DIFFERENCE METHODS IN THE STUDY OF PARABOLIC EQUATIONS

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INTRODUCTION

An analysis of the solutions of difference analogs to boundary-value and initial-value problems for certain differential equations can lead to very natural and quite satisfactory existence theorems for the solutions of the corresponding differential problems. In most cases the solution \( v \) of the difference analog is given explicitly by the equation it must satisfy and the data, or at least its existence is easily demonstrated. A function \( u \) is exhibited as a "limit" of solutions \( v \) of the difference analog as the mesh size goes to zero. Using properties of these approximating functions, it is then shown that the function \( u \) has the derivatives required of a solution of the differential problem, and that these derivatives are "limits" of the corresponding difference quotients of \( v \). From the form of the difference analog and the definition of derivative, it follows that \( u \) is a solution of the differential problem.

Before discussing this method further we need some terminology. Suppose that a function \( w \) is determined by another function \( f \). For example, \( w \) may be the solution of an initial-value problem with initial data \( f \). We say that \( w \) is weakly stable on a set \( X \) with respect to \( f \) if

\[
\sup_X |w| \leq (\text{const.}) \max_{x \in A} \sup_{x \in A} |D^\alpha f|,
\]

where \( \{D^\alpha f : \alpha \in A\} \) is \( f \) and some of its derivatives, and the second supremum is taken over the entire domain of definition of the function \( f \). Similarly, we say that \( w \) is strongly stable on \( X \) with respect to \( f \) if

\[
\sup_X |w| \leq (\text{const.}) \sup_X |f|.
\]
Stability theorems for the solution \( v \) of a difference analog and for some of its difference quotients may carry over immediately into analogous theorems for the solution \( u \) of the differential problem and for some of its derivatives. Suppose that a difference quotient \( \Delta v \) of \( v \) satisfies a stability theorem with a constant which is independent of the mesh size. Suppose that a corresponding derivative \( Du \) of \( u \) is the "limit" of this difference quotient as the mesh size goes to zero. Then \( Du \) will satisfy a stability theorem analogous to that satisfied by \( \Delta v \).

The success of the simple but crude existence theorem outlined in the first paragraph requires the assumption of rather strong differentiability conditions on the data. Weak stability theorems with respect to the data for several of the difference quotients of \( v \) are used to establish the existence of the "limits" involved and to show the differentiability of \( u \). However, appropriate strong stability theorems for \( v \) and some of its difference quotients, when extended to analogous theorems for \( u \) and some of its derivatives, make it possible to weaken greatly these restrictions on the data.

Fritz John [3] used this method to study the following initial-value problem for a quasi-linear parabolic equation:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= a(x,t)\frac{\partial^2 u}{\partial x^2} + b(x,t)\frac{\partial u}{\partial x} + e(x,t,u), \\
(1) &\quad \text{for } -\infty < x < +\infty, \quad 0 < t \leq T; \\
(2) &\quad u(x,0) = f(x), \quad \text{for } -\infty < x < +\infty.
\end{align*}
\]

For the linear problem,

\[
(1) \quad e(x,t,u) = c(x,t)u + s(x,t),
\]

he proves existence and uniqueness of the solution \( u(x,t) \) of (I) under the conditions,

\[
(2) \quad 0 < \alpha \leq a,
\]

\[
(3) \quad a, \frac{\partial a}{\partial x}, \frac{\partial^2 a}{\partial x^2}, b, \frac{\partial b}{\partial x}, c, s, \text{ and } f \text{ are uniformly}
\]

\[
\quad \text{uniformly bounded.}
\]
continuous and bounded for $-\infty < x < +\infty$, $0 < t \leq T$. He shows that in this case
\[ u, \sqrt{t} \frac{\partial u}{\partial x}, \text{ and } t \frac{\partial^2 u}{\partial x^2} \]
are uniformly continuous and bounded for $-\infty < x < +\infty$, $0 < t \leq T$.

He also presents criteria for the stability of very general forward-difference analogs of (I). He constructs a fundamental solution for the difference analog, and he discusses generalized solutions of (I).

Murray Ritterman, a student of John, used in his thesis [7] methods similar to John's to study a boundary-value problem analogous to (I):

\[ \begin{align*}
\left( \frac{\partial u}{\partial x} \right) &= a(x,t) \frac{\partial^2 u}{\partial x^2} + b(x,t) \frac{\partial u}{\partial x} + e(x,t,u), \\
&\quad \text{for } 0 < x < 1, \quad 0 < t \leq T; \\
u(x,0) &= f(x), \quad \text{for } 0 \leq x \leq 1; \\
\frac{\partial u}{\partial t}(0,t) &= \alpha_0(t) \frac{\partial u}{\partial x}(0,t) + \beta_0(t,u(0,t)), \\
\frac{\partial u}{\partial t}(1,t) &= \alpha_1(t) \frac{\partial u}{\partial x}(1,t) + \beta_1(t,u(1,t)),
\end{align*} \]

for $0 \leq t \leq T$.

The unusual boundary conditions given here have as a special case the simple first boundary conditions,

\[ \begin{align*}
u(0,t) &= g_0(t) \\
u(1,t) &= g_1(t)
\end{align*} \]

for $0 \leq t \leq T$, with consistency conditions,

\[ g_0(0) = f(0), \quad g_1(0) = f(1). \]

Ritterman proves existence of the solution of (II) with the linear equation, (I), and the consistent first boundary conditions, (4), (5), under the following conditions:

(6) $f \in C^2[0,1]$; 
(7) $g_0, g_1 \in C^1[0,T]$; 
(8) $0 < \alpha \leq a$; 
(9) $a \in C^3(\mathbb{R})$; $b, c \in C^2(\mathbb{R})$; $s, t \partial^1 \frac{\partial s}{\partial t} \in C(\mathbb{R})$. 
Here $0 \leq \theta < 1$, and

$$ R = \{(x,t): 0 < x < 1; 0 < t \leq T\}. $$

In the first part of this paper we will consider a simplification of the boundary-value problem (II) and a corresponding linear integro-differential equation (problem (I\sigma) on page 6). As John and Ritterman did with problems (I) and (II), we will analyze the solution $v$ of a forward-difference analog to prove existence theorems for our differential problem. However, our method of studying $v$ is basically different from that of John and Ritterman. They considered the difference quotients of $v$ as solutions of difference problems similar to that satisfied by $v$ itself. This method is much more involved and less satisfactory for the boundary-value problem. Accordingly, Ritterman can treat only the case of $C^2$ initial data. We will separate the variables and express $v$ at each time level as a linear combination of eigenvectors for the simple forward-difference analog of the heat equation. (This representation is standard. See \[1\], page 35.) This allows us to obtain stability theorems for $v$ and some of its difference quotients by analyzing the resulting coefficients. We will thus be able to prove an existence theorem for our problem, requiring that the initial data be merely continuous and that it satisfy the consistency condition (5). However this representation is satisfactory only if the coefficients in the differential equation are independent of $x$ and if no first $x$ derivatives are involved. Hence we must make these strong restrictions. We also will not consider the quasi-linear problem.

In part two we will establish an a-priori bound on the error obtained in approximating the solution of a boundary-value problem for a simple heat equation by a forward difference analog. This discussion is entirely analogous to one used by Wasow [9] in connection with the Dirichlet problem.
In a rectangle for Laplace's equation.

In part three we will prove an existence theorem for the initial-value problem for a non-linear integro-differential equation. (See problem (VI) on page 34.) Here again we will obtain our results by analyzing a difference analog.

We will also discuss the uniqueness of solutions and the convergence of the solutions of the approximating difference analogs to the actual solutions. We will obtain only very unsatisfactory results along these lines in part three.

We need some notation. Consider a function $F(x, y, z)$ defined on a set $A$. We say that

$$F \in C^{i,j,k}(A)$$

if for $0 \leq r \leq i$, $0 \leq p \leq j$, $0 \leq s \leq k$,

$$\frac{\partial^{r+p+s}}{\partial x^r \partial y^p \partial z^s} F(x, y, z)$$

exists and is continuous in $A$. At boundary points of $A$, the derivative will denote "one-sided" derivative.
I. Statement of the Problem and Reduction to Vanishing Boundary Data

Problems involving integro-differential equations arise in the study of physical situations in which there are "hereditary" effects such as hysteresis. The solution to such a problem at a certain time depends directly on the solution at previous times. The analysis of our problem will exhibit and use the fact that solutions of a difference analog of the heat equation, like the solutions of the heat equation itself, are "smooth", even though the initial data may not be. This would not be the case if the solution at all time depended directly on the initial data. Therefore we are led to consider the following modified form of an integro-differential equation.

Set, for \( \sigma \geq 0 \),

\[
L(u) = -\frac{\partial}{\partial t}u(x,t) + a(t)\frac{\partial^2}{\partial x^2}u(x,t) + b(t)u(x,t)
+ \left\{ \begin{array}{ll}
0, & \text{for } 0 < t \leq \sigma, \\
\int_{0}^{t} c(t, \tau)u(x, \tau) \, d\tau, & \text{for } t \geq \sigma. 
\end{array} \right.
\]

We will consider the following problem:

\[
\begin{aligned}
& L(u) = s(x, t), & \text{for } 0 < x < 1, & 0 \leq t \leq T; \\
& u(x, 0) = f(x), & \text{for } 0 \leq x \leq 1; \\
& (I_{\sigma}) \quad u(0, t) = g_{0}(t), \\
& u(1, t) = g_{1}(t), & \text{for } 0 \leq t \leq T.
\end{aligned}
\]

We will assume that the consistency condition,

\[
\begin{aligned}
& f(0) = g_{0}(0) \\
& f(1) = g_{1}(0),
\end{aligned}
\]

holds.
First we will reduce the problem to one with vanishing boundary data. To do this we will assume that

\[(3) \quad g_0, g_1 \in C^2[0,T].\]

We will suppose, throughout the discussion, that the coefficients in the differential equation satisfy

\[(4) \quad 0 < \alpha \leq a;\]

\[(5) \quad a, b, c \in C^1[0,T], \text{ as functions of } t;\]

\[c \in C^1[\sigma,T], \text{ as a function of } \tau.\]

To accomplish the reduction to vanishing boundary data we need to find a function \(\mu(x,t)\) satisfying

\[(6) \quad \begin{cases} 
\mu(0,t) = g_0(t), \\
\mu(1,t) = g_1(t), \text{ for } 0 \leq t \leq T,
\end{cases}\]

and such that if we set

\[(7) \quad G(x,t) = -\mu(x,t),\]

then \(G(x,t)\) will satisfy

\[(8) \quad \frac{\partial^j}{\partial x^j} G(0,t) = \frac{\partial^j}{\partial x^j} G(1,t) = 0, \text{ for } 0 \leq j \leq 2, 0 \leq t \leq T.\]

We may assign \(\frac{\partial}{\partial x} \mu(0,t)\) and \(\frac{\partial}{\partial x} \mu(1,t)\) arbitrarily, so we set

\[(9) \quad f'(0) = \frac{\partial}{\partial x} \mu(0,t) \neq \frac{\partial}{\partial x} \mu(1,t) = f'(1).\]

Then (7) and (8) will determine, step by step, \(\frac{\partial^j}{\partial x^j} \mu(0,t)\) and \(\frac{\partial^j}{\partial x^j} \mu(1,t)\), for \(2 \leq j \leq 4\). For example,

\[0 = G(0,t) = \frac{\partial}{\partial t} \mu(0,t) + a(t) \frac{\partial^2}{\partial x^2} \mu(0,t) + b(t) \mu(0,t)\]

requires, using (6) and (9),

\[\frac{\partial^2}{\partial x^2} \mu(0,t) = \frac{1}{a(t)} \left\{ g_1'(t) - b(t) g_0(t) \right\} - \int_{\sigma}^{t} c(0,\tau) g_0(\tau) d\tau, \quad 0 \leq t \leq \sigma,\]

\[-\int_{\sigma}^{t} c(0,\tau) g_0(\tau) d\tau, \quad \sigma \leq t \leq T.\]
Because of (3), (4), and (5), the values of $\frac{\partial^j}{\partial x^j} \mu(0, t)$ and $\frac{\partial^j}{\partial x^j} \mu(1, t)$, for $0 \leq j \leq 4$, thus prescribed will be in $C^1[0, T]$. We can find a polynomial in $x$ with linear combinations of these ten functions as coefficients which will be a satisfactory $\mu(x, t)$. It will be in $C^1[0, T]$ as a function of $t$, and the $G(x, t)$ defined by (7) will be $C^\infty$ in $x$. These $x$ derivatives will be continuous, $0 \leq x \leq 1$, $0 \leq t \leq T$.

