METRICS DEFINED BY DIFFERENTIAL FORMS

by

Howard Edward Taylor

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The properties of a metric and the geometry associated with it may be determined from the differential form

\[ d\sigma = f(u,v) \mid dw \mid \quad w = u + iv \quad (1) \]

where \( \sigma \) is arc length in the metric. It is the purpose of this paper to study methods of describing the geodesics, distance between two points, and the locus of points at a fixed distance from a fixed point determined by metrics defined by (1) where \( f(u,v) \) is a real, positive, single valued function, continuous on the \( w \)-plane except at certain isolated points or on certain regular arcs. By geodesic we shall mean the curve of shortest length between two fixed points; that is, the curve along which \( \int_{w_1}^{w_2} f(u,v) \mid dw \mid \) is a minimum. When the form of the geodesic is known, the integral of \( d\sigma \) along the geodesic from \( w_1 \) to \( w_2 \) is the distance between these two points. If the integral along a geodesic from \( w_1 \) to \( w_2 \) is set equal to a constant \( \rho \) the equation of the locus of all points \( w \) at a distance \( \rho \) from \( w_1 \) is obtained.

The curve along which \( \int_{w_1}^{w_2} f(u,v) \mid dw \mid \) is a minimum is found by use of the following:

1. **Theorem:** Let \( G(x,y,p) \) be a given function of the three independent variables \( x, y, p \) which is continuous together with its partial derivatives of the first and second order when \( (x,y) \) lies in a given region \( S \) and \( p \) has

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any value whatever. Let two fixed points \( A(a,b) \)
and \( B(c,d) \) be joined by a curve \( C: y = g(x) \) lying
in \( S \) with \( y, y', y'' \) all continuous \( a \leq x \leq c \).

A necessary condition that \( y = g(x) \) give the integral
\[
\int_a^c G(x, y, y') \, dx
\]
an extreme (or stationary) value is
that \( y = g(x) \) be the solution of
\[
\frac{d}{dx} \frac{2G}{\frac{dy}{dx}} - \frac{2G}{\frac{dy}{dx}} = 0
\]
(2)
with \( g(a) = b \) and \( g(c) = d \).

This is the well known Euler-Lagrange equation.

The solutions of (2) are known as the extremals of the integral.

If it happens that \( G(x, y, y') \) does not contain \( x \) explicitly, the
first integration of the Euler-Lagrange equation can be written
at once and the condition that \( y = g(x) \) give an extreme value of
the integral is
\[
G - y \frac{\partial G}{\partial y} = K, \text{ a constant}
\]
(3)

It cannot in general be expected that explicit analytic
expressions in closed form can be obtained for the distance
between any two points \( v_1 \) and \( v_2 \) by the method described above.
In cases where explicit formulas cannot be obtained from integra-
tion, other devices sometimes give the desired result, while in
some cases approximations can be used to good advantage. Some of
the methods will be illustrated by four examples.

1. Metric defined by
\[
d\sigma = \frac{|dr|}{|W|}
\]
Here the function \( f(u,v) \) is \( \frac{1}{\sqrt{u^2 + v^2}} \) and it is desired to make
\[
\int_{v}^{u} \frac{\sqrt{u^2 + v^2}}{\sqrt{u^2 + v^2}}
\]
a minimum.
It is more convenient here and in all the examples to use the representation \( w = re^{i\theta} \). Then (4) becomes
\[
\int_{r_1}^{r_2} \frac{\sqrt{1+r^2\theta'^2}}{r} \, dr
\]
Let \( G(r,\theta,\theta') = \frac{\sqrt{1+r^2\theta'^2}}{r} \). Then \( \frac{\partial G}{\partial \theta} = \frac{r^2 \theta'}{r \sqrt{1+r^2\theta'^2}} \) and \( \frac{\partial G}{\partial \theta'} = 0 \)

The extremals of integral (4) are given by
\[
\frac{d}{dr} \left( \frac{r^2 \theta'}{r \sqrt{1+r^2\theta'^2}} \right) = 0
\]

So that if \( \theta = g(r) \) satisfies
\[
\frac{r \theta'}{\sqrt{1+r^2\theta'^2}} = C \quad g(r_1) = \theta_1 \quad g(r_2) = \theta_2
\]
it will represent the extremal through \((r_1, \theta_1)\) and \((r_2, \theta_2)\).

The solution of (5) is \( a\theta = \log br \) where \( a = \frac{\log r_2}{\theta_2 - \theta_1} \)

and \( \log b = \frac{\log r_2}{\theta_2 - \theta_1} - \log r_1 \)

So the extremals are the logarithmic spirals \( br = e^{a\theta} \) where \( w = re^{i\theta} \).

Consider the transformation defined by \( x = \log r \quad y = \theta \)
under which \( \int \frac{\sqrt{1+r^2\theta'^2}}{r} \, dr \) becomes \( \int \sqrt{dx^2 + dy^2} \)

The extremals \( br = e^{a\theta} \) or \( \log r = a\theta + c \) transform into \( x = ay + c \). It is well known that the curves \( x = ay + c \) are the minimal curves for \( \int \sqrt{dx^2 + dy^2} \) and hence the extremals \( br = e^{a\theta} \) are the geodesics for the metric defined by (4).

