \int_{-\frac{1}{8}}^{\frac{1}{8}} \frac{d\eta}{i\eta - z} \right|$$

$$\leq \frac{1}{2\pi} \int_{-\frac{1}{8}}^{\frac{1}{8}} \frac{a_k + \delta_k}{|a_k + \delta_k + i\eta - z|\,|i\eta - z|}\, d\eta$$

$$\leq \frac{1}{2\pi} \int_{-\frac{1}{8}}^{\frac{1}{8}} \frac{a_k + \delta_k}{\rho^2}\, d\eta$$

$$= \frac{a_k + \delta_k}{8\pi\rho^2} \qquad (z \in N,\ k \geq k) \quad .$$

Thus

(28)
$$\begin{cases} \lim\limits_{k \to \infty} \dfrac{1}{2\pi i} \displaystyle\int_{\gamma_k} \dfrac{d\zeta}{\zeta - z} = \dfrac{1}{2\pi i} \displaystyle\int_{-\gamma} \dfrac{d\zeta}{\zeta - z} \qquad (z \in S) \\ \text{and the convergence is subuniform in } S. \end{cases}$$

Now for z exterior to C we have

$$\int_{C_k + \gamma_k - \gamma_{k+1}} \frac{d\zeta}{\zeta - z} = 0 \qquad (k \geq 1) \quad .$$

Thus for z exterior to C we have from (28)

$$\sum_1^\infty \frac{1}{2\pi i}\int_{C_k}\frac{d\zeta}{\zeta - z} = \lim_{n\to\infty}\sum_1^n\left(\frac{1}{2\pi i}\int_{C_k+\gamma_k-\gamma_{k+1}}\frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_{\gamma_k-\gamma_{k+1}}\frac{d\zeta}{\zeta - z}\right)$$

$$= \lim_{n\to\infty}\sum_1^n \frac{1}{2\pi i}\int_{\gamma_{k+1}-\gamma_k}\frac{d\zeta}{\zeta - z}$$

$$= \lim_{n\to\infty}\frac{1}{2\pi i}\int_{\gamma_{n+1}}\frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_{\gamma_1}\frac{d\zeta}{\zeta - z}$$

$$= \frac{1}{2\pi i}\int_{-\gamma}\frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_{\gamma_1}\frac{d\zeta}{\zeta - z} \quad .$$

Thus from (28)

$$(29)\quad \begin{cases} \displaystyle\sum_1^\infty \frac{1}{2\pi i}\int_{\gamma_k}\frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i}\int_{\gamma_1}\frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i}\int_{\gamma}\frac{d\zeta}{\zeta - z} = 0 \\ \text{for } z \text{ exterior to } C \text{ and the convergence is subuniform in } S. \end{cases}$$

Now define $\Gamma_m = C_m + C_{m+1} + \gamma_m - \gamma_{m+2}$. Then for z interior to C we have z interior to some Γ_m . Then

$$\int_{C_k+\gamma_k-\gamma_{k+1}}\frac{d\zeta}{\zeta - z} = 0 \qquad (k \neq m,\ m+1,\ k \geq 1) \quad \text{and}$$

$$\int_{\Gamma_m}\frac{d\zeta}{\zeta - z} = 1 \ .$$

Thus for z interior to C we have from (28)

$$\sum_{1}^{\infty} \frac{1}{2\pi i}\int_{\gamma_k} \frac{d\zeta}{\zeta - z} = \lim_{n\to\infty} \sum_{1}^{n} \left(\frac{1}{2\pi i}\int_{C_k+\gamma_k-\gamma_{k+1}} \frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_{\gamma_k-\gamma_{k+1}} \frac{d\zeta}{\zeta - z}\right)$$

$$= \frac{1}{2\pi i}\int_{C_m+\gamma_m-\gamma_{m+1}} \frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i}\int_{C_{m+1}+\gamma_{m+1}-\gamma_{m+2}} \frac{d\zeta}{\zeta - z}$$

$$- \lim_{n\to\infty} \sum_{1}^{n} \frac{1}{2\pi i}\int_{\gamma_k-\gamma_{k-1}} \frac{d\zeta}{\zeta - z}$$

$$= \frac{1}{2\pi i}\int_{\Gamma_m} \frac{d\zeta}{\zeta - z} + \lim_{n\to\infty} \frac{1}{2\pi i}\int_{\gamma_{n+1}} \frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_{\gamma_1} \frac{d\zeta}{\zeta - z}$$

$$= 1 + \frac{1}{2\pi i}\int_{-\gamma} \frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_{\gamma_1} \frac{d\zeta}{\zeta - z} .$$

Thus from (28)

$$(30) \quad \begin{cases} \displaystyle\sum_{1}^{\infty} \frac{1}{2\pi i}\int_{C_k} \frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i}\int_{\gamma_1} \frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i}\int_{\gamma} \frac{d\zeta}{\zeta - z} = 1 \\ \text{for } z \text{ interior to } C \text{ and the convergence is subuniform in } S. \end{cases}$$

Combining (29) and (30), we see that Lemma 1 has been proved.

It should be noted that in particular (24), with the suitable interpretation of $\int_C \frac{d\zeta}{\zeta - z}$, is satisfied.

Now for each n, choose a rational function $R_n(z)$ with poles only on C_n such that

$$(31) \qquad \left| \frac{1}{2\pi i} \int_{C_n} \frac{d\zeta}{\zeta - z} - R_n(z) \right| < \frac{\varepsilon}{2^{n+1}} \quad \text{for } z \notin D_n \ .$$

Using Runge's pole-pushing process, let $\emptyset_n(z)$ be a function, holomorphic in S, obtained by pushing the poles of $R_n(z)$ out toward x = 0 through the channel $\Delta_n = \bigcup_{n}^{\infty} D_k$ in such a way that

$$(32) \qquad |R_n(z) - \emptyset_n(z)| < \frac{\varepsilon}{2^{n+1}} \quad \text{for } z \in S - \Delta_n \ .$$

Thus

$$(33) \qquad \left| \frac{1}{2\pi i} \int_{C_n} \frac{d\zeta}{\zeta - z} - \emptyset_n(z) \right| < \frac{\varepsilon}{2^n} \quad \text{for } z \in S - \Delta_n \ .$$

Now the series

$$\sum_{n}^{\infty} \frac{1}{2\pi i} \int_{C_k} \frac{d\zeta}{\zeta - z}$$

is, as we have seen, subuniformly convergent in $S - \Delta_n$; thus from (33) we see that

$$\emptyset(z) = \sum_{1}^{\infty} \emptyset_n(z)$$

converges subuniformly in S to a holomorphic function $\emptyset$. Also from (33)

$$(34) \qquad \left| \emptyset(z) - \sum_{1}^{\infty} \frac{1}{2\pi i} \int_{C_k} \frac{d\zeta}{\zeta - z} \right| < \varepsilon \text{ for } z \in S - D \ .$$

We note that the functions $\frac{1}{2\pi i}\int_{\gamma_1}\frac{d\zeta}{\zeta - z}$ and $\frac{1}{2\pi i}\int_{\gamma}\frac{d\zeta}{\zeta - z}$ are holomorphic in S. Thus if we set

$$f(z) = \phi(z) + \frac{1}{2\pi i}\int_{\gamma_1}\frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i}\int_{\gamma}\frac{d\zeta}{\zeta - z} ,$$

we see from (26), (27), and (34) f(z) satisfies (23).

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