Set

$$\overline{f}(x) = f(x) - \mu(x, 0).$$

Suppose that we can solve the following problem:

$$\begin{cases}
\begin{align*}
\dot{\overline{u}}(x, t) &= s(x, t) + G(x, t), & 0 < x < l, 0 < t \leq T; \\
\overline{u}(x, 0) &= \overline{f}(x), & 0 \leq x \leq l; \\
\overline{u}(0, t) &= \overline{u}(l, t) = 0, & 0 \leq t \leq T;
\end{align*}
\end{cases}
$$

where $G(x, t)$ satisfies (7), and $\overline{f}(x)$ is defined by (10). Note in particular that by (2), (6), and (10),

$$\overline{f}(0) = \overline{f}(1) = 0.$$  

Then the function,

$$u(x, t) = \overline{u}(x, t) + \mu(x, t),$$

will be a solution of (L).

To obtain our first existence theorem we will assume stronger conditions on the inhomogeneous term than those satisfied by $G(x, t)$. Later we will reduce the conditions to include those satisfied by $G$.

We have reduced problem (L) with conditions (2)-(5) to a problem of the form,

$$\begin{cases}
\begin{align*}
\dot{u}(x) &= S(x, t), & 0 < x < l, 0 < t \leq T; \\
u(x, 0) &= f(x), & 0 \leq x \leq l; \\
u(0, t) &= u(l, t) = 0, & 0 \leq t \leq T;
\end{align*}
\end{cases}
$$

where

$$f(0) = f(1) = 0.$$  

It is this problem that we will study in the next few sections.
II. The Difference Analog and Representation of its Solution

We need to introduce some notation. Choose a positive integer \( N \) and a positive constant \( \mu \) such that if we set
\[
h = N^{-\lambda}, \quad k = \mu h^2,
\]
then
\[
\sigma = n_0 k, \quad \text{for some integer } n_0.
\]

We will consider a difference equation for functions defined on the grid
\[
R_{hk} = \{(ih, nk) : 0 \leq ih \leq 1, \ 0 \leq nk \leq T \}.
\]
For such a function \( v \), we set
\[
\begin{align*}
\Delta_x v_{in} &= \frac{v_{i+1,n} - v_{in}}{h}, \\
\Delta^2_x v_{in} &= \frac{v_{i+1,n} - 2v_{in} + v_{i-1,n}}{h^2}, \\
\Delta_t v_{in} &= \frac{v_{i,n+1} - v_{in}}{k}.
\end{align*}
\]
(16)

We will also set
\[
c_{nj} = c(nk,jk), \quad a_n = a(nk), \quad \text{etc.,}
\]
and we will make the notational convention,
\[
\sum_{j=n_0}^{n} = 0, \quad \text{for } n \neq n_0.
\]

Consider the following forward-difference analog of (II\( \sigma \));
(III\( \sigma \))
\[
\begin{align*}
v_{i,n+1} &= v_{in} + ka_n \Delta^2_x v_{in} + kb_n v_{in} + kS_{in} \\
&\quad + k^2 \sum_{j=n_0}^{n} c_{nj} v_{ij}, \quad 0 < i < N, \ 0 \leq nk \leq T; \\
v_{i0} &= f(ih), \quad 0 \leq i \leq N; \\
v_{0n} &= v_{Nn} = 0, \quad 0 \leq nk \leq T.
\end{align*}
\]

We will use the standard technique of Duhamel's
principle to reduce (III\(\sigma\)) to the homogeneous case. So consider the solution \(v_{\text{in}}\) of (III\(\sigma\)) with \(S = 0\). Note that \(v_{\text{in}}\) is given explicitly in terms of the solution at previous time levels and the initial data.

Express \(v_{10}\) in a Fourier sine sum:

\[
(18) \quad v_{10} = f(ih) = \sum_{p=1}^{N-1} d_p \sin \pi ph,
\]

where

\[
(19) \quad d_p = 2h \sum_{i=1}^{N-1} f(ih) \sin \pi ph.
\]

Note that

\[
(20) \quad k A^2 \sin \pi ph = -4 A^2 \sin^2 \left( \frac{\pi ph}{2} \right) \sin \pi ph.
\]

Hence we can put \(v_{\text{in}}\) in the form,

\[
(21) \quad v_{\text{in}} = \sum_{p=1}^{N-1} \beta_{np} \sin \pi ph.
\]

Put the expression (21) in (III\(\sigma\)) with \(S = 0\), applying (20):

\[
(22) \quad v_{1,n+1} = \sum_{p=1}^{N-1} \lambda_{np} \beta_{np} \sin \pi ph + k \sum_{p=1}^{N-1} \beta_{np} \sin \pi ph
\]

where

\[
(23) \quad \lambda_{np} = 1 - 4 A^2 \sin^2 \left( \frac{\pi ph}{2} \right).
\]

Comparing (21) and (22) we obtain equations satisfied by the \(\beta_{np}\):

\[
(24) \quad \beta_{n+1,p} = \lambda_{np} \beta_{np} + k b_n \beta_{np} + k^2 \sum_{j=n_0}^{n-1} c_{nj} \beta_{jp}.
\]

Introduce the following notation, noting (4) and (5),

\[
(25) \quad \begin{cases} 
0 < \alpha \leq a \leq A; \\
|b|, |c| \leq A.
\end{cases}
\]

We need the following lemma which was stated by John [3] in a different context:
LEMMA 1. Suppose that $0 < \mu < (2A)^{-1}$. Then there exists a constant $M = M(\alpha, A, \mu)$ such that for $0 \leq \phi \leq 1$,

$$|\alpha_n| = |1 - 4 \mu a_n \sin^2 \frac{1}{2} \phi| \leq \exp[-M(\phi)^2].$$

Assume henceforth that $\mu$ is fixed, commensurable with $\sigma$, and that $0 < \mu < (2A)^{-1}$.

Applying Lemma 1 to equation (24) and noting (18) and (23), we obtain

$$E^2 \exp\{-M(\phi)^2\} \leq P_{n+1} \leq P_n,$$

(28)

$$|\beta_{n+1, p}| \leq \left\{ \begin{array}{ll}
\exp[-M(\phi)^2] + kA & |\beta_{np}| \\
+ kA \sum_{j=n}^{n-1} |\beta_{jp}| \\
|\beta_{0p}| = |d_p|,
\end{array} \right.$$  

We will now obtain a general bound on $|\beta_{np}|$ in terms of $|d_p|$ and then a better one in case $\sigma > 0$ in (III$^\sigma$).

LEMMA 2. Let

$$H(p, h) = \exp[-M(\phi)^2] + kA(1 + T).$$

Then there exists a constant $Q = Q(\alpha, A, T, \mu)$, such that for $0 \leq \phi \leq 1$, and $0 \leq n_k \leq T$,

(30) $$H(p, h)^n \leq Q.$$  

There exists a constant $p_0 = p_0(\alpha, A, T, \mu)$, such that for $0 \leq \phi \leq 1$, and $p \geq p_0$,

(31) $$H(p, h)^n \leq \exp\left[-\frac{Mp^2 kn}{2M} \right].$$  

Proof. By (29) we have

$$H(p, h) = \exp[-M(\phi)^2] \left\{ 1 + \exp[M(\phi)^2] kA(1 + T) \right\}.$$  

Hence for $0 \leq \phi \leq 1$, since $k = \mu h^2$, $\mu h^2$,

$$H(p, h) \leq \exp[-M(\phi)^2] \left\{ 1 + k e^M A(1 + T) \right\}^2,$$

(32) $$\leq \exp[-M(\phi)^2 \mu^{-1}] \exp[k e^M A(1 + T)] .$$

Let $R = \max [ -Mp^2 \mu^{-1} + e^M A(1 + T) ]$. Then by (32)

$$H(p, h)^n \leq \exp[Rkn] \leq \max(1, e^{RT}) = Q,$$

which is (30).

Let $p_0$ be such that for $p \geq p_0$. 
(33) $e^M A(1 + T) < \frac{M^2}{2i}\lambda$.

Then for $0 \leq ph \leq 1$, $0 \leq nk \leq T$, and $p > p_o$, (32) combined with (33) gives (31). QED.

**LEMMA 3**

(34) $|\beta_{np}| \leq \begin{cases} H(p,h)^n |d_p|, & \text{for } H(p,h) \geq 1, \\ |d_p|, & \text{for } H(p,h) \leq 1. \end{cases}$

And therefore by Lemma 2, inequality (30), for $0 \leq nk \leq T$, and $0 \leq ph \leq 1$, we have

(35) $|\beta_{np}| \leq Q|d_p|.$

**Proof**

**Case 1:** $H(p,h) \geq 1$. (34) holds for $n=0$. Suppose it holds for $0 \leq n \leq r$ with $(r+1)k \leq T$. Then by (26),

$$|\beta_{r+1,p}| \leq \left\{ \exp\left[-M(ph)^2\right] + kA \right\} H(p,h)^r |d_p|$$

$$+ k^2A \sum_{j=n_0}^{r-1} H(p,h)^j |d_p|$$

$$\leq \left\{ \exp\left[-M(ph)^2\right] + kA(1+T) \right\} H(p,h)^r |d_p|$$

$$= H(p,k) |d_p| \leq |d_p|.$$ QED.

**Case 2:** $H(p,h) \leq 1$.

(34) holds for $n=0$. Suppose that it holds for $0 \leq r \leq n$ with $(r+1)k \leq T$. Then by (26),

$$|\beta_{r+1}| \leq \left\{ \exp\left[-M(ph)^2\right] + kA \right\} |d_p| + k^2A \sum_{j=n_0}^{r+1} |d_p|$$

$$\leq \left\{ \exp\left[-M(ph)^2\right] + kA(1+T) \right\} |d_p|$$

$$= H(p,k) |d_p| \leq |d_p|.$$ QED.

**LEMMA 4**

For $0 \leq ph \leq 1$, $0 < nk \leq \sigma$, and $p > p_o$, (36) $|\beta_{np}| \leq |d_p| \exp[-\frac{M^2 kn}{2\lambda}].$

**Proof**

By (26) we have for $0 < nk \leq \sigma$,

(37) $|\beta_{np}| \leq \left\{ \exp\left[-M(ph)^2\right] + kA^n \right\} |d_p|$

$$\leq H(p,h)^n |d_p|.$$

Inequalities (31) and (37) prove this lemma. QED.
**Lemma 5** For $0 \leq \phi h \leq 1$, $s \leq nk \leq T$, and $p > p_o$,

$$|\beta_{np}| \leq |d_p| \exp[-\frac{M(p)^2}{2p}]$$

**Proof** Recall that $\sigma = n_0 k$. Hence by (31), it will suffice to show that for $0 \leq \phi h \leq 1$, $s \leq nk \leq T$, and $p > p_o$,

$$|\beta_{np}| \leq |d_p| H(p, h)^{n_0}$$

This is true for $n = n_0$ by Lemma 4. Suppose that it holds for $n_0 \leq n \leq r$, with $(r+1)k \leq T$. Then by (26)

$$|\beta_{np}| \leq \left\{ \exp[- M(p h)^2] + kA + k^2 A \sum_{j=n_c}^{r-1} H(p, h)^{n_0} |d_p| \right\}$$

$$\leq H(p, h) H(p, h)^{n_0} |d_p|$$

$$\leq H(p, h)^{n_0} |d_p|,$$

since by (31) $H(p, h) \leq 1$ for $p > p_o$, $0 \leq \phi h \leq 1$. QED.

