II

UPPER SEMI-CONTINUOUS COLLECTIONS OF THE SECOND TYPE

CONSIDER a compact and connected space S satisfying Axioms 0, 1 and 2 and in which there are no contiguous points.

A collection G of continua is said to be an *upper semi-continuous collection of the first type* (type 1) provided it is true that (1) the continua of the collection G are mutually exclusive and (2) if g is a continuum of the collection G and $g_1, g_2, g_3, \cdots$ is a sequence of continua of G and, for each n, A_n and B_n are points of g_n and the sequence $A_1, A_2, A_3, \cdots$ converges to a point of g, then for every infinite subsequence of $B_1, B_2, B_3, \cdots$ there is an infinite subsequence of that subsequence converging to a point of g.

A collection G of continua is said to be an *upper semi-continuous collection of the second type* (type 2) provided it is true that (1) if g is a continuum of the collection G and $g_1, g_2, g_3, \cdots$ is a sequence of distinct continua of G and, for each n, A_n and B_n are points of g_n and the sequence $A_1, A_2, A_3, \cdots$ converges to a point of g, then for every infinite subsequence of $B_1, B_2, B_3, \cdots$ there is an infinite subsequence of that subsequence converging to a point of g, (2) no two elements of G have more than one point in common, (3) if a point is common to two elements of G it is itself an element of G and there exist at least two non-degenerate elements of G containing it, (4) if the point P is common to two elements of G and g is a non-degenerate element of G containing P then there exists a region R containing P such that no component of $R-P$ contains both a point of g and a point of some other element of G that contains P and such that furthermore no point of R belongs both to a connected subset of $R-P$ that contains a point of g and to a connected subset of R that contains P but no other point of g.

Let G denote some definite upper semi-continuous collection of continua of type 2 filling up the space S.

DEFINITION. A degenerate continuum of the collection G which is a point belonging to some non-degenerate continuum of G is called a *junction element of G.*

THEOREM 1. *No junction element of G is a point of uncountably many different continua of the collection G and if there exist infinitely many distinct continua $g_1, g_2, g_3, \cdots$ of G all containing the same point g then the sequence of point sets $g_1, g_2, g_3, \cdots$ converges to the point g.*

Proof. Suppose g is a point which is an element of G and $g_1, g_2, g_3, \cdots$ is a sequence of distinct and non-degenerate continua of the collection G all containing g. There exists a sequence of points $P_1, P_2, P_3, \cdots$ converging to the point g and such that, for each n, P_n belongs to g_n. Hence if $X_1, X_2, X_3, \cdots$ is another sequence of points such that, for each n, X_n belongs to g_n and $n_1, n_2, n_3, \cdots$ is an ascending sequence of natural numbers then, since G is upper semi-continuous, some subsequence of $X_{n_1}, X_{n_2}, X_{n_3}, \cdots$ converges to g. It follows that the sequence $X_1, X_2, X_3, \cdots$ converges to g. Let H denote the set of all continua of G that contain g. Since every sequence of continua of the set H converges to a *point*, H is countable.

THEOREM 2. *If g is a non-degenerate element of G there do not exist uncountably many elements of G which are points of g.*

Proof. Suppose there is an uncountable set H of elements of G such that every element of H is a point of g. For each point h of the set H, let g_h denote a definite non-degenerate continua of the set G which contains h but which is distinct from g. Let W denote the collection of all the continua g_h for all points h of H. Since W is an uncountable collection of non-degenerate point sets there exist a point k belonging to H, a sequence $h_1, h_2, h_3, \cdots$ of distinct points of H and a sequence of points $P_1, P_2, P_3, \cdots$ such that (1) g_k is distinct from g, (2) for each n, P_n belongs to g_{h_n} and (3) the sequence $P_1, P_2, P_3, \cdots$ converges to a point P belonging to g_k but distinct from k. There exists an ascending sequence of natural numbers $n_1, n_2, n_3, \cdots$ such that $h_{n_1}, h_{n_2}, h_{n_3}, \cdots$ converges to some point X. Since the points $h_1, h_2, h_3, \cdots$ all belong to g so must X. Therefore, since G is upper semi-continuous, some subsequence of $P_{n_1}, P_{n_2}, P_{n_3}, \cdots$ converges to X. But this is impossible since X is distinct from P.

