TOPOLOGY OF GROUPS

by

George Piranian

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Preface

The present paper is self-contained and can be read without previous knowledge of either group theory or topology. Except for some alterations in the approach to elementary topology and some minor omissions elsewhere, the first three chapters of Pontrjagin's treatise are faithfully covered.

No attempt is made to avoid "small theorems" that are not necessary for development on later pages.

The author hopes that the inclusion of a very detailed table of contents will facilitate the reading of passages in which references to other paragraphs occur. Paragraphs whose serial numbers stand in brackets contain definitions, the others, theorems or examples. Occasionally this rule is violated to avoid pedantic splitting of the text.
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Factor groups of (locally) compact topological groups are (locally) compact.

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Center

The center is a normal subgroup of \( G \).

Central normal subgroups.

Discrete normal subgroups of a connected topological group are central normal subgroups.

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Chains of order \( n \). Sets \( F^* \).

The intersection of the sets \( F^* \) is the set \( \{ e \} \).

Construction of the open subgroup \( H \)

If in (20.6) \( G \) is compact, \( H \) can be chosen so that it is normal.

Topological 0-dimensional groups have no connected subsets containing more than one element.

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Local isomorphism

If \( N \) is a discrete normal subgroup of \( G \), \( G \) and \( G/N \)
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If \( G \) and \( G' \) are connected and locally isomorphic, there exists a group \( H \) such that \( G \) and \( G' \) are isomorphic with the factor groups \( H/N \) and \( H/N' \) (\( N \) and \( N' \) discrete).

Example: Additive group of real numbers

References
INTRODUCTION

According to Poincaré, the theory of groups underlies many of the ideas in Euclid's geometry\(^1\). The modern career of the theory begins with Lagrange, who in *Réflexions sur la résolution algébrique des équations*, a group of papers published in the *Mémoires of the Berlin Academy* during the years 1770 and 1771\(^2\), "practically established ... the theorem that the order of a subgroup is a divisor of the order of the group"\(^3\). Abel used groups to prove that equations of degree higher than the fourth can not generally be solved in terms of radicals. Galois, the first to apply the term "group" in a technical sense (1830), determined by means of groups the solvability of algebraic equations.

Cauchy's publications (1815, 1844, 1845-46) referred to "systems of conjugate substitutions". Cauchy gradually came to consider the subject as more than a servant of the theory of equations and gave it independent status as the theory of abstract groups.

To Jordan goes the credit for recognizing the significance of quotient groups\(^4\).

Sophus Lie, collaborating with Felix Klein and Friedrich Engel, investigated particularly the continuous groups.

More recently Elie Cartan has examined the relation between the idea of continuous abstract groups and the basic concepts of topology. Many of the properties of certain large classes of groups are logical consequences of the fact that these groups can be topologized so that the resulting spaces

\(^1\) Josephine E. Burns,\(^2\) Lagrange,\(^3\) Cajori,\(^4\) Miller.
satisfy certain conditions, notably the Hausdorff second axiom of countability.

The present paper is taken largely from Pontrjagin's *Topological Groups*, which gives a monographic account of recent developments in this branch of group theory.
I. ABSTRACT GROUPS

1. The concept of groups

[1.1] A set G of elements over which a binary operation is defined constitutes a group, provided the operation (which we shall call multiplication, regardless of how it may be defined, in any special case) satisfies the group axioms:

i) With every ordered pair \( a, b \) of elements in G the multiplication associates a third element \( c \) in G. We express the relation in this form: \( ab = c \).

ii) The multiplication is associative; i.e., if \( a, b, c \) are any elements in G, \( (ab)c = a(bc) \). (We shall simply write the product as \( abc \).)

iii) G contains a right identity \( e \): an element such that \( ae = a \) for every element \( a \) in G.

iv) Every element \( a \) in G has a right inverse: an element \( a^{-1} \) in G with the property \( aa^{-1} = e \).

If the multiplication is commutative, the group is abelian.

If the group G contains only a finite number \( n \) of elements, it is a finite group of order \( n \). Otherwise, it is infinite.

(1.2) A right inverse element \( a^{-1} \) of the element \( a \) is also a left inverse element of \( a \), and a right identity is also a left identity.

For by (iv) of [1.1] \( a^{-1} \) has a right inverse \( b \), and we can write

\[ a^{-1}a = (a^{-1}a)e = (a^{-1}a)(a^{-1}b) = a^{-1}(aa^{-1})b = a^{-1}e = a^{-1}b = e, \]

so that \( a^{-1}a = e \); i.e., \( a^{-1} \) is a left inverse of \( a \). We have now \( ea = (aa^{-1})a = a \), and \( e \) is a left identity.

(1.3) We consider the equations

\[ ax = b, \]
\[ ya = b, \]
where a and b are any two fixed elements in G. If x and y are elements in G that constitute solutions of the equations, we have

\[ x = a^{-1}ax = a^{-1}b, \]
\[ y = yan^{-1} = ba^{-1}, \]

so that each of the equations has at most one solution in G.

Moreover, since \( a(a^{-1}b) = b \) and \( (ba^{-1})a = b \), the values that we have found for x and y actually constitute solutions.

(1.3.1) It follows, in particular, that the identity is unique, since it is a solution of the equation \( ax = a \); and that the inverse of the element a is unique, since it satisfies the equation \( ax = e \).

(1.5.2) It is now obvious that if we define, inductively,

\[ a^m = a^{m-1}a, \quad (m = 1, 2, 3, \ldots) \]

with \( a^0 = e \), and \( a^{-m} = (a^{-1})^m \), we have \( a^{p+q} = a^p a^q \) and \( (a^p)^q = a^{pq} \) for every pair of positive or negative integers p and q.

(1.4) If for some particular element a there exists an integer m such that \( a^m = e \), there exists a least positive integer n with this property, and we say that the element a has the finite order n.

If no such integer exists, we say that the element is free.

(1.4.1) If a is of order n and \( a^n = e \), n is a divisor of r. For we can write \( r = pn + q \), where \( 0 < q < n \), and this gives \( a^r = a^{pn + q} = a^{pn} a^q = a^q \).

Therefore \( a^r = e \) implies \( q = 0 \) or \( q > n \); and since \( 0 < q < n \), \( q = 0 \), q. e. d.

(1.5) Examples

(1.5.1) Let the elements in G be the set of all real numbers, and let the product of the elements a and b be defined to be the element \( ab \). Clearly, G is an abelian group - the additive group; its identity is the element zero, and the inverse of the element a is the element -a.

(1.5.2) If the set G is composed of all complex numbers other than zero, and if the product of two elements a and b is defined to be the arithmetical product \( ab \), then G is an abelian group with the identity 1; the inverse of a is \( \overline{a} \).
If the elements of $G$ are all the $n$-rowed square matrices $\|a_{ij}\|$ whose elements are complex numbers and whose determinants differ from zero, and if the product of two matrices $\|a_{ij}\|$ and $\|b_{ij}\|$ is defined to be the matrix $\|c_{ij}\|$, where $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$, then $G$ is a nonabelian group (for $n>1$) with the identity element $e = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker delta ($\delta_{ij} = 1$ when $i=j$, $\delta_{ij} = 0$ when $i \neq j$).

2. The subgroup and the factor group

[2.1] Notation: If $G$ is any group, and $A$ and $B$ are subsets of the set $G$, we denote by $AB$ the set of all products $xy$, where $x$ and $y$ belong to $A$ and $B$, respectively. We write $A^2 = A$, $A^{m+1} = A^m A$ ($m \geq 1$), $A^{-m} = (A^{-1})^m$, $A^0 = \{e\}$.

By $Ab$ we denote the set $AB$ where $B$ is composed of the single element $b$.

Obviously, $Ae = eA = A$, $G^{-1} = G$, $AG = G$, and $G^n = G$ ($n \geq 0$).

[2.2] If $G$ is a group and $H$ is a subset of the elements of $G$ with the property that the elements of $H$ satisfy the group axioms [1.1] under the binary operation of $G$, then $H$ is a subgroup of $G$.

(2.2.1) If $H$ is a subgroup, we have obviously

\begin{align*}
&i) \quad HH^{-1} \subseteq H, \\
&ii) \quad H^2 \subseteq H \quad \text{and} \quad H^{-1} \subseteq H.
\end{align*}

We show that the validity of either of these two statements is also sufficient for $H$ to be a group.

If the element $a$ is in $H$, (i) guarantees that $a^{-1}$ is in $H$, and therefore also that $e$ is in $H$. If $a$ and $b$ are any two elements in $H$, then $b^{-1}$ is in $H$, and therefore $ab = a(b^{-1})^{-1}$ is in $H$, and conditions [1.1] are satisfied.

In the case of (ii): if $a$ and $b$ are in $H$, then $ab$ is in $H$; and if $a$ is in $H$, then $a^{-1}$ is in $H$, and by the first part, $aa^{-1}$ is in $H$. Therefore $H$ is a subgroup.
If $G$ is a group and a one of its elements, then the subset $H$ containing the element $a^n$ for each integral value of $n$ is obviously a subgroup of $H$. If $H = G$, we say that $G$ is a cyclic group; and if in addition $a$ is a free element (cf. [1.4]), then $G$ is a free group.