**III. Bounds on the Coefficients $d_p$**

Lemmas 3, 4, and 5 give us bounds on the coefficients in the eigenvector expansions of $v$ in terms of the coefficients $d_p$ in the eigenvector expansion of the initial data $f$. We will now obtain some bounds on the $d_p$ when $f$ has special properties.

If $f \in C^r[0, 1]$, let

$$F_r = \max_{0 \leq s \leq r} \max_{0 \leq x \leq 1} |f(s)(x)|.$$  

(39)

We need a few more lemmas which will be proved in the appendix.

**Lemma 6**

$$\left( j + \frac{1}{2} \right) \int_{(j-h)^S}^{(j+\frac{1}{2})} (x-jh)^s \sin px \, dx = \begin{cases} 2 \sin \pi jh \left( \frac{h}{2} \right)^{s+1} B_s(p, h), \\ \text{for } s \text{ even;} \\ 2 \cos \pi jh \left( \frac{h}{2} \right)^{s+1} B_s(p, h), \\ \text{for } s \text{ odd.} \end{cases}$$

(40)
\begin{equation}
\left\{ \begin{array}{l}
(j+1)h \\
(x-jh)^{s} \sin \pi px \ dx \\
(j-h)h
\end{array} \right. = \begin{cases} 
-2 \sin \pi p j h \left( \frac{h}{2} \right)^{s+1} B_{s}(p, h), & \text{for } s \text{ odd;} \\
2 \cos \pi p j h \left( \frac{h}{2} \right)^{s+1} B_{s}(p, h), & \text{for } s \text{ even.}
\end{cases}
\end{equation}

Here
\begin{equation}
B_{s}(p, h) = \begin{cases} 
\sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n)! (2n+s+1)} \left( \frac{\pi p h}{2} \right)^{2n} , & s \text{ even} \\
\sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n+1)! (2n+s+2)} \left( \frac{\pi p h}{2} \right)^{2n+1} , & s \text{ odd.}
\end{cases}
\end{equation}

**Lemma 7**
\begin{equation}
B_{0}(p, h) > \frac{1}{2} , \text{ for } 0 \leq ph \leq 1.
\end{equation}

**Lemma 8** There exist constants \(M_{s}\) such that
\begin{equation}
|B_{s}(p, h)| \leq M_{s} , \text{ for } 0 \leq ph \leq 1, \quad s=0, 1, 2, \ldots
\end{equation}

**Lemma 9** Let \(f \in C^{r}[0, 1]\), with \(f(j)(0) = f(j)(1) = 0\) for \(0 \leq j \leq r-2\). Then for \(0 \leq ph \leq 1\),
\begin{equation}
\sum_{j=1}^{N-1} f(jh) \sin \pi ph j h = O(F_{p} p^{-r}),
\end{equation}
\begin{equation}
\sum_{j=1}^{N-1} f(jh) \cos \pi ph j h = O(F_{p} p^{-r}).
\end{equation}

**Proof** Consider \(r=0\).
\begin{equation}
|h \sum_{j=1}^{N-1} f(jh) \sin \pi ph j h| \leq h \sum_{j=1}^{N-1} F_{0} \leq F_{0} ;
\end{equation}
\begin{equation}
|h \sum_{j=1}^{N-1} f(jh) \cos \pi ph j h| \leq h \sum_{j=1}^{N-1} F_{0} \leq F_{0} .
\end{equation}
So the claim holds for \(r=0\). Suppose that it holds for \(0 \leq r \leq s-1\). Let \(f(x)\) satisfy the hypotheses for \(r=s\). Note that since \(f(j)(0) = f(j)(1) = 0\), for \(0 \leq j \leq r-2\), integration by parts \(s\) times gives
\begin{equation}
\int_{0}^{1} f(x) \sin \pi px \ dx = \begin{cases} 
\frac{f(s-1)(x) \sin \pi px}{(\pi p)^{s}} \bigg|_{0}^{1} + \frac{f(s)(x) \sin \pi px}{(\pi p)^{s}} \bigg|_{0}^{1} \\
\text{or}
\end{cases}
\end{equation}
\begin{equation}
\int_{0}^{1} f(x) \cos \pi px \ dx = \begin{cases} 
\frac{f(s-1)(x) \cos \pi px}{(\pi p)^{s}} \bigg|_{0}^{1} + \frac{f(s)(x) \cos \pi px}{(\pi p)^{s}} \bigg|_{0}^{1} \\
\end{cases}.
Hence
\[\int_0^1 f(x) \sin \pi px \, dx, \quad \int_0^1 f(x) \cos \pi px \, dx = O(F_s p^{-s}).\]

Also
\[f(x) = \frac{x^{s-1} p(s-1)(p)}{(s-1)!} = O(F_s x^{s-1}),\]
\[f(x) = \frac{(1-x)^{s-1} p(s-1)(p)}{(s-1)!} = O(F_s (1-x)^{s-1}),\]
where \(0 < \Theta_1 < x < \Theta_2 < 1.\)

So
\[
\int_0^{\frac{1}{2} h} f(x) \sin \pi px \, dx, \quad \int_0^{\frac{1}{2} h} f(x) \cos \pi px \, dx, \quad \int_0^{\frac{1}{2} h} f(x) \sin \pi px \, dx,
\]
(49)
and
\[
\int_0^{\frac{1}{2} h} f(x) \cos \pi px \, dx \text{ are all } O(F_s h^S) = O(F_s p^{-S}) \text{ for } 0 \leq ph \leq 1.
\]

We will complete the proof of (45). The proof of (46) is entirely analogous.

(48) and (49) give
\[
O(F_s p^{-S}) = \int_0^1 f(x) \sin \pi px \, dx = \sum_{j=1}^{N-1} \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} f(x) \sin \pi px \, dx
\]
(50)

Also
\[f(x) = \sum_{r=0}^{s-1} \frac{(x-jh)^r}{r!} f(r)(jh) + O(F_s |x-jh|^S),\]
for \(0 \leq x \leq 1.\)

So
\[
\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} f(x) \sin \pi px \, dx = \sum_{r=0}^{s-1} \frac{f(r)(jh)}{r!} \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} (x-jh)^r \sin \pi px \, dx
\]
+ \(O(F_s h^{s+1}).\)

Applying Lemma 6 to this last expression we obtain, for \(0 \leq ph \leq 1,\)
\[
\int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} f(x) \sin \pi px \, dx = \sum_{r=0}^{s-1} \frac{f(r)(jh)}{r!} \frac{h^2}{2} B_r(p, h) O(jh)
\]
(51)
+ \(h O(F_s p^{-S}).\)
where \( \varphi_r(x) = \begin{cases} \sin px, & \text{for } r \text{ even} \\ \cos px, & \text{for } r \text{ odd} \end{cases} \) (52)

Combining (50) and (51), we obtain

\[
(53) \quad B_0(p,h) h \sum_{j=1}^{N-1} f(jh) \sin px \varphi_r(jh) + O(F_s p^{-S}) = - \sum_{r=1}^{s-1} \frac{1}{s!} \frac{h^r}{r!} B_r(p,h) h \sum_{j=1}^{N-1} f(r)(jh) \varphi_r(jh) .
\]

\( f(r)(x) \) satisfies the inductive hypotheses for \( s-r \), so by (52)

\[
(54) \quad h \sum_{j=1}^{N-1} f(r)(jh) \varphi_r(jh) = O(F_s p^{s-r}), \quad 1 \leq r \leq s-1 .
\]

Lemma 8 with (53) and (54) gives for \( 0 \leq ph \leq 1 \),

\[
(55) \quad B_0(p,h) h \sum_{j=1}^{N-1} f(jh) \sin px = \sum_{r=1}^{s-1} O(F_s h^r p^{s-r}) + O(F_s p^{-S}) = O(F_s p^{-S}) .
\]

Since by Lemma 7, \( B_0(p,h) \geq \frac{1}{3} \) for \( 0 \leq ph \leq 1 \), (55) gives (45). QED.

In this last section we have generalized the method used by Douglas [2] to obtain Lemma 9 for \( r=1 \), two space variables.

IV. Weak Stability Lemmas and an Existence Theorem

The results of the last two sections will give us weak stability theorems for the solution \( v_{in} \) of (III\( \sigma \)) and for some of its difference quotients. These bounds will be for a fixed mesh ratio \( \mu \) but will be independent of the mesh size.

**Lemma 10** Let \( f(x) \in C^6[0,1] \) with \( f^{(j)}(0) = f^{(j)}(1) = 0 \) for \( 0 \leq j \leq 4 \). Let \( v_{in} \) be the corresponding solution of (III\( \sigma \)) with \( S = 0 \). Then for fixed \( \mu \), \( 0 < \mu < (2A)^{-1}, 0 \leq i \leq N-r \),

\[
(56) \quad \Delta^r_x v_{in} = O(F_0), \quad \text{for } 0 \leq r \leq 4 .
\]

independent of the mesh size.

**Proof** By equation (21) on page 10, inequality (35) on
page 12, and Lemma 9 we have

\[ v_{in} = \sum_{p=1}^{N-1} \beta_{np} \sin p\pi h \]

with

\[ |\beta_{np}| \leq q |d_p| = O(F_6 p^{-6}). \]

Therefore for \( 0 \leq i \leq N-r \),

\[ \Delta^r_{x} v_{in} = \sum_{p=1}^{N-1} \beta_{np} \Delta^r_{x} \sin p\pi h. \]

Applying (58) to this gives

\[ |\Delta^r_{x} v_{in}| \leq \sum_{p=1}^{N-1} |\beta_{np}| (\pi p)^r = \sum_{p=1}^{N-1} O(F_6 p^{r-6}). \]

Therefore for \( 0 \leq r \leq 4, 0 \leq i \leq N-r \),

\[ \Delta^r_{x} v_{in} = O(F_6) \sum_{p=1}^{N-1} p^{-2} = O(F_6). \quad \text{QED.} \]

**LEMMA 11.** Let \( f(x) \) satisfy the conditions of Lemma 10. Let \( S(x,t) \in C^6,1(\mathbb{R}), \) and let it satisfy

\[ \frac{\partial^j}{\partial x^j} S(0,t) = \frac{\partial^j}{\partial x^j} S(1,t) = 0, \text{ for } 0 \leq j \leq 4, 0 \leq t \leq T. \]

Finally, let \( v_{in} \) be the corresponding solution of (III\( \sigma \)). Then

\[ \Delta^r_{x} v_{in} = O(1), \text{ for } 0 \leq i \leq N-r, 0 \leq r \leq 4; \]

\[ \Delta_{t} \Delta^r_{x} v_{in} = O(1), \text{ for } 0 \leq i \leq N-r, 0 \leq r \leq 2; \]

\[ \Delta_{t} \Delta_{t} v_{in} = O(1). \]

The bounds here depend on \( f(x), S(x,t), \) the coefficients in the differential equation, \( \mu \), and \( T \); but they do not depend on the mesh size.