Definition. If the element g of G is not a junction element of G then g is said to be the *sequential limit element* of the sequence $g_1, g_2, g_3, \cdots$ of elements of G provided it is true that if $h_1, h_2, h_3, \cdots$ is a convergent subsequence of the sequence of *point sets* $g_1, g_2, g_3, \cdots$ then (1) the sequence of point sets $h_1, h_2, h_3, \cdots$ converges to a subset L of the point set g and (2) if L contains a point of g which is a junction element of G then L is that point and, for every region R containing L, there exists a number m such that, for every n greater than m, h_n is a subset of some component of $R-L$ that contains some point of g.

If the element g of G is a junction element of G then g is said to be the *sequential limit element of the sequence* $g_1, g_2, g_3, \cdots$ of elements of G provided it is true that (1) the sequence of *point sets* $g_1, g_2, g_3, \cdots$ converges to the point g

and (2) if x is a non-degenerate continuum belonging to G and containing the point g there exists a region R containing g such that no continuum of the sequence $g_1, g_2, g_3, \cdots$ contains a point which lies in a component of $R-g$ that contains a point of x.

A sequence of elements of G is said to *converge* to the element g if g is the sequential limit element of that sequence.

DEFINITION. The element g is said to be a *limit element* of the set H of elements of G if g is the sequential limit element of some sequence of distinct elements of H.

The following theorem may be easily proved.

THEOREM 3. *If H and K are sets of elements of G, every limit element of $H+K$ is a limit element either of H or of K.*

A subset of G is said to be *closed* if it contains all of its limit elements.

The set D of elements of G is said to be a domain of elements of G if no element of D is a limit element of a set of elements of G no one of which belongs to D. In other words, D is a domain if no element of D is a limit element of $G-D$ that is to say if $G-D$ is closed.

THEOREM 4. *If H and K are subcollections of G and each element of K is a limit element of H then every limit element of K is a limit element of H.*

Proof. Suppose g is an element of G which is a limit element of K. There exists a sequence $k_1, k_2, k_3, \cdots$ of continua of K converging to a subset L of g such that either $k_1, k_2, k_3, \cdots$ are all junction elements of G or none of them is and such that (1) if g is not a junction element of G and L contains a point of g which is a junction element of G then L is that point and, for every region R containing L, there exists a number m such that, for every n greater than m, k_n is a subset of some component of $R-L$ that contains some point of g and (2) if g is a junction element of G then if x is a non-degenerate element of G containing g there exists a region W_x containing g such that no continuum of the sequence $k_1, k_2, k_3, \cdots$ contains a point which lies in a component of $W_x - g$ that contains a point of x.

For each n, there exists a sequence $h_{n1}, h_{n2}, h_{n3}, \cdots$ of continua of H converging to a subset L_n of k_n such that (1) if k_n is not a junction element of G and L_n contains a point of g which is a junction element of G then it is that point and for every region R containing L_n there exists a number m_{nR} such that, for every i greater than m_{nR}, h_{ni} is a subset of some component of $R - L_n$ that contains some point of k_n and (2) if k_n is a junction element of G and x is a non-degenerate element of G containing k_n there exists a region W_{xn} containing k_n such that no continuum of the sequence $h_{n1}, h_{n2}, h_{n3}, \cdots$ contains a point which lies in a component of $W_{xn} - k_n$ that contains a point of x.

Case 1. Suppose that neither g nor L is a junction element of G. Then there exists an ascending sequence of numbers $j_1, j_2, j_3, \cdots$ such that the sequence of continua $h_{1j_1}, h_{2j_2}, h_{3j_3}, \cdots$ has, as its sequential limiting set, a subset of L. The element g of G is the sequential limit element of the sequence $h_{1j_1}, h_{2j_2}, h_{3j_3}, \cdots$ of elements of G.