To determine \( D(w_1, w_2) \), the distance from \( w_1 \) to \( w_2 \) we evaluate the integral
\[
D(w_1, w_2) = \int_{w_1}^{w_2} \frac{\sqrt{1+r^2\theta'^2}}{r} \, dr
\]
where \( C \) is \( \log br \) and \( w_1 = r_1 e^{i\theta_1} \) and \( w_2 = r_2 e^{i\theta_2} \)

\[
D(w_1, w_2) = \int_{r_1}^{r_2} \frac{1}{\sqrt{1 + \frac{v^2}{a^2} - r^2}} \, dr = \sqrt{1 + \frac{v^2}{a^2}} \log \frac{r_2}{r_1}
\]

But \( a = \frac{\log \frac{r_2}{r_1}}{\theta_2 - \theta_1} \) and so

\[
D(w_1, w_2) = \sqrt{\left(\frac{\theta_2 - \theta_1}{a}\right)^2 + \log \frac{r_2}{r_1}} = \sqrt{(\theta_2 - \theta_1)^2 + \left(\log r_2 - \log r_1\right)^2}
\]

From this we notice that the origin is at an infinite distance from any other point in the \( w \)-plane as was to be expected from the form of \( d\sigma \). Also as \( r_2 \to \infty \) or \( \theta_2 \to \infty \) the distance from \( r_1, \theta_1 \) becomes infinite.

To determine the locus of all points at a distance \( \rho \) from the fixed point \( w_1 = r_1 e^{i\theta_1} \), set \( w_2 = r e^{i\theta} \) and \( D(w_1, w_2) = \rho \) so that

\[
\rho = \sqrt{(\theta - \theta_1)^2 + \left(\log r - \log r_1\right)^2}
\]

Then \( (\theta - \theta_1)^2 + \left(\log \frac{r}{r_1}\right)^2 = \rho^2 \) is the equation of the non-Euclidean circle, center at \( w_1 = r_1 e^{i\theta_1} \) and non-Euclidean radius \( \rho \).

In this equation the extreme values of \( \theta \) are seen to be \( \theta_1 - \rho \) and \( \theta_1 + \rho \) occurring when \( r = r_1 \). To find the circumference \( M \) of a non-Euclidean circle it is necessary to integrate \((4)\)

from \( \theta_1 - \rho \) to \( \theta_1 + \rho \) along \( \log \frac{r}{r_1} = \sqrt{\rho^2 - (\theta - \theta_1)^2} \) and add to this result the integral \((4)\) evaluated from \( \theta_1 - \rho \) to \( \theta_1 + \rho \) along \( \log \frac{r}{r_1} = -\sqrt{\rho^2 - (\theta - \theta_1)^2} \)

However, along both of these curves the integrand of \((4)\) is the same, since

\[
\frac{d\rho}{d\theta} = \frac{r(\theta - \theta_1)}{\log \frac{r}{r_1}} \quad \text{and} \quad \frac{\sqrt{r^2 + \left(\frac{d\rho}{d\theta}\right)^2}}{r} = \sqrt{1 + \left(\frac{\theta - \theta_1}{\log \frac{r}{r_1}}\right)^2}
\]
So the circumference $M$ is given by

$$M = 2 \int_{\theta_1}^{\theta_2} \sqrt{1 + \left( \frac{\theta - \theta_1}{\log \frac{r}{r_1}} \right)^2} \, d\theta = 2 \int_{\theta_1}^{\theta_2} \sqrt{1 + \left( \frac{\theta - \theta_1}{\log \frac{r}{r_1}} \right)^2} \, d\theta $$

$$M = 2 \rho \sin^{-1} \left. \frac{\theta - \theta_1}{\rho} \right|_{\theta = \theta_1, \rho} = 2 \pi \rho$$

2. Metric defined by $d\sigma = \frac{|d\mathbf{w}|}{1 + |\mathbf{w}|^2}$.

Again it is more convenient to use $w = re^{i\theta}$ so that

$$d\sigma = \frac{\sqrt{1 + r^2 \theta'^2}}{1 + r^2} \, dr$$

and the integral to be made a minimum is

$$\int \frac{\sqrt{1 + r^2 \theta'^2}}{1 + r^2} \, dr$$

Applying the Euler-Lagrange equation with $G(r, \theta, \theta') = \frac{\sqrt{1 + r^2 \theta'^2}}{1 + r^2}$ we see the condition for an extremal is that $\theta = g(r)$ satisfy

$$\frac{r^2 \theta'}{(1 + r^2)^{3/2}} = C$$

$$g(r_1) = \theta_1 \quad g(r_2) = \theta_2$$

(7)

A solution of (7) is

$$\theta = \int \frac{C (1 + r^2)}{r^2 \sqrt{1 - \frac{C^2}{(1 + r^2)^2}}} \, dr + K$$

The integration can be effected by the substitution $2u = (r - \frac{1}{r})$ and $\theta - K = \sin^{-1} \left[ \frac{r - \frac{1}{r}}{\sqrt{1 - \frac{C^2}{(1 + r^2)^2}}} \right]$ represents an extremal.

Simplifying, $r - \frac{1}{r} = F \sin (\theta - K)$ where $F = \frac{\sqrt{1 - \frac{C^2}{c^2}}}{c}$ and the extremals are

$$r^2 - Fr \sin (\theta - K) - 1 = 0$$

(8a)

or in rectangular coordinates

$$x^2 + y^2 + Ax + By - 1 = 0$$

(8b)
This family intersects the unit circle at the same points as the line \( Ax + By = 0 \), that is, at the extremities of a diameter, hence the differential form \( d\sigma = \frac{|du|}{1 + |u|^2} \) leads to the elliptic geometry (or geometry of the sphere). If the \( w \)-plane is projected stereographically on a sphere of unit diameter which is tangent to the \( w \)-plane at the origin, the circles meeting the unit circle at the extremities of a diameter are projected into the great circles of the sphere. If \( x, y, z \) are the coordinates of a point on the sphere, the relation between these coordinates and the coordinates \((u, v)\) of the corresponding point on the plane are

\[
\begin{align*}
    u &= \frac{x}{1-z} \\
    v &= \frac{y}{1-z} \\
    w &= \frac{z}{1+u^2+v^2}
\end{align*}
\]

So that

\[
\begin{align*}
    du &= \frac{(1-z)dx + x
dz}{(1-z)^2} \\
    dv &= \frac{(1-z)dy + y
dz}{(1-z)^2}
\end{align*}
\]

\[
\begin{align*}
    du^2 + dv^2 &= \frac{dx^2 + dy^2 + dz^2}{(1-z)^2} \\
    dx^2 + dy^2 + dz^2 &= \frac{du^2 + dv^2}{(1+u^2+v^2)^2}
\end{align*}
\]

This result shows that an element of arc along a curve on the \( w \)-plane is transformed to an element of arc along a great circle.

