A PROOF OF GREEN'S LEMMA

by

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The proof of Green's lemma for a rectangular region
R:  \( a \leq x \leq b, \ c \leq y \leq d \), for integration in the sense of
Lebesgue, is a familiar one, as is the extension of the rectangular
case to that for regions which are plurisegments. There have
been published several proofs of the lemma for less restricted
regions, each of which is approximated by a sequence of interior
polygonal regions.¹

This paper presents a relatively simple method of
determining such a sequence of polygons interior to a region
bounded by a simple closed rectifiable curve and employs the
theorem of Helly in order to prove Green's lemma for such a
region.

We first prove the lemma for a rectangular region:

**Theorem 1.** If \( Q(x,y) \) is an absolutely continuous
function of \( x \), \( a \leq x \leq b \), for almost all \( y \), \( c \leq y \leq d \), if
\( \frac{\partial}{\partial x} Q(x,y) \) is integrable (L) in \( R:  a \leq x \leq b, \ c \leq y \leq d \), and
if \( C \) is the boundary of \( R \), then

\[
\int \int_R \frac{\partial}{\partial x} Q(x,y) \, dx \, dy = \int_C Q(x,y) \, dy.
\]

M. R. Hestenes, "An analogue of Green's theorem in the
pp. 300-311.
W. T. Reid, "Green's lemma and related results," Amer.
Proof: Because of the integrability of $Q(x,y)$ the theorem of Fubini applies:

$$\int \int_{R} \frac{2}{\partial x} Q(x,y) dx dy = \int_{a}^{b} dy \int_{a}^{b} \frac{2}{\partial x} Q(x,y) dx.$$ 

Since $Q(x,y)$ is absolutely continuous in $x$ for almost all $y$, then

$$\int_{a}^{b} \frac{2}{\partial x} Q(x,y) dx = Q(b,y) - Q(a,y)$$

for almost all $y$. Hence:

$$\int \int_{R} \frac{2}{\partial x} Q(x,y) dx dy = \int_{c}^{d} \left[ Q(b,y) - Q(a,y) \right] dy$$

$$= \int_{c}^{d} Q(b,y) dy - \int_{d}^{c} Q(a,y) dy$$

$$= \int_{c}^{d} Q(x,y) dy.$$  

q.e.d.

Corollary. If $R$ is a two-dimensional plurisegment with boundary $C$, and if the conditions of Theorem 1 on $Q(x,y)$ and $\frac{2}{\partial x} Q(x,y)$ hold for each segment of $R$, then

$$\int \int_{R} \frac{2}{\partial x} Q(x,y) dx dy = \int_{C} Q(x,y) dy.$$ 

Since $R = \sum_{i} R_{i}$, where each $R_{i}$ is a rectangle with sides parallel to the coordinate axes, an inductive argument may be used to prove the corollary quite easily.

Now if $C$ is a simple closed rectifiable curve, it may be defined by the coordinate functions $x = x(t), y = y(t), 0 \leq t \leq 1$, which are continuous and of bounded variation on the closed interval $[0,1]$.

We shall show that the Jordan interior $D$ of $C$ contains
a sequence of simply connected polygonal regions $R_n$ each of whose boundary $P_n$ is defined by the coordinate functions $x = X_n(t), y = Y_n(t), 0 \leq t \leq 1$, such that:

I. $R_n \subseteq R_{n+1}$.

II. $\lim_{n \to \infty} |R_n| = |D|$, where $|R_n|$ and $|D|$ denote the measure of $R_n$ and $D$ respectively.

III. Every linear arc of $P_n$ is parallel to one of the axes.

IV. For $0 \leq t \leq 1$, $X_n(t)$ and $Y_n(t)$ are continuous and uniformly of bounded variation.

V. A one-to-one correspondence may be established between points of $C$ and points of $P_n$ such that for increasing values of $t$ the order along both curves in is the same sense.