Case 2. Suppose that L is, but g is not, a junction element of G. There exists a sequence of connected domains $D_1, D_2, D_3, \cdots$ closing down on L. There exists an ascending sequence of natural numbers $j_1, j_2, j_3, \cdots$ such that, for each n, k_{j_n} lies in some connected subset T_n of $D_n - L$ that contains a point of g. For each n there exists a connected domain I_n containing k_{j_n} and lying wholly in $D_n - L$ and there exists a natural number i_n such that $h_{j_n i_n}$ is a subset of I_n. The sequence of continua $h_{j_1 i_1}, h_{j_2 i_2}, \cdots$ converges to the point L. Suppose R is a region containing L. There exists a number δ such that, for every n greater than δ, D_n is a subset of R. If $n>\delta$, I_n+T_n is a connected subset of $R-L$ containing $h_{j_n i_n}$ and some point of g. Therefore the element g of G is the sequential limit element of the sequence of elements $h_{j_1 i_1}, h_{j_2 i_2}, h_{j_3 i_3}, \cdots$.

Case 3. Suppose that g is a junction element of G and that there exists a number q such that if $n>q$, k_n does not contain g. Let $x_1, x_2, x_3, \cdots$ denote the non-degenerate elements of G that contain g. There exist an infinite sequence $j_1, j_2, j_3, \cdots$ of natural numbers all greater than q and a sequence $D_1, D_2, D_3, \cdots$ of domains closing down on the point g such that (1) for each n, k_{j_n} is a subset of D_n and of $S-\overline{D}_{n+1}$, (2) no matter what natural number n may be, no continuum of the sequence $k_{j_1}, k_{j_2}, k_{j_3}, \cdots$ lies in a component of D_n-g that contains a point of x_n. For each n there exists a connected domain I_n containing k_{j_n} and lying wholly in $D_n \cdot (S-\overline{D}_{n+1})$ and there exists a number i_n such that $h_{j_n i_n}$ is a subset of I_n. No continuum of the sequence $h_{j_1 i_1}, h_{j_2 i_2}, \cdots$ contains a point of a component of D_n-g that contains a point of x_n. For if $h_{j_m i_m}$ contained a point of a connected subset T_m of D_n-g containing a point of x_n then m would necessarily be equal to or greater than n and I_m+T_m would be a connected subset of D_n-g containing k_{j_m} and a point of x_n. It follows that the element g of G is the sequential limit element of the sequence $h_{j_1 i_1}, h_{j_2 i_2}, \cdots$. Hence g is a limit element of H.

Case 4. Suppose that g is a junction element of G and that there exists an infinite sequence of natural numbers $j_1, j_2, j_3, \cdots$ such that, for each n, k_{j_n} is a non-degenerate continuum containing g. There exist an infinite subsequence $i_1, i_2, i_3, \cdots$ of the sequence $j_1, j_2, j_3, \cdots$ and a sequence of domains $D_1, D_2, D_3, \cdots$ closing down on g such that, for each n, $k_{i_{n+1}}$ is a subset of D_n and there is no component of $D_n - g$ containing both a point of k_{i_n} and a point of some other continuum of G that contains the point g.

Suppose first that L_{i_n} is identical with g. Then there exists a number t_n such that $h_{i_n t_n}$ lies in some component of $D_n - g$ that contains a point of k_{i_n}.

Suppose secondly that L_{i_n} is not identical with g. Let P_n denote some point of L_{i_n} distinct from g. If $n > 1$ there exists a connected domain U_n lying in D_{n-1} and containing P_n but no point of any continuum of the sequence $x_1, x_2, x_3, \cdots$ except k_{i_n}. There exists a number t_n such that $h_{i_n t_n}$ lies in D_{n-1} and intersects U_n.

Therefore, whether or not L_{i_n} is identical with g, the continuum $h_{i_n t_n}$ lies in a component of $D_{n-1} - g$ that contains a point of k_{i_n}. It follows that if m is any natural number there exists a number δ_m such that if $n > \delta_m$ then $h_{i_n t_n}$ is not a subset of any component of $D_{\delta_m} - g$ that contains a point of x_m. Hence g is the sequential limit element of the sequence $h_{i_1 t_1}, h_{i_2 t_2}, \cdots$.

The following theorem may be easily established.