If for every pair of elements $a$ and $b$ in $M$ the relation $a \sim b$ either holds or is false, the relation is an equivalence, provided

i) $a \sim a$ for all $a$,

ii) $a \sim b$ implies $b \sim a$,

iii) $a \sim b$ and $b \sim c$ implies $a \sim c$.

It is well-known that an equivalence relation in $M$ divides $M$ into disjoint subsets that exhaust $M$.

If $G$ is a group and $H$ a subgroup, and if $a$ and $b$ are any two elements of $G$, we say that $a \sim b$ provided $ab^{-1}$ is in $H$.

Since $H$ is a subgroup of $G$, it contains $e$, so that $aa^{-1}$ is in $H$ for all $a$ in $G$; i.e., $a \sim a$.

If $ab^{-1}$ is in $H$, then $(ab^{-1})^{-1} = ba^{-1}$ is in $H$. Therefore $a \sim b$ implies $b \sim a$.

If $ab^{-1}$ is in $H$ and $bc^{-1}$ is in $H$, then $ab^{-1}bc^{-1} = ac^{-1}$ is in $H$, and therefore $a \sim b$ and $b \sim c$ implies $a \sim c$.

We have shown that the relation $a \sim b$ satisfies the conditions of [2.4], and therefore that the relation divided $G$ into classes of equivalent elements. Each of these classes is a right coset of $H$ relative to $G$.

If $A$ is any right coset of $H$, and if $a$ is in $A$, then $A = Ha$.

For if $x$ is in $A$, $xa^{-1}$ is in $H$, and therefore $x$ is in $Ha$; and if $y$ is in $Ha$, then $ya^{-1}$ is in $H$, and therefore $y$ is in $A$, q.q.d.

If $b$ belongs to the coset $B$, then by (2.5.1), $B = HB$. Since the cosets exhaust $G$, every set $Hb$ is a right coset of $H$.

Similarly, defining that $a \sim b$ provided $a^{-1}b$ is in $H$, we determine left cosets of the form $bH$. 
[2.6] A subgroup $N$ of $G$ is an invariant or normal subgroup of $G$ provided for every $a$ in $G$ the relation $a^{-1}Na = N$ holds.

We note first that since $N$ contains the element $e$, the sets $aN$ and $Na$ have the element $e$ in common, so that if $Na$ is also a left coset, it is identical with $aN$.

In the definition of normal subgroups we can replace the relation $a^{-1}Na \subseteq N$ by the relation $a^{-1}Na = N$. For corresponding to every element $a$ in $G$ there is an element $a^{-1}$ in $G$, and therefore the definition requires $aNa^{-1} \subseteq N$, i.e. $a^{-1}(aN^{-1})a = a^{-1}Na$, $N \subseteq a^{-1}Na$, and together with $a^{-1}Na \subseteq N$ this proves that $a^{-1}Na = N$.

It follows then that if $N$ is a normal subgroup of $G$, $Na = a^{-1}Na = aNa = N$ for all $a$ in $G$, i.e. every right coset of $N$ is a left coset of $N$.

Conversely, if $Na = aN$ for every $a$ in $G$, $a^{-1}Na = a^{-1}aNa = N$, i.e. $N$ is a normal subgroup.

[2.7] If as in [2.1] we define the product $AB$ of two sets of elements $A$ and $B$ to be the set of all products $ab$ where $a$ is in $A$ and $b$ is in $B$, and if $N$ is a normal subgroup of a group $G$, then the set of all cosets of $N$ constitutes the factor group $G/N$ of $G$ by $N$.

To show that the factor group satisfies the requirements

[1.1] for a group, we need only point out that

i) the product $C$ of two elements $AB$ is an element, since, by the normality of $N$, $aNa = aNa = abN = abN = C$;

ii) since multiplication is associative in $G$, the same holds in $G/N$;

iii) since $Ne$ contains the identity element in $G$, $AEa = A$, and therefore $AN = A$, i.e. $N$ is the identity element in $G/N$;

iv) since $(Na)(a^{-1}N) = NaN = N$, every element $Na$ in $G/N$ has the inverse $a^{-1}N = Na^{-1}$. 
[2.8] If $G$ has no normal subgroups other than the two trivial ones consisting of the element $e$ and of all elements of $G$, respectively, $G$ is a simple group.

(2.9) Examples

(2.9.1) The additive group of real numbers is abelian, and therefore all its subgroups are normal. Since $na \neq 0$ when $a \neq 0$ and $n \neq 0$, each of its subgroups generated by a single element other than 0 is a free cyclic group.

Let $H$ be the subgroup of all rational numbers $h$. All cosets of $H$ are composed of elements of the form $h+a$, where $a$ is a fixed real number. If $a$ and $b$ are two real numbers, the product of the elements $a$ and $b^{-1}$ is the element $a-b$; therefore $Ha$ and $Hb$ are different cosets of $H$ except when $a-b$ is a rational number.

(2.9.2) We consider the group $G$ of non-singular $n$-rowed square matrices of (1.5.3).

The elements whose determinants equal unity form a subgroup.

The elements with $a_{ij} = a_{ji}$ ($i = 1, 2, \ldots, n; j = 1, 2, \ldots, n$), i.e. the symmetric elements, form a subgroup.

The subgroup generated by the element $\|a_{ij}\|$ where $a_{ij} = 0$ for $i \neq j$ and $a_{ij} = e^{i \pi / q}$ for $i = j$ ($q$ rational fractions in their lowest terms) is a cyclic group; its order is the least common multiple of $2q_1, 2q_2, \ldots, 2q_n$.

Not all subgroups of $G$ are normal. For example, the subgroup $H$ composed of the element

$$
\begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 1 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 1 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0
\end{pmatrix}
$$
and the identity element is not normal. For the right coset containing the matrix

\[
\begin{pmatrix}
2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

contains as its other element

\[
\begin{pmatrix}
0 & \cdots & 0 \\
0 & \cdots & 1 \\
0 & \cdots & 0 \\
1 & \cdots & 0 \\
\end{pmatrix}
\]

whereas the left coset containing the first of these two elements contains as its other element

\[
\begin{pmatrix}
0 & \cdots & 0 & 2 \\
0 & \cdots & 1 & 0 \\
0 & \cdots & 1 & 0 \\
0 & \cdots & 1 & 0 \\
\end{pmatrix}
\]

3. **Mappings of groups**

[3.1] A mapping \( f \) of a group \( G \) on a group \( G' \) is isomorphic (or is an isomorphism) if it is one-to-one and if, for all \( x \) and \( y \) in \( G \), \( f(xy) = f(x)f(y) \). Clearly the inverse of an isomorphic map is also isomorphic.

[3.1.1] If such an isomorphic mapping exists between \( G \) and \( G' \), \( G \) and \( G' \) are said to be isomorphic.

[3.1.2] An isomorphic mapping of \( G \) on itself is an automorphism. Since the identity transformation is an automorphism, it is clear that if the product of two automorphisms is defined to be the automorphism resulting from the successive application of the two automorphisms, the set of all automorphisms of \( G \) constitutes a group.
For some fixed element $a$ in the group $G$ we determine the inner automorphism $f_a$ of $G$ by defining, for any element $x$ in $G$, the map $f_a$:

$$f_a(x) = axa^{-1}.$$ 

To show that this map is an automorphism we observe that it has the inverse $(f_a)^{-1} = f_{a^{-1}}$ (for $a^{-1}(axa^{-1})a = x$), so that the map is one-to-one, and that in the group $G$

$$f(x)f(y) = axa^{-1}aya^{-1} = axya^{-1} = f(xy).$$

(3.1.4) The set of the inner automorphisms of $G$ is a subgroup of the group of all automorphisms of $G$. For $f_a(f_b(x)) = abxb^{-1}a^{-1} = ab(ab)^{-1} = f_{ab}(x)$; and since $f_a$ has been shown to have an inverse which is also an inner automorphism, condition (2.2.11) is satisfied.

(3.2) The mapping of a group $G$ into a group $G^*$ is a homomorphism if $g(xy) = g(x)g(y)$.

The set of elements of $G$ which are mapped on the identity $e^*$ of $G^*$ is called the kernel of the homomorphism.

(3.2.1) If $g$ is a homomorphism of $G$ on $G^*$, then $g(e) = e^*$, and

$$g(x^{-1}) = (g(x))^{-1}.$$ For $g(x)g(e) = g(xe) = g(x)$, so that $g(e) = e^*$; it follows now that $g(x)g(x^{-1}) = e^*$, and therefore $g(x^{-1}) = (g(x))^{-1}$.