VI. For $0 \leq t \leq 1$, $\lim_{n \to \infty} X_n(t) = x(t)$ and $\lim_{n \to \infty} Y_n(t) = y(t)$ uniformly.

The existence of such polygons, which will be established by a sequence of lemmas, makes possible the proof of:

**Theorem 2.** If $C$ is a simple closed rectifiable curve with Jordan interior $D$, if $\frac{\partial^2}{\partial x^2} Q(x,y)$ is integrable (L) in $D$, and if $Q(x,y)$ is continuous in $R = C + D$ and absolutely continuous in $x$ for almost all $y$ in $D$, then

$$\iint_D \frac{\partial^2}{\partial x^2} Q(x,y) dx dy = \int_C Q(x,y) dy.$$
Proof: The corollary to Theorem 1 applies in particular to simply connected plurisegments and thus to each $R_n$, so that

$$\int \int_{R_n} \frac{\partial}{\partial x} Q(x,y) dx dy = \int_{P_n} Q(x,y) dy.$$ 

Hence

$$\lim_{n \to \infty} \int \int_{R_n} \frac{\partial}{\partial x} Q(x,y) dx dy = \lim_{n \to \infty} \int_{P_n} Q(x,y) dy. \quad (1)$$

Since $\frac{\partial}{\partial x} Q(x,y)$ is integrable in $D$, then for $E \subset D$,

$$\int \int_{E} \frac{\partial}{\partial x} Q(x,y) dx dy$$

is absolutely continuous. Since $\lim_{n \to \infty} |D - R_n| = 0$,

it then follows that

$$\lim_{n \to \infty} \int \int_{R_n} \frac{\partial}{\partial x} Q(x,y) dx dy = \int \int_{D} \frac{\partial}{\partial x} Q(x,y) dx dy. \quad (2)$$

It remains to show that

$$\lim_{n \to \infty} \int_{P_n} Q(x,y) dy = \int_{C} Q(x,y) dy. \quad (3)$$

Let $Q(x(t), y(t)) = Q(t)$, $Q(X_n(t), Y_n(t)) = Q_n(t)$. There is no loss of generality in assuming that positive sense along $P_n$ is for increasing values of $t$. Then

$$\int_{P_n} Q(x,y) dy = \int_{0}^{1} Q_n(t) dY_n(t),$$
and since the sense along $C$ is the same as that for $P_n$, 
\[ \int_C Q(x,y)dy = \int_0^1 Q(t)dy(t). \]

Since \( \lim_{n \to \infty} X_n(t) = x(t) \), \( \lim_{n \to \infty} Y_n(t) = y(t) \) uniformly, and since $Q(x,y)$ is continuous in the closed region $R$ and thus uniformly continuous, then \( \lim_{n \to \infty} Q_n(t) = Q(t) \) uniformly. Because of the uniform bounded variation of \( \{Y_n(t)\} \) there exists an upper bound $T_Y$ of the total variations of all the $Y_n(t)$. Hence for each $\varepsilon > 0$, there exists $n_0(\varepsilon)$, independent of $t$, such that for $n > n_0$, 
\[ |Q_n(t) - Q(t)| < \frac{\varepsilon}{T_Y}, \quad 0 \leq t \leq 1. \]

Thus:
\[ \left| \int_0^1 Q_n(t)dy_n(t) - \int_0^1 Q(t)dy(t) \right| \leq \int_0^1 |Q_n(t) - Q(t)| \left| dy_n(t) \right| \]
\[ < \int_0^1 \frac{\varepsilon}{T_Y} \left| dy_n(t) \right| = \frac{\varepsilon}{T_Y} \int_0^1 \left| dy_n(t) \right| = \varepsilon \]

\[ \lim_{n \to \infty} \int_0^1 Q_n(t)dy_n(t) = \lim_{n \to \infty} \int_0^1 Q(t)dy(t). \quad (4) \]
By the theorem of Helly:

$$\lim_{n \to \infty} \int_0^1 Q(t) dY_n(t) = \int_0^1 Q(t) dy(t).$$

From (4) and (5) follows (3), which with (2) and (1) proves the theorem. q.e.d.

We now establish the existence of the sequence \( \{R_n\} \) of polygonal regions.