**Proof.** Note first that \( S(x,t) \) satisfies the conditions of Lemma 10 for each fixed \( t \). Also Lemma 10 is valid for \( \sigma = 0 \). Therefore we can apply a standard Duhamel-principle argument to obtain (59) from Lemma 10.

Rewrite the differential equation in (III\( \sigma \)).
\begin{align}
\Delta_t v_{in} &= a_n \Delta_x^2 v_{in} + b_n v_{in} + S_{in} + k \sum_{j=n}^{n_0} c_{nj} v_{ij}.
\end{align}

Since for each \( t \), \( S(x,t) \in C^2[0,1] \) as a function of \( x \), we can apply \( \Delta_x^r \) \((r=1,2)\) to equation (62) and obtain a quantity which by (59) is bounded independent of the mesh size. This proves (60).

Also \( a(t), b(t), c(t, z) \), and \( S(x, t) \) for \( 0 \leq x \leq 1 \), are in \( C^1[0,T] \) as functions of \( t \), and \( c(t, z) \) is bounded. Hence upon applying \( \Delta_t \) to equation (62) we obtain, by (60), a quantity which is bounded independent of the mesh size. QED.

**Theorem 1.**

Let \( f(x) \) and \( S(x,t) \) satisfy the conditions given in the statement of Lemma 11. Then there exists a solution \( u(x,t) \) of (11\( \gamma \)) with

\[
\begin{align*}
&u(x,t), \frac{\partial}{\partial x} u(x,t), \frac{\partial^2}{\partial x^2} u(x,t) \text{ and } \frac{\partial}{\partial t} u(x,t)
\end{align*}
\]

uniformly Lipschitz continuous for \( 0 \leq x \leq 1 \), and \( 0 \leq t \leq T \). 

**Proof.**

We set \( h_s = 2^{-s} \) and pick \( \mu = \frac{1}{2^n} \) such that for \( k_s = \mu h_s^2 \) and for \( s \) sufficiently large, \( \sigma' = n_s k_s \) for integers \( n_s \).

Let \( v_s(x,t) \) denote the solution of (11\( \gamma \)) corresponding to \( h_s \) and \( k_s \). \( v_s(x,t) \) is defined on the set

\[
R_s = \left\{(ih_s, nk_s) : 0 \leq ih_s \leq 1, 0 \leq nk_s \leq T \right\}.
\]

Note that \( R_{s+1} \supset R_s \). Let

\[
R_\infty = \bigcup_{s=1}^{\infty} R_s , \text{ a countable set dense in } \mathbb{R}.
\]

By Lemma 11, the sequence of functions \( v_s(x,t) \) are uniformly bounded. Pick an infinite set of integers \( I \) such that

\[
\lim_{s \to \infty} v_s(x,t) = u(x,t)
\]

\( s \in I \).
exists for \((x, t) \in R_{\infty}\). From the bounds given by Lemma 11, it can be shown (see John [3], pp. 174-175) that \(u(x, t)\) has the following properties: (1) \(u(x, t)\) is uniformly continuous on \(R_{\infty}\). (2) After extending it to all of \(R\) by continuity, it has two \(x\) derivatives and one \(t\) derivative. (3) These derivatives are uniformly Lipschitz-continuous in \(R\), and hence \(u(x, t) \in C^2(R)\). And finally, (4) for \((x, t) \in R_{\infty}\),

\[
\lim_{s \to \infty} \frac{\Delta_x}{\Delta_s} v_s(x, t) = \frac{\partial}{\partial x} u(x, t), \text{ for } r = 1, 2; \text{ and }\]

\[
\lim_{s \to \infty} \frac{\Delta_t}{\Delta_s} v_s(x, t) = \frac{\partial}{\partial t} u(x, t).
\]

John treats a pure differential equation, and the above considerations suffice to show that the difference analog carries over into the differential problem and that the \(u(x, t)\) so produced is a solution of the differential problem. We are considering an integro-differential equation, so we must show that the Riemann-sum approximations go over into the appropriate integral.

Let \(p_0 \in I\) be large enough so that the corresponding mesh contains a row at \(t = \sigma\), and set \(k_o = k_{p_0}\). Let \(p\) be an arbitrary integer in \(I\), \(p > p_0\). Set \(k = k_p\).

We claim that for \((x, t) \in R_{p_0}\),

\[
k \sum_{\sigma \leq jk < t} c(t, \sigma, jk) v_p(x, jk) = k_o \sum_{\sigma \leq nk_o < T} c(t, \sigma, nk_o) v_p(x, nk_o + \sigma) + O(n)
\]

independent of the choice of \(p \in I, p > p_0\).

Consider the portion of the first sum corresponding to an interval \([nk_o, (n+1)k_o)\) with \(\sigma \leq nk_o < T\). It is

\[
k_{\sigma/k_o} \sum_{i=0}^{k_{\sigma/k_o} -1} c(t, nk_o + ik) v_p(x, nk_o + ik).
\]

But since \(c(t, \tau) \in C^1[\tau, T]\) as a function of \(\tau\), and since
\[ \Delta_t v_p(x, t) \text{ is bounded independent of } p \text{ on all of } \mathbb{R}, \]

\[ c(t, nk_o + ik)v_p(x, nk_o + ik) = c(t, nk_o)v_p(x, nk_o) + O(k_o), \]

independent of \( p > p_o \), for \( 0 \leq i < k/o k. \)

With (68), (67) becomes

\[ k_o/k \rightarrow 1 \]

\[ k \sum_{i=0}^{k_o/k} [c(t, nk_o)v_p(x, nk_o) + O(k_o)] \]

Hence

\[ k\sum_{\sigma \leq jk < t} c(t, jk) v_p(x, jk) = \sum_{\sigma \leq nk_o < t} k \sum_{i=0}^{k_o/k} [c(t, nk_o + ik)v_p(x, nk_o + ik)] \]

This gives (66) immediately.

Now for fixed \( p_o \), by equation (63), the limit of the right-hand side of equation (66) exists as \( p \rightarrow \infty \) with \( p \in I \). This gives

\[ \lim_{p \rightarrow \infty} k \sum_{\sigma \leq jk < t} c(t, jk) v_p(x, jk) = k_o \sum_{\sigma \leq nk_o < t} c(t, nk_o)u(x, nk_o) + O(k_o). \]

Since \( c(t, \omega)u(x, \omega) \) is a uniformly Lipschitz-continuous function of \( \omega \) on \([\sigma, T]\), and since the right-hand side of (70) is a Riemann sum for its integral, (70) gives us

\[ \lim_{p \rightarrow \infty} k \sum_{\sigma \leq jk < t} c(t, jk) v_p(x, jk) = \int_{\sigma}^{t} c(t, \omega)u(x, \omega)d\omega + O(k_o). \]

Equations (63), (64), (65), and (71) show that the \( u(x, t) \) given by (63) is a solution of problem \((II_\omega)\). Comments at the top of page 19 show that \( u(x, t) \) satisfies the rest of
the claims of Theorem 1.

In the next sections we will produce some strong stability theorems which we will use to weaken the hypotheses in our existence theorem.

V. Strong Stability Theorems

**Theorem 2.** Let $v_{in}$ be a solution of (III). Let $P_0$ and $S_0$ denote upper bounds on $|f(x)|$ and $|S(x,t)|$, respectively, over their respective domains. Then for $0 < \mu < (2A)^{-1}$,

$$v_{in} = 0(P_0 + S_0),$$

where the bound depends on the coefficients in the equation, and on $T$ but not on $f$, $S$, or the mesh size.

**Proof.** This type of proof is standard. Set

$$M_n = \max_{0 \leq j \leq n} |v_{ij}|.$$  

Rewrite the difference equation in (III):

$$v_{i,n+1} = (1-\mu a_n) v_{in} + \mu a_n(v_{i+1,n} + v_{i-1,n}) + k b_n v_{in} + k^2 \sum_{j=0}^{n-1} c_{nj} v_{ij} + k S_{in}.$$  

Recall that $0 < \mu \leq a$, and $|a|, |b|, |c| \leq A$. For $0 < \mu \leq (2A)^{-1}$, the first two coefficients are positive. Hence for $0 \leq n \leq T$, and $\mu$ this small, we have, noting (73),

$$|v_{i,n+1}| \leq [1 + kA(1 + T)]M_n + kS_n, \quad 1 \leq i \leq N.$$  

Therefore, since $v_{0n} = v_{Nn} = 0$,

$$\begin{cases} 
M_{n+1} \leq [1 + kA(1 + T)]M_n + kS_n \\
M_0 = P_0
\end{cases}$$

This gives

$$M_n \leq [1 + kA(1 + T)]M_0 + kS_0 \sum_{j=0}^{n-1} [1 + kA(1 + T)]^j$$

$$\leq M_0 e^{(A+AT)kn} + S_0 \frac{e^{(A+AT)kn} - 1}{e^{AT} - 1} = 0(P_0 + S_0).$$  

QED.
Theorem 2 shows the strong stability of the solution \(v\) with respect to \(f(x)\) and \(S(x,t)\) on the entire region \(\overline{R}\). We will now show the strong stability of all \(x\)-difference quotients of \(v\) with respect to \(f(x)\) on compact subsets of \(R\) when \(\sigma'\) is positive and \(S = 0\).

**Theorem 3.** Let \(v_{in}\) be the solution of \((III_\sigma)\) with vanishing \(S\) and with \(\sigma'\) strictly positive. Then independent of the mesh size, for \(0 \leq i \leq N-r\),

\[
\Delta_x^{r} v_{in} = \begin{cases} 
0(F_0) \left( (nk)^{-\sigma'/(r+1)} \right), & 0 < nk \leq \sigma; \\
0(F_0) \left( \sigma^{-\sigma'/(r+1)} \right), & \sigma \leq nk \leq T.
\end{cases}
\]

**Proof.** First we will prove the following simple fact:

For \(r > 0\), \(0 < nk \leq T\), and fixed \(M > 0\),

\[
\sum_{p=0}^{N-1} p^r e^{-Mp^2 kn} = O[(nk)^{-1/2}(r+1)].
\]

Here again the bound is independent of \(h = N^{-1}\).

Consider, for \(0 < nk \leq T\),

\[
Q(\xi) = \xi^r e^{-Mk n \xi^2}.
\]

A simple calculation shows that \(\max_{0 \leq \xi} Q(\xi) = O(\sqrt{\frac{r}{Mkn}})\), and

that \(\sum_{p=0}^{N-1} p^r e^{-Mp^2 kn} < \sum_{p=0}^{\infty} O(p) < \int_0^{\infty} O(\xi) d\xi + 2 \max_{0 \leq \xi} Q(\xi)\)

\[= O[(nk)^{-1/2}(r+1)] + O[(nk)^{-1/2}]\]

\[= O[(nk)^{-1/2}(r+1)], \text{ for } nk \leq T.\]

As in the proof of Lemma 10, we have for \(0 \leq i \leq N-r\)

\[
|\Delta_x^{r} v_{in}| \leq \sum_{p=1}^{N-1} |B_{np}| (a_p)^r.
\]

Consider first \(0 < nk \leq \sigma\). By Lemma 4 and (77) we have for \(0 \leq i \leq N-r\),

\[
|\Delta_x^{r} v_{in}| \leq \sum_{p=0}^{N-1} |d_p| (a_p)^r \exp \left[ - \frac{M p^2 kn}{\lambda} \right] + \sum_{p=1}^{p_e} |d_p| (a_p)^r Q
\]

Also by inequality (47) on page 14,

\[
|d_p| \leq 2F_0.
\]
Combining (76), (78), and (79), we obtain the first half of (75).