It is well known that the minimal curves of a sphere are the great circles, hence (8b) represents a geodesic, since the great circles are the projections of the curves of (8b).
A determination of the distance between \( w_1 \) and \( w_2 \) by the same method used in the first example would involve the determination of the two constants in (8a) so that the geodesic passes through \( w_1 \) and \( w_2 \). The integral (6) would then have to be evaluated along this geodesic from \( w_1 \) to \( w_2 \). This procedure rapidly becomes unmanageable and some other method of determining \( D(w_1, w_2) \) must be sought. For this purpose consider the transformation 
\[ z = \frac{w - a}{1 + aw} \]

Here \( w = \frac{z + a}{1 - az} \) and \( dw = \frac{(1 + az)}{(1 - az)^2} dz \)

Let \( a = c + id \) and \( z = x + iy \) we have
\[ \frac{|dw|}{1 + |w|^2} = \frac{|dz|}{1 + |z|^2 + |z + a|^2} = \frac{|dz|}{1 + |z|^2} \]

Now also consider the transformation 
\[ z = (e^{i\phi}) w \]

Here \( w = e^{i\phi}z \) and \( 1 + |w|^2 = 1 + |z|^2 \)

So it has been shown that under the transformation
\[ z = e^{i\phi} \left( \frac{w - a}{1 + aw} \right) \] (9)

the differential form \( \frac{|dw|}{1 + |w|^2} \) is invariant, and so the distance from \( w_1 \) to \( w_2 \) in the metric defined by this form can be obtained by the following device:
Consider the geodesic through \( w_1 \) and \( w_2 \) (ie, the circle through these points meeting the unit circle at the extremities of a diameter). Transform the \( w \)-plane by the transformation

\[
z' = \frac{w - w_1}{1 + \overline{w_1} w}
\]

This transformation carries \( w_1 \) to the origin and \( w_2 \) to a point \( z'_2 \) where \( z'_2 = \frac{w_2 - w_1}{1 + \overline{w_1} w_2} \) and the original geodesic is carried into the straight line through the origin and the point \( z'_2 \). Now apply the transformation \( z = e^{-i\arg z'}(z') \)

This transformation carries \( z'_2 \) to \( z_2 \) where

\[
z_2 = e^{-i\arg z'_2} \quad z'_2 = |z'_2|
\]

and the straight line through the origin and the point \( z'_2 \) is carried into the positive real axis.

So the transformation

\[
z = e^{-i\arg(w_2 - w_1)} \left( \frac{w - w_1}{1 + \overline{w_1} w} \right)
\]

which is the product of (10a) followed by (10b), carries \( w_1 \) to the origin and \( w_2 \) to the point \( z_2 = \frac{|w_2 - w_1|}{1 + \overline{w_1} w_2} \) on the real axis. It is of the type (9) so the differential form is invariant under the transformation and hence the distance from the point \( w_1 \) to \( w_2 \) is the same as the distance from the origin along the real axis to \( z_2 \).

\[
D(0, z_2) = \int_0^z \frac{d\omega}{1 + \omega^2} = \tan^{-1} \frac{\omega}{\omega_0} = \tan^{-1} z_2
\]

So the distance from \( w_1 \) to \( w_2 \) is

\[
D(w_1, w_2) = \tan^{-1} \frac{|w_2 - w_1|}{1 + \overline{w_1} w_2}
\]
It is of interest to note here that the maximum distance between any two points \( w_1 \) and \( w_2 \) is \( \frac{\pi}{2} \).

The locus of all points \( w \) at a non-Euclidean distance \( \rho \) from \( w_1 \) is

\[
\rho = \tan^{-1} \frac{|w - w_1|}{1 + w^2 w_1^2}
\]

To determine the circumference of the circle, transform \( w \) to the origin by

\[
z = \frac{w - w_1}{1 + w^2 w_1^2}
\]

and the equation of the circle becomes

\[|z| = \tan \rho \quad \text{and} \quad \text{the circumference is}
\]

\[
M = \int_0^{\frac{\pi}{2}} \frac{\tan \theta}{1 + \tan^2 \theta} d\theta = 2 \pi \sin \rho \cos \rho
\]

Since the differential form is invariant under this transformation, \( M \) is the circumference of any circle of radius \( \rho \).

From this evaluation of \( M \) it is seen that as the radius \( \rho \) increases from zero to its limit \( \frac{\pi}{2} \) the circumference of the circles increases from zero to \( \pi \) and then decreases to zero when \( \rho = \frac{\pi}{2} \). This occurrence and the phenomenon of the limited distance between any two points is to be expected since the geometry defined by the form

\[
d\sigma = \frac{|dw|}{1 + |w|^2}
\]

in which the geodesics are the circles of the family which meet the unit circle at the extremities of a diameter is the well known elliptic geometry, or the geometry of the sphere. On the sphere the greatest possible distance between two points occurs when the points are at the opposite ends of a diameter of the sphere, that is at a distance \( \pi \gamma \) where \( \gamma \) is the radius of the sphere. On the sphere of unit diameter the curve which is made up of all the points at a distance \( \rho \) from a given point has a length which grows from zero as \( \rho \) increases.
until \( \rho \) is \( \frac{\pi}{4} \) when the curve becomes an equator of the sphere of length \( \pi \). Thereafter the length of the curve decreases as \( \rho \) increases until \( \rho \) becomes \( \frac{\pi}{2} \) and the curve has degenerated into one point.