We note that there is no loss of generality in assuming that the parameter \( t \) defining \( C \) over the interval \([0,1]\) is the arc-length measured along \( C \) from a point \( q_0 \in C \). For, since \( C \) is rectifiable, if its length is \( L \), then \( C \) permits parameterization in terms of its arc-length \( s \) from a point \( q_0 : (X_0, Y_0) \); that is, \( C \) may be defined by \( X = X(s), Y = Y(s), \) \( 0 \leq s \leq L \), these functions being continuous and of bounded variation. Then if \( t = L^{-1} s, x = L^{-1} X, y = L^{-1} Y \), the resulting functions \( x = x(t) = L^{-1} X(L^{-1} s), y = y(t) = L^{-1} Y(L^{-1} s) \), define \( C \) for \( 0 \leq t \leq 1 \), and are continuous and of bounded variation; and it is readily seen that \( t \) will be the arc-length of \( C \) measured from \( q_0: (x_0, y_0) \), \( x_0 = L^{-1} X_0, y_0 = L^{-1} Y_0 \), in the \( x-y \) coordinate system.

The plane can be covered by a net composed of a sequence of lattices \( \{S_k\} \), each of whose lines is of the form
\[ x = j 2^{-k}, \quad y = j 2^{-k}, \quad j = 0, \pm 1, \pm 2, \ldots \]

Then each mesh of the lattice \( S_k \) is a square with length of side \( 2^{-k} \), and each intersection of lattice lines has rational coordinates.

Let \( 0 \) be any point of \( D \) with irrational coordinates. Then \( 0 \) is an interior point of a mesh of each \( S_k \), and for \( k \) sufficiently large the mesh \( m \) of \( S_k \) of which \( 0 \) is an interior point will be contained in \( D \). Consider the union \( m' \) of \( m \) with those meshes contained in \( D \) having common mesh sides with \( m \), the union \( m'' \) of \( m' \) with those meshes contained in \( D \) having common mesh sides with meshes of \( m' \), and so on. Then there results a connected union \( M_k \subset D \) of meshes of \( S_k \). Furthermore, \( M_k \) is simply connected, because it is so determined that its boundary is a Jordan curve contained in \( D \).

Thus, for \( k \) sufficiently large, there exists a simply connected union \( M_k \subset D \) of meshes of \( S_k \), having \( 0 \) as an interior point, and \( M_k \) is the union of the greatest number of meshes of \( S_k \) contained in \( D \) which is simply connected and has \( 0 \) as interior point.

**Definition 1.** The maximal polygonal region, with respect to \( 0 \), of the lattice \( S_k \), in \( D \), is the closed region \( M_k \).

**Definition 2.** The maximal polygon \( P_k \), with respect to \( 0 \), of \( S_k \) in \( D \) is the boundary of \( M_k \).

It will be shown that \( \{ M_k \} \) contains a subsequence \( \{ M_{k_n} \} \) having the properties which, assumed for \( \{ R_n \} \), enabled the proof of Theorem 2.
By definition the maximal polygons have property III. And it follows from the definition of $M_k$ that property I applies. For, since every mesh of $S_k$ is the square union of 4 meshes of $S_{k+1}$ and since 0 is an interior point of both $M_k$ and $M_{k+1}$, then $M_{k+1}$ is obtained by the addition of meshes of $S_{k+1}$ to $M_k$.

The method of proof of the remaining properties involves showing that the lengths of all the maximal polygons are uniformly bounded, so that the coordinate functions are uniformly of bounded variation; that any point $p \in D$ will be, for $k$ sufficiently large, an interior point of $M_k$, so that property II holds; and that a one-to-one correspondence preserving order can be established between points of $C$ and $P_{k_n}$ such that, for each $k_n$, the distance between any two corresponding points of $C$ and $P_{k_n}$ is $O(2^{-n})$, and thus the coordinate functions of the $P_{k_n}$ approach $x(t)$ and $y(t)$ uniformly.

We consider now a set of points of $C$ such that the arc-length between two successive points along $C$ is, for a given integer $n \geq 2$, $2^{-n}$. Let $q_j^{(n)} \in C$ be the point determined by $t = j2^{-n}$, $j = 1, 2, \ldots, 2^n$. Let $C_j^{(n)} \subset C$ be the arc determined by $j2^{-n} \leq t \leq (j+1)2^{-n}$. 
Lemma 1. If \( \lambda_k \) is the arc-length of the maximal polygon \( P_k \) with respect to 0, of \( S_k \) in D, then every point \( p \subseteq P_k \) is within the distance \( 2^{-k+1} \) of a point \( q \subseteq C \), and \( \lambda_k \leq 24 \).