THEOREM 5. *If the sequence $H_1, H_2, H_3, \cdots$ of elements of G converges to the element L of G and, for each n, K_n is an element of G such that either H_n is a point of the continuum K_n or K_n is a point of the continuum H_n then the sequence $K_1, K_2, K_3, \cdots$ converges to L.*

THEOREM 6. *In order that the element g of G should be a limit element of the subcollection H of G it is necessary and sufficient that every domain of elements of G that contains g should contain an element of H distinct from g.*

Proof. This condition is clearly necessary. It will be shown that it is sufficient. Suppose g is not a limit element of H. Let K denote the set consisting of all limit elements of H together with all elements of H distinct from g. Let R

denote the set $G-K$. No element of R is a limit element of $G-R$. For if an element x of R were a limit element of $G-R$, that is to say of K then, by Theorems 3 and 4, x would be a limit element of H. Therefore R is a domain of elements of G. But R contains g but no element of H distinct from g.

The subcollections H and K of G are said to be *mutually separated* if no continuum of either of them is a subset of a continuum of the other one and neither of them contains a limit element of the other one.

A subcollection of G is said to be *connected* if it is not the sum of two mutually separated collections.

EXAMPLES. Suppose that the straight line intervals AB and BC have only the point B in common and that AB and BC are both continua of the collection G. Then the point B is also an element of G. The *point sets* AB and BC are connected and have a point B in common and therefore the *point set* $AB+BC$ is connected. But neither of the point sets AB and BC is a *subset* of the other one and neither of the elements AB and BC of G' is a limit element of the other one. Therefore the set of *elements of* G consisting of AB and BC is *not* connected. The *point set* $AB+BC$ is identical with the point set $AB+B+BC$. But the set whose *elements* are AB and BC is quite different from the set whose elements are the three continua AB, B and BC. Indeed the latter set is a connected set of elements of G. For if it is the sum of two sets, one of them (call it H) contains B. The other one, K, contains at least one of the continua AB and BC. But B is a subset of each of these continua. Hence H and K are not mutually separated.

If, in this example, AB, B and BC are the only elements

[1] No element of G is a limit element of a single element of G or of any finite set of elements of G.

of G then the point B is a domain of elements of G and so is AB, as well as BC. For no one of these elements is a limit element of any set of elements of G. But in the space S whose elements are the *points* of the continuum $AB+B+BC$, the point B is not a domain nor is AB, BC or any other continuum except the whole of S.

In the theory of upper semi-continuous collections of type 1, in order that the element g of G should be a limit element of the subcollection H of G it is necessary and sufficient that the point set g should contain a limit point of the point set H^*-g. This condition is neither necessary nor sufficient here. To see that it is not sufficient consider again the collection whose elements are AB, B and BC. Here the point B is a limit point of the point set $BC-B$ but the element B is not a limit element of the element BC. To see that it is not necessary consider the following example.

In a Cartesian plane let O denote the origin of coordinates and let A denote the point (1, 0). There exists a sequence $P_1, P_2, P_3, \cdots$ whose terms are the points between O and A whose abscissas are rational numbers. For each n, let A_n denote a point with the same abscissa as P_n but with an ordinate equal to $1/n$, let B_n denote a point whose abscissa is that of P_1 but whose ordinate is $-1/n$ and let P_nA_n and P_1B_n denote straight line intervals with endpoints as indicated. Let S' denote the dendron obtained by adding together the straight line interval OA, all the intervals P_1A_1, P_2A_2, P_3A_3, $\cdots$ and all the intervals P_1B_1, P_1B_2, P_1B_3, $\cdots$. Let G' denote the collection whose elements are OA, the intervals of the sequences P_1A_1, P_2A_2, P_3A_3, $\cdots$ and P_1B_1, P_1B_2, P_1B_3, $\cdots$ and the points of the sequence $P_1, P_2, P_3, \cdots$. The collection G' is an upper semi-continuous collection of type 2 filling up the space S'. Let g

denote the interval OA and let H denote the set whose elements are the points of the sequence $P_1, P_2, P_3, \cdots$. The element g is a limit element of the set H of elements of G'. But it is not true that some point of g is a limit point of the point set H^*-g. Indeed there is no such point set since H^* is a subset of g.