3. **Metric defined by** \( d\sigma = \frac{|du|}{\sqrt{1 - |u|^2}} \n\)

Writing \( u = re^{i\theta} \), the integral to be made a minimum is

\[
\int_{\gamma} \frac{\sqrt{1 + r^2 \theta'^2}}{1 - r^2} \, dr \quad \gamma \, = \, r_1 \, e^{i\theta_1}, \, \gamma \, = \, r_2 \, e^{i\theta_2} \tag{11}
\]

Applying the Euler-Lagrange equation, the condition for an extremal is that \( \theta = g(r) \) satisfy

\[
\frac{r^2 \theta'}{(1 - r^2) \sqrt{1 + r^2 \theta'^2}} = \mathcal{C} \quad g(r_1) = \theta_1, \quad g(r_2) = \theta_2 \tag{12}
\]

The solution of (12) is

\[
\theta = \int \frac{c(1 - r^2)}{r^2 \sqrt{r^2 - c^2(1 - r^2)^2}} \, dr + K
\]

Here the integration is effected by the substitution

\[
2u = -r - \frac{1}{r} \quad \text{and} \quad \theta - K = \sin^{-1} \left[ \frac{-r - \frac{1}{r}}{\sqrt{1 + \frac{4c^2}{r^2}}} \right] \tag{13a}
\]

represents an extremal. This can be written

\[
r + \frac{1}{r} + M \sin (\theta - K) = 0
\]

which in rectangular coordinates is

\[
x^2 + y^2 + Ax + By + 1 = 0 \tag{13b}
\]

This is the family of circles which are orthogonal to the unit circle, hence the form \( \frac{|du|}{\sqrt{1 - |u|^2}} \) defines the hyperbolic geometry. It is well known that the minimal curves in the hyperbolic geometry are the circles which are orthogonal to the unit circle. So (13b) represents a geodesic.
Again to find the distance between two points \( w_1 \) and \( w_2 \) by integration is impractical, but the method of the preceding example may be applied. In this case, the transformation

\[
z = e^{i\theta} \left( \frac{w - a}{1 - a\overline{w}} \right)
\]

can be shown to leave the differential form \( \frac{|dw|}{1 - |w|^2} \) invariant, and the distance from \( w_1 \) to \( w_2 \) may be obtained by subjecting the \( w \)-plane to the transformation

\[
z = e^{-i\arg\left( \frac{w_2 - w_1}{1 - \overline{w}_2w_1} \right)} \left( \frac{w_1}{1 - \overline{w}_2w_1} \right)
\]

This carries \( w_1 \) to the origin, \( v_2 \) to a point \( z_2 = \frac{|w_2 - w_1|}{|1 - \overline{w}_2w_1|} \) and the original geodesic into the positive real axis, so that the distance from \( w_1 \) to \( w_2 \) is the distance from the origin to \( z_2 \) (\( z_2 \) real > 0).

\[
D(0, z_2) = \int_{0}^{z_2} \frac{d\omega}{1 - \omega^2} = \frac{1}{2} \log \frac{1 + z_2}{1 - z_2}
\]

So the distance \( D(w_1, w_2) = \frac{1}{2} \log \frac{|1 - w_1w_2| + |w_2 - w_1|}{|1 - w_1w_2| - |w_2 - w_1|} \) (14)

One fact of interest here is that any \( w_1 \) interior to the unit circle is at an infinite distance from any \( w_2 \) on the unit circle. Such a fact is expected from the form \( \frac{|dw|}{1 - |w|^2} \) which is singular along \( |w| = 1 \).

The locus of all points at a distance \( \rho \) from \( w_1 \) is

\[
\rho = \frac{1}{2} \log \frac{|1 - w_1w| + |w - w_1|}{|1 - w_1w| - |w - w_1|}
\]

(15)

To simplify this, transform \( w_1 \) to the origin by \( z = \frac{w - w_1}{1 - w_1\overline{w}} \) (which leaves distance invariant) and the locus is

\[
\frac{1}{2} \log \frac{|1 + z|}{1 - |z|} = \rho \quad \Rightarrow \quad |z| = \frac{e^{2\rho} - 1}{e^{2\rho} + 1}
\]
Finally on transforming back, the locus of all points \( w \) at a distance \( \rho \) from \( w_0 \) is 
\[
\frac{|w - w_0|}{|1 - \overline{w} w_0|} = \frac{e^{2\rho} - 1}{e^{2\rho} + 1}
\]

The circumference \( C \) of the non-Euclidean circle, center at \( w_0 \), and radius \( \rho \) is the same as the circumference of the circle 
\[
|z| = \frac{e^{2\rho} - 1}{e^{2\rho} + 1}
\]
since distance is invariant under the given transformation. Let \( z = re^{i\theta} \), then

\[
C = \int_0^{2\pi} \sqrt{r^2 + (dr/d\theta)^2} d\theta = 2\pi \left[ \frac{r^2 - 1}{e^{2\rho} + 1} \right]^{1/2} \left[ 1 - \frac{(e^{2\rho} - 1)^2}{(e^{2\rho} + 1)^2} \right]^{1/2}
\]

\[
C = 2\pi \frac{(e^{2\rho} - 1)(e^{2\rho} + 1)}{4e^{2\rho}} = 2\pi \sinh \rho \cosh \rho
\]

Here, the circumference increases without limit as \( \rho \) increases.