Proof: Since each \( C_j^{(k)} \) has arc-length \( 2^{-k} \), \( C_j^{(k)} \) is contained in a square \( T_j^{(k)} \) with length of side \( 2^{-k} \). Since, in one dimension, any interval of length \( 2^{-k} \) will be strictly interior to the union of three abutting meshes of any lattice with mesh length \( 2^{-k} \), it follows that every \( T_j^{(k)} \) will be strictly interior to the square union \( \bigcup_{j}^{(k)} \) of \( 3^2 = 9 \) meshes of \( S_k \). Then the length \( L_j^{(k)} \) of all mesh lines in \( \bigcup_{j}^{(k)} \) will be \( 24(2^{-k}) \).
\[ \bigcup_k = \sum_{j=1}^{p^k} \bigcup_j \] is a connected union of meshes of \( S_k \) such that 
\[ C \subseteq \bigcup_j \] and 
\[ L(k) \leq 2^k 24(2^{-k}) = 24, \] where \( L(k) \) is the length of all the mesh lines of \( S_k \) in \( \bigcup_j \).

I say: \( P_k \subseteq \bigcup_j \). For, if not, there exists at least one point \( p \in P_k \cdot (R - \bigcup_j) \). Hence \( p \) is the boundary point of both \( P_k \) and of 1, 2, or 3 meshes \( m_i \) of \( S_k \), \( m_i \notin M_k \).

Then the union of \( M_k \) with the \( m_i \) results in the simply connected union \( M_k' \subseteq D \) of meshes of \( S_k \) such that \( 0 \in M_k \subseteq M_k' \).

Hence \( M_k \) is not the maximal polygonal region of \( S_k \) in \( D \). This contradiction makes the assumption that \( P_k \notin \bigcup_j \) invalid.

Thus \( \lambda_k \leq L(k) \leq 48 \), since \( P_k \) is composed of mesh lines of \( \bigcup_j \).

Now suppose there exists \( p \in P_k \) such that no point of
0 is within the distance $2^{-k}1$ of $p$. Then the closed circular region $C(p, 2^{-k} \sqrt{2})$ with center $p$ and radius $2^{-k} \sqrt{2}$ is such that $C(p, 2^{-k} \sqrt{2}) \subset D$. Furthermore $p$ is an interior point of a simply connected union $M \subset C(p, 2^{-k} \sqrt{2})$ of meshes of $S_k$. $M \neq M_k$.

Then $M'_k = M_k + M$ is a simply connected union of meshes of $S_k$ such that $0 \in M_k \subset M'_k \subset D$, so that $M_k$ is not the maximal polygonal region of $S_k$ in $D$. Hence every $p \in P_k$ is within the distance $2^{-k+1}$ of a point $q \in C$. q.e.d.

**Lemma 2.** For each point $p \in D$ there exists an integer $k_p$ such that $p$ is an interior point of $M_{k_p}$.
Proof: Since $D$ is simply connected and $p, 0 \in D$, then $p$ and $0$ are end-points of a simple continuous polygonal arc $\gamma \subset D$. Since each $q \in \gamma$ is an interior point of $D$, there exists $\mathcal{C}_q$ such that $\mathcal{C}(q, \mathcal{C}_q) \subset D$. Then for $k_q$ sufficiently large that the length of side of each mesh of $S_{k_q}$ is

$$2^{-k_q} < \sqrt{2} \rho_q,$$

$q$ will be an interior point of a simply connected union $U_q \subset \mathcal{C}(q, \mathcal{C}_q)$ of meshes of $S_{k_q}$.

The set of all such $U_q$ covers $\gamma$, a closed, bounded set, and by the theorem of Heine-Borel-Lebesgue, there exists a finite subset $\left\{ U_{q_i} \right\}$, $i = 1, \ldots, n$, covering $\gamma$. Let $k_p = \max_{i=1}^n k_{qi}$.

Then $U = \sum_{i=1}^n U_{q_i}$ is a connected union of meshes of $S_{k_p}$ such that
\[ \gamma \subset U, \text{ and hence } p, 0 \in \gamma \text{ are interior points of } U. \] From the
definition of \( M_k \) it follows that \( U \subset M_k \).

q.e.d.