Similarly, by Lemma 5 and (77) we have for $0 < \sigma' \leq nk \leq T$, and $0 \leq i \leq N-r$,

$$|\Delta^r_xv| \leq \sum_{p=0}^{N-1} |a_p| (\pi p)^r \exp\left[-\frac{M_1^2 \sigma'}{2} \right] + \sum_{p=1}^{P_0} |d_p| (np)^r \frac{Q}{3}$$

Then (76), (79) and (80) give us the second half of (75). QED.

The last two theorems give us the strong stability theorems we wanted for $v_{in}$ and some of its difference quotients.

Now we will obtain analogous results for the solution $u(x,t)$ of (III$\sigma$).

**Theorem 4.** Let $f(x)$ satisfy the conditions of Theorem 1, so that there exists a corresponding solution $u(x,t)$ of (III$\sigma$).

Suppose that $\sigma'$ is positive and that $S(x,t) = 0$. Then

(81) $u(x,t) = 0(F_0)$, for $0 \leq x \leq 1$, and $0 \leq t \leq T$;

(82) $\frac{\partial^r}{\partial x^r} u(x,t) = \begin{cases} 0(F_0) \left( t^{r-1} (r+1) \right), & \text{for } 0 < t \leq \sigma' \\ 0(F_0) \left( t^{r-1} (r+1) \right), & \text{for } \sigma' \leq t \leq T, \end{cases}$

for $0 \leq x \leq 1$, and $r = 1, 2$;

(83) $\frac{\partial}{\partial t} u(x,t) = \begin{cases} 0(F_0) \left( t^{-\frac{3}{4}} \right), & \text{for } 0 < t \leq \sigma' \\ 0(F_0) \left( t^{-\frac{3}{4}} \right), & \text{for } \sigma' \leq t \leq T, \end{cases}$

for $0 \leq x \leq 1$.

The bounds here are of course independent of $f(x)$.

**Proof.** The solution $u(x,t)$ in Theorem 1 is the limit on a dense set of a sequence of solutions of the difference equation (III$\sigma$) with $\sigma' > 0$ and $S = 0$, as the mesh size goes to zero. The derivatives of $u(x,t)$ in (82) are similarly "limits" of the corresponding difference quotients of the sequence of difference solutions. The bounds on these approximating quantities which are given in Theorems 3 and 4 are independent of the mesh size. Also $u(x,t)$ and its first two derivatives in (78) are uniformly continuous in $R$. Therefore $u(x,t)$ satisfies (81) and (82). Finally, (83)
follows from the form of the differential equation in (II).

QED.

VI. Improved Weak Stability Theorems

If \( \sigma = 0 \) or \( S(x,t) \) is present in our problem, we do not get satisfactory strong stability theorems. However we can improve our weak stability theorems. As usual, we will do this first for the difference solution.

**Lemma 13.** Suppose that \( f \in C^4[0,1] \) and \( S(x,t) \in C^4,0(\mathbb{R}) \). Suppose further that for \( r = 0,1,2, \) and \( 0 \leq t \leq T \),

\[
\frac{d^r}{dx^r} f(x) = \frac{d^r}{dx^r} S(x,t) = 0, \text{ at } x = 0,1.
\]

Let \( v_{\text{in}} \) be the corresponding solution of (III') with \( \sigma \geq 0 \). Then

\[
(84) \quad \frac{d^r}{dx^r} v_{\text{in}} = O(F_4 + S_4), \text{ for } r = 0,1,2, \text{ and } 0 \leq i \leq N-r, \text{ and } 0 \leq nk \leq T.
\]

Here as usual the bound is independent of the mesh size. \( S_4 \) denotes a bound on the first four \( x \) derivatives of \( S \).

**Proof.** First, suppose that \( S = 0 \). By Lemma 9 with \( r = 4 \),

\[
\frac{d^4}{dx^4} = O(F_4 + S_4) \text{ for } 0 \leq ph \leq 1.
\]

Then by exactly the same argument as in the proof of Lemma 10, we obtain (84) with \( S_4 = 0 \).

Since we have not required that \( \sigma \) be positive and since \( S(x,t) \) satisfies for fixed \( t \) the same conditions as \( f(x) \), a typical simple Duhamel-principle argument shows that (84) holds in the general case.

**Theorem 5.** Let \( f(x) \) and \( S(x,t) \) satisfy the conditions of Theorem 1, so that there exists a corresponding solution \( u(x,t) \) of (II'). Then for \( 0 \leq x \leq 1, \) and \( 0 \leq t \leq T \),

\[
(85) \quad u(x,t), \frac{\partial}{\partial t} u(x,t), \frac{\partial^r}{\partial x^r} u(x,t) = O(F_4 + S_4), \text{ } r = 1,2.
\]

**Proof.** This proof is the same as that of Theorem 4. Here we use (84) where we used Theorems 3 and 4 in the proof of Theorem 4.
VII. Refined Existence and Stability Theorems for (IIσ)

THEOREM 6. Let \( f(x) \) and \( S(x, t) \) satisfy the conditions of Lemma 13. Then there exists a corresponding solution \( u(x, t) \) of (IIσ) for \( \sigma \geq 0 \). This solution will satisfy the stability properties (85). The functions in (85) are continuous in \( R \).

Proof. Let \( \{f_n(x)\}_{n=1}^{\infty} \) and \( \{S_n(x, t)\}_{n=1}^{\infty} \) be sequences of functions satisfying the conditions of Theorem 1 and converging uniformly with their first four \( x \) derivatives to \( f(x) \), \( S(x, t) \) and their corresponding derivatives. Let \( \{u_n(x, t)\}_{n=1}^{\infty} \) be the corresponding solutions of (IIσ), which exist by Theorem 1. Then by Theorem 5 and familiar arguments \( u_n(x, t) \) converges uniformly to a function \( u(x, t) \), and the first two \( x \) derivatives of the \( u_n(x, t) \) converge uniformly to the corresponding derivatives of \( u(x, t) \). By the form of (IIσ), \( \frac{\partial}{\partial t} u_n(x, t) \) converges uniformly to \( \frac{\partial}{\partial t} u(x, t) \). Hence, \( u(x, t) \) is a solution of (IIσ) for \( f(x) \) and \( S(x, t) \), and satisfies (85).

THEOREM 7. Let \( \sigma > 0 \), and suppose that \( S(x, t) \) satisfies the conditions in Theorem 6. Suppose further that

\[
f(x) \in C[0, 1], \quad \text{and} \quad f(0) = f(1) = 0.
\]

Then there exists a solution \( u(x, t) \) of (IIσ) corresponding to \( f(x) \) and \( S(x, t) \). This solution will satisfy (81) - (83) if \( S = 0 \). Also \( u \in C(R) \), and \( \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x^2} \) are continuous for \( 0 \leq t \leq T \).

Proof. Let \( \{f_n(x)\}_{n=1}^{\infty} \) be a sequence of functions satisfying the conditions of Theorem 6 and converging uniformly to \( f(x) \). Let \( u_n(x, t) \) be a solution of (IIσ) for \( f_n(x) \) and \( S(x, t) \). Such solutions exist by Theorem 6. Then the functions \( u_n(x, t) - u_m(x, t) \) are solutions of (IIσ) corresponding to initial data \( f_n(x) - f_m(x) \) and with vanishing inhomogeneous term. Therefore we can apply Theorem 5 and standard arguments to show that \( u_n(x, t) \) converges uniformly to a function \( u(x, t) \), and that its first \( t \) and first two \( x \) derivatives converge uniformly on compact subsets of the region to the corresponding
derivatives of \( u(x, t) \). This shows that \( u(x, t) \) satisfies the requirements of the theorem.

VIII. Existence Theorems for \((\mathcal{I}_\alpha)\)

Recall the reduction of problem \((\mathcal{I}_\alpha)\) to problem \((\mathcal{II}_\alpha)\), which was done on page 8. Note in particular that the \( G(x, t) \) constructed there satisfies the requirements made on \( S(x, t) \) in Theorems 6 and 7, and that \( \mu(x, 0) \in \mathcal{C}^\infty[0, 1] \). Theorem 7 gives the following result for \((\mathcal{I}_\alpha)\).

**Theorem 8.** Suppose that conditions (2), (3), and (4) hold, and that \( f \in \mathcal{C}[0, 1] \), \( s(x, t) \in \mathcal{C}^4, 0(R) \). Suppose further that

\[
\frac{\partial^r}{\partial x^r} s(0, t) = \frac{\partial^r}{\partial x^r} s(1, t) = 0, \text{ for } r = 0, 1, 2.
\]

Then there exists a corresponding solution of \((\mathcal{I}_\alpha)\) for \( \alpha \) positive.

Problem \((\mathcal{I}_\alpha)\) will reduce to a problem \((\mathcal{I}_\beta)\) satisfying the conditions of Theorem 6, if

\[
(86) \quad \bar{f}''(x) = \bar{f}'(x) - \frac{\partial^2}{\partial x^2} \mu(x, 0), \quad \bar{f}'(x) = f'(x) - \frac{\partial}{\partial x} \mu(x, 0)
\]

vanishes at \( x = 0 \) and \( 1 \). Noting the equation at the bottom of page 7 and an analogous one for \( x = 1 \), we see that \((86)\) holds if and only if the data satisfies the differential equation at the corners. Thus we are led to the following analog to Theorem 6, for \((\mathcal{I}_\alpha)\).

**Theorem 9.** Suppose that the conditions of Theorem 8 are satisfied and that \( f \in \mathcal{C}^4[0, 1] \). Suppose further that the differential equation in \((\mathcal{I}_\alpha)\) is satisfied at the corners; i.e., suppose that

\[
(87) \quad g^a_0(0) = a(0) f''(0) + b(0) g^a_0(0), \text{ and } g^a_1(0) = a(0) f''(1) + b(0) g^a_1(0).
\]

Then there exists a corresponding solution of \((\mathcal{I}_\alpha)\) for \( \alpha > 0 \).

Note that in effect we required that the differential equation be satisfied at the corners, in Theorem 6 also.
IX. Convergence and Uniqueness Theorems

**Theorem 10.** Suppose that \( u(x,t) \) is a solution of \((I_\sigma)\) with
\[
\begin{align*}
  u(x,t), \quad \frac{\partial}{\partial t} u(x,t), \quad \text{and} \quad \frac{\partial^2}{\partial x^2} u(x,t)
\end{align*}
\]
uniformly continuous on \( \bar{R} \). Then the solution \( v_{\text{in}} \) of the corresponding difference analog, \((II_{\sigma})\) with boundary data \( g_0 \) and \( g_1 \) instead of \( 0 \), converges to \( u(x,t) \) as \( h \to 0 \).

**Proof.** This type of proof is standard. \( u_{\text{in}} = u(\text{i}h, nk) \) satisfies the same difference problem as \( v_{\text{in}} \) with the addition of an inhomogeneous term which is \( o(1) \) as \( h \to 0 \).

**Corollary.** Any solution of \((I_\sigma)\) which satisfies the conditions given in Theorem 10 is unique in that class of functions.

**Theorem 11.** Let \( \sigma' > 0 \). Then there is at most one solution \( u(x,t) \) of \((I_{\sigma'})\) satisfying the following conditions:

(i) \( u(x,t) \) is continuous in \( \bar{R} \);

(ii) \( \frac{\partial}{\partial t} u(x,t) \), and \( \frac{\partial^2}{\partial x^2} u(x,t) \) are continuous for \( 0 \leq x \leq 1 \), and \( 0 \leq t \leq T \) for each \( t \geq 0 \).

