RICE UNIVERSITY

TWO APPLICATIONS OF RUNGE'S TECHNIQUES ON APPROXIMATION

by

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SECTION I

Given $\mu(r)$ for $r \geq 0$ with $0 < \mu(r) \uparrow \infty$, G. R. MacLane [1, Thm. 3, Coroll.] has constructed a function $\varnothing(z)$, meromorphic in $|z| < \infty$, with the properties

1. the image under $w = \varnothing(z)$ of an unbounded curve: $z = \gamma(t)$ ($0 < t < \infty$, $\gamma$ continuous, $\limsup_{t \to \infty} |\gamma(t)| = \infty$) is dense on $|w| < \infty$, and

2. the Nevanlinna characteristic of $\varnothing$ satisfies

$$T(r) \leq \mu(r) \log r.$$  

The function $\varnothing$ was constructed geometrically by specifying the Riemann surface of its inverse as a covering of the sphere. For a given sequence $\{a_n\}_{n=1}^\infty$ of points on $|w| < \infty$ such that

3. $\{a_n\}_{n=1}^\infty$ is dense on $|w| < \infty$,

the construction defines an expanding sequence $\{C_n\}_{n=1}^\infty$ of analytic curves with the property that

4. $|\varnothing(z) - a_n| < \frac{1}{n}$ for $z$ on $C_n$ ($n \geq 1$).

It is clear that (1) follows from (3) and (4).

It should be noted that (2) is as strong as possible. For a rational function clearly cannot satisfy (1), and the characteristic of a function which is meromorphic in $|z| < \infty$ and non-rational satisfies [3, p. 218]

$$\lim_{r \to \infty} \frac{T(r)}{\log r} = \infty.$$
The object of this section is to construct, by giving an explicit formula, a function possessing the essential properties of the function $\phi$. More specifically, let $\{a_n\}_{n=1}^{\infty}$ be a sequence satisfying (3) and $a_n \neq 0 \ (n \geq 1)$, and choose $\{\varepsilon_n\}_{n=1}^{\infty}$ with $\varepsilon_1 < 1$ and $\varepsilon_n \downarrow 0$. We will define a double sequence $\{(\varphi_n, r_n)\}_{n=1}^{\infty}$ satisfying

$$0 < \varphi_1 < r_1 < \varphi_2 < r_2 < \ldots \uparrow \infty$$

and such that if we let

$$\phi(z) = \sum_{k=1}^{\infty} a_k \left( \frac{\varphi_{k+1}}{\varphi_k - z} - \frac{\varphi_k}{\varphi_k - z} \right),$$

then $\phi(z)$ is meromorphic in $|z| < \infty$ and

$$|\phi(z) - a_n| < \varepsilon_n \quad \text{for} \quad |z| = r_n \quad (n \geq 1).$$

Then since $\phi(z)$ has poles only on the $\{|z| = \varphi_n\}$ and has only one pole on each $\{|z| = \varphi_n\}$, (2) will be satisfied if $\varphi_n \uparrow \infty$ sufficiently fast.

The idea of the proof of (7) is as follows. For a fixed $n$ and $|z|$ sufficiently large

$$\frac{\varphi_n}{\varphi_n - z} \approx 1$$

where we are using "$\sim$" to mean "approximately equal to"; and for a fixed $|z|$ and $n$ sufficiently large
\[
\frac{\varphi_n}{\varphi_{n-1}} \approx 1
\]

Thus if \( \left\{(\varphi_n, r_n)\right\}_1^\infty \) is properly chosen,

\[
\frac{\varphi_{n+1}}{\varphi_{n+1-z}} - \frac{\varphi_n}{\varphi_{n-z}} \approx \begin{cases} 
1 & \text{for } |z| = r_n \\
0 & \text{for } \frac{|z|}{\varphi_n} \text{ small or } \frac{|z|}{\varphi_{n+1}} \text{ large.}
\end{cases}
\]

Thus in particular

\[
\frac{\varphi_{n+1}}{\varphi_{n+1-z}} - \frac{\varphi_n}{\varphi_{n-z}} \approx \begin{cases} 
1 & \text{for } |z| = r_n \\
0 & \text{for } |z| = r_k, \ k \neq n
\end{cases}
\]

Now let \( g(r) = \frac{1}{\pi} \int_0^{2\pi} \log \left| \frac{1}{1 - re^{i\theta}} \right| d\theta \). We would like to let

\[
(8) \quad B = \max_{0 \leq r \leq 1} g(r)
\]

and to establish that

\[
(9) \quad g(r) \leq g(1) \text{ for } r \geq 1.
\]

It is elementary that \( \left| \frac{1}{1 - re^{i\theta}} \right| \) is a decreasing function of \( r \) for \( r \geq 1 \). Thus to establish (8) and (9), it is sufficient to prove the continuity of \( g(r) \) at \( r = 1 \). To this end we note that

\[
|1 - re^{i\theta}|^2 = 1 - 2r \cos \theta + r^2 = (1 - r \cos \theta)^2 + r^2 \sin^2 \theta
\]

Thus

\[
|1 - re^{i\theta}| \leq r |\sin \theta| \leq \frac{1}{2} |\sin \theta| \text{ for } r \geq \frac{1}{2}
\]
\[
\log \left| \frac{1}{1 - \cos \theta} \right| \leq \log(2|\csc \theta|) = \log (2|\csc \theta|).
\]

Therefore, if we choose \( r_n \to 1 \), we have from Lebesgue's Dominated Convergence Theorem

\[
g(r_n) \to g(1).
\]