4. Metric defined by 
\[
d\sigma = \frac{|dz|}{1 + |w|}
\]

With \( w = re^{i\theta} \) the integral to be made a minimum can be written in the form 
\[
\int \frac{\sqrt{r^2 + r'^2}}{1 + r} d\theta
\]

(16)

It is apparent that under the transformation \( z = e^{i\beta} w \) the form 
\[
\frac{|dz|}{1 + |w|}
\]
is invariant, so that a rotation leaves the set of extremals of the integral (16) invariant, and it is sufficient to consider the family of extremals through the point \( w_0 = (r_0, 0) \).

Since the integral of (16) does not contain \( \theta \) explicitly we see from (3) the condition for an extreme value of (16) is that \( r \) be a function which satisfies
\[ \frac{r^2}{(1+r)^2 r^2 + r^4} = C \]

It is expedient to introduce the parameter \( \psi \), the angle from the radius vector produced through a point on a curve to the tangent to the curve. Since \( \cot \psi = \frac{r}{r'} \), \( \sin \psi = \pm \frac{r}{\sqrt{r'^2 + r^2}} \)

and (17) becomes \( \frac{r}{1+r} \sin \psi = C \)

If we require the curve to pass through \((r_0, 0)\) making an angle \( \psi_0 \) with the polar axis, where \(-\frac{\pi}{2} \leq \psi_0 \leq \frac{\pi}{2}\), \( C \) becomes \( \frac{r_0}{1+r_0} \sin \psi_0 \) and the condition that the curve through \((r_0, 0)\) at an angle \( \psi_0 \) be an extremal is that

\[ \frac{r}{1+r} \sin \psi = \frac{r_0}{1+r_0} \sin \psi_0 \] (18)

We will define the angle \( \alpha \) such that \( \sin \alpha = \frac{r_0}{1+r_0} \sin \psi_0 \)

with \( \alpha \) always in the same quadrant as \( \psi_0 \). Then the coordinate \( r \) of any point on the extremal passing through \((r_0, 0)\) with \( \psi = \psi_0 \)

there is given by \( r = \frac{\sin \alpha}{\sin \psi - \sin \alpha} \) (19)

The coordinate \( \theta \) of any point on this extremal can also be expressed in terms of \( \psi \) since \( d\theta = \frac{dr}{r} \tan \psi \)

\[ d\theta = \frac{-\cos \psi \sin \alpha}{(\sin \psi - \sin \alpha)^2} d\psi \]

and \( dr = \frac{-\cos \psi}{\sin \psi - \sin \alpha} d\psi \)

So \( d\theta = \frac{-\sin \psi}{\sin \psi - \sin \alpha} d\psi = -d\psi - \sin \alpha \frac{d\psi}{\sin \psi - \sin \alpha} \) (20)
Consider the integral \[ I = \int \frac{d\psi}{\sin \psi - \sin \alpha} \]

There are two cases:

**Case 1.** \( 0 \leq \psi \leq \frac{\pi}{2} \) \( 0 \leq \alpha \leq \sin^{-1} \frac{r_0}{1 + r_0} \)

Let \( t = \tan \frac{\psi}{2} \) so that

\[ I = \int \frac{2 dt}{(\sin \alpha) t^2 + 2t - \sin \alpha} = \frac{2}{\sin \alpha} \int \frac{dt}{-t^2 + \frac{2t}{\sin \alpha} - 1} \]

We shall be interested in the range of \( t \) for which the integrand is positive. Since the curve represented by the denominator as a function of \( t \) is a parabola opening downward with two real roots (since \( b^2 - 4ac = 4 \csc^2 \alpha - 4 > 0 \) \( t \) will be restricted to lie between the roots of \(-t^2 + 2(\csc \alpha)t - 1\) which are \( \tan \frac{\alpha}{2} \) and \( \cot \frac{\alpha}{2} \), that is

\[ 0 < \tan \frac{\alpha}{2} < t < \cot \frac{\alpha}{2} \]

From this inequality we can determine the range of the parameter \( \psi \)

\[ \tan \frac{\alpha}{2} < \tan \frac{\psi}{2} < \cot \frac{\alpha}{2} \]

\[ \frac{\alpha}{2} < \frac{\psi}{2} < \frac{\pi}{2} - \frac{\alpha}{2} \]

\[ \alpha < \psi < \pi - \alpha \]

We then have

\[ I = \frac{2}{\sin \alpha} \int \frac{d(\tan \frac{\psi}{2})}{(\tan \frac{\psi}{2} - \tan \frac{\alpha}{2})(\cot \frac{\alpha}{2} - \tan \frac{\psi}{2})} \]

where both terms in the denominator are positive.

\[ I = \frac{2}{\sin(\cot \frac{\alpha}{2} - \tan \frac{\alpha}{2})} \log \frac{\tan \frac{\psi}{2} - \tan \frac{\alpha}{2}}{\cot \frac{\alpha}{2} - \tan \frac{\psi}{2}} = \frac{1}{\cos \alpha} \log \frac{\sin \frac{\psi - \alpha}{2}}{\cos \frac{\psi + \alpha}{2}} + K \]
Case 2. \(-\frac{\pi}{2} \leq \psi \leq 0\) \hspace{1cm} \(-\sin^{-1}\frac{r_o}{r} \leq \alpha \leq 0\)

As in Case 1., \(I = \frac{2}{\sin \alpha} \int \frac{d(t)}{t^2 + \frac{2t}{\sin \alpha} - 1}\)

We again are interested in the range of \(t\) for which the integrand is positive, and so \(t\) must lie between the roots of the denominator \(\cot \frac{x}{2}\) and \(\tan \frac{x}{2}\) which in this case are both negative, that is

\[
\cot \frac{x}{2} < \tan \frac{\psi}{2} < \tan \frac{x}{2} \leq 0
\]

From this inequality the range of \(\psi\) in Case 2. is found

\[
-\frac{\pi}{2} - \frac{x}{2} < \frac{\psi}{2} < \frac{x}{2} \leq 0
\]

\[-(\pi + \alpha) < \psi < \alpha \leq 0\]