**Proof.** Suppose that \( u(x,t) \) and \( w(x,t) \) are two solutions of \((I_{\sigma'})\), satisfying (i) and (ii). Pick \( t_0 \) with \( 0 < t_0 < \sigma' \). We showed in the previous section the existence of a solution \( z(x,t;t_0) \) of the following problem:

\[
\begin{align*}
  \frac{\partial}{\partial t} z(x,t,t_0) &= 0, \quad \text{for} \quad 0 \leq x \leq 1, \quad t_0 \leq t \leq T; \\
  z(x,t_0,t_0) &= u(x,t_0) - w(x,t_0), \quad \text{for} \quad 0 \leq x \leq 1; \\
  z(0,t_0,t_0) &= z(1,t_0,t_0) = 0, \quad \text{for} \quad t_0 \leq t \leq T.
\end{align*}
\]

By the construction of \( z(x,t;t_0) \), we have (See Theorem 4)

\[
|z(x,t;t_0)| \leq (\text{const}) \max_{0 \leq x \leq 1} |z(x,t_0; t_0)|,
\]

and \( z(x,t_0;t_0) \) satisfies the conditions of Theorem 10 with \( t \) replaced by \( t - t_0 \). The function \( u(x,t) - w(x,t) \) is a solution of problem (A). By Theorem 10, \( u - w = z(x,t;t_0) \). Therefore for \( t_0 \leq t \leq T \),

\[
|u(x,t) - w(x,t)| \leq (\text{const}) \max_{0 \leq x \leq 1} |z(x,t_0; t_0)|.
\]

Finally, by the uniform continuity of \( u(x,t) - w(x,t) \),

\[
\max_{0 \leq x \leq 1} |z(x,t_0; t_0)| \to 0, \quad \text{as} \quad t_0 \to 0, \quad \text{and} \quad u(x,t) = w(x,t).
\]

This proof is due to Fritz John. See [3].
X. Differentiability of the Solutions

In this section we will establish further differentiability properties of the solutions of \((I_\sigma)\) when \(\sigma > 0\) and there is no inhomogeneous term.

**Lemma 14.** Suppose that \(\sigma > 0\) and that \(S(x,t) = 0\). Let

\[
\begin{align*}
\begin{cases}
a(t), b(t) \in C^s[0,T] \\
c(t,\tau) \in C^{s-1}[\sigma,T]x[\sigma,T]
\end{cases}
\]
\]

Let \(v_{in}\) be the corresponding solution of \((III_\sigma)\). Then, where defined,

\[
\Delta_x^q \Delta_t^r v_{in} = 0(F_q) \quad \text{for} \quad 0 < t_0 \leq nk \leq T, \quad 0 \leq r \leq s+1, \quad \text{all} \quad q.
\]

Here the bound depends on the coefficients, \(a, t_0\), and \(T\), but not on \(f(x)\).

**Proof.** We will prove this by induction on \(s\). The result will follow from Theorem 3 and the form of \((III_\sigma)\). Apply \(\Delta_x^q\) to \(\Delta_t^v v\) given in \((III_\sigma)\) for any \(q\). This gives \(\Delta_x^q \Delta_t^v v\) as a linear combination of \(x\) differences of \(v_{in}\) which, for \(0 < t_0 \leq nk \leq T\), are \(0(F_q)\) by Theorem 3. Hence (89) holds for \(s = 0\).

Suppose that (89) holds for \(0 \leq s \leq j\), and that (88) holds for \(s = j\). Apply \(\Delta_x^q \Delta_t^{j+1} v\) to \(\Delta_t^v v\) given by \((III_\sigma)\). The resulting expression for \(\Delta_x^q \Delta_t^{j+2} v_{in}\) is \(0(F_{q+j})\), for \(t_0 \leq nk \leq T\), by the inductive hypothesis, properties (88) and Theorem 3. Hence (89) holds for \(s = j+1\).

**QED.**

**Lemma 15.** Let conditions (88) be satisfied. Let \(g_0, \varphi \in C^{s+2}[0,T]\).

Consider again the reduction of \((I_\sigma)\) to \((II_\sigma)\). First,

\[
\mu(x,t) \in C^\infty, s+1(\bar{R}); \quad G(x,t) \in C^\infty, s(\bar{R}).
\]

Let \(q\) be any positive integer. We can construct a \(G(x,t)\) so that the solution \(v_{in}\) of \((III_\sigma)\) with inhomogeneous term \(G\), will satisfy

\[
\Delta_x^q \Delta_t^r v_{in} = 0(F_{0+1}) \quad \text{for} \quad 0 < t_0 \leq t \leq T, \quad \text{and} \quad
\]

\[
0 \leq r \leq q, 0 \leq j \leq s+1.
\]

**Proof.** Statement (90) follows from the definition of \(\mu\) and \(G\). We can extend the determination of boundary conditions made on
\( \mu(x,t) \) on page 7, to obtain a \( G(x,t) \) satisfying
\[
\frac{\partial^j}{\partial x^j} G(x,t) = 0 \text{ at } x = 0, 1, \text{ for } 0 \leq j \leq q.
\]
Then by Lemma 9 with \( r = q+2 \), the Fourier coefficients \( d \) in the eigenvector expansions of \( G(\text{i}h, t) \) for fixed \( t \), satisfy
\[
d_p = O(p^{-q-2}).
\]
Applying a Duhamel-principle argument, arguments such as were used in obtaining Lemma 13, and Lemma 14, we obtain
\[
A^j v_{\infty} = O(t_0 + 1) \text{ for } t_0 \leq t \leq T, \text{ } 0 \leq j \leq q.
\]
Statement (91) then follows as in the proof of Lemma 14. QED.

**Theorem 12.** Suppose that \( \sigma > 0 \), that conditions (88) are satisfied, and that \( g_0, g_1 \in C^{s+2}[0,T] \).
Suppose that \( f(0) = g_0(0) \) and \( f(1) = g_1(0) \) with \( f \in C[0,1] \).
Then by Theorems 8 and 11, there exists a unique solution \( u \) of (I_1) in the class described in Theorem 11. This \( u(x,t) \) satisfies
\[
(92) \quad u(x,t) \in C^{\infty \cdot s}[0,1] \times (0,T].
\]
**Proof** We will show that (92) holds by showing that for any positive integer \( q \),
\[
(93) \quad u(x,t) \in C^{q \cdot s}[0,1] \times (0,T].
\]
Pick a \( G(x,t) \) as in Lemma 15, so that the solution \( v_{\infty} \) of the difference analog of the corresponding problem (I_0^q) satisfies (91). We will show first that the solution \( \overline{u}(x,t) \) of (I_0^q) satisfies (93).

A sequence of solutions \( v_{\infty} \) of the difference analog of (I_0^q) converges on \( R_0^\infty \) to \( \overline{u}(x,t) \). Using the fact that these functions \( v_{\infty} \) satisfy (91) independent of the mesh size, and arguing as in the proof of Theorem 1, we can show that
\[
\overline{u}(x,t) \in C^{q \cdot s}[0,1] \times [t_0,T], \text{ for any } t_0 > 0.
\]
This implies that \( \overline{u}(x,t) \) satisfies (93). Therefore by (90)
\[
u(x,t) = \overline{u}(x,t) + \mu(x,t) \text{ satisfies } (93) \text{ also.}
\]
Since \( q \) was an arbitrary positive integer, (92) holds. QED.
PART 2

AN A-PRIORI ERROR ESTIMATE FOR THE NUMERICAL INTEGRATION
OF A BOUNDARY-VALUE PROBLEM FOR THE HEAT EQUATION

We will obtain an a-priori bound on the error arising in the use of a simple forward-difference analog to approximate the solution of the following problem:

\[
\begin{aligned}
\frac{\partial}{\partial t}u(x, t) &= \frac{\partial^2}{\partial x^2}u(x, t), \quad \text{for } 0 < x < 1, \quad 0 < t \leq T; \\
u(x, 0) &= f(x), \quad \text{for } 0 \leq x \leq 1; \\
u(0, t) &= u(1, t) = 0, \quad \text{for } 0 \leq t \leq T.
\end{aligned}
\]

We will assume that

1. \(f(x) \in C^3[0, 1]\), and
2. \(f(0) = f(1) = 0\).

Using the same notation as in Part 1, we will consider the following difference analog:

\[
\begin{aligned}
\Delta_t v_{in} &= \Delta_x v_{in}, \quad 1 \leq i \leq N-1, \quad 0 \leq nk \leq T; \\
v_{i0} &= f(ih), \quad 0 \leq i \leq N \\
v_{0n} &= v_{Nn} = 0, \quad 0 \leq nk \leq T.
\end{aligned}
\]

Our analysis requires the choice of \(\mu\) with \(0 < \mu < \frac{1}{4}\).

As an example we will take \(\mu = .45\).

The technique we will use follows very closely the one used by Wasow [9] in obtaining a similar result for the numerical solution of the Dirichlet problem in a rectangle for Laplace's equation. Wasow, Juncosa, and Young [4, 5, 6, 9] have published papers treating this problem for (IV) with more general \(f(x)\). However, they do not give explicit bounds, and those established by Juncosa and Young only hold for \(t \geq t_o > 0\).

The solution of (IV) is

\[
(3) \quad u(x, t) = \sum_{p=1}^{\infty} c_p e^{-\frac{p^2}{\mu}} t \sin px,
\]
where

\[
(4) \quad c_p = 2 \int_0^1 f(x) \sin px \, dx.
\]

Integration by parts gives
Therefore
(6) \[ |c_p| \leq \frac{2K}{(\pi P)^3}, \]

where
(7) \[ K = |f''(1) - f''(0)| + \int_0^1 |f'''(x)| \, dx. \]

The solution of (V) is
(8) \[ v_{in} = \sum_{p=1}^{N-1} d_p \lambda_p^n \sin \pi nh, \]

where \( d_p \) is given by (19) on page 10, and
(9) \[ \lambda_p = 1 - 4\mu \sin^2 \left( \frac{\pi h}{2} \right). \]

As Wasow states,
(10) \[ d_p = c_p + \sum_{j=1}^{\infty} (c_{2Nj+p} + c_{2Nj-p}). \]

Therefore (8) becomes
(11) \[ v_{in} = \sum_{p=1}^{N-1} c_0 \lambda_p^n \sin \pi nh + \sum_{p=N+1}^{\infty} c_p \lambda_q^n \sin \pi q nh, \]

where \( q_p \epsilon \{1, 2, \ldots, N\} \), and in particular, by (9) and the choice of \( \lambda \),
(12) \[ |\lambda_q^n \sin \pi q nh| \leq 1. \]

Now we want to compare the solutions of (IV) and (V).
Set \( u_{in} = u(\text{ih}, nk) \) and \( z_{in} = u_{in} - v_{in} \).
Since \( nk = nh^2 \mu \), we have by (3) and (11),
(13) \[ z_{in} = \sum_{p=1}^{N-1} c_p \left[ e^{-\pi n^2 \mu} - \lambda_p^n \right] \sin \pi nh \]
\[ + \sum_{p=N+1}^{\infty} c_p \left[ e^{-\pi n^2 \mu} \sin \pi nh - \lambda_q^n \sin \pi q nh \right]. \]

Consider the second sum in (13), noting (12).
\[ | \sum_{p=N}^{\infty} c_p e^{-x(p)^2} \sin(p \omega) - \lambda_p^n \sin(p \omega) | \leq 2 \sum_{p=N+1}^{\infty} |c_p| \leq 4K_n^{-3} \sum_{p=N+1}^{\infty} \]

\[ < 4K_n^{-3} \int_{N}^{\infty} y^{-3} \, dy = 2K_n^{-3} \frac{1}{N-2} = (2K_n^{-3}) \frac{1}{N} \]

Now we must treat the first sum in (13).

\[ a^n - b^n = (a - b) \sum_{j=0}^{n-1} a^{n-j-1} b^j. \]

Let

\[ \Phi(x) = e^{-\left(\frac{\pi x}{2}\right)^2} \mu - (1 - 4\mu \sin(\frac{\pi x}{2})). \]

Then by (9), (15), and (16)

\[ \sum_{p=1}^{N-1} c_p [e^{-\left(\frac{\pi x}{2}\right)^2} \mu - \lambda_p^n] \sin(p \omega) = \sum_{p=1}^{N-1} c_p \Phi(p) \sum_{j=0}^{n-1} e^{-\left(\frac{\pi x}{2}\right)^2} \mu (n-1-j) \lambda_p^n \sin(p \omega). \]

We need to establish explicit, if crude, bounds on \(|\Phi(p)|, |\lambda_p^n|\).