We define the sequence \( \{c_n\}_0^\infty \) by

\[
\begin{align*}
  c_0 &= \frac{1}{2}B + \log|a_1| + \log 6 \\
  c_1 &= B + \frac{1}{2}g(1) + \sum_{k=1}^{2} \log |a_k| + \log 18 \\
  c_n &= B + g(1) + \sum_{k=n-1}^{n+1} \log |a_k| + \log 30 \quad (n \geq 2).
\end{align*}
\]

For notational convenience we will assume that \( M(r) \) is continuous and strictly increasing. This is no restriction since for any \( M(r) \) given for \( r \geq 0 \) with \( 0 < M(r) \to \infty \), we may choose a continuous, strictly increasing \( \mu'(r) \) with \( 0 < \mu'(r) < \mu(r) \) \( (r \geq 0) \) and such that \( \mu' \to \infty \). Then

\[
\mu'(r) \log r < \mu(r) \log r.
\]

We may also assume without restriction that \( \mu(0) < 2 \).

Now define \( r_0 \) and the sequence \( \{(\varrho_n, r_n)\}_1^\infty \) by

\[
\begin{align*}
  r_0 &= \max \left\{ \mu^{-1}(2c_0), \epsilon \right\} \\
  \varrho_1 &= \max \left\{ r_0 \left( 1 + \frac{2|a_1|}{\epsilon_1} \right), \mu^{-1}(2c_1) \right\} \\
  r_1 &= \varrho_1 \left( 1 + \max \left\{ \frac{2^3|a_1|}{\epsilon_1}, 2 \right\} \right) \\
  \varrho_n &= \max \left\{ r_{n-1} \left( 1 + \max \left[ \frac{2^{n+1}|a_{n-1}|}{\epsilon_{n-1}}, \frac{2^n|a_n|}{\epsilon_n}, 2 \right] \right), \mu^{-1}(2c_n), \mu^{-1}(2n) \right\} \\
  r_n &= \varrho_n \left( 1 + \max \left\{ \frac{(n-1)2^{n+2} \sum_{k=1}^{n} |a_k|}{\epsilon_n}, 2 \right\} \right) \quad (n \geq 2).
\end{align*}
\]
Note that \( \{ (\varphi_n, r_n) \}_{1}^{\infty} \) satisfies (5).

Upon taking (6) as our definition of the function \( \varnothing \) we can prove

**Theorem 1.** (6) converges subuniformly to a meromorphic function \( \varnothing \) and satisfies (7) and (2).

Then we have

**Corollary.** The function \( \varnothing(z) \) is meromorphic in \( |z| < \infty \) and has the properties (1) and (2).

Now for \( |z| \leq r_{k-1} \) (\( k \geq 1 \)) we have from (5)

\[
\left| \frac{\varnothing_{k+1}}{\varnothing_{k+1} - z} - \frac{\varnothing_{k}}{\varnothing_{k} - z} \right| = \left| \frac{z(\varnothing_{k} - \varnothing_{k+1})}{(\varnothing_{k+1} - z)(\varnothing_{k} - z)} \right|
\]

\[
\leq \frac{r_{k-1}(\varnothing_{k+1} - \varnothing_{k})}{(\varnothing_{k+1} - r_{k-1})(\varnothing_{k} - r_{k-1})}
\]

\[
= \frac{\varnothing_{k}}{\varnothing_{k} - r_{k-1}} - \frac{\varnothing_{k+1}}{\varnothing_{k+1} - r_{k-1}}
\]

\[
< \frac{\varnothing_{k}}{\varnothing_{k} - r_{k-1}} - 1
\]

\[
= \frac{1}{r_{k-1}} - 1
\]

So from (11)

**(12)**

\[
\left| \frac{\varnothing_{k+1}}{\varnothing_{k+1} - z} - \frac{\varnothing_{k}}{\varnothing_{k} - z} \right| \leq \frac{\varepsilon_k}{2^k |a_k|} \quad \text{for} \quad |z| \leq r_{k-1} \quad (k \geq 1).
\]
We may now prove that (6) converges subuniformly in $|z| < \infty$. Let $K$ be a compact subset of $|z| < \infty$, and choose $r_n$ such that $K \subset |z| < r_n$. Let

$$R_n(z) = \sum_{k=1}^{n} a_k \left( \frac{\phi_{k+1}}{\phi_{k+1} - z} - \frac{\phi_k}{\phi_k - z} \right).$$

Observe that from (12) we have

$$\sum_{m=1}^{\infty} \left| a_k \left( \frac{\phi_{k+1}}{\phi_{k+1} - z} - \frac{\phi_k}{\phi_k - z} \right) \right| < \varepsilon_m \quad z \in K \quad (m > n).$$

Thus since $\varepsilon_n \downarrow 0$ we see that

$$\phi(z) = R_n(z) + \sum_{n+1}^{\infty} a_k \left( \frac{\phi_{k+1}}{\phi_{k+1} - z} - \frac{\phi_k}{\phi_k - z} \right)$$

is the sum of the rational function $R_n(z)$ and the infinite sum which is uniformly convergent on $K$. Thus $\phi(z)$ is meromorphic in $|z| < \infty$.