(3.3) If $g$ is a homomorphic map of the group $G$ on the group $G^*$, and if $N$ is the kernel of the homomorphism, then $N$ is a normal subgroup of $G$, and $G^*$ is isomorphic with the factor group $G/N$. In other words: if $X$ is the set of all elements in $G$ which are mapped on $x^*$ in $G^*$, then $X$ is a coset of $N$ with respect to $G$, and the one-to-one relation thus obtained between the elements of $G/N$ and $G^*$ is an isomorphism.

This isomorphism is called the natural isomorphism between $G/N$ and $G^*$.

To show that the set of elements $N$ constitutes a group under the multiplication in $G$. 

i) If \( x \) and \( y \) belong to \( N \), then \( g(xy) = g(x)g(y) = e^*e^* = e^* \) so that \( xy \) is in \( N \).

ii) Clearly, the associative law holds in \( N \), since it holds in \( G \).

iii) We have shown in (3.2.1) that \( g(e) = e^* \) so that \( N \) has an identity.

iv) \( g(x^{-1}) = (g(x))^{-1} \) by (3.2.1); therefore if \( g(x) = e^* \), \( g(x^{-1}) = (e^*)^{-1} = e^* \), and if \( x \) is in \( N \), so is \( x^{-1} \).

To show that \( N \) is a normal subgroup of \( G \):

If \( x \) is in \( N \) and \( a \) in \( G \), then

\[
g(a^{-1}xa) = g(a^{-1})g(x)a = g(a^{-1})g(a) = e^*;
\]

therefore \( a^{-1}xa \) belongs to \( N \) for all \( x \) in \( N \) and \( a \) in \( G \), and therefore \( N \) is normal.

It remains to demonstrate the isomorphism of \( G^* \) with \( G/N \).

If \( a^* \) is in \( G^* \) and \( A \) is the set of elements in \( G \) which are mapped on \( a^* \), we have, for any pair of elements \( a \) and \( a^1 \) in \( A \),

\[
g(a'a^{-1}) = g(a')g(a^{-1}) = g(a')(g(a))^{-1} = a^*(a^*)^{-1} = e^*;
\]

i.e., \( a'a^{-1} \) is in \( N \), and \( a \) and \( a^1 \) belong to the same coset of \( N \).

Conversely, suppose that \( x \) belongs to the same coset of \( N \) as does \( a \); then \( xa^{-1} \) is in \( N \), and therefore

\[
g(x)(a^*)^{-1} = g(x)g(a^{-1}) = g(xa^{-1}) = e^*;
\]

and therefore \( g(x) = a^* \). Therefore \( A \) is a coset of \( N \), and there exists a one-to-one correspondence between cosets of \( N \) and elements of \( G^* \). Every element of the group \( G/N \) is a coset of \( N \) and therefore to every element \( A \) in \( G/N \) corresponds to the element \( f(A) = a^* \) in \( G^* \), where \( A = Na \) and \( g(a) = a^* \).

If \( A \) and \( B \) are elements in \( G/N \), and \( a \) is in \( A \) and \( b \) in \( B \), then

\[
f(A)f(B) = g(a)g(b) = g(ab) = f(AB).
\]

This completes the proof of the theorem.
[3.4] If \( H \) is a normal subgroup of a group \( G \), we define the natural homomorphism of \( G \) on its factor group \( G/H \) by associating with every element \( x \) of \( G \) the element \( g(x) = x \) in \( G/N \) that contains \( x \).

To show that a homomorphism is thus defined:

If \( a \) is in the element \( A \) of \( G/H \), and \( b \) is in the element \( B \) of \( G/H \), then \( g(a) = A \), \( g(b) = B \); but the binary operation in the factor group is defined (cf. [2.7]) so that \( AB \) is the set of products of elements \( x \) and \( y \) belonging to \( A \) and \( B \), respectively; therefore \( AB \) contains \( ab \); i.e., \( g(a)g(b) = AB = g(ab) \).

(5.5) We note two simple facts regarding a homomorphism \( g \) of a group \( G \) on a group \( G^* \).

(3.5.1) If the kernel \( N \) of \( g \) is composed of the identity element alone, \( g \) is an isomorphism, since then every coset of \( N \) contains exactly one element.

(3.5.2) If \( H \) is a (normal) subgroup of \( G \), then \( g(H) \) is a (normal) subgroup of \( G^* \).

To show that \( g(H) \) is a subgroup, it is by (2.2.11) sufficient to show that if \( x^* \) and \( y^* \) are any two elements in \( g(H) \), then \( x^*y^{*-1} \) is an element in \( g(H) \). By hypothesis, there exist elements \( x \) and \( y \) in \( H \) such that \( g(x) = x^* \) and \( g(y) = y^* \), so that \( x^*y^{*-1} = g(x)g(y)^{-1} \) = \( g(x)g(y^{-1}) = g(xy^{-1}) \);

i.e., \( x^*y^{*-1} \) is the map of an element \( (xy^{-1}) \) in \( H \), and therefore it belongs to \( H^* \).

If \( H \) is normal, let \( x^* \) be any element in \( G^* \), and let \( x^* = g(x) \). \( x^{-1}Hx = H \), and therefore \( x^{*-1}g(H)x^* = g(x^{-1}Hx) = g(H) \), q.e.d.

(5.6) Examples

(3.6.1) The additive group of real numbers is mapped on itself automorphically by the mapping \( f(x) = kx \) (\( k \neq 0 \)). Since the
additive group is commutative, its only inner automorphism is the identity transformation.

(3.6.2) The additive group of real numbers is mapped isomorphically on the multiplicative group of positive real numbers by $g(x) = e^x$.

(3.6.3) Let $G$ be the group of matrices of (1.5.3), and $G^*$ the multiplicative group of complex numbers other than zero. If $a$ is a matrix of $G$, and $|a|$ the determinant of $a$, the relation $g(a) = |a|$ maps $G$ on $G^*$ homomorphically. The kernel of $G$ is the subgroup of matrices $a$ with the property $|a| = 1$.

4. The center and the commutator subgroup

[4.1] Two elements $a$ and $b$ of the group $G$ commute if $ab = ba$.

[4.2] An element $z$ of the group $G$ is central if it commutes with every element in $G$ ($xz = xz$, $x^{-1}zx = z$). The set $Z$ of all central elements is the center of the group $G$.

(4.2.1) $Z$ is a subgroup of $G$. For

i) if $z$ and $z'$ are any two elements in $Z$, we have, for all $x$ in $G$, $xz'z = xz'z = xz'x$; i.e. $zz'$ is in $Z$;

ii) the associative property follows from the fact that $G$ is a group;

iii) $e$ commutes with all elements in $G$, by (1.2); therefore $e$ is in $Z$;

iv) since $xz = xz$ for all $x$ in $G$ and all $z$ in $Z$, we have, $(xz)^{-1} = (zx)^{-1}$, i.e. $z^{-1}x^{-1} = x^{-1}z^{-1}$. Therefore, for all $y$ in $G$, $z^{-1}y = yz^{-1}$, and $z^{-1}$ is in $Z$.

(4.2.2) Every subgroup $H$ of the group $Z$ is a normal subgroup of $G$.

For if $h$ is an element in $H$, it is an element in $Z$, and this implies that, for all $x$ in $G$, $x^{-1}hx = h$, i.e. $xh^{-1}x$ is in $H$.

The subgroups of $Z$ are central normal subgroups of $G$.

[4.3] Obviously, $a$ and $b$ commute if and only if
ab(ba)^{-1} \cdot aba^{-1}b^{-1} = e_1 \cdot aba^{-1}b^{-1} is the commutator of a and b.

(4.3.1) The set Q containing each element q of G which is the commutator of two elements, together with all products \(q_1q_2\ldots q_m\) \((m = 1, 2, \ldots)\) of such commutators, is a normal subgroup of G; it is called the \textit{commutator subgroup} of the group G.

i) If \(x = q_1q_2\ldots q_m\) and \(y = q_1'q_2'\ldots q_m'\), then \(xy = q_1q_2\ldots q_m'q_1'q_2'\ldots q_m'\); i.e., xy is in Q.

ii) Multiplication is associative.

iii) e commutes with every element, therefore e belongs to Q.

iv) If \(q = aba^{-1}b^{-1}\), then \(q^{-1} = bab^{-1}a^{-1}\); i.e., \(q^{-1}\) is also a commutator. Therefore if \(x = q_1q_2\ldots q_m\), \(x^{-1} = q_m^{-1}q_2^{-1}q_1^{-1}\), and \(x^{-1}\) is an element in Q.

It remains to show that the subgroup Q is normal, i.e.,

that if \(x = q_1q_2\ldots q_m\) and c is any element in G, \(c^{-1}xc\) is in Q.

Now if \(q = aba^{-1}b^{-1}\), \(c^{-1}qc = (c^{-1}ac)(c^{-1}bc)(c^{-1}a^{-1}c)(c^{-1}b^{-1}c) = (c^{-1}ac)(c^{-1}bc)(c^{-1}ac)^{-1}(c^{-1}bc)^{-1}\); i.e., \(c^{-1}qc\) is a commutator.