Lemma 3. \[ \lim_{k \to \infty} |M_k| = |D|. \]

Proof: Since \( \{M_k\} \) is an expanding sequence of sets,
\[ |M_k| = |\sum_{i=1}^{k} M_i| ; \text{ and since each } M_k \subset D, |M_k| < |D|. \] Thus
\[ |\sum_{i=1}^{\infty} M_i| = \lim_{k \to \infty} |M_k| \leq |D|. \]

By Lemma 2, every \( p \in D \) is a point of \( \sum_{i=1}^{\infty} M_i \). Thus
\[ |D| \leq |\sum_{i=1}^{\infty} M_i| = \lim_{k \to \infty} |M_k|. \]

Hence \( \lim_{k \to \infty} |M_k| = |D| \).

q.e.d.

We determine now a subsequence \( \{M_{k_n}\} \) of \( \{M_k\} \) which
possesses the remaining desired properties for the regions approxi-
mating \( D \). Consider for each \( n \geq 2 \), the arc \( C_j^{(n)} \) and a corre-
sponding arc \( \gamma_j^{(n)} \) determined by \( t \in [0, 1] \cdot C \left[ (j-1)2^{-n}, (j+2)2^{-n} \right] \).
$c_j^{(n)}$ and $\gamma_j^{(n)}$ are disjoint closed sets, so that there exists $d_j^{(n)} = \text{dist}(c_j^{(n)}, \gamma_j^{(n)}) > 0$. Let $d^{(n)} = \min_{j=1,\ldots,2^n} d_j^{(n)}$. Let $I_j^{(n)}$ denote the interior of $C(q_j^{(n)}, 3^{-1}d^{(n)})$. Since each $q_j^{(n)}$ is a limit point of $D$, there exists a point $r_j^{(n)} \in D, I_j^{(n)}$. By Lemma 2 there is an integer $m_j$ such that $r_j^{(n)}$ is an interior point of the maximal polygonal region $M_{m_j}$. Choose $m_d$ such that $2^{-md} < 18^{-(18)}$. Let $k_n = \max_{j=1,\ldots,2^n} (m_j, m_d, k_{n-1})$. Then each $r_j^{(n)}$ is an interior point of $M_{k_n}$, and since $k_n \geq k_{n-1}$, the sequence $\{M_{k_n}\}$ is, for increasing values of $n$, an expanding sequence of polygonal regions. From this and Lemmas 1 and 3 it follows immediately that properties I-III apply to $\{M_{k_n}\}$. We use the $\{k_n\}$ just selected to prove the remaining properties.

Lemma 4. For each integer $n \geq 2$ there corresponds to each $q_j^{(n)} \in C$ a point $p_j^{(n)} \in P_{k_n}$ such that:
(a) \[ |p_j^{(n)} - q_j^{(n)}| < 2^{-n}. \]

(b) For increasing values of \( j \) the order of the \( p_j^{(n)} \) along \( P_k \) is in the same sense as that of the \( q_j^{(n)} \) along \( C \).

**Proof:** Since \( r_j^{(n)} \) is an interior point of \( M_{k_n} \) and \( q_j^{(n)} \) is exterior to \( M_{k_n} \), the line segment with end-points \( q_j^{(n)} \) and \( r_j^{(n)} \) has at least one point in common with \( P_{k_n} \), so that there are points of \( P_{k_n} \) within the distance \( 3^{-1d(n)} \) of \( q_j^{(n)} \).

Since \( P_{k_n} \) is a closed curve, the set of values \( \left| q_j^{(n)} - p \right| \) is closed, and, because of the boundedness of \( R \), the set is bounded. Hence the set has a greatest lower bound belonging to the set; that is, there exists at least one point \( p' \in P_{k_n} \) such that \( \left| q_j^{(n)} - p \right| \geq \left| q_j^{(n)} - p' \right| \). Let one of the points \( p' \) of \( P_{k_n} \) nearest \( q_j^{(n)} \) be taken, and denote this point by \( p_j^{(n)} \).
Let $T_j^{(n)}$ be the line segment with end-points $p_j^{(n)}$ and $q_j^{(n)}$. Since $T_j^{(n)}$ and $C$ are closed sets, then $T_j^{(n)} \cdot C$ is closed, and the set of values $\left| p_j^{(n)} - q \right|_{q \in T_j^{(n)} \cdot C}$ is closed and bounded, and therefore has as an element its greatest lower bound.