Now for $|z| \geq r_n \quad (n \geq 2)$ we have for $k \leq n$

$$\left| \frac{\phi_k}{\phi_k - z} \right| \leq \frac{\phi_k}{r_n - \phi_k} = \frac{1}{r_n - 1} - \frac{1}{\phi_k - 1}.$$ 

Thus from (11)

$$\left| \frac{\phi_k}{\phi_k - z} \right| \leq \frac{\varepsilon_n}{(n-1)^{2n+2} \sum_{1}^{n} |a_n|} \leq \frac{\varepsilon_n}{(n-1)^{2n+2} (|a_k| + |a_{k-1}|)} \quad (n \geq 2)$$

and for $|z| \geq r_n \quad (n \geq 2)$ and $k < n$ we have
\[ (13) \quad \left| \frac{\varphi_{k+1}}{\varphi_{k+1} - z} - \frac{\varphi_k}{\varphi_k - z} \right| \leq \frac{\epsilon_n}{(n-1)2^{n+2}(|a_{k+1}| + |a_k|)} + \frac{\epsilon_n}{(n-1)2^{n+2}(|a_k| + |a_{k-1}|)} < \frac{\epsilon_n}{(n-1)2^{n+1}|a_k|}. \]

Now for \(|z| = r_n\), \((n \geq 1)\) we have from (11)

\[ \left| \frac{\varphi_{n+1}}{\varphi_{n+1} - z} - 1 \right| = \left| \frac{1}{\frac{\varphi_{n+1}}{z} - 1} \right| \leq \frac{1}{\frac{\varphi_{n+1}}{r_n} - 1} \leq \frac{\epsilon_n}{2^{n+2}|a_n|}. \]

Also from (11) for \(|z| = r_1\)

\[ \left| \frac{\varphi_1}{\varphi_1 - z} \right| \leq \frac{\varphi_1}{r_1 - \varphi_1} = \frac{1}{r_1 \varphi_1 - 1} \leq \frac{\epsilon_1}{2^{|a_1|}}. \]

and for \(|z| = r_n\) \((n \geq 2)\)

\[ \left| \frac{\varphi_n}{\varphi_n - z} \right| \leq \frac{\varphi_n}{r_n - \varphi_n} = \frac{1}{r_n \varphi_n - 1} \leq \frac{\epsilon_n}{2^{n+2}|a_n|}. \]

Thus for \(|z| = r_n\) \((n \geq 1)\) we have

\[ (14) \quad \left| \frac{\varphi_{n+1}}{\varphi_{n+1} - z} - \frac{\varphi_n}{\varphi_n - z} - 1 \right| \leq \left| \frac{\varphi_{n+1}}{\varphi_{n+1} - z} - 1 \right| + \left| \frac{\varphi_n}{\varphi_n - z} \right| \leq \frac{\epsilon_n}{2^{n+1}|a_n|}. \]

Combining (12), (13), and (14) we have for \(|z| = r_1\)
\[ |\varphi(z) - a_1| \leq |a_1| \left| \frac{\varphi_2}{\varphi_{2-z}} - \frac{\varphi_1}{\varphi_{1-z}} - 1 \right| + \sum_{2}^{\infty} |a_k| \left| \frac{\varphi_{k+1}}{\varphi_{k+1-z}} - \frac{\varphi_{k}}{\varphi_{k-z}} \right| \]

\[ \leq \frac{\varepsilon_1}{2^2} + \sum_{2}^{\infty} \frac{\varepsilon_k}{2^k} \]

\[ < \varepsilon_1 \]

and for \(|z| = r_n\ (n \geq 2)\)

\[ |\varphi(z) - a_n| \leq \sum_{1}^{n-1} |a_k| \left| \frac{\varphi_{k+1}}{\varphi_{k+1-z}} - \frac{\varphi_k}{\varphi_{k-z}} \right| + |a_n| \left| \frac{\varphi_{n+1}}{\varphi_{n+1-z}} - \frac{\varphi_n}{\varphi_{n-z}} - 1 \right| \]

\[ + \sum_{n+1}^{\infty} |a_k| \left| \frac{\varphi_{k+1}}{\varphi_{k+1-z}} - \frac{\varphi_k}{\varphi_{k-z}} \right| \]

\[ \leq \sum_{1}^{n-1} \frac{\varepsilon_n}{(n-1)2^{n+1}} + \frac{\varepsilon_n}{2^{n+1}} + \sum_{n+1}^{\infty} \frac{\varepsilon_k}{2^k} \]

\[ = \sum_{1}^{\infty} \frac{\varepsilon_k}{2^k} \]

\[ < \varepsilon_n \]

Thus (7) has been proved.

Now for \(|z| \leq \varphi_n\ (n \geq 1)\) we have from (11)

\[ \left| \frac{\varphi_{n+1}}{\varphi_{n+1-z}} - 1 \right| = \left| \frac{1}{\varphi_{n+1}} - 1 \right| \leq \frac{1}{\varphi_{n+1}} - 1 \leq \frac{1}{\varphi_n} - 1 \]

\[ < \frac{1}{\varphi_n} - 1 \leq \frac{1}{2}. \]
Thus

\[ | \frac{\varphi_{n+1}}{\varphi_{n+1} - z} | < \frac{3}{2} \quad \text{for} \quad |z| \leq \varphi_n \quad (n \geq 1). \]

Also from (11) for \( |z| \geq \varphi_n \quad (n \geq 2) \)

\[ \left| \frac{\varphi_{n-1}}{\varphi_{n-1} - z} \right| \leq \frac{\varphi_{n-1}}{\varphi_n - \varphi_{n-1}} = \frac{1}{\varphi_n - 1} = \frac{1}{\varphi_n - 1} \leq \frac{1}{2} \]