In the theory of upper semi-continuous collections of type 1, if H is a subcollection of G then in order that H should be closed it is necessary and sufficient that H^* should be closed. Here this condition fails as to sufficiency but not as to necessity. In the space S' of the last example, there exists an infinite ascending sequence of distinct natural numbers $n_1, n_2, n_3, \cdots$ such that the sequence of points $P_{n_1}, P_{n_2}, P_{n_3}, \cdots$ converges to the point P_1. Let H denote the set whose elements are P_1A_1 and the intervals of the sequence $P_{n_1}A_{n_1}, P_{n_2}A_{n_2}, P_{n_3}A_{n_3}, \cdots$. The point set H^* is closed but the set H of elements of G is not closed since OA is a limit element of H which does not belong to it. Hence the condition in question is not sufficient.

Again, let H denote the subcollection of G' whose elements are the intervals of the sequence $P_1B_1, P_1B_2, P_1B_3, \cdots$. The point set H^* is closed but the point P_1 is a limit element of H which does not belong to it.

The following theorem holds true.

THEOREM 7. *If T is a closed point set and H is the set of all elements of G that contain one or more points of T then H^* is closed.*

If the upper semi-continuous collection G is of type 1 and H is a subcollection of G then in order that H should be connected it is necessary and sufficient that H^* should be. But if, in the last example, H denotes the collection whose elements are the intervals OA and P_1A_1, H^* is connected but H is not. So this condition is not sufficient here.

It is however easily seen to be necessary. Furthermore the following theorem holds true.

THEOREM 8. *If T is a connected point set and H is the set of all continua of the collection G that contain one or more points of T then H is a connected set of elements of G.*

Proof. Suppose, on the contrary, that H is the sum of two mutually separated sets H_1 and H_2. Suppose the point sets $T \cdot H_1^*$ and $T \cdot H_2^*$ have a point P in common. The point P belongs to a continuum h_1 of H_1 and a continuum h_2 of H_2. Since H_1 and H_2 are mutually separated, h_1 and h_2 are distinct and non-degenerate. Hence P is an element of G. It belongs to one of the sets H_1 and H_2 and it is a subset both of the continuum h_1 of H_1 and of the continuum h_2 of H_2. This involves a contradiction. It follows that $T \cdot H_1^*$ and $T \cdot H_2^*$ are mutually exclusive. Therefore a continuum of the set H belongs to H_i $(i=1, 2)$ if, and only if, it has a point in common with $T \cdot H_i^*$.

Suppose now that one of the sets $T \cdot H_1^*$ and $T \cdot H_2^*$ contains a point X which is a limit point of the other one. Suppose $T \cdot H_1^*$ does. If X does not belong to G it is a point of a continuum g_X of G and g_X is a limit element of H_2, contrary to the supposition that H_1 and H_2 are mutually separated. If X does belong to G then it belongs to H_1 and if C_g denotes the set of all non-degenerate continua of G that contain X then C_g is a subset of H_1. Either X or some element of the set C_g is a limit element of the set H_2. Thus the supposition that Theorem 8 is false leads to a contradiction.

If the collection G is of type 1 and D is a domain containing the element g of G there exists a domain W containing g and such that every point set of the collection G that contains a point of W is a subset of D. But if D denotes the set of all points of the dendron S' whose ordinates are numerically less than $1/10$, D is a domain containing the continuum OA and no matter what point set W may be containing OA, regardless of whether it is a domain, the continua P_1A_1, P_2A_2, P_3A_3, $\cdots$, $P_{10}A_{10}$ and P_1B_1, P_1B_2, P_1B_3, $\cdots$, P_1B_{10} all belong to G and contain points of W but no one of them is a subset of D. However, the following proposition holds true.

THEOREM 9. *If D is a domain containing the element g of the collection G there exists a domain W containing g such that if there are any continua of the collection G which contain points of W but which are not subsets of D then there are only a finite number of such continua and each of them contains a point of g which is a junction element of G.*

Proof. There exists a sequence of domains D_1, D_2, D_3, $\cdots$ closing down on the point set g. Suppose that, for each n, D_n contains a point P_n of $S-g$ belonging to some continuum g_n of G which is not a subset of D. There exists a sequence of distinct natural numbers n_1, n_2, n_3, $\cdots$ such that the sequence of points P_{n_1}, P_{n_2}, P_{n_3}, $\cdots$ converges to some point P. For each i, g_{n_i} contains a point X_{n_i} of $S-D$. The point P necessarily belongs to g. Since G is upper semi-continuous it follows that there exists a subsequence m_1, m_2, m_3, $\cdots$ of the sequence n_1, n_2, n_3, $\cdots$ such that X_{m_1}, X_{m_2}, X_{m_3}, $\cdots$ converges to a point of g. But this is impossible since g is a subset of the domain D and no point of this sequence belongs to D. Hence there exists a number m such that every continuum of G which contains a point of D_m-g is a subset of D.