Thus there exists a point $\Pi_j^{(n)} \subseteq T_j^{(n)} \cdot C$ which is the point of $C$ on the line segment nearest $p_j^{(n)}$. It follows that $p_j^{(n)}$ is a point of $p_k^{(n)}$ nearest $\Pi_j^{(n)}$. 
Now $d(n)$ was so selected that, for $j \neq k$,

$$d(n) \leq \left| q_j(n) - q_k(n) \right|. \text{ In particular } d(n) \leq \left| q_j(n) - q_{j+1}(n) \right| \leq 2^{-n}.$$

Thus

$$\left| q_j(n) - p_j(n) \right| \leq \left| r_j(n) - q_j(n) \right| < 3^{-1}d(n) < 2^{-n}.$$

The points $\{p_j(n)\}$ are geometrically distinct, because for $j \neq k$:

$$\left| p_j(n) - q_j(n) \right| \geq \left| q_j(n) - q_k(n) \right| - \left| q_j(n) - p_j(n) \right| - \left| p_k(n) - q_k(n) \right|$$

$$> d(n) - \frac{d(n)}{3} - \frac{d(n)}{3} = \frac{d(n)}{3} > 0.$$

$\mathcal{P}_j(n)$ was selected so that $\mathcal{P}_j(n) \subseteq c_j(n) + c_j(n)$, and similarly $\mathcal{P}_{j-1}(n) \subseteq c_j(n) + c_{j+1}(n)$. Hence there is an arc $\alpha \subset \mathcal{C}$.
with end-points \( \pi_j^{(n)} \) and \( \pi_{j+1}^{(n)} \) such that no points of the set \( \{ q_j^{(n)} \} \) other than \( q_j^{(n)} \) and \( q_{j+1}^{(n)} \) belong to \( \alpha \).

For simplicity, let \( \pi_1 = p_j^{(n)} \), \( \pi_2 = p_{j+1}^{(n)} \), \( \pi_1 = \pi_j^{(n)} \), \( \pi_2 = \pi_{j+1}^{(n)} \), \( q_1 = q_j^{(n)} \), \( q_2 = q_{j+1}^{(n)} \), \( c_1 = c_j^{(n)} \).

Since \( n \geq 2 \), there will always be at least four geometrically distinct points \( p_1 \) of the set \( \{ p_j^{(n)} \} \). Now \( p_1 \) and \( p_2 \) are end-points of two simple polygonal arcs \( \mathcal{P}_1, \mathcal{P}_2 \), \( \mathcal{P}_1 + \mathcal{P}_2 = \mathcal{P}_n \). Then at least one of these arcs, say \( \mathcal{P}_2 \), possesses a third point \( p_3 \in \{ p_j^{(n)} \} \) as an element.

I say: \( p_1 \) and \( p_2 \) are the only points of \( \{ p_j^{(n)} \} \) belonging to \( \mathcal{P}_1 \). Suppose the contrary: \( p_4 \in \mathcal{P}_1 \), \( p_4 \neq p_1, p_2 \).
Consider the closed bounded region $\overline{S}$ whose boundary is the Jordan curve $p_1 \overline{p_2} \overline{p_1}$. $\mathcal{J}_2$ has only the points $p_1$ and $p_2$ in common with $\mathcal{J}_1$, since $\mathcal{J}_1 + \mathcal{J}_2 = P_{kn}$, a simple curve; $\mathcal{J}_2$ has no points common to the interior of either of the line segments $\overline{p_1 p_2}$ or $\overline{p_1 p_2}$, since $p_1$ is a point of $P_{kn}$ nearest $\overline{p_1}$ and similarly for $p_2$ and $\overline{p_2}$. Therefore $\mathcal{J}_2 \subset D - S$. Now $\mathcal{J}_2$ is a continuous arc with end-points $p_1$ and $p_2$. Consequently it must cross the interior of the line segment $\overline{p_4 p_4}$. But the point of intersection will be a point of $P_{kn}$ nearer to $\overline{p_4}$ than $p_4$ is, contrary to the method of selecting $p_4$ and $\overline{p_4}$. Thus only the end-points of $\mathcal{J}_1$ belong to $\{p_{(n)}\}$.