Since Q is a group, it follows that

\[c^{-1}xc = c^{-1}q_1q_2\ldots q_mc = (c^{-1}q_1c)(c^{-1}q_2c)\ldots(c^{-1}q_mc)\]

is in Q, and the proof is complete.

(4.4) The factor group G/Q is commutative; and if N is any other normal subgroup of G such that G/N is commutative, then Q is a subgroup of N.

If A and B are any two cosets of Q, \(ABA^{-1}B^{-1}\) is also a coset of Q, and since it contains the commutator \(aba^{-1}b^{-1}\), it is identical with Q; therefore the element \(ABA^{-1}B^{-1}\) of the group G/Q is the identity element of the group, and this implies that A and B commute in G/Q. Since A and B are arbitrary, G/Q is commutative.

We show now that if N is any normal subgroup of G such that Q is not a subgroup of N, G/Q is not commutative; If N is such a
subgroup, there exist at least two elements $a$ and $b$ such that $aba^{-1}b^{-1}$ is not an element of $N$ (otherwise, $N$ would contain all commutators, contrary to hypothesis). If $A$ and $B$ are the cosets of $N$ containing $a$ and $b$, respectively, the coset $ABA^{-1}B^{-1}$ cannot be identical with $N$, since $N$ does not contain $aba^{-1}b^{-1}$; i.e., $ABA^{-1}B^{-1}$ is not the identity of the group $G/N$, and $A$ and $B$ do not commute; therefore $G/N$ is not a commutative group, q.e.d.

(4.4.1) If $N$ is a normal subgroup of $G$, and $Q$ the commutator subgroup of $N$, $Q$ is a normal subgroup of $G$.

That $Q$ is a subgroup of $G$ follows from the fact that $N$ is a subgroup of $G$. If $q$ is a commutator of two elements $a$ and $b$ in $N$, we have, for any $c$ in $G$,

$$c^{-1}ac, c^{-1}aba^{-1}b^{-1}c$$

$$= (c^{-1}ac)(c^{-1}bc)(c^{-1}ac)^{-1}(c^{-1}bc)^{-1}$$

and since $N$ is a normal subgroup of $G$, $c^{-1}ac$ and $c^{-1}bc$ are in $N$, together with their inverses. By [4.3], this means that $c^{-1}qc$ is the commutator of two elements in $N$, and therefore that it belongs to $Q$. By [2.6] this proves that $Q$ is a normal subgroup of $G$.

[4.5] If $G$ is any group, and if for some $n$ the group $Q_n$ is the identity of $G$, where $Q_i$ is the commutator subgroup of $G$, and $Q_{i+1}$ is the commutator subgroup of $Q_i$ ($i = 1, 2, \ldots$), then the group $G$ is solvable.

(4.6) Example: Let $G$ be the group of matrices in (1.5.3), and let $Z$ be the subgroup of elements in which $a_{ij} = a_{ji}$ ($i = j$) and $a_{ij} = 0$ ($i \neq j$). Clearly, $Z$ is a central normal subgroup of $G$. We show that $Z$ is actually the center of $G$. Let $||a_{ij}||$ be any element in the center, and let $||b_{ij}||$ be the matrix

$$\begin{bmatrix} 0 & \ldots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \ldots \\ 0 & \ddots & \vdots \\ \vdots & \ldots & 0 \end{bmatrix}$$
Then $||a_{ij}|| \cdot ||b_{ij}|| = ||b_{ij}|| \cdot ||a_{ij}||$ implies $a_{ij} = a_{ji}$

$(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$. Now choose as $||c_{ij}||$ the matrix

$$
\left|
\begin{array}{cccc}
0&1&0&0 \\
0&0&1&0 \\
0&0&0&1 \\
\vdots&\vdots&\vdots&\vdots \\
0&0&0&1 \\
\end{array}
\right|
$$

Then $||a_{ij}|| \cdot ||c_{ij}|| = ||c_{ij}|| \cdot ||a_{ij}||$ implies $2a_{12} = a_{21}$, i.e. $a_{12} = 0$. Similarly, it can be shown that $a_{ij} = 0$ for all cases where $i \neq j$.

It remains to be shown that $a_{ij} = a_{11}$ $(i = 2, 3, \ldots, n)$. If $||d_{ij}||$ is the matrix

$$
\left|
\begin{array}{cccc}
0&1&0&0 \\
0&0&1&0 \\
0&0&0&1 \\
\vdots&\vdots&\vdots&\vdots \\
0&0&0&1 \\
\end{array}
\right|
$$

the relation $||a_{ij}|| \cdot ||d_{ij}|| = ||d_{ij}|| \cdot ||a_{ij}||$ gives, together with the fact that $a_{ij} = 0$ $(i \neq j)$, that $a_{11} = a_{22}$. Similarly we show that $a_{12} = a_{23}$, etc., and we have proved that $Z$ is the center of the group $G$.

It is obvious that all the commutators $aba^{-1}b^{-1}$ in $G$ are matrices with determinants equal to unity.

5. Intersection and product of subgroups. Direct product of groups.

(5.1) If $M$ is a set of (normal) subgroups of $G$ and $D$ the intersection of all the subgroups in $M$ (i.e. $D$ is the set of those elements in $G$ that are contained in each subgroup in $M$) then $D$ is a (normal) subgroup of $G$. For if $a$ and $b$ are elements in $D$, every subgroup in $M$ contains $ab^{-1}$, by (2.2.1), and therefore $ab^{-1}$ is in $D$; consequently $D$ is a subgroup of $G$. If all subgroups in $M$ are normal, we have, for all $x$ in $G$ and all $a$ in $D$, that $x^{-1}ax$ belongs to each subgroup in $M$ (since $a$ belongs to each subgroup in $M$),
and therefore $x^{-1}ax$ is in $D$, q.e.d.

[5.2] If $R$ is a product of elements of the group $G$, and $M$ is the set of all subgroups of $G$ which contain $R$, then the minimal (normal) subgroup of $G$ which contains $R$ is the intersection of all (normal) subgroups in $M$.

(5.3) If $H$ is a subgroup and $N$ a normal subgroup of the group $G$, then their intersection $H \cap N = D$ is a normal subgroup of the group $H$.

That $D$ is a group and therefore a subgroup of $H$ follows from (5.1). Now, if $h$ is any element in $H$ and $n$ an element in $D$, $h^{-1}nh$ is in $N$, since all factors are contained in $H$; but also $h^{-1}n$ is in $N$, since $N$ is normal. Therefore $h^{-1}nh$ is in $D$, and $D$ is a normal subgroup of $H$.

(5.4) If $H$ is a (normal) subgroup and $N$ a normal subgroup of the group $G$, then $HN = NH$, and this product is a (normal) subgroup of $G$.

Since $N$ is a normal subgroup of $G$, its left cosets are also right cosets, and therefore $HN = NH$. If $a$ and $b$ are elements of $HN$, we can write $a = hn$, $b = h'nx$, where $h$ and $h'$ belong to $H$, $n$ and $n'$ to $N$. We have then

$$ab^{-1} = hnn'^{-1}h'^{-1} = hh'^{-1}(h'hnn'^{-1}h'^{-1});$$

since $N$ is normal, the factor in parentheses is an element $n''$ in $N$, and we have $ab^{-1} = hh'^{-1}n''$, and this clearly belongs to $HN$. It follows from (2.2.1) that $HN$ is a subgroup of $G$.

If $H$ is a normal subgroup of $G$ we have, for any $x$ in $G$ and any $a$ in $HN$,

$$x^{-1}ax = x^{-1}hnx = (x^{-1}hx)(x^{-1}nx),$$

and therefore $x^{-1}ax$ belongs to $HN$, and $HN$ is normal, q.e.d.

(5.4.1) If $N_1$, $N_2$, ..., $N_k$ are normal subgroups of $G$, then it follows from (5.4), by induction, that $N_1N_2\ldots N_k$ is a normal subgroup of $G$. 
(5.5) If \( H \) is a subgroup and \( N \) a normal subgroup of \( G \), and if \( D = H \cap N \) and \( P = HN \), then the factor group \( H/D \) is isomorphic with the factor group \( P/N \).

To establish a one-to-one correspondence between the two factor groups \( H/D \) and \( P/N \): every element \( A \) of the group \( H/D \) is of the form \( A = Da \), where \( a \) is in \( H \). If we write \( A' = Na \), it follows from the fact that \( D \) is in \( N \) that \( Da \) is in \( Na \), i.e. \( A \) is in \( A' \).

Therefore every element of the group \( H/D \) is contained in at least one element of the group \( P/N \); it is contained in only one since the cosets of \( N \) in \( P \) are mutually exclusive.

Conversely, any element \( B' \) of the factor group \( P/N \) can be written in the form \( B' = bN = Nb \), where \( b \) is in \( H \). If we write \( B = Db \), it follows from the fact that \( D \) is in \( N \), that \( Db \) is in \( Nb \), i.e. \( B \) is in \( B' \). Therefore every element of \( P/N \) contains at least one element of \( H/D \).