So \(I = \frac{2}{\sin \alpha} \int \frac{d(tan \frac{\psi}{2})}{(tan \frac{\psi}{2} - tan \frac{x}{2})(tan \frac{\psi}{2} - cot \frac{x}{2})}\)

where both factors in the denominator are positive.

\[I = \frac{2}{\sin(\tan \frac{\psi}{2} - cot \frac{x}{2})} \log \frac{\tan \frac{\psi}{2} - cot \frac{x}{2}}{\tan \frac{\psi}{2} - tan \frac{x}{2}} = \frac{1}{\cos \alpha} \log \frac{\sin \frac{\psi - \alpha}{2}}{\cos \frac{\psi + \alpha}{2}} + k_2\]

Returning now to (20) and integrating we find that for \(\theta\) to be the coordinate of a point on an extremal it must be represented by

Case 1. \(0 \leq \psi \leq \frac{\pi}{2}\) \hspace{1cm} \(\theta = -\psi - \tan \alpha \log \frac{\sin \frac{\psi - \alpha}{2}}{\cos \frac{\psi + \alpha}{2}} + k\)

If the extremal is to go through \((x_0, 0)\) with \(\psi = \psi_0\) at that point, then

\[k = \psi_0 + \tan \alpha \log \frac{\sin \frac{x_0 - \alpha}{2}}{\cos \frac{x_0 + \alpha}{2}}\]
and so \[ \theta = \psi_0 - \psi + \tan \alpha \log \frac{\cos \frac{\psi - \alpha}{2} \sin \frac{\psi_0 - \alpha}{2}}{\sin \frac{\psi - \alpha}{2} \cos \frac{\psi_0 - \alpha}{2}} \]

Case 2. \[ -\frac{\pi}{2} \leq \psi_0 \leq 0 \] \[ \theta = -\psi - \tan \alpha \log \frac{\sin \frac{\psi_0 - \alpha}{2}}{\cos \frac{\psi_0 + \alpha}{2}} + K' \]

If the extremal is to go through \((r_0, 0)\) with \(\psi = \psi_0\) at that point, then \[ K' = \psi_0 + \tan \alpha \log \frac{\sin \frac{\alpha - \psi_0}{2}}{\cos \frac{\psi_0 + \alpha}{2}} \]

and so \[ \theta = \psi_0 - \psi + \tan \alpha \log \frac{\sin \frac{\alpha - \psi_0}{2}}{\cos \frac{\psi_0 + \alpha}{2} \sin \frac{\alpha - \psi}{2}} \]

Combining these results with (19), the extremal passing through \((r_0, 0)\) with \(\psi = \psi_0\) at that point is given by the parametric equations

\[ r = \frac{\sin \alpha}{\sin \varphi - \sin \alpha} \]

\[ \theta = \psi_0 - \psi + \tan \alpha \log \frac{\cos \frac{\psi_0 - \alpha}{2} \sin \frac{\psi_0 - \alpha}{2}}{\cos \frac{\psi_0 + \alpha}{2} \sin \frac{\psi - \alpha}{2}} \] \hspace{1cm} (21a)

where \[ 0 \leq \psi_0 \leq \frac{\pi}{2} \] \[ \sin \alpha = \frac{r_0}{r_0 + r} \sin \varphi_0 \] \[ 0 \leq \alpha \leq \sin \frac{r_0}{1 + r_0} \] \[ \alpha < \varphi < \pi - \alpha \]

\[ r = \frac{\sin \alpha}{\sin \varphi - \sin \alpha} \]

\[ \theta = \psi_0 - \psi + \tan \alpha \log \frac{\cos \frac{\alpha - \psi_0}{2} \sin \frac{\alpha - \psi_0}{2}}{\cos \frac{\psi_0 + \alpha}{2} \sin \frac{\psi - \alpha}{2}} \] \hspace{1cm} (21b)

where \[ -\frac{\pi}{2} \leq \psi_0 \leq 0 \] \[ \sin \alpha = \frac{r_0}{r_0 + r} \sin \varphi_0 \] \[ -\sin \frac{r_0}{1 + r_0} \leq \alpha \leq 0 \] \[ -(\pi + \omega) < \psi < \omega \]

In equations (21a) as \(\varphi \to \alpha\) \[ r \to \infty \] \[ \theta \to \infty \]

as \(\varphi \to \pi - \alpha\) \[ r \to \infty \] \[ \theta \to -\infty \]
r attains a minimum value when \( \psi = \frac{\pi}{2} \) this value being \( \frac{\sin \alpha}{1 - \sin \alpha} \).

In equations (21b) as \( \psi \to \infty \) \( r \to \infty \) \( \theta \to -\infty \)

as \( \psi \to -(\pi/2) \) \( r \to \infty \) \( \theta \to \infty \)

Here \( r \) has a minimum value when \( \psi = -\frac{\pi}{2} \) the minimum being \( \frac{\sin \alpha}{1 - \sin \alpha} \). From equations (21a) and (21b) it is apparent that the extremals are symmetric about the polar axis, since replacing \( \psi \) by \( -\psi \), \( \alpha \) by \(-\alpha \) and \( \psi \) by \(-\psi \) does not change \( r \) but replaces \( \theta \) by \(-\theta \). In case \( \psi_0 = 0 \) we see from (18) that \( \psi \) must be either 0 or \( \pi \) and the extremal in this case is \( \theta = 0 \). Hence it is true that any extremal through the origin making an angle \( \theta \) with the polar axis is represented by \( \theta = \theta_0 \).