We are now in a position to prove

\[ m(r, \infty) = \frac{1}{2\pi} \int_0^{2\pi} \log |\varphi(re^{i\theta})| \ d\theta < \frac{1}{2} \mu(r) \log r \quad (r \geq 0). \]

In the following computation we will use the facts that

\[ \log \prod_{i=1}^{p} \alpha_i \leq \sum_{i=1}^{p} \log \alpha_i \quad \text{and} \]

\[ \log \sum_{i=1}^{p} \alpha_i \leq \sum_{i=1}^{p} \log \alpha_i + \log p. \]

For \( 0 \leq r \leq r_0 \) we have from (12)

\[ |\varphi(re^{i\theta})| \leq \sum_{k=1}^{\infty} |a_k| \left| \frac{\varphi_{k+1}}{\varphi_{k+1} - re^{i\theta}} - \frac{\varphi_k}{\varphi_k - re^{i\theta}} \right| \]

\[ < \sum_{k=1}^{\infty} \frac{\varepsilon_k}{2^k} \]

\[ < 1 \]
Thus

\[(18) \quad m(r, \infty) = 0 \text{ for } 0 \leq r \leq r_0\]

For \(r_0 < r \leq \rho_1\) we have again from (12)

\[
\sum_{2}^{\infty} |a_k| \left| \frac{\varphi_{k+1}}{\varphi_{k+1-re^{i\theta}}} - \frac{\varphi_k}{\varphi_{k-re^{i\theta}}} \right| < \sum_{2}^{\infty} \frac{\varepsilon_k}{2^k} < 1
\]

Thus for \(r_0 < r \leq \rho_1\) we have from (15)

\[
\log^+ |\varphi(re^{i\theta})| \leq \log^+ \left( |a_1| \left| \frac{\varphi_2}{\varphi_2-re^{i\theta}} - \frac{\varphi_1}{\varphi_1-re^{i\theta}} \right| \right)
\]

\[
+ \log^+ \sum_{2}^{\infty} |a_k| \left| \frac{\varphi_{k+1}}{\varphi_{k+1-re^{i\theta}}} - \frac{\varphi_k}{\varphi_{k-re^{i\theta}}} \right| + \log 2
\]

\[
\leq \log^+ \left( \left| \frac{\varphi_1}{\varphi_1-re^{i\theta}} \right| + \frac{3}{2} \right) + \log^+ |a_1| + \log 2
\]

\[
\leq \log^+ \left| \frac{1}{1-re^{i\theta}} \right| + \log^+ |a_1| + \log 6
\]

Thus from (9) and (10), for \(r_0 < r \leq \rho_1\)

\[
\frac{1}{2\pi} \int_{0}^{2\pi} \log^+ |\varphi(re^{i\theta})| \, d\theta \leq \frac{1}{2\pi} \int_{0}^{2\pi} \log^+ \left| \frac{1}{1-re^{i\theta}} \right| \, d\theta
\]

\[
+ \log^+ |a_1| + \log 6
\]

\[
\leq \frac{1}{2}B + \log^+ |a_1| + \log 6
\]

\[= c_0,\]
and we have

\[ m(r, \infty) \leq c_0 \text{ for } r_0 < r \leq \rho_1 \].

For \( \rho_1 < r \leq \rho_2 \) we have as before from (12)

\[ \sum_{3}^{\infty} |a_k| \left| \frac{\varphi_{k+1}}{\varphi_{k+1} - re^{i\theta}} - \frac{\varphi_k}{\varphi_k - re^{i\theta}} \right| < \sum_{3}^{\infty} \frac{\varepsilon_k}{2^k} < 1. \]

Thus for \( \rho_1 < r \leq \rho_2 \) it follows from (15) and (16) that

\[ + \log |\varphi(re^{i\theta})| \leq \log \left( |a_1| \left| \frac{\varphi_2}{\varphi_2 - re^{i\theta}} - \frac{\varphi_1}{\varphi_1 - re^{i\theta}} \right| \right) + \log \left( |a_2| \left| \frac{\varphi_3}{\varphi_3 - re^{i\theta}} - \frac{\varphi_2}{\varphi_2 - re^{i\theta}} \right| \right) + \log \left( \sum_{3}^{\infty} |a_k| \left| \frac{\varphi_{k+1}}{\varphi_{k+1} - re^{i\theta}} - \frac{\varphi_k}{\varphi_k - re^{i\theta}} \right| + \log 3 \right) \]

\[ \leq \log \left( \left| \frac{\varphi_2}{\varphi_2 - re^{i\theta}} \right| + \left| \frac{\varphi_1}{\varphi_1 - re^{i\theta}} \right| \right) + \log \left( \left| \frac{\varphi_2}{\varphi_2 - re^{i\theta}} \right| + \frac{3}{2} \right) + \sum_{1}^{2} + |a_k| + \log 3 \]

\[ \leq 2 \log \left| \frac{1}{1 - \frac{r}{\varphi_2}} \right| + \log \left| \frac{1}{1 - \frac{r}{\varphi_1}} \right| + \sum_{1}^{2} + |a_k| + \log 18, \]
and from (8), (9), and (10) we have for \( \rho_1 < r \leq \rho_2 \)

\[
\frac{1}{2\pi} \int_0^{2\pi} \log |\phi(re^{i\theta})| \, d\theta \leq \frac{1}{\pi} \int_0^{2\pi} \log \left| \frac{1}{1 - \frac{r}{\rho_2}} \right| \, d\theta \\
+ \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{1}{1 - \frac{r}{\rho_1}} \right| \, d\theta + \sum_{l=1}^{2} \log |a_k| + \log 18
\]

\[
\leq B + \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{1}{1 - \frac{r}{\rho_1}} \right| \, d\theta + \sum_{l=1}^{2} \log |a_k| + \log 18
\]

\[= c_1.
\]