Suppose now that the element \( C' \) of \( P/N \) contains the elements \( A \) and \( B \) of \( H/D \). Then \( AB^{-1} \) is contained in \( C'C'^{-1} = N \); also, by [2.5], \( AB^{-1} \) is in \( N \), since \( A \) and \( B \) are cosets of \( D \) in \( H \). Therefore \( AB^{-1} \) is contained in \( D \), the identity element of the group \( H/D \); i.e., \( A = B = D \).

This establishes the existence of a one-to-one correspondence between the elements of the groups \( H/D \) and \( P/H \).

Now let \( A \) and \( B \) be two elements of \( H/D \), and \( A' \) and \( B' \) the two elements of \( P/N \) that contain \( A \) and \( B \), respectively. Then \( A'B' \) contains \( AB \), and the isomorphism is established.

[5.6] If \( H \) and \( K \) are two normal subgroups of the group \( G \), and if \( HK = G \) and \( H \cap K \) contains only the element \( e \), \( G \) is decomposed into the direct product of \( H \) and \( K \).

(5.6.1) If \( G \) is decomposed into the direct product of \( H \) and \( K \), every element of \( H \) commutes with every element of \( K \), and every element of
H commutes with every element of K, and every element of G can be represented uniquely in the form \(hk\), where \(h\) is in \(H\) and \(k\) in \(K\).

If \(h\) is in \(H\) and \(k\) in \(K\), we have, from the normality of \(H\) and \(K\), that \(hk^{i-1}\) is in \(K\) and \(kh^{-1}k\) is in \(H\). Therefore the commutator

\[ q = h(kh^{-1}k^{-1})(hk^{-1})k^{-1} \]

is in both \(H\) and \(K\); i.e., \(q = e\), and \(h\) and \(k\) commute.

If \(x\) is any element of \(G\), it follows from the relation \(G = HK\) that \(x = hk\) (\(h\) in \(H\), \(k\) in \(K\)). Suppose also \(x = h'k'\)
(\(h'\) in \(H\), \(k'\) in \(K\)). Then we have

\[ h'k' = h'k, \]

\[ h^{-1}hk'k'^{-1} = h^{-1}h'k'k'^{-1}, \]

\[ kk'^{-1} = h^{-1}h'. \]

Since the element on the left side of the last equation is in \(K\), the other in \(H\), this gives \(kk'^{-1} = e = h^{-1}h'\), and since inverses are unique, \(k = k'\), \(h = h'\), q.e.d.

[5.7] From two groups \(H\) and \(K\) composed of elements \(\{h\}\) and \(\{k\}\), respectively, we construct the direct product \(G\) composed of elements of the form \((h,k)\), and with the law of multiplication

\( (h,k)(h',k') = (hh', kk') \). Obviously, \(G\) is a group. Its identity is the element \((e, e')\) (\(e\) the identity in \(H\), \(e'\) that in \(K\)), and the inverse of \((h,k)\) is \((h^{-1}, k^{-1})\).

We shall show that this definition of a direct product is equivalent with that of [5.6].

(5.7.1) If

i) \(G\) can be decomposed into a direct product of normal subgroups \(H\) and \(K\) (cf. [5.6]),

ii) \(H'\) and \(K'\) are groups isomorphic with \(H\) and \(K\), respectively,

iii) \(G'\) is the direct product of the group \(H'\) and \(K'\),
then the group $G'$ is isomorphic with the group $G$. For if $f$ and $g$ are isomorphic mappings of $H'$ on $H$ and of $K'$ on $K$, respectively, the relation $s((h',k')) = f(h')g(k')$ determines an isomorphic map of $G'$ on $G$ (the fact that the map is one-to-one is a consequence of (5.6.1)).

(5.7.2) If $G$ is the direct product of groups $H$ and $K$, and if $H'$ and $K'$ are the sets of elements in $G$ which are of the form $(h,e')$ and $(e,k)$, respectively, then

1) $H'$ and $K'$ are normal subgroups of $G$;

2) $G$ is decomposed into the direct product of $H'$ and $K'$;

and

3) $H'$ and $K'$ are isomorphic with $H$ and $K$, respectively.

To prove 1): if $(h,e)$ and $(h',e')$ are two elements of $H'$, then $(h,e')(h'e')^{-1} = (hh'^{-1}, e')$, and since this is in $H'$, (2.2.1) applies and $H'$ is a subgroup. To show normality, we point out that if $(a,b)$ is any element of $G$, we have $(a,b)^{-1}(h,e')(a,b) = (a^{-1}ha, e')$, and the element on the right is in $H'$, similarly we can prove that $K'$ is a normal subgroup.

To prove 2): if $(h,e') = (e,k)$, then $h = e$ and $k = e'$. Therefore $H' \cap K'$ contains only the identity; also the product of the two groups coincides with $G$, since each element $(h,k)$ of $G$ can be written in the form $(h,e')(e,k)$. Therefore, $G$ is decomposed into $H'$ and $K'$.

To prove 3): the isomorphism between $H$ and $H'$ is established if we associate with every element $h$ of $H$ the element $(h,e)$ of $H'$; with a similar argument for $K$ and $K'$, the proof is complete.

(5.7.3) If $G$ is decomposed into the direct product of normal subgroups $H$ and $K$, $H$ is isomorphic with the factor group $G/K$. 
For $H\cap K = \{e\}$, and $HK = G$, and by (5.5), $H/e$ is isomorphic with $G/K$. Therefore $H$ is isomorphic with $G/K$.

(5.8) If $G$ is a group and $G_1, G_2, \ldots$ a countable set of normal subgroups of $G$, $G$ is said to be decomposable into the direct product of the subgroups of the set $G_1, G_2, \ldots$ provided
1) the minimal subgroup of $G$ (cf. (5.2)) which contains all the subgroups $G_1, G_2, \ldots$ coincides with $G$;
2) if $H$ denotes the minimal normal subgroup of $G$ which contains the subgroups $G_1, G_2, \ldots G_{n-1}, G_{n+1}, G_{n+2}, \ldots$ then the subgroups $H_1, H_2, \ldots$ have in common no elements other than the identity $e$.

5.8.1) With the notation of (5.8), $G$ can be decomposed into the direct product of $G_n$ and $H_n$.

For by (5.4) the product $G_n H_n$ is a normal subgroup, and it contains all the subgroups $G_i$; by (1) of (5.8), $G_n H_n = G$. Let $G_n'$ be the set of elements common to $H_1, H_2, \ldots H_{n-1}, H_{n+1}, H_{n+2}, \ldots$. Since $H_n$, the only set $H_i$ which may possibly not contain $G_n$, is missing from this sequence, $G_n' \subseteq G_n$. From (ii) of (5.8) it follows that $G_n' \cap H_n = \{e\}$, since $G_n' \cap H_n$ is the intersection of all sets $H_n$, Therefore $G_n' \cap H_n = \{e\}$, and $G$ can be decomposed into the direct product of $G_n$ and $H_n$.

(5.8.2) Every element of $G_i$ commutes with every element of $G_j$ ($i \neq j$), and every element $x$ in $G$ can be represented in the form $x = x_1 x_2 \ldots x_n$, where $x_i$ is in $G_i$ and $n$ is a finite number depending on $x$; moreover, this representation is unique.

From (5.8.1), (5.6.1), and the fact that $G_i \subseteq H_j$ we deduce the commutativity of the elements of the groups $G_i$ and $G_j$, respectively. We observe that the set $G'$ of all products $x = x_1 x_2 \ldots x_n$, where $x_i$ is in $G_i$ ($i = 1, 2, \ldots, n; n = 1, 2, \ldots$), is a normal subgroup of the
group $G$, and that every group $G_k$ is contained in $G^*$.  

It follows now from $(1)$ of $(5.8)$ that $G^* = G$, and every

element $x$ in $G$ can be written $x = x_1 x_2 \cdots x_n$ with $x_i$ in $G_i$.

If now $x_1 x_2 \cdots x_n = x'_1 x'_2 x'_3 \cdots x'_n$, we deduce from (5.8.1) and

(5.6.1) that $x_1 = x'_1$. Repetition of the process gives $x_k = x'_k$, etc.,

and the uniqueness is proved.

[5.9] From a countable set of groups $G_1, G_2, \ldots$, we construct a new


group $G$, the direct product of the groups $G_1, G_2, \ldots$. The elements

of $G$ are the sequences

$$x = \{x_1, x_2, \ldots\}$$

where $x_i$ is an element of $G_i$, and all but a finite number of the

$x_i$'s are the identities of their respective groups. The product of

the two elements $x$ and $y$ in $G$ we define to be the sequence

$$xy = \{x_1 y_1, x_2 y_2, \ldots\}.$$

It is obvious that the group $G$ does not depend on the way in

which the group $\{G_i\}$ are numbered; i.e., if in the sequence $\{G_i\}$

a finite or infinite number of groups exchange position, the new

group $G'$ will be isomorphic with $G$. It is also clear that $G$ has

the identity $e = \{e_1, e_2, \ldots\}$, and that $x$ has the inverse element

$$x^{-1} = \{x_1^{-1}, x_2^{-1}, \ldots\}.$$  

By steps similar to those of (5.7.1) and (5.7.2), the

equivalence of [5.9] and [5.9] can be shown.