Suppose now there exist infinitely many distinct continua h_1, h_2, h_3, $\cdots$ of the set G such that, for each n, h_n contains both a point B_n of g and a point C_n not belonging to D. There exists an infinite sequence of distinct natural numbers n_1, n_2, n_3, $\cdots$ such that the sequence B_{n_1}, B_{n_2}, B_{n_3}, $\cdots$

converges to some point B. The point B necessarily belongs to g. Hence there exists an infinite subsequence $m_1, m_2, m_3, \cdots$ of the sequence $n_1, n_2, n_3, \cdots$ such that $C_{n_1}, C_{n_2}, C_{n_3}, \cdots$ converges to some point of g. But this is impossible.

THEOREM 10. *If D is a domain containing at least one continuum of the collection G then the collection of all continua of G that lie wholly in D is a domain of elements of G.*

Proof. Suppose g is an element of R, the set of all continua of G that lie in D. By Theorem 9 there exists a domain W containing g such that (1) D contains every point set of the collection G that contains a point of W but no point of g, (2) of the point sets of the collection G that intersect g all but a finite number are subsets of D. Suppose g is a limit element of a subcollection H of G. Then there are infinitely many elements of H each containing a point of W. Hence there are infinitely many of them lying wholly in D and therefore belonging to R. Hence R is a domain of elements of G.

DEFINITION. The sequence $D_1, D_2, D_3, \cdots$ of domains of elements of G is said to *close down* on the set K of elements of G if (1) K is the set of all elements of G which belong to every domain of this sequence, (2) for each n, $\overline{D}_{n+1}$ is a subset of D_n and (3) for every domain R of elements of G such that K is a subcollection of R there exists a number n such that D_n is a subcollection of R.

THEOREM 11. *If g is an element of G which neither is a junction element of G nor contains one and $H_1, H_2, H_3, \cdots$ is a sequence of domains (of points) closing down on the point set g and $D_1, D_2, D_3, \cdots$ is a sequence of domains of elements of G such that, for each n, D_n contains g and D_n^* is a subset of H_n then the sequence $D_1, D_2, D_3, \cdots$ closes down on the element g.*

THEOREM 12. *If D is a domain of elements of G and H is a domain of points and g is an element of G belonging to*

D and lying in H then there exists a connected domain Q of elements of G such that g belongs to Q, $\overline{Q}$ is a subset of D and Q^* is a subset of H.

Proof. There are two cases to be considered.

Case 1. Suppose g is not a junction element of G. Let C_g denote the set of all junction elements of G which are not elements of D but which are points of g. The set C_g is finite. If P is a point of g not belonging to C_g there exists a domain W_P (of points) lying in H and containing P such that every element of G that contains a point of $\overline{W}_P$ belongs to D. If P is a point of C_g there exists a domain T_P (of points) containing P and lying in H such that if x is any continuum of the collection G that contains a point lying in a component of $T_P - P$ that contains a point of g then x belongs to D. There exists a domain N_P (of points) containing P such that $\overline{N}_P$ is a subset of T_P. Let Q_P denote the set of all points y of N_P such that y belongs to a component of $N_P - P$ that contains a point of g. The point set Q_P is a domain. With the help of the Borel-Lebesgue Theorem and the fact that $P + g \cdot Q_P$ is identical with $g \cdot N_P$ it may be seen that there exists a finite set Z of domains covering $g - C_g$ such that if z is any domain of the collection Z there exists a point P of g such that z is identical with Q_P or with W_P according as P is or is not a point of the set C_g. Let Q denote the set of all elements x of G such that x and g belong to a connected set of elements of $G - C_g$ all, except g, lying in Z^*. It may be shown that Q is a connected domain of elements of G and that $\overline{Q}$ is a subset of D.