Thus

(20) \[ M(r, \omega) \leq c_1 \text{ for } \rho_1 < r \leq \rho_2. \]

Now for \( \rho_n < r \leq \rho_{n+1} \) \((n \geq 2)\) we have from (12) and (13), if we interpret \( \sum_{l=1}^{0} ( ) = 0 \),

\[
\sum_{l=1}^{n-2} |a_k| \left| \frac{\phi_{k+1}}{\phi_{k+1} - re^{i\theta}} - \frac{\phi_k}{\phi_k - re^{i\theta}} \right| \leq \sum_{l=1}^{n-2} \frac{\varepsilon_{n-1}}{(n-2)^2} < 1 \text{ and }
\]

\[
\sum_{n+2}^{\infty} |a_k| \left| \frac{\phi_{k+1}}{\phi_{k+1} - re^{i\theta}} - \frac{\phi_k}{\phi_k - re^{i\theta}} \right| \leq \sum_{n+2}^{\infty} \frac{\varepsilon_k}{2^k} < 1.
\]
Thus

\[ \log |\varphi(r e^{i\theta})| \leq \sum_{l=1}^{n-2} |a_k| \left| \frac{\varphi_{k+1}}{\varphi_{k+1} - r e^{i\theta}} - \frac{\varphi_k}{\varphi_k - r e^{i\theta}} \right| \]

\[ + \sum_{n-l}^{n+1} \log \left( |a_k| \left| \frac{\varphi_{k+1}}{\varphi_{k+1} - r e^{i\theta}} - \frac{\varphi_k}{\varphi_k - r e^{i\theta}} \right| \right) \]

\[ + \log \sum_{n+2}^{\infty} |a_k| \left| \frac{\varphi_{k+1}}{\varphi_{k+1} - r e^{i\theta}} - \frac{\varphi_k}{\varphi_k - r e^{i\theta}} \right| + \log 5 \]

\[ \leq \sum_{n-l}^{n+1} \log \left| \frac{\varphi_{k+1}}{\varphi_{k+1} - r e^{i\theta}} - \frac{\varphi_k}{\varphi_k - r e^{i\theta}} \right| + \sum_{n-l}^{n+1} \log |a_k| \]

\[ + \log 5 \]

So from (15) and (16) we have for \( \varphi_n < r < \varphi_{n+1} \) (\( n \geq 2 \))

\[ \log |\varphi(r e^{i\theta})| \leq \log \left( \left| \frac{\varphi_n}{\varphi_n - r e^{i\theta}} \right| + \frac{1}{2} \right) + \log \left( \left| \frac{\varphi_{n+1}}{\varphi_{n+1} - r e^{i\theta}} \right| + \frac{\varphi_n}{\varphi_n - r e^{i\theta}} \right) \]

\[ + \log \left( \left| \frac{\varphi_{n+1}}{\varphi_{n+1} - r e^{i\theta}} \right| + \frac{3}{2} \right) + \sum_{n-l}^{n+1} \log |a_k| + \log 5 \]

\[ \leq 2 \log \left| \frac{1}{1 - \frac{r e^{i\theta}}{\varphi_n}} \right| + 2 \log \left| \frac{1}{\varphi_n} \right| \]

\[ + \sum_{n-l}^{n+1} \log |a_k| + \log 30 \]
Thus for \( \varphi_n < r \leq \varphi_{n+1} \) (n \geq 2) we have from (8), (9), and (10)

\[
\frac{1}{2\pi} \int_0^{2\pi} \log |\varphi(re^{i\theta})| \, d\theta \leq \frac{1}{n} \int_0^{2\pi} \log \left| \frac{1}{1 - \varphi_{n+1}} \right| \, d\theta \\
+ \frac{1}{n} \int_0^{2\pi} \log \left| \frac{1}{1 - \varphi_n} \right| \, d\theta \\
+ \sum_{n-1}^{n+1} \log |a_k| + \log 30 \\
\leq B + \frac{1}{n} \int_0^{2\pi} \log \left| \frac{1}{1-e^{i\theta}} \right| \, d\theta \\
+ \sum_{n-1}^{n+1} \log |a_k| + \log 30 \\
= c_n,
\]

and

(21) \quad m(r, \omega) \leq c_n \text{ for } \varphi_n < r \leq \varphi_{n+1} \text{ (n \geq 2)}.\]

Thus from (18) through (21) and (11)

\[
m(r, \omega) = 0, \quad 0 \leq r \leq r_0
\]

and

\[
m(r, \omega) \leq \begin{cases} 
\frac{1}{2} \mu(r_0) < \frac{1}{2} \mu(r) \log r, & r_0 < r \leq \varphi_1 \\
\frac{1}{2} \mu(\varphi_n) < \frac{1}{2} \mu(r) \log r, & \varphi_n < r \leq \varphi_{n+1} \text{ (n \geq 1)},
\end{cases}
\]
and we see that (17) has been proved.