As \( r \to \infty \) very simple asymptotic expressions for \( \theta \) in terms of \( r \) can be obtained. When we say "\( g(x) \) is asymptotically represented by \( Af(x) \)" as \( x \to \infty \) which is written \( g(x) \sim Af(x) \) \( x \to \infty \) we mean

\[
\frac{|g(x)|}{|f(x)|} - A = o(1) \quad \text{as} \quad x \to \infty
\]

Because of the symmetry of the extremals about the polar axis, it is sufficient to consider only the cases in which \( \theta > 0 \), which are

i. \( 0 < \psi_0 \leq \frac{\pi}{2} \); \( 0 < \alpha < \psi \leq \psi_0 \)

ii. \( -\frac{\pi}{2} \leq \psi < 0 \); \( -(\pi + \alpha) \leq \psi \leq \psi_0 < 0 \)

For \( 0 < \psi_0 \leq \frac{\pi}{2} \) \( \alpha < \psi < \psi_0 \) consider the expression

\[
-\tan \alpha \log \frac{r}{r_0} = \tan \alpha \log \frac{\sin \psi - \sin \alpha}{\sin \psi} = \tan \alpha \log \frac{\cos \frac{\alpha + \psi}{2} \sin \frac{\psi - \alpha}{2}}{\cos \frac{\alpha - \psi}{2} \sin \frac{\psi + \alpha}{2}}
\]
Subtract this from the θ of equation (21a)

$$\theta - \tan \alpha \log \frac{r}{r_0} = \psi_0 - \psi + \tan \alpha \log \frac{\cos^2 \frac{\psi + \delta}{2}}{\cos^2 \frac{\psi_0 + \delta}{2}}$$  \hspace{1cm} (22)

Now for all $\psi_0$ in the range under consideration and all $\psi$ in the prescribed interval

$$\left| \tan \alpha \log \frac{\cos^2 \frac{\psi + \delta}{2}}{\cos^2 \frac{\psi_0 + \delta}{2}} \right| < M$$

$\psi_0 - \psi$ is also bounded. Divide (22) by $\tan \alpha \log \frac{r}{r_0}$

$$\left( \frac{\theta}{\tan \alpha \log \frac{r}{r_0}} - 1 \right) = \frac{\psi_0 - \psi + \tan \alpha \log \frac{\cos^2 \frac{\psi + \delta}{2}}{\cos^2 \frac{\psi_0 + \delta}{2}}}{\tan \alpha \log \frac{r}{r_0}}$$  \hspace{1cm} (23)

But the right hand member of (22) is bounded with the bound independent of $\psi$, hence as $r \to \infty$ the right hand member of (23) approaches zero, with the result that $\theta \sim \tan \alpha \log \frac{r}{r_0}$ as $r \to \infty$ uniformly in $\psi$. For $-\frac{\pi}{2} \leq \psi_0 < 0$ and $-\frac{\pi}{2} < \psi < \psi_0$ consider the expression

$$\tan \alpha \log \frac{r}{r_0} = \tan \alpha \log \frac{-\sin \frac{\psi + \delta}{2}}{-\sin \frac{\psi_0 + \delta}{2}} = \tan \alpha \log \frac{\cos \frac{\psi + \delta}{2}}{\cos \frac{\psi_0 + \delta}{2}}$$

Subtract this from the θ of equation (21b).

$$\theta - \tan \alpha \log \frac{r}{r_0} = \psi_0 - \psi + \tan \alpha \log \frac{\sin^2 \frac{\psi + \delta}{2}}{\sin^2 \frac{\psi_0 + \delta}{2}}$$  \hspace{1cm} (24)

For a fixed $\psi_0$ in the range under consideration and for all $\psi$ in the prescribed interval $\tan \alpha \log \frac{\sin^2 \frac{\psi + \delta}{2}}{\sin^2 \frac{\psi_0 + \delta}{2}}$ is bounded.
Now \[ \frac{\theta}{\tan^{-1}\frac{r}{\alpha}} - 1 = \frac{\psi_0 \tan^{-1}\frac{r}{\alpha}}{\psi_0 \tan^{-1}\frac{r}{\alpha}} \]

(25)

Since the right hand member of (24) is bounded with the bound being independent of \( \psi \), as \( r \to \infty \) the right member of (25) approaches zero with the result that \( \theta \sim \tan^{-1}\frac{r}{\alpha} \) uniformly in \( \psi \).

This can be written \( \theta \sim -\tan^{-1}\frac{r}{\alpha} \) and since in this case \( \alpha < 0 \) and in the first case \( \alpha > 0 \), the result is that along the part of the extremal for which \( \theta \) is positive

\[ \theta \sim |\tan^{-1}\frac{r}{\alpha}| \quad r \to \infty \]

(26a)

\[ r \sim r_0 e^{\cot \theta} \quad r \to \infty \]

(26b)

This result is valid only for each initial angle \( \psi_0 \neq 0 \).

It has already been noted that \( \theta = 0 \) is the extremal corresponding to \( \psi_0 = 0 \). In that case \( \alpha = 0 \) and (26a) still applies. The part of the extremal for which \( \theta \) is negative is the reflection of the part given by (26b).

That the extremals (21a) and (21b) are geodesics can be shown by reference to the sufficient condition for a weak relative minimum stated in Bliss:\(^2\)

Let \( f(r,\theta) = 0 \) be an arc without corners joining \( v \), and \( v \), and having the properties:

1. It is an extremal.

\(^2\) Calculus of Variations, Chicago, 1925, p. 157
2. \( \frac{\partial^2 G(\theta, r, \theta^1)}{\partial r^1} > 0 \) at every set of values \( \theta, r, r^1 \) on it.

3. It contains no point conjugate to \( w_1 \), where by a conjugate point of \( w_1 \) we mean a point of contact of the arc with the envelope of the family of the extremals through \( w_1 \).