Case 2. Suppose g is a junction element of G. Let C_g denote the set of all non-degenerate elements of G, if there are any, which are not elements of D but which contain the point g. The set C_g is finite. There exists a domain W_1 containing g such that no point of W_1 lies both in a connected subset of W_1-g that contains a point of C_g^* and in a connected subset of $W_1-W_1\cdot C_g^*+g$ that contains g. There exists a domain W_2 lying in W_1 and containing g such that if x is any element of the set G which contains a point of the component of $W_1-W_1\cdot C_g^*+g$ that contains g then x belongs to D. There exists a domain W_3 containing g and lying in H such that $\overline{W}_3$ is a subset of W_2. Let Q denote the set of all elements x of G such that x and g belong to a connected set of elements of G all lying in W_3 and containing no point of C_g^* other than the point g. It may be seen that Q is a connected domain of elements of G and that $\overline{Q}$ is a subset of D.

THEOREM 13. *If the elements of G are called "points" and every domain of elements of G is called a "region" and the "point" x is said to be contiguous to the "point" y if and only if either x is an ordinary point of the continuum y or y is an ordinary point of the continuum x, the axioms of* Σ_c' *all hold true for this interpretation of point, region and contiguity.*

With the help of the preceding theorems it is easy to see that all of the axioms of Σ_c' except Axiom 1 hold true for this interpretation. It will be shown that Axiom 1 also holds.

Proof. Let T denote the set of all continua g such that g is either a junction element of G or a non-degenerate continuum of G that contains one. It may be shown that the set T is countable. Hence there exists a sequence $g_1, g_2, g_3, \cdots$ whose terms are the continua of T. By Theorem 81 of Chapter I of P. S. T., there exists a sequence $Z_1, Z_2, Z_3, \cdots$ such that (1) for each n, Z_n is a subcollection of G_n covering S and Z_{n+1} is a subcollection of Z_n, (2) if H and K are two mutually exclusive closed point sets there exists a number m such that if x and y are intersecting regions of Z_m and x intersects H then y contains no point of K. For each natural number n, let Q_n denote the set of all domains D of elements of G such that

(1) for some finite subcollection H of Z_n that properly covers some continuum of the set G, D^* is a subset of H^*, (2) D contains no element of the sequence $g_1, g_2, g_3, \cdots$ of subscript less than n.

With the help of Theorem 12 it may be shown that there exists a sequence $\beta_1, \beta_2, \beta_3, \cdots$ such that (1) for each m, β_m is a sequence $D_{m1}, D_{m2}, D_{m3}, \cdots$ of domains of elements of G that closes down on g_m, (2) the domain D_{mn} contains no element of the sequence $g_1, g_2, g_3, \cdots$ distinct from g_m and of subscript less than n, (3) there exists a finite subcollection H_{mn} of Z_n properly covering g_m and such that D^*_{mn} is a subset of H^*_{mn}.

For each n, let G'_n denote the set whose elements are the domains of the set Q_n and those of the sequence $D_{1n}, D_{2n}, D_{3n}, \cdots$. It is clear that the sequence $G'_1, G'_2, G'_3, \cdots$ satisfies, with respect to "point" and "region," all the conditions required of $G_1, G_2, G_3, \cdots$ under (1) and (2) in the statement of Axiom 1. It will be shown that it also satisfies those required under (3). Suppose R is any domain whatsoever with respect to G, x is an element of R and y is an element of R either identical with x or not. There exists a connected domain D with respect to G containing x and such that $\overline{D}$ is a subset of $R-y+x$. If H is any domain of ordinary points containing the point set x there exists a number δ such that if $n>\delta$ then the point set H contains the point set obtained by adding together all the regions (in the original sense) of the set G_n that contain points of x. It follows, with the help of Theorem 11, that if D is any domain of elements of the set G containing the element x then there exists a number δ'_n such that if $n>\delta'_n$ and Q is a domain of the set G'_n that contains x then $\overline{Q}$ is a subset of D.

Thus all of the conditions of Axiom 1, except (4), are satisfied here. But space is compact. Hence (4), also, is fulfilled.