Now since $\varphi_1 > 1$, we have for $\varphi_n < r \leq \varphi_{n+1}$ \,(n \geq 1)

$$N(r, \infty) = \int_0^r \frac{n(t, \infty) - n(0, \infty)}{t} \, dt + n(0, \infty) \log r$$

$$\leq n \int_1^r \frac{dt}{t}$$

$$= n \log r$$

Thus for $\varphi_n < r \leq \varphi_{n+1}$ \,(n \geq 1) we have from (11)

$$N(r, \infty) \leq \frac{1}{2} \mu(\varphi_n) \log r < \frac{1}{2} \mu(r) \log r$$

Then since it is clear that $N(r, \infty) = 0$, for $r \leq \varphi_1$, we have

$$N(r, \infty) < \frac{1}{2} \mu(r) \log r \quad (r \geq 0)$$

This result combined with (17) yields (2).
SECTION II

G.R. MacLane has proved

**Theorem 2.** Let \( f(z) \) be holomorphic in \( |z| < 1 \) and suppose there exists a dense set \( \Theta \) on \([0, 2\pi]\) such that

\[
(22) \quad \int_0^1 (1 - r) \log |f(re^{i\theta})| \, dr < \infty \quad \forall \theta \in \Theta.
\]

Then any component of \( \{z \mid |f(z)| = c\} \) which is not compact must tend, at each end, to a definite point of \( \{|z| = 1\} \).

The purpose of this section is to construct a meromorphic function with "wobbly" level curves, which indicates that some condition, such as (22), is necessary to conclude that level curves end at points. More specifically, we will construct a function, holomorphic in \( |z| < 1 \), which has a level curve one component of which tends, at one end, to an arc on \( \{|z| = 1\} \).

Let \( S \) denote the partially open square \( \{z = x + iy \mid 0 < x < 1, \ 0 \leq y \leq 1\} \). Let \( l = a_1 > a_2 > \ldots > a_n > 0 \) be a given sequence. Define

\[
\Gamma = \left( \bigcup_{l=1}^{\infty} \{z \mid x = a_n, \ 0 < y \leq \frac{3}{4}\} \right) \cup \{z \mid 0 < x \leq 1, \ y = 0\}
\]

\[
\Gamma^* = \left( \bigcup_{l=1}^{\infty} \{z \mid x = \frac{a_{n+1} + a_n}{2}, \ \frac{1}{4} \leq y < 1\} \right) \cup \{z \mid 0 < x \leq \frac{a_1 + a_2}{2}, \ y = 1\}
\]

![Figure 1](image-url)
We can prove

**Theorem 3.** Given \( \epsilon > 0 \). There exists a function \( f(z) \), holomorphic in \( S \), such that

\[
\begin{align*}
|f(z) - 1| < \epsilon, & \quad z \in \Gamma \\
|f(z)| < \epsilon, & \quad z \in \Gamma^* 
\end{align*}
\]

(23)

From Theorem 3 the desired result easily follows. We choose \( 0 < \epsilon < \frac{1}{2} \) and let \( \Gamma = \{ z \mid |f(z)| < \frac{1}{2} \} \). Then \( \Gamma^* \) is contained in some component, say \( \Gamma' \), of \( \Gamma \). We let \( A \) denote the boundary of \( \Gamma' \) and note that \( A \) is a component of the level curve \( \{ z \mid |f(z)| = \frac{1}{2} \} \). Note also that \( \Gamma \cap (\Gamma' \cup A) = \emptyset \). We now let \( \Psi' \) be a conformal map of \( |z| < 1 \) onto the interior of \( S \). Then \( \Psi' \) has a continuous extension \( \Psi \) to \( |z| < 1 \) and \( \Psi \) gives a one to one correspondence between \( \{ |z| = 1 \} \) and the boundary of \( S \). Thus letting \( \Psi(z) = f(\Psi(z)) \) we have

**Corollary.** The function \( \Psi(z) \) is holomorphic in \( |z| < 1 \) and the component \( \Psi^{-1}(A) \) of the level curve \( \{ z \mid |\Psi(z)| = \frac{1}{2} \} \) tends, at one end, to an arc on \( \{ |z| = 1 \} \).

The proof of Theorem 3 is an extension of a familiar argument. If there existed a rectifiable Jordan curve \( C \) which contained \( \Gamma \) in its interior and \( \Gamma^* \) in its exterior, then we would have

(24)

\[
F(z) = \frac{1}{2\pi i} \int_C \frac{dz}{3 - z} = \begin{cases} 0 & z \in \Gamma^* \\ 1 & z \in \Gamma \end{cases}
\]

We could then find a rational function \( R(z) \), with poles only on \( C \), approximating \( F(z) \) on \( \Gamma \) and \( \Gamma^* \) and, using Runge's pole-pushing process, obtain a function \( f(z) \), holomorphic in \( S \), approximating \( R(z) \) on \( \Gamma \) and \( \Gamma^* \). Then Theorem 3 would be proved.
But clearly no such rectifiable curve can exist. We will define a curve \( C \) which contains \( \Gamma \) in its interior and \( \Gamma^* \) in its exterior and prove that the improper integral \( \int_C \frac{d\zeta}{\zeta - z} \) "converges" (in a suitable sense) and satisfies (24). It will then be a simple matter to approximate \( \int_C \frac{d\zeta}{\zeta - z} \) on \( \Gamma \) and \( \Gamma^* \) by a function holomorphic in \( S \).