Then \( f(r, \theta) = 0 \) is such that the value of integral (16) from \( w_1 \) to \( w_2 \) along \( f(r, \theta) \) is less than the value of (16) from \( w_1 \) to \( w_2 \) along any other arc distinct from \( f(r, \theta) = 0 \) joining \( w_1 \) and \( w_2 \) and having its elements \( \theta, r, r^1 \) all in a sufficiently small neighborhood of the elements of \( f(r, \theta) = 0 \).

The curves (21a) and (21b) satisfy condition 1. Furthermore they are asymptotically equivalent to logarithmic spirals, hence have no envelope, and (21a) and (21b) satisfy condition 3.

\[
\frac{\gamma^2 G}{\partial r^1} = \frac{r^1}{(1 + r)(r^2 + r^2)^{1/2}} \quad \text{which is always positive.}
\]

Hence the curves satisfy condition 2. So (21a) and (21b) represent geodesics since they are the only extremals of integral (16).

To determine the distance between two points, it is necessary to consider only the case \( 0 < \psi \leq \frac{\pi}{2} \); \( 0 < \pi < \psi \leq \frac{\pi}{2} \) because of the symmetrical character of the geodesics.

Consider the problem of determining the distance from \( w_0 : (r, \theta, 0) \) to \( w: (r, \theta) \). \( D(w_0, w) = \int_{\theta_0}^{\theta_1} \frac{\sqrt{r^2 + r^2}}{1 + r} \, d\theta \) where
the integration is along the geodesic. Write the integral as

\[
\int \frac{\sqrt{r^2 + r'^2}}{r} \, \frac{dr}{1 + r} \frac{d\tan \psi}{r}
\]

and using the fact that along a geodesic \( \frac{r}{1 + r} = \frac{\sin \alpha}{\sin \psi} \)

\[
\frac{d}{dr} \tan \psi = \frac{-\sin \psi \, d\psi}{\sin \psi - \sin \alpha}
\]

and on any curve \( \frac{\sqrt{r^2 + r'^2}}{r} = \frac{1}{\sin \psi} \)

the integral can be written

\[
\sin \alpha \int \frac{-d\psi}{\sin \psi (\sin \psi - \sin \alpha)} = \int \frac{d\psi}{\sin \psi} - \int \frac{d\psi}{\sin \psi - \sin \alpha}
\]

So

\[
D = \log \tan \frac{\psi}{2} - \frac{1}{\cos \alpha} \log \frac{\sin \frac{\psi - \alpha}{2}}{\cos \frac{\psi + \alpha}{2}} + K
\]

Since \( D = 0 \) when \( r = r_o ; \theta = 0 \)

\[
K = -\log \tan \frac{\psi_0}{2} + \frac{1}{\cos \phi} \log \frac{\sin \frac{\psi_0 - \alpha}{2}}{\cos \frac{\psi_0 + \alpha}{2}}
\]

\[
D(w_0, w) = \log \frac{\tan \frac{\psi}{2}}{\tan \frac{\psi_0}{2}} + \frac{1}{\cos \phi} \log \frac{\sin \frac{\psi - \alpha}{2}}{\cos \frac{\psi + \alpha}{2}}
\]

(27)

The distance \( D(w_0, w_1) \) can be written as

\[
D(w_0, w_1) = \log \frac{\tan \frac{\psi_1}{2}}{\tan \frac{\psi_0}{2}} + \frac{1}{\cos \phi} \log \frac{\sin \frac{\psi_0 - \alpha}{2}}{\cos \frac{\psi + \alpha}{2}}
\]

where \( \psi_1 \) is the value of the parameter \( \psi \) at the point \( w_1 : (r_1, \theta) \) along the geodesic from \( w_0 : (r_o, 0) \) to \( w_1 \) with initial angle \( \psi_0 \). This expression is too complicated for useful analysis, so an asymptotic expression for \( D \) as \( r \to \infty \) is obtained in the same manner as for the extremals.
Consider
\[
\frac{1}{\cos \alpha} \log \frac{r}{r_0} = \frac{1}{\cos \alpha} \log \frac{\cos \frac{\alpha + \psi_0}{2} \sin \frac{\psi - \alpha}{2}}{\cos \frac{\alpha - \psi_0}{2} \sin \frac{\psi + \alpha}{2}}
\]

Subtract this expression from (27)
\[
D - \frac{1}{\cos \alpha} \log \frac{r}{r_0} = \log \frac{\tan \frac{\psi}{2}}{\tan \frac{\psi_0}{2}} + \frac{1}{\cos \alpha} \log \frac{\cos \frac{\psi + \alpha}{2}}{\cos \frac{\psi_0 + \alpha}{2}}
\]

For any \( \psi \) in the range under consideration and all \( \psi \) in the prescribed interval, the right hand member of (28) is bounded.

Now
\[
D - \frac{1}{\cos \alpha} \log \frac{r}{r_0} = \log \frac{\tan \frac{\psi}{2}}{\tan \frac{\psi_0}{2}} + \sec \alpha \log \frac{\cos \frac{\psi + \alpha}{2}}{\cos \frac{\psi_0 + \alpha}{2}}
\]

Since the right hand member of (28) is bounded the right hand member of (29) approaches zero as \( r \to \infty \) so that
\[
D(\psi_0, \psi) \sim \sec \alpha \log \frac{r}{r_0}, \quad r \to \infty
\]

Here \( D \) represents the numerical value of the non-Euclidean distance from \((r_0, 0)\) along the geodesic with \( \psi = \psi_0 \) at the initial point to the point \((r, \theta)\).

If \( D \) is set equal to a constant \( \rho \),
\[
r \sim r_0 e^{\rho \cos \alpha}
\]

is an asymptotic expression for the variation of the coordinate \( r \) of a point at a distance \( \rho \) from \((r_0, 0)\) as the initial angle \( \psi_0 \) varies.