(5.10) Examples

(5.10.1) If $G$ is a countable commutative group all of whose elements

except $e$ are of prime order $\rho$, then $G$ can be decomposed into the

direct product of a countable number of cyclic subgroups $H$ of order $\rho$:

$$H = \{e, a, a^2, \ldots a^{\rho-1}\}.$$  

(5.10.2) Let $G'$ be the subgroups of matrices of (1.5.3) which have

real elements and positive determinants. Let $Z$ be the subgroup of
$C'$ determined by $a_{ij} = 0 \ (i \neq j), \ a_{ii} = a_{ij} > 0 \ (i = 1, 2, \ldots, n)$.

Let $\mathcal{Q}$ be the set of matrices of $C'$ whose determinants are equal to unity. It is obvious that $Z \cap \mathcal{Q} = \{e\}$ and that every matrix in $C'$ can be expressed as the product of a matrix in $Z$ by a matrix in $\mathcal{Q}$; $C'$ is decomposed into the direct product of $Z$ and $\mathcal{Q}$.
II. TOPOLOGICAL SPACES

In the first chapter, we examined those aspects of the operation of multiplication which do not depend on the properties of the special sets of elements over which the operation is usually defined, but only on certain axioms. Validity of those axioms we regard as the characterizing attribute of multiplication among all binary operations.

We shall now apply a similar treatment to the operation of passing to a limit. To this end, we shall construct an abstract space which has those properties which in other spaces permit the operation in question. The construction of an abstract space for this special purpose will enable us to avoid the multitude of explicit detail, usually arithmetic in nature, that so often clutters the scene and obstructs the view when we consider passage to a limit in specific cases.
6. The concept of a topological space.

[6.1] Different authors have used sets of axioms based on apparently distinct concepts, and have arrived at topological spaces which have the same properties. We shall use the axioms of Hausdorff (6):

If $R$ is a set of objects, called points, and if subsets $U_x$, $V_x$, ... of $R$, called neighborhoods of $x$, are associated with each point $x$ in $R$, then we say $R$ is a topological space, provided the following conditions are satisfied:

i) Each point $x$ in $R$ has at least one neighborhood, and $x$ is contained in each of its neighborhoods.

ii) If $U_x$ and $V_x$ are two neighborhoods of $x$, $x$ has a neighborhood $W_x$ contained in $U_x \cap V_x$.

iii) If $y$ is any point in $U_x$, $y$ has a neighborhood $U_y$ contained in $U_x$.

iv) If $x \neq y$, there exists a neighborhood $U_x$ which does not contain the point $y$. (7)

Henceforth the letter $R$ shall denote a topological space.

[6.2] $x$ is an interior point of the set $E$ in $R$ if one of its neighborhoods $U_x$ is contained in $E$. The set $E$ is an open set if all its points are interior points.

(6.2.1) It follows from axiom (iii) of [6.1] that every neighborhood is an open set.

(6.2.2) A necessary and sufficient condition that there exists a neighborhood $U_x$ such that every point of $U_x$ (except possibly $x$) has a certain property $P$ is that there exists an open set $E$

(6) Grundzüge der Mengenlehre, 1914, Chapters VII, VIII.

(7) This axiom is weaker than that given by Hausdorff: if $x \neq y$, there exist two neighborhoods $U_x$ and $V_y$ that do not intersect. It is interesting to note that the example in (6.6.4) does not satisfy this requirement.
containing \( x \) such that every point of \( E \) (except possibly \( x \)) has the property \( P \).

The necessity of the condition follows immediately from (6.2.1), the sufficiency from \([6.2]\).

Similarly, a necessary and sufficient condition that every neighborhood of \( x \) containing more than one point contain a point with a certain property \( Q \) is that every open set containing \( x \) and at least one other point contain a point with property \( Q \).

The necessity of the condition follows from 6.2, the sufficiency from (6.2.1).

[6.2.3] As is customary, we denote by \( E_1 \cup E_2 \) the sum of the sets \( E_1 \) and \( E_2 \), i.e. the set of points contained in \( E_1 \) or \( E_2 \); the sum of a set of sets \( \{E_i\} \) is the set of points each of which is contained in at least one of the sets \( E_i \).

By \( E_1 \cap E_2 \) we denote the intersection of \( E_1 \) and \( E_2 \), i.e. the set of points contained in both \( E_1 \) and \( E_2 \); the intersection of a set of sets \( \{E_i\} \) is the set of points contained in each of the sets \( E_i \).

If two sets have common points, we say that they intersect.

The complement of a set \( E \) is the set of points not in \( E \); if \( E_1 \) and \( E_2 \) do not intersect \( (E_1 \cap E_2 = \emptyset) \) and \( E_1 \cup E_2 = \mathbb{R} \), then \( E_1 \) and \( E_2 \) are each other's complements.

By \( E - a \) we denote the set of points in \( E \) distinct from \( a \), by \( A - B \) the set of points that are in \( A \) but not in \( B \).

(6.2.4) The sum of an aggregate of open sets is an open set. For if \( x \) is a point of the aggregate, it is a point of one of the sets, and therefore the aggregate contains one of its neighborhoods.

[6.3] For any set \( E \) in \( \mathbb{R} \) we define the closure \( \overline{E} \) by the criterion that the point \( a \) belongs to \( \overline{E} \) if every neighborhood of \( a \) contains
at least one point of $E$.

(6.3.1) It follows immediately that if $E_1 \supset E_2$, then $E_1 \supset \overline{E}_2$.

(6.3.2) $\overline{E} \supset E$. For if $x$ is a point of $E$, each neighborhood of $x$ contains at least one point of $E$—namely, $x$—and therefore $x$ is in $\overline{E}$.

(6.3.3) $\overline{E} = \overline{E}$. Clearly, $\overline{E} \supset \overline{E}$. Now, if $x$ is not a point of $\overline{E}$, some neighborhood $U_x$ contains no points of $\overline{E}$. But by (iii) of $[6.1]$, every point $y$ in $U_x$ has a neighborhood $U_y$ contained in $U_x$, and $U_y$ will contain no points of $E$. It follows that no point $y$ in $U_x$ is a point of $\overline{E}$, and therefore $x$ is not a point of $\overline{E}$; i.e., $\overline{E} \supset \overline{E}$. Therefore $\overline{E} = \overline{E}$.

(6.3.4) If $E$ contains only one point, $\overline{E} = \overline{E}$. For if $E = \{x\}$ and $y \notin x$, there exists a neighborhood $U_y$ which does not contain $x$, by axiom (iv) of $[6.1]$. On the other hand, every neighborhood of $x$ contains $x$, by axiom (i) of $[6.1]$. This proves the proposition.

(6.4) $E$ is a closed set if $E = \overline{E}$.

(6.4.1) For all $E$, $\overline{E}$ is a closed set, by (6.3.3).

(6.4.2) If $E_1$ is the complement of $E_2$, a necessary and sufficient condition for $E_1$ to be open is that $E_1$ is closed.

If $E_1$ is not closed, every neighborhood of some point $x$ not in $E_1$ contains points of $E_1$; i.e., every neighborhood of some point in $E_2$ contains points not in $E_1$, and $E_2$ cannot be open.

If $E_2$ is closed and $x$ is not a point of $E_1$, some neighborhood of $x$ contains no points of $E_1$, since $E_1 = \overline{E}_2$; i.e., if $x$ is a point of $E_2$, some neighborhood of $x$ is contained in $E_2$; therefore $E_2$ is open.

(6.4.3) If $\{E\}$ is a finite or infinite set of closed sets, then the intersection $D$ of these sets is a closed set. For by (6.4.2) the complement of each set in $\{E\}$ is open, and therefore the set composed of all points not contained in any of the sets of $\{E\}$
is open, and therefore the set $C$ composed of all points not contained in any of the sets of $B$ is open, by (6.2.4). But $G$ is the complement of $D$, and therefore $D$ is closed.

(6.4.4) If $E_1$ and $E_2$ are closed sets, $E_1 \cup E_2$ is a closed set.

If $x$ is a point of $E_1 \cup E_2$, every neighborhood of $x$ contains at least one point of $E_1$ or of $E_2$. Suppose now that $U_x$ contains no point of $E_1$. Then every neighborhood of $x$ contained in $U_x$ fails to contain points of $E_1$, and therefore by hypothesis contains points of $E_2$. Therefore $x$ is in $E_2$, and since $E_2$ is closed, $x$ is in $E_2$. If on the other hand every neighborhood of $x$ contains points of $E_1$, $x$ is in $E_1$, and therefore in $E_1$.

We have shown that if $x$ is in $E_1 \cup E_2$, $x$ is in $E_1 \cup E_2$; therefore $E_1 \cup E_2$ is closed, q.e.d.

(6.4.5) $\overline{E_1 \cup E_2} = \overline{E_1} \cup \overline{E_2}$.