Now define

\[
\delta_n = \begin{cases} 
\min \left\{ \frac{a_1 - a_2}{4}, \frac{1}{8} \right\}, & n = 1 \\
\min \left\{ \frac{a_{n-1} - a_n}{4}, \frac{a_n - a_{n+1}}{4}, \frac{1}{8} \right\}, & n \geq 2
\end{cases}
\]

and let

\[
C_n = \left\{ z \mid a_{n+1} + \delta_{n+1} \leq x \leq a_n + \delta_n, \ y = -\frac{1}{8} \right\} \\
\cup \left\{ z \mid a_{n+1} + \delta_{n+1} \leq x \leq a_n - \delta_n, \ y = \frac{1}{8} \right\} \\
\cup \left\{ z \mid x = a_n + \delta_n, \ \frac{1}{8} \leq y \leq \frac{3}{4} \right\} \cup \left\{ z \mid x = a_n - \delta_n, \ \frac{1}{8} < y \leq \frac{3}{4} \right\} \\
\cup \left\{ z \mid |z - a_n - \frac{3}{4} i| = \delta_n, \ y > \frac{3}{4} \right\} \quad (n \geq 1),
\]

\[
\gamma_n = \left\{ z \mid x = a_n + \delta_n, \ -\frac{1}{8} < y < \frac{1}{8} \right\} \quad (n \geq 1), \text{ and}
\]

\[
\gamma = \left\{ z \mid x = 0, \ -\frac{1}{8} < y < \frac{1}{8} \right\}.
\]

The curve \( \gamma \) will be considered as oriented with the negative direction of the imaginary axis. We will consider \( \gamma_n \) and \( C_n \) oriented as shown in the following diagram.
We also define
\[ C = \bigcup_{1}^{\infty} c_n \cup \gamma_1 \cup \gamma \],
\[ D_n = \left\{ z \left| \text{dist} (z, c_n) < \frac{\delta_n}{2} \right. \right\} \quad (n \geq 1), \quad \text{and} \]
\[ D = \bigcup_{1}^{\infty} D_n \].

Note from (25) it is clear that \( \Gamma \) is interior and \( \Gamma^* \) exterior to \( C \). Note also that \( C \subset D \) and \( (\Gamma \cup \Gamma^*) \cap D = \emptyset \).

We now write formally
\[ F(z) = \frac{1}{2\pi i} \int_{C} \frac{d\gamma}{3 - z} = \sum_{1}^{\infty} \frac{1}{2\pi i} \int_{C_n} \frac{d\gamma}{3 - z} + \frac{1}{2\pi i} \int_{\gamma_1} \frac{d\gamma}{3 - z} + \frac{1}{2\pi i} \int_{\gamma} \frac{d\gamma}{3 - z} \]

and prove

Lemma 1. (26) converges subuniformly in \( S - C \) to a function \( F(z) \) which satisfies
\[ F(z) = \begin{cases} 0, & z \text{ exterior to } C \\ 1, & z \text{ interior to } C \end{cases} \]
Write \( \Im = \gamma + i\eta \). Now choose \( z_0 \in S - C \) and let \( N \) be a disc about \( z_0 \) and \( k_0 \) an integer such that \( x - a_k - \delta_k \geq \varphi > 0 \) for \( z = x + iy \in N \) and \( k \geq k_0 \). Then

\[
\left| \frac{1}{2\pi i} \int_{\gamma_k} \frac{d\zeta}{3 - \zeta} - \frac{1}{2\pi i} \int_{-\gamma} \frac{d\zeta}{3 - \zeta} \right| = \frac{1}{2\pi} \int_{\gamma} \frac{d\eta}{a_k + \delta_k + i\eta - z} - \int_{\gamma} \frac{d\eta}{i\eta - z} \\
\leq \frac{1}{2\pi} \int_{\gamma} \frac{a_k + \delta_k}{|a_k + \delta_k + i\eta - z||i\eta - z|} \, d\eta \\
\leq \frac{1}{2\pi} \int_{\gamma} \frac{a_k + \delta_k}{\varphi^2} \, d\eta \\
= \frac{a_k + \delta_k}{8\pi \varphi^2} \quad (z \in N, \ k \geq k_0) .
\]

Thus

\[
\lim_{k \to \infty} \frac{1}{2\pi i} \int_{\gamma_k} \frac{d\zeta}{3 - \zeta} = \frac{1}{2\pi i} \int_{-\gamma} \frac{d\zeta}{3 - \zeta} \quad (z \in S)
\]

(28)

and the convergence is subuniform in \( S \).

Now for \( z \) exterior to \( S \) we have

\[
\int_{C_k + \gamma_k - \gamma_{k+1}} \frac{d\zeta}{3 - \zeta} = 0 \quad (k \geq 1) .
\]
Thus for $z$ exterior to $C$ we have from (28)

$$
\sum_{k=1}^{\infty} \frac{1}{2\pi i} \oint_{C_k} \frac{d\zeta}{\zeta - z} = \lim_{n \to \infty} \sum_{k=1}^{n} \left( \frac{1}{2\pi i} \oint_{C_k + \gamma_k - \gamma_{k+1}} \frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i} \oint_{\gamma_k - \gamma_{k+1}} \frac{d\zeta}{\zeta - z} \right)
$$

$$
= \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{2\pi i} \oint_{\gamma_k - \gamma_{k+1}} \frac{d\zeta}{\zeta - z}
$$

$$
= \lim_{n \to \infty} \frac{1}{2\pi i} \oint_{\gamma_{n+1} - \gamma_1} \frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i} \oint_{\gamma_1} \frac{d\zeta}{\zeta - z}
$$

$$
= \frac{1}{2\pi i} \oint_{\gamma} \frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i} \oint_{\gamma} \frac{d\zeta}{\zeta - z}.
$$

Thus from (28)

$$
\left\{ \sum_{k=1}^{\infty} \frac{1}{2\pi i} \oint_{\gamma_k} \frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i} \oint_{\gamma_1} \frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i} \oint_{\gamma} \frac{d\zeta}{\zeta - z} \right\} = 0
$$

(29)

for $z$ exterior to $C$ and the convergence is subuniform in $S$.