**Lemma 1**

\[ \Phi(0) = \Phi'(0) = \Phi''(0) = \Phi'''(0) = 0 \]

\[ |\Phi''(x)| < \Phi, \text{ for } 0 \leq x \leq 1, \quad 0 < \mu < \frac{1}{4}. \]

**Proof**

\[ \Phi(x) = e^{-\left(\frac{\pi x}{2}\right)^2} \mu - (1 - 2\mu + 2\mu \cos \pi x). \]

\[ \Phi'(x) = -2\mu x^2 e^{-\left(\frac{\pi x}{2}\right)^2} \mu + 2\mu \pi \sin \pi x. \]

\[ \Phi''(x) = (4\mu^2 x^2 - 2\mu x^2) e^{-\left(\frac{\pi x}{2}\right)^2} \mu + 2\mu \pi^2 \cos \pi x. \]

\[ \Phi'''(x) = (12 \mu^4 x^4 - 8 \mu^3 x^3) e^{-\left(\frac{\pi x}{2}\right)^2} \mu - 2\mu^3 \pi^3 \sin \pi x. \]

These formulas prove (19). (20) is immediate. \(\text{QED.}\)

For the particular value \(\mu = 0.45\), one can easily show that \(\Phi = 400\) will do in (20).

By Lemma 1 of part 1, there exists an \(M = M(\mu)\) such that

\[ |\lambda_p^n| \leq e^{-M(\mu)^2}, \text{ for } 0 \leq \phi \leq 1. \]

And for \(\mu = 0.45\), a satisfactory \(M\) is 0.23.
Lemma 1 gives us

\begin{equation}
Q(\phi) \leq \frac{1}{4!} (\phi)^4 \Phi \quad \text{for } 0 \leq \phi \leq 1.
\end{equation}

Consider the first sum in (13), employing (17), (21) and (22).

\begin{equation}
\sum_{p=1}^{N-1} c_p Q(\phi) \sum_{j=0}^{n-1} e^{-n\phi^2} (n-l)^j \lambda_p \sin np\phi
\end{equation}

\begin{equation}
< \frac{\Phi}{4!} \sum_{p=1}^{N-1} c_p |Q(\phi)| n e^{-M(\phi)^2(n-l)}
\end{equation}

Noting (6) we see that this quantity is bounded in turn by

\begin{equation}
\frac{\Phi}{4!} \sum_{p=1}^{N-1} c_p |Q(\phi)| n e^{-M(\phi)^2(n-l)}
\end{equation}

**Lemma 2.** For \( 0 \leq nk \leq T \),

\begin{equation}
h \sum_{p=1}^{N-1} (ph)n e^{-M(\phi)^2(n-l)} \leq C(M, \mu).
\end{equation}

**Proof.** The quantity on the left-hand side of (25) is

\begin{align*}
h \sum_{p=1}^{N-1} (ph) &+ h \sum_{p=1}^{N-1} (ph)(n-l)e^{-M(\phi)^2(n-l)} \\
&< 1 + 2h \max_{0 \leq y} [y(n-l)e^{-M(\phi)^2(n-l)}] + (n-l) \int_0^1 y e^{-M(\phi)^2(n-l)} dy \\
&< 1 + 2h \left( \frac{n-1}{2M} \right)^{1/2} + \frac{1}{M} \int_0^\infty y e^{-y^2} dy \\
&\leq 1 + 2h \left( \frac{n-1}{2M} \right)^{1/2} \left( \frac{1}{2} \right) + \frac{1}{M} = C(M, \mu).
\end{align*}

QED.

For \( \mu = 0.45 \), and \( M = 0.23 \), \( C(M) < 3 + \sqrt{T} \).

Combining (24) and (25) we obtain the following result:

**Theorem.** The error \( z_\in \) made by approximating the solution \( u_\in \) of (IV) by the solution \( v_\in \) of (V) satisfies for \( 0 < \mu < \frac{1}{2} \), the following inequality:

\begin{equation}
|z_\in| \leq \frac{2K [C(M, \mu) \Phi + 4!]}{4!} h^2.
\end{equation}

In particular, for \( \mu = 0.45 \) and the crude estimates on \( M \) and \( \Phi \) given above,

\begin{equation}
|z_\in| \leq 3(3 + \sqrt{T})K h^2,
\end{equation}

where \( K \) is given by (7).
PART 3
AN INITIAL-VALUE PROBLEM FOR
AN INTEGRO-DIFFERENTIAL EQUATION

In this section we will outline the proof of an existence theorem for the following problem:

\[
\begin{align*}
\frac{\partial}{\partial t} u(x, t) &= a(x, t) \frac{\partial^2}{\partial x^2} u(x, t) + \int_0^t g[t, u(x, \zeta), \frac{\partial u}{\partial \zeta}(x, \zeta)] d\zeta, \\
u(x, 0) &= f(x), \quad \text{for } -\infty < x < +\infty, \; 0 < t \leq T.
\end{align*}
\]

We could have included a term, \( c[x, t, u(x, t)] \), and we could have let \( g = g[x, t, \zeta, u(x, \zeta)], \) However, problem (VI) contains the factors which cause the most difficulty, namely the dependence of \( g \) on \( t \) and on \( \frac{\partial u}{\partial \zeta} \).

We will say that a function \( F(y, z, w) \) defined on a set \( D \) is in \( C^{p, q, k}(D) \), if

\[
\frac{\partial^{p+q+s}}{\partial y^p \partial z^q \partial w^s} F \quad \text{exists and is continuous and bounded on } D \text{ for } 0 \leq p \leq i, \; 0 \leq q \leq j, \; 0 \leq s \leq k.
\]

Let \( R = \{ (x, t); \; -\infty < x < +\infty, \; 0 < t \leq T \} \).

We will assume that the following conditions hold,

\begin{align*}
(1) & \quad 0 < \alpha \leq a \leq A; \\
(2) & \quad a \in C^{2, 2}(\mathbb{R}); \; g \in C^{2, 1, 1}(\mathbb{R}); \; f \in C^4(-\infty, +\infty).
\end{align*}

We may relax the condition that \( g \) be bounded. We only require that its derivatives referred to in (2) be bounded.

Let \( h > 0, \; 0 < \mu \leq (2A)^{-1} \) and \( k = \mu h^2 \).

Our method of proof will be similar to that used by John in studying (I). (See page 2 and [3].) However, we will not obtain any strong stability theorems or consider any hypotheses weaker than \( f \in C^4 \). We will establish bounds on difference quotients of the solution \( v \) of the following difference analog:

\[
\begin{align*}
\begin{cases}
\Delta^2 v_{i n+1} = & v_{i n} + k a n \Delta^2 v_{i n} + k^2 \sum_{j=0}^{n-1} g(nk, v_{i j}, \Delta v_{i j}), \\
v_{i 0} = & f(ih), \quad -\infty < ih < +\infty, \; 0 \leq nk < T.
\end{cases}
\end{align*}
\]

Apply \( \Delta^4 \) to (VII).
\[ \Delta_{t}v_{in,n+1} = \Delta_{t}v_{in} + k a_{i,n+1} \Delta_{t}v_{in} + k \Delta_{t}a_{i,n} \Delta_{t}^{2}v_{in} + k g[(n+1)k v_{in}, \Delta_{t}v_{in}] + k^{2} \sum_{j=0}^{n-1} \Delta_{t}g(n k v_{ij}, \Delta_{t}v_{ij}), \]

where \( \Delta_{t}g \) denotes differencing only with respect to the first variable in \( g \). We also have from (VII)

\[ \Delta_{t}v_{i0} = a_{i0} \Delta_{t}x_{v_{i0}} = a_{i0} \Delta_{t}f(ih). \]

We need some more notation. Let \( g_{i} = \frac{\partial g}{\partial t} \), etc.

Let \( ||w||_{n} = \sup_{0 \leq j \leq n} |w_{in}| \), and \( ||w|| = \sup_{i} |w_{i}| \), wherever \( w \) is defined.

Since \( f \in C^{2} \), equation (4) gives us

\[ ||\Delta_{t}v||_{0} = O(1). \]

Because \( g \) has bounded derivatives,

\[ g(t,v_{in}, \Delta_{t}v_{in}) = g(0,0,0) + O(1 + ||v||_{n} + ||\Delta_{t}v||_{n}), \]

for \( 0 \leq t \leq T \).

Since \( v_{i0} = f(ih) \) is bounded,

\[ ||v_{in}|| = O(1 + ||\Delta_{t}v||_{n}). \]

(2), (3), (6) and standard arguments (See [1], page 12) show that for \( 0 < \mu < (2A)^{-1} \),

\[ ||\Delta_{t}v_{in+1}|| \leq ||\Delta_{t}v||_{n} + k 0(1 + ||v||_{n} + ||\Delta_{t}v||_{n} + ||\Delta_{t}v||_{n}). \]

Equations (1), (6), and (VII) give

\[ ||\Delta_{t}^{2}v_{in}|| = O(1 + ||v||_{n} + ||\Delta_{t}v||_{n}). \]

By combining (7), (8), and (9) we obtain

\[ ||\Delta_{t}v||_{n+1} \leq [1 + k 0(1)] ||\Delta_{t}v||_{n}. \]

Expressions (5) and (10) show that

\[ ||\Delta_{t}v||_{n} = O(1) \text{ for } 0 \leq nk < T. \]

Then equations (7), (9), and (11) give

\[ ||v||_{n}, ||\Delta_{t}^{2}v||_{n} = O(1), \text{ for } 0 \leq nk < T. \]
Now apply $\Delta_t$ to equation (3).

$$\Delta_t^{2} \nu_{i,n+1} = \Delta_t^{2} \nu_{i,n} + k a_{i,n+2} \Delta_t^{2} \nu_{i,n} + 2k \Delta_t a_{i,n+1} \Delta_t^{2} \nu_{i,n} + k \Delta_t a_{i,n+1} \Delta_t^{2} \nu_{i,n} +$$

(13) $$+ 2k \tilde{g}_1 + k \tilde{g}_2 \Delta_t \nu_{i,n} + k \tilde{g}_3 \Delta_t^{2} \nu_{i,n} + k^2 (n-1) \tilde{g}_{11} ,$$

where a bar denotes an intermediate value assigned by the mean-value theorem or, in the case of $\tilde{g}_{11}$, an average of such values.