The proof in (6.4.4) shows that if $x$ is in $\overline{E_1 \cup E_2}$, $x$ is in $\overline{E_1} \cup \overline{E_2}$; i.e. $\overline{E_1 \cup E_2} \subseteq \overline{E_1} \cup \overline{E_2}$. Also $\overline{E_1 \cup E_2} \supseteq \overline{E_1} \cup \overline{E_2}$; for $E_1 \cup E_2 \supseteq E_1$ and $E_2 \cup E_2 \supseteq E_2$, and by (6.3.1), $\overline{E_1 \cup E_2} \supseteq \overline{E_1} \cup \overline{E_2}$.

(6.4.6) By induction, the propositions (6.4.4) and (6.4.5) can obviously be extended to any finite number of closed sets.

(6.4.7) The intersection of any finite number of open sets is an open set. This follows from (6.4.6), the fact that the complement of the intersection of an aggregate of sets is the sum of the complements of the sets of the aggregate, and the fact that the complement of a closed set is an open set, by (6.4.2).

[6.5] The point $a$ is a limit point of the set $E$ if every neighborhood $U_a$ contains a point of $E$ distinct from $a$.

(6.5.1) If $a$ is a limit point of $E$, every neighborhood of $a$ contains infinitely many points of $E$. For let $U_a$ be a neighborhood of $a$, and $x$ a point of $E$ in $U_a$. Then, by (ii) and (iv) of (6.1), $U_a$
contains a neighborhood \( V_a \) which does not contain \( x \). By hypothesis, \( V_a \) contains a point \( y \) of \( E \), distinct from \( a \). Repeating the argument indefinitely, we find an infinite sequence of points of \( E \) that are distinct from \( a \), and contained in \( U_a \).

(6.5.2) A closed set contains all its limit points; and a set containing all its limit point is closed.

(6.5.3) A necessary and sufficient condition for \( a \) to be a limit point of \( E \) is that \( a \) be contained in \( \overline{E - a} \). For if \( a \) is a limit point of \( E \), every neighborhood of \( a \) contains points of \( \overline{E - a} \), and therefore \( a \) is contained in \( \overline{E - a} \). If on the other hand \( a \) is contained in \( \overline{E - a} \), every neighborhood of \( a \) contains points of \( \overline{E - a} \), and therefore \( a \) is a limit point of \( E \).

(6.5.4) A topological space without limit points is said to be discrete.

(6.6) Examples

(6.6.1) Any set of points can be topologized by assigning to each point \( x \) the point \( x \) itself as neighborhood. It is clear that such a space is discrete, and that all its sets are both open and closed.

(6.6.2) Any metric space can be topologized by assigning to each point \( x \) as neighborhoods all open spheres \(|x' - x| < r| \) containing the point.

(6.6.3) Let \( R \) be the Cartesian plane, and let the neighborhoods of \((x_0, y_0)\) be all the sets \((x - x_0)^2 + (y - y_0)^2 < r^2\); in addition, let the point \((x_0, y_0)\) be its own neighborhood provided \( x_0 \) and \( y_0 \) are not both rational numbers.

Then \( \overline{E} = R \) provided \( E \) contains all the points whose coordinates are both rational. If \( E \) contains none of these points, \( \overline{E} = E \).

(6.6.4) Let \( R \) be any infinite set, and let every set \( U \) in \( R \) be a neighborhood of \( x \), provided it contains \( x \), and provided it contains
all but a finite number of the points in R. It is easy to see that in this space every finite set is closed since its complement is open, and that the only closed infinite set in R is R itself. (cf. footnote on [6.1]).

7. Continuous mapping and homeomorphism

[7.1] A mapping $g$ of a topological space $R$ on a topological space $R'$ is continuous provided, for every point $a$ in $R$ and every neighborhood $U'$ of the point $g(a)$ in $R'$, there exists a neighborhood $U_a$ in $R$ such that $g(U_a) \subseteq U'$. 

[7.2] A mapping $g$ of a topological space $R$ on a topological space $R'$ is topological provided it is one-to-one and continuous and its inverse $g^{-1}$ is also continuous.

[7.2.1] Two topological spaces are homeomorphic or topologically equivalent if one can be mapped on the other topologically.

[7.2.2] If $R$ is a topological space with a system $\Sigma$ of neighborhoods, and if $R'$ is the space composed of the set of points contained in $R$, with the system $\Sigma'$ constructed by assigning to each point all the sets of the system $\Sigma$ which contain it, then $R$ and $R'$ are homeomorphic.

It is easy to see that the system $\Sigma'$ satisfies the Hausdorff axioms in [6.1], since the system $\Sigma$ satisfies in particular axiom (iii). Let $g$ be the one-to-one mapping under which each point $x'$ in $R$ is the image of the corresponding point $x$ in $R$. If $U'$ is a neighborhood of $x'$ in $R'$, and if $U' = g(U)$ and $x' = g(x)$, $U$ contains a neighborhood $V$ of $x$, by (iii) of [6.1], and $g(V) \subseteq U$; i.e. $g$ is a continuous mapping. It is obvious that its inverse is also continuous.

[7.2.3] If a topological space has a property $P$, $P$ is a topological property provided it is invariant under all topological mappings.
A necessary and sufficient condition for a mapping $g$ of $R$ on $R'$ to be continuous is that, for every set in $R$, $g(E) \subseteq g(E')$.

Suppose that $g$ is a continuous mapping, and suppose that $x$ is a point of $E$ in $R$ and $x' = g(x)$. Let $U'$ be any neighborhood of $x'$; then there exists a neighborhood $U_x$ such that $g(U_x) \subseteq U'$. But every neighborhood of $x$ contains points of $E$, and therefore every neighborhood $U'$ contains points of $g(E)$. It follows that $x'$ is a point of $g(E)$ and, since $E$ and $x$ are arbitrary, that $g(E) \subseteq g(E')$, for all sets $E$.

Suppose on the other hand that $g(E) \subseteq g(E')$ for every set $E$ in $R$. Let $a$ be any point in $R$, $U'$ a neighborhood of $g(a)$ in $R'$, $F'$ the complement of $U'$, and $F$ the set of points in $R$ of which $F'$ is the image. By (6.2.1) and (6.4.2), $F' = \overline{F}$. Now, since $F'$ does not contain $g(a)$, $F$ does not contain $a$. To show that $g$ is a continuous map, we have to demonstrate that $a$ has a neighborhood $U_a$ such that $g(U_a) \subseteq U'$.

We have $g(F) \subseteq g(F') = \overline{F'}$; i.e. $F$ does not contain the point $a$; therefore some neighborhood of $a$ has no points in $F$, and is mapped into $U'$, q.e.d.

A necessary and sufficient condition for $g$ to map $R$ on $R'$ continuously is that if $E'$ is an open set in $R'$, the sum of all sets $E$ with the property $g(E) \supseteq E'$ is open.

To show that the condition is necessary: Let $E'$ be any open set in $R'$, and $E$ the set of points of which $E'$ is the image. Let $a$ be any point of $E$; let $a' = g(a)$, and let $U'$ be a neighborhood of $a'$ contained in $E'$ ($E'$ is open). Now, if $g$ is continuous, there exists a neighborhood $U_a$ such that $g(U_a) \subseteq U'$. Since $U' \subseteq E'$, $U_a \subseteq E$, and therefore $a$ is an interior point of $E$. Since $a$ is arbitrary, $E$ is open, q.e.d.

To show the sufficiency of the condition: Let $E'$ be an open
set and $U'$ a neighborhood of $a'$ contained in $E'$, and let $E$ be the set of points of which $E'$ is the image. Then the set of points of which $U'$ is the image is an open set contained in $E$; if $g(a) = a'$, $a$ has a neighborhood $U_a$ such that $g(U_a) \subset U'$, and $g$ is a continuous map.

(7.3.2) Recalling that by (6.4.2) the complement of an open set is closed, and conversely, we see immediately that in (7.3.1) the word "open" can be replaced by the word "closed".

(7.4) It follows immediately from (7.3) that a necessary and sufficient condition for a one-to-one map to be topological is that $g(F) = g(F)$, for every set $E$.

(7.4.1) The properties of being open, or closed, are topological properties of sets.

(7.5) Examples

(7.5.1) Let $R$ be the Cartesian n-space in which every sphere is a neighborhood of every point it contains; in addition, let the origin $O$ be its own neighborhood.

Let $R'$ be the Cartesian n-space with the same system of neighborhoods, except that the origin is not its own neighborhood.

Then the mapping $g(x_1, x_2, \ldots, x_n) = (x_1', x_2', \ldots, x_n')$ ($x_i' = x_i; i = 1, 2, \ldots, n$) maps $R$ on $R'$ continuously, but not topologically. For the inverse map is not continuous, since no neighborhood of the origin $O'$ in $R'$ is mapped into that neighborhood of $O$ in $R$ which consists solely of the point $0$.