Now define $\Gamma_m = \gamma_m + \gamma_{m+1} + \gamma_m - \gamma_{m+2}$. Then for $z$ interior to $C$ we have $z$ interior to some $\Gamma_m$. Then

$$
\oint_{\gamma_k + \gamma_k - \gamma_{k+1}} \frac{d\zeta}{\zeta - z} = 0 \quad (k \neq m, m+1, k \geq 1) \quad \text{and}
$$

$$
\oint_{\Gamma_m} \frac{d\zeta}{\zeta - z} = 1.
$$
Thus for \( z \) interior to \( C \) we have from (28)

\[
\sum_{k=1}^{\infty} \frac{1}{2\pi i} \int_{\gamma_k} \frac{d\zeta}{\zeta - z} = \lim_{n \to \infty} \sum_{k=1}^{n} \left( \frac{1}{2\pi i} \int_{c_k + \gamma_k - \gamma_{k+1}} \frac{d\zeta}{\zeta - z} \right) \left( \frac{d\zeta}{\zeta - z} \right)_{\gamma_{k+1}}
\]

\[
= \frac{1}{2\pi i} \int_{c_m + \gamma_m - \gamma_{m+1}} \frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i} \int_{c_{m+1} + \gamma_{m+1} - \gamma_{m+2}} \frac{d\zeta}{\zeta - z}
\]

\[- \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{2\pi i} \int_{\gamma_{k+1} - \gamma_k - 1} \frac{d\zeta}{\zeta - z}
\]

\[
= \frac{1}{2\pi i} \int_{\Gamma_m} \frac{d\zeta}{\zeta - z} + \lim_{n \to \infty} \frac{1}{2\pi i} \int_{\gamma_{n+1} - \gamma_1} \frac{d\zeta}{\zeta - z}
\]

\[
= 1 + \frac{1}{2\pi i} \int_{-\gamma} \frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i} \int_{\gamma_1} \frac{d\zeta}{\zeta - z}
\]

Thus from (28)

\[
\left\{ \begin{array}{l}
\sum_{k=1}^{\infty} \frac{1}{2\pi i} \int_{c_k} \frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i} \int_{\gamma_1} \frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i} \int_{\gamma} \frac{d\zeta}{\zeta - z} = 1
\end{array} \right.
\]

(30)

for \( z \) interior to \( C \) and the convergence is subuniform in \( S \).

Combining (29) and (30), we see that Lemma 1 has been proved.

It should be noted that in particular (24), with the suitable interpretation of \( \int_{c} \frac{d\zeta}{\zeta - z} \), is satisfied.
Now for each \( n \), choose a rational function \( R_n(z) \) with poles only on \( C_n \) such that

\[
\left| \frac{1}{2\pi i} \int_{C_n} \frac{d\zeta}{\zeta - z} - R_n(z) \right| < \frac{\epsilon}{2^{n+1}} \quad \text{for } z \neq D_n.
\]

Using Runge's pole-pushing process, let \( \phi_n(z) \) be a function, holomorphic in \( S \), obtained by pushing the poles of \( R_n(z) \) out toward \( x = 0 \) through the channel \( \Delta_n = \bigcup_{n} D_k \) in such a way that

\[
|R_n(z) - \phi_n(z)| < \frac{\epsilon}{2^{n+1}} \quad \text{for } z \in S - \Delta_n.
\]

Thus

\[
\left| \frac{1}{2\pi i} \int_{C_n} \frac{d\zeta}{\zeta - z} - \phi_n(z) \right| < \frac{\epsilon}{2^n} \quad \text{for } z \in S - \Delta_n.
\]

Now the series

\[
\sum_{n} \frac{1}{2\pi i} \int_{C_k} \frac{d\zeta}{\zeta - z}
\]

is, as we have seen, subuniformly convergent in \( S - \Delta_n \); thus from (33) we see that

\[
\phi(z) = \sum_{n=1}^{\infty} \phi_n(z)
\]

converges subuniformly in \( S \) to a holomorphic function \( \phi \). Also from (33)

\[
\left| \phi(z) - \sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_{C_k} \frac{d\zeta}{\zeta - z} \right| < \epsilon \quad \text{for } z \in S - D.
\]
We note that the functions \( \frac{1}{2\pi i} \int_{\gamma_1} \frac{d\zeta}{\zeta - z} \) and \( \frac{1}{2\pi i} \int_{\gamma} \frac{d\zeta}{\zeta - z} \) are holomorphic in \( S \). Thus if we set

\[
f(z) = \phi(z) + \frac{1}{2\pi i} \int_{\gamma_1} \frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i} \int_{\gamma} \frac{d\zeta}{\zeta - z},
\]

we see from (26), (27), and (34) \( f(z) \) satisfies (23).

2. G. R. MacLane, On the asymptotic values of functions holomorphic in the unit disc and of reasonable growth, in preparation.