We will treat (13) as we did (3). From (1), (2), (3), and (6) we obtain

(14) $$|\Delta_t^{2} \nu_{i,n}| = 0 [1 + |\Delta_t^{2} \nu_{i,n}| + |\Delta_t^{2} \nu_{i,n}| + |\Delta_t^{2} \nu_{i,n}| + |\nu_{i,n}|].$$

From this, (11), and (12), it follows that

(15) $$|\Delta_t^{2} \nu_{i,n}| = 0(1 + ||\Delta_t^{2} \nu||_n).$$

Finally, with the same arguments as those used in obtaining (8), equations (13), (15), etc., give

(16) $$||\Delta_x^{2} \nu||_{n+1} \leq [1 + k 0(1)] ||\Delta_x^{2} \nu||_n.$$

We must show now that

(17) $$||\Delta_t^{2} \nu||_0 = 0(1).$$

$$\Delta_t \nu_{i,0} = a_{i,0} \Delta_x^{2} \nu_{i,0} ; \quad \Delta_t \nu_{i,1} = a_{i,1} \Delta_x^{2} \nu_{i,1} + k g(k, \nu_{i,0}, \Delta_t \nu_{i,0}).$$

$$\Delta_x^{2} \nu_{i,1} = \Delta_x^{2} (\nu_{i,0} + k a_{i,1} \Delta_x \nu_{i,0}).$$

Employing these we have

$$\Delta_t^{2} \nu_{i,0} = \frac{1}{k} [\Delta_t^{2} \nu_{i,1} - \Delta_t \nu_{i,0}] = \frac{1}{k} [a_{i,1} \Delta_x^{2} \nu_{i,0} - a_{i,0} \Delta_x^{2} \nu_{i,0}],$$

(18) $$+ a_{i,1} \Delta_x^{2} (a_{i,0} \Delta_x \nu_{i,0} + g(k, \nu_{i,0}, \Delta_t \nu_{i,0})).$$

Expression (17) follows immediately from (2) and the fact that $\nu_{i,0} = f(ih)$. From (16) and (17) we obtain

(19) $$||\Delta_t^{2} \nu||_n = 0(1), \quad \text{for} \quad 0 \leq nk < T.$$

Then equations (15) and (19) give

(20) $$||\Delta_x^{2} \Delta_t \nu||_n = 0(1), \quad \text{for} \quad 0 \leq nk < T.$$

Now we have to bound $\Delta_x \nu_{i,n}, \Delta_x \Delta_t \nu_{i,n},$ and $\Delta_x \Delta_x \nu_{i,n}$. 


Applying $\Delta_x$ to (VII), we obtain

\[
(21) \quad \Delta_x a_{V_{\text{in}}} = \Delta_x (a_{\text{in}} \Delta^2_{V_{\text{in}}}) + k \sum_{j=0}^{n-1} \Delta_x g(nk, v_{ij}, \Delta_t v_{ij}).
\]

Using (1) and (2) as before, (21) shows that

\[
(22) \quad |\Delta_x \Delta^2_{V_{\text{in}}}| = 0(|\Delta_x \Delta_{V_{\text{in}}}| + |\Delta^2_{V_{\text{in}}}| + |\Delta_{V_{\text{in}}}| + 1).
\]

Since $\Delta_x v_{10} = \Delta_x f(ih) = 0(1),$

\[
(23) \quad |\Delta_x v_{\text{in}}| = 0(1 + |\Delta_x \Delta_{V_{\text{in}}}|).
\]

Equation (22) with (23) and (12) gives

\[
(24) \quad |\Delta_x \Delta^2_{V_{\text{in}}}| = 0(1 + |\Delta_x \Delta_{V_{\text{in}}}|).
\]

Now apply $\Delta_x$ to (3).

\[
\Delta_x \Delta_{V_{\text{in}}}, n+1 = \Delta_x a_{V_{\text{in}}} + k a_{i+1, n+1} \Delta^2_{x t} +
\]

\[
+ k \Delta_x a_{i, n+1} \Delta^2_{V_{\text{in}}} + k \Delta_t a_{i+1, n} \Delta^2_{V_{\text{in}}} +
\]

\[
+ k \Delta_t a_{i, n} \Delta^2_{V_{\text{in}}} + k \Delta^2_{x t} + k \Delta^2_{V_{\text{in}}} +
\]

\[
+ k^2 \sum_{j=0}^{n-1} (\Delta_x g_{ij} + \Delta^2_{x t} v_{ij}).
\]

By the same argument as used to get (8) and (16),

\[
(25) \quad |\Delta_x \Delta^2_{V_{\text{in}}}, n+1| \leq [1 + k 0(1)] |\Delta_x \Delta_{V_{\text{in}}}| +
\]

\[
+ k 0(|\Delta_x \Delta_{V_{\text{in}}}| + |\Delta^2_{x t} + |\Delta^2_{V_{\text{in}}}| + |\Delta_{V_{\text{in}}}| +
\]

\[
+ |\Delta_x \Delta_{V_{\text{in}}}| + |\Delta_{V_{\text{in}}}|).
\]

Equation (25), combined with (24), (23), (20), and (12), yields

\[
(26) \quad |\Delta_x \Delta_{V_{\text{in}}}|, n+1 \leq [1 + k 0(1)] |\Delta_x \Delta_{V_{\text{in}}}|.
\]

Finally,

\[
(27) \quad |\Delta_x \Delta_{V_{\text{in}}}| = \Delta_x (a_{i0} \Delta^2_{V_{\text{in}}}) = 0(1).
\]

So (26) and (27) give

\[
(28) \quad |\Delta_x \Delta_{V_{\text{in}}}|, n = 0(1), \text{ for } 0 \leq nk < T.
\]
Recalling (11), (12), (19), (20), (23), (24) and (28), we have the following lemma:

**Lemma** Let $a$, $g$, and $f$ satisfy properties (1) and (2). Let $v_{in}$ be the corresponding solution of (VII). Then

$$v, \Delta_x v, \Delta_t v, \Delta_x \Delta_t v, \Delta^2_x v, \Delta^2_x \Delta_t v,$$

are uniformly bounded on $\mathbb{R}$, independent of the mesh size. Here we have assumed of course that $0 < \mu \leq (2A)^{-1}$.

**Theorem 1** Let $a$, $g$, and $f$ satisfy properties (1) and (2). Then there exists a solution $u(x,t)$ of (VI) with

$$u(x,t), \frac{\partial}{\partial x} u(x,t), \frac{\partial}{\partial t} u(x,t), \text{ and } \frac{\partial^2}{\partial x^2} u(x,t)$$

uniformly Lipschitz continuous and bounded in $\mathbb{R}$.

**Proof** The proof of this is almost the same as that of Theorem 1 in part 1, page 18. To handle the integral in (VI) one must use the uniform Lipschitz continuity of $g, u, \text{ and } \frac{\partial u}{\partial t}$.

Unfortunately, it seems unlikely that $\Delta^2 t v_{in}$ can be shown to be bounded by continuing the procedure used to obtain the Lemma. And we have been unable to establish the convergence of the solutions of (VII) to a solution of (VI) unless the solution $u(x,t)$ satisfies differentiability conditions which are stronger than those guaranteed by Theorem 1. Therefore, we also have no uniqueness theorem for the solution we have exhibited in Theorem 1. The best we can do is the following:

**Theorem 2** Let $a$, and $g$ satisfy properties (1) and (2). Suppose that $u(x,t)$ is a solution of (VI) with the quantities (30) as well as

$$\frac{\partial^2}{\partial t^2} u(x,t), \frac{\partial^3}{\partial t^3} u(x,t), \text{ and } \frac{\partial^2}{\partial x \partial t} u(x,t)$$

uniformly continuous and bounded in $\mathbb{R}$. Then the solution $v_{in}$ of the corresponding difference analog (VII) converges to $u(x,t)$ as the mesh size goes to zero. And finally, $u(x,t)$ is unique among this class of solutions.
\[ \frac{\partial u}{\partial t} = a(x,t) \frac{\partial^3 u}{\partial t \partial x^2} + \frac{\partial a(x,t)}{\partial t} \frac{\partial^2 u}{\partial x^2} + g(t,u(x,t), \frac{\partial u(x,t)}{\partial t}) + \int_0^t g_1(t, u(x,t), \frac{\partial u(x,t)}{\partial t}) \, dt. \]

By the uniform continuity of all quantities involved, and the mean value theorem, this gives

\[ \Delta^2_t u_{in} = a_{i,n+1} \Delta_t x_v_{in} + \Delta_t \Delta^2_t x_v_{in} + g((n+1)k, u_{in}, \Delta_t u_{in}) + \]

\[ + k \sum_{j=0}^{n-1} \Delta_t g(nk, u_{ij}, \Delta_t u_{ij}) + o(1). \]

Let \( z_{in} = u_{in} - v_{in} \). Then by (3) and (32),

\[ \Delta^2_t z_{in} = a_{i,n+1} \Delta_t x_z_{in} + \Delta_t \Delta^2_t x_z_{in} + g_2 z_{in} + g_3 \Delta_t z_{in} + \]

\[ + k \sum_{j=0}^{n-1} [g_{12} z_{ij} + g_{13} \Delta_t z_{ij}] + o(1). \]

From this it follows, with arguments similar to those used above, that

\[ ||\Delta_t z||_{n+1} = [1 + o(k)] ||\Delta_t z||_n + o(k) ||z||_n \]

\[ + o(k) ||\Delta^2_t z||_n + o(k) \]

\[ = [1 + o(k)] ||\Delta_t z||_n + o(k). \]

Noting that \( ||\Delta_t z||_0 = o(1) \), we obtain from (34)

\[ ||\Delta_t z||_n = [1 + o(k)]^n o(1) + o(k) \sum_{j=0}^{n-1} [1 + o(k)]^j \]

\[ = o(1). \]

Since \( z_{i0} = 0 \), this proves the theorem. \( \text{Q.E.D.} \)
Appendix

**Lemma 6.**

\[ \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} (x-jk)^s \sin \pi p x \, dx \]

\[ = \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left[ \frac{d^n}{dx^n \sin \pi x} \right]_{x=jh} (x-jh)^{n+s} \, dx \]

\[ = \int_{(j-\frac{1}{2})h}^{(j+\frac{1}{2})h} \left\{ \sin \pi p jh \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (\pi p)^{2n} (x-jh)^{2n+s} \right. \]

\[ + \cos \pi p jh \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (\pi p)^{2n+1} (x-jh)^{2n+s+1} \] \[ \left. \right\} \, dx \]

\[ = \sin \pi p jh \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{(\pi p)^{2n}}{(2n+s+2)} (x-jh)^{2n+s+1} \right\}_{x=j\frac{1}{2}h} \]

\[ + \cos \pi p jh \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{(\pi p)^{2n+1}}{(2n+s+2)} (x-jh)^{2n+s+2} \right\} \]

If \( s \) is even, this equals

\[ 2 \sin \pi p jh \sum_{n=0}^{\infty} \frac{(-1)^n (\pi p)^{2n}}{(2n)! (2n+s+1)!} (\frac{h}{2})^{2n+s+1} = 2 \sin \pi p jh (\frac{h}{2})^{s+1} B_s(p,h). \]

If \( s \) is odd, it equals

\[ 2 \cos \pi p jh \sum_{n=0}^{\infty} \frac{(-1)^n (\pi p)^{2n+1}}{(2n+1)! (2n+s+2)!} (\frac{h}{2})^{2n+s+2} = 2 \cos \pi p jh (\frac{h}{2})^{s+1} B_s(p,h). \]

This proves (40). The proof of (41) is analogous.

**Lemma 7.**

\[ B_0(p,h) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left( \frac{\pi ph}{2} \right)^{2n} \]

\[ = 1 - \frac{1}{3!} \left( \frac{\pi ph}{2} \right)^2 + \frac{1}{5!} \left( \frac{\pi ph}{2} \right)^4 - \cdots. \]

Therefore for \( 0 \leq ph \leq 1 \),

\[ B_0(p,h) > 1 - \frac{1}{6} z^2 = 1 - \frac{2}{3} = \frac{1}{3}. \]

**Lemma 8.** This follows directly from the definition of \( B_s(p,h) \).
Bibliography