(7.5.2) Let $R$ be the Cartesian n-space in which every sphere of positive rational radius and with center $(x_1, x_2, \ldots, x_n)$ ($x_i$ rational; $i = 1, 2, \ldots, n$) is a neighborhood of every point it contains. Let $R'$ be the Cartesian n-space in which every sphere $\sum (x_i - a_i)^2 < r^2$ is a neighborhood of the point $(a_1, a_2, \ldots, a_n)$. 
We observe that $R$ contains only a denumerable infinity of neighborhoods, while $R^*$ has a non-denumerable infinity of neighborhoods; moreover, that the transformation $g(x_1, x_2, \ldots, x_n) = (x_1', x_2', \ldots, x_n')$ $(x_i' = x_i; i = 1, 2, \ldots, n)$ maps no neighborhood of the point in $R$ with an irrational coordinate on a neighborhood in $R$. Nevertheless, it is easily verified that the transformation is topological, so that the two spaces are homeomorphic.

7.5.3) If $R$ contains no isolated points, while $R^*$ contains at least one isolated point $p^*$, $R$ and $R^*$ are not topologically equivalent.

For $\{p^*\}$ is an open set in $R^*$, and it cannot be mapped one-to-one on any open set in $R$; by (7.4), $R$ and $R^*$ are not homeomorphic.

(A point is an isolated point of its space if it constitutes a neighborhood.)

8. Subspace

[8.1] Let $R$ be a topological space and $R^*$ a subset of the points of $R$. We topologize $R^*$ by defining as the set of neighborhoods of $x$ in $R^*$ the sets of points $U_x^* = U_x \cap R^*$, where $U_x$ is any neighborhoods of $x$ in the space $R$. It is easily verified that the system of neighborhoods satisfies the Hausdorff axioms in [6.1]. $R^*$ is said to be a subspace of $R$.

[8.1.1] The relation $g(x) = x$ maps $R^*$ on $R$ continuously. For if $U_x$ is any neighborhood of the point $x$ in $R$, the neighborhood $U_x^* = U_x \cap R^*$ is mapped on $U_x$, and by [7.1] $g(x)$ is a continuous mapping.

[8.1.2] If $g$ maps $R$ into $R^*$ continuously and $g(R) \subset R^* \subset R^*$, then $g$ maps $R$ into $R^*$ continuously.

Again, this follows immediately from the particular way in which the neighborhoods of $R^*$ are chosen (cf. [8.1]).

[8.2] We say that a set $B$ in $R^*$ is open relative to $R^*$ if for every
point x of E there exists a neighborhood $U_x^* \subseteq E^*$ in $E^*$. $E^*$ is defined to be the set $\overline{E} \cap R^*$; i.e., the closure of E in $R$ with respect to $R^*$ is the set of all points in $R^*$ with the property that each of their neighborhoods in $R$ contains a point of $E$; since $E$ is in $R^*$, the set $E^*$ can also be defined as follows: a point belongs to $E^*$ if it is in $R$ and if each of its neighborhoods in $R^*$ contains a point of $E$.

$E$ is closed relative to $R^*$ if $E = \overline{E}$.

(8.2.1) If $E$ is open (closed) in $R$, $E \cap R^*$ is open (closed) relative to $R^*$, and $E \cap R^* \subseteq \overline{E} \cap R^*$. The assertion concerning open sets follows at once from the choice of neighborhoods in $R^*$, as does the statement on closures; from the latter follows the statement on closed sets.

(8.2.2) Every set $E^*$ in $R^*$ that is relatively open (closed) is the intersection of $R^*$ with an open (closed) set in $R$. If $E = \overline{E} = \overline{E} \cap R^*$, $E$ is the intersection of $R^*$ and $\overline{E}$. If $E$ is open with respect to $R^*$, its complement is closed, and is therefore the intersection of $R^*$ with a closed set $F$; it follows then that $E$ is the intersection of $R^*$ with the complement of $F$.

(8.3) Examples

(8.3.1) Let $R$ be the set of all real numbers $x$, and let the neighborhoods of $x$ be the sets of numbers $y$ ($|y-x|<\epsilon$). Let $R^*$ be the set of numbers $-\frac{1}{2} < x < \frac{1}{2}$. In this case, the subspace $R^*$ is homeomorphic with $R$; for the relation $g(x) = \tan x$ maps $R^*$ on $R$ topologically.

(8.3.2) Let $R$ be the set of real numbers, as in (8.3.1), and let $R^*$ be the subset of rational real numbers. The map $g(x) = x$ of $R^*$ on $R$ is not topological. For example, the set of points $0.3, 0.33, 0.333, \ldots$ is closed in $R^*$ but not in $R$. 
9. Connectedness

[9.1] A topological space $R$ is connected if it is not the sum of two non-null, non-intersecting closed subsets.

(9.1.1) We show that the following definition is equivalent to

[9.1]: A topological space $R$ is connected if it is not the sum of two non-null, non-intersecting open subsets.

For suppose $R$ is not connected. Then $R = A \cup B$, where $A$ and $B$ are both closed and $A \cap B = \emptyset$. Since the complement of $A$ is open, $B$ is open. Similarly, $A$ is open, since it is the complement of $B$. Therefore $R = A \cup B$ where $A$ and $B$ are both closed and $A \cap B = \emptyset$.

Similarly, we show that if $R = A \cup B$ where $A$ and $B$ are open and $A \cap B = \emptyset$, $A$ and $B$ are both closed.

[9.2] A set $E$ in a space $R$ is connected if it cannot be decomposed into the sum of two non-null, non-intersecting sets $A$ and $B$ which are such that $\overline{A} \cap \overline{B} \cap E$ is a null set.

(9.2.1) The sum $E$ of any aggregate of connected sets each of which contains the point $a$ is a connected set.

Suppose $E$ is not connected; then we can write $E = A \cup B$, where $\overline{A} \cap \overline{B} \cap E = \emptyset$ and $A$ and $B$ are not null. We may suppose that $a$ is in $A$; let $b$ be a point in $B$, and suppose that $P$ is one of the sets of $M$ which contain $b$. If we write $A' = A \cap P$, $B' = B \cap P$, $A'$ and $B'$ are two non-intersecting, non-null sets and $A' \cup B' = P$. By (6.3.1) $A' \subseteq \overline{A}$ and $B' \subseteq \overline{B}$. Therefore

$$\overline{A'} \cap \overline{B'} \cap P \subseteq \overline{A} \cap \overline{B} \cap E$$

and since the expression on the right by hypothesis is a null-set, $P$ is not connected, contrary to hypothesis.

[9.3] If $a$ is a point in $R$ and $K$ is the sum of all connected sets in $R$ that contain the point $a$, $K$ is the component of $a$ in the space $R$. 
It follows from (9.2.1) that K is connected. It follows at once from the definition that K is not a non-trivial subset of any other connected set containing the point a.

(9.3.1) The component K of the point a in R is a closed set. To prove this, it is clearly sufficient to show that K is connected, since this will imply that K is connected.

Suppose K is not connected. Then K = A ∪ B where A ∩ B ∩ K = 0 and A and B are non-null non-intersecting closed sets. If we write A' = A ∩ K, B' = B ∩ K, and suppose that a is contained in A', then B' is empty; for otherwise, K would not be connected, since then A' ∩ B' ∩ K ⊂ A ∩ B ∩ K = 0. Since B' is empty, we have K ⊂ A; moreover,

\[ A \cap B \cap K = K \cap B \cap K = K \cap B \cap K \]

and since B is not empty and is contained in K, the hypothesis that A ∩ B ∩ K = 0 is contradicted, and the theorem is proved.

(9.4) If R is connected, and if R can be mapped on R' continuously, then R' is connected. For if R' were not connected, it would be the sum of two non-null non-intersecting sets, and by (7.3.1) the same would be true of R, and R would not be connected, contrary to hypothesis.

9.4.1) It follows at once that connectedness is a topological property.

10. Regularity. The second axiom of countability

[10.1] A topological space R is regular if in addition to the Hausdorff axioms in [6.1] the neighborhood system of R satisfies the axiom of Victoria(8); Every neighborhood U_x of x contains a neighborhood V_x of x such that every point of the complement of V_x is interior to U_x.

(8) Victoria, L.
(10.1.1) \( R \) is regular if and only if every neighborhood \( U_x \) of \( x \) contains a neighborhood \( V_x \) of \( x \) such that \( V_x \subset U_x \).

Suppose that \( U_x \) contains a neighborhood \( V_x \) such that every point not in \( U_x \) is interior to the complement of \( V_x \). Clearly, no point of the complement of \( U_x \) is in \( V_x \); suppose then that such a point is in \( \overline{V}_x \); then each of its neighborhoods contains points of \( V \), contrary to hypothesis. Therefore \( \overline{V}_x \subset U_x \).

Suppose on the other hand that \( \overline{V}_x \subset U_x \), and that \( y \) is any point in the complement of \( U_x \). If every neighborhood of \( y \) contains points of \( V_x \), \( y \) is in \( \overline{V}_x \), which is impossible, since \