Now suppose that with respect to $0$ the topological index of $P_{kn}$ is such that the positive sense along $\mathcal{J}_1$ is from $p_1$ to $p_2$. Then positive sense along $\mathcal{J}_2$ is from $p_2$ to $p_1$.
The curve \( p_2 \rho_2 \overrightarrow{p_1p_1} \approx \overrightarrow{p_2p_2} \) is a Jordan curve with 0 in its interior. Hence in this curve the positive sense along \( \rho_2 \) is from \( p_2 \) to \( p_1 \), so that the positive sense along \( \approx \) is from \( \Pi_1 \) to \( \Pi_2 \). Since \( \left| q_1 - q_2 \right| \geq d(n) \), \( \left| q_1 - \Pi_1 \right| < 3^{-1}d(n) \), \( \left| q_2 - \Pi_2 \right| < 3^{-1}d(n) \), it follows that, with respect to the interior \( D \) of \( C \), the positive sense along \( C_1 \) is from \( q_1 \) to \( q_2 \).

A similar argument holds: If the positive sense along \( \rho_1 \) is from \( p_2 \) to \( p_1 \), then the positive sense along \( C_1 \) is from \( q_2 \) to \( q_1 \).

Thus for each \( C_j \) there corresponds an arc \( \rho_j(n) \subset P_{kn} \) such that the sense along \( C_j \) from \( q_j(n) \) to \( q_{j+1} \) is the same as that along \( \rho_j(n) \) from \( p_j(n) \) to \( p_{j+1}(n) \), and the only points of \( \left\{ p_j(n) \right\} \) belonging to \( \rho_j(n) \) are its end-points, \( p_j(n) \) and \( p_{j+1}(n) \).

Then since \( \sum_{j=1}^{\infty} C_j(n) = C \) and \( \sum_{j=1}^{\infty} \rho_j(n) = P_{kn} \), the lemma is proved.

q.e.d.

As in Lemma 4, let \( \rho_j(n) \) be that arc of \( P_{kn} \) having end-points \( p_j(n) \) and \( p_{j+1}(n) \) such that no other point of \( \left\{ p_j(n) \right\} \) belongs to \( \rho_j(n) \). Let \( \lambda_j(n) \) be the arc-length of \( \rho_j(n) \).
Lemma 5. \( \lambda_j^{(n)} \leq 2^{-n} \).

Proof: We treat \( C_j^{(n)} \) as we did \( C \) in Lemma 1. Subdivide \( C_j^{(n)} \) into \( 2^{k_n-n} \) subarcs, each of length \( 2^{-k_n} \). Then \( C_j^{(n)} \) can be covered by \( 2^{k_n-n} \) squares, each of which is the square union \( \bigcup_{i,j}^{k_n} \) of nine meshes of \( S_{k_n} \).

Then \( C_j^{(n)} \subseteq \bigcup_{i,j}^{k_n} = \bigcup_j^{(k_n)} \), and the total length \( L_j^{(k_n)} \) of all the mesh lines of \( \bigcup_j^{(k_n)} \) is

\[
L_j^{(k_n)} \leq 2^{k_n-n}24(2^{-k_n}) = 2^{-n}24.
\]

The length of side of each \( \bigcup_{i,j}^{(k_n)} \) is \( 3(2^{-k_n}) \). Since each \( \bigcup_{i,j}^{(k_n)} \) has a point of \( C_j^{(n)} \) in its interior and since \( 2^{-k_n} < 18^{-1}d(n) \), then every point of \( \bigcup_j^{(k_n)} \) is within the distance \( 3(2^{\frac{1}{2}2^{-k_n}} < 6(2^{-k_n}) < 3^{-1}d(n) \) of a point of \( C_j^{(n)} \).
Let \( V = \bigcup_{j=1}^{\infty} (k_n)^j \cup \bigcup_{j=0}^{\infty} (k_n)^j \cup \bigcup_{j=1}^{\infty} (k_n)^j \).

\[
W = \sum_{m=1 \atop m \neq j-1, j, j+1}^{2^n} (k_m).
\]

Then \( P_{k_n} \subset W \cup V, \ (n) \subset V, \ (n) \subset W \). Every point of \( W \) is within the distance \( 3^{-1}(n) \) of \( (n) \). Since \( \text{dist}(C_j(n), (n)) \geq d(n) \), then \( \text{dist}(W, C_j(n)) \geq 2/3 d(n) \).
Now \( I_j^{(n)} \subset V \). For, assume the contrary: There exists \( p \in I_j^{(n)} \). Then \( p \) is without the distance \( 2/3 d_j^{(n)} \) of \( C \). Both of the line segments \( p_j^{(n)} \Pi_j^{(n)} \) and \( p_{j+1}^{(n)} \Pi_{j+1}^{(n)} \) are within the distance \( 3^{-1} d_j^{(n)} \) of \( C_j^{(n)} \). Hence \( p \) is without the distance \( 3^{-1} d_j^{(n)} \) of each of the line segments.

\( p \) is a boundary point of the simple connected region \( S \), described in Lemma 4, such that the only boundary points of \( S \) within the
distance $-1d(n)$ of $p$ are points of $\mathcal{P}_j^{(n)}$, which is composed of mesh lines of $S_{k^n}$, each with length less than $16^{-1d(n)}$. Then necessarily $p$ is a boundary point of at least one mesh $m \in S$, such that $m \cup M_{k^n}$ is simply connected, contrary to the fact that $M_{k^n}$ is the maximal polygonal region of $S_{k^n}$. Hence $\mathcal{P}_j^{(n)} \subset V$.

Then $\lambda_j^{(n)} = L_j^{(k^n)} + L_j^{(k^n)} + L_{j+1}^{(k^n)} < 2^{-n/2}$.

q.e.d.

**Lemma 6.** The subsequence of maximal polygonal regions $\{M_{k^n}\}$ and their boundaries $\{P_{k^n}\}$ possess properties I-VI.

**Proof:** Property I holds, because of the definition of $\{M_{k^n}\}$.

Property II holds, by Lemma 3.

Property III holds, by definition of $\{P_{k^n}\}$.

Each $q_j^{(n)} \subset C$ has coordinates $x = x(j2^{-n}), y = y(j2^{-n})$.

Let the coordinates of the corresponding points $p_j^{(n)} \subset P_{k^n}$ be the values of $X_n(t), Y_n(t)$ for $t = j2^{-n}$. Let $s_{k^n}(p)$ denote the arc-length of $P_{k^n}$ from $p_0^{(n)} = p_2^{(n)}$ to the point $p$ in the sense along $P_{k^n}$ described for increasing values of $j$. Since each $P_{k^n}$
is a simple closed curve, \( X_n \) and \( Y_n \) are continuous functions of \( s_k^{(n)}(p) \). For \( p \leq \mathcal{C}^{(n)}_j \) let

\[
t - j2^{-n} = \frac{2^{-n}}{\lambda_j^{(n)}} ( s_k^{(n)}(p) - s(p_j^{(n)}) ).
\]

Then for \( p = p_{j+1}^{(n)} \), \( t - j2^{-n} = 2^{-n} \)

\[
t = (j+1)2^{-n}.
\]

Thus \( t \) is a linear function of \( s_k^{(n)}(p) \), and therefore continuous, such that \( t = j2^{-n} \) for \( p = p_j^{(n)} \). Then \( X_n(t), Y_n(t) \) are continuous for \( 0 \leq t \leq 1 \), and Lemma 1 insures uniform bounded variation of the functions. Thus property IV holds.

Lemma 4 and the functional relationship (1) establish property V.

Now consider any \( t_o \in [0,1] \). Then \( t_o \) determines a point \( q_o \in \mathcal{C}_o^{(n)} \) and a point \( p_o \in \mathcal{C}_o^{(n)} \).

\[
|q_o - q_j^{(n)}| \leq 2^{-n}. \text{ By Lemma } 4 \quad |q_j^{(n)} - p_j^{(n)}| < 2^{-n}.
\]

And by Lemma 5 \( |p_o - p_j^{(n)}| < 2^{-n}72 \). Then

\[
|q_o - p_o| \leq |q_o - q_j^{(n)}| + |q_j^{(n)} - p_j^{(n)}| + |p_o - p_j^{(n)}|
\]

\[
\leq 2^{-n}74.
\]

Hence:

\[
|X_n(t_o) - x(t_o)| < 2^{-n}74, \quad |Y_n(t_o) - y(t_o)| < 2^{-n}74. \quad (2)
\]

Since the inequalities (2) hold for all \( t_o \in [0,1] \), then property VI holds.

q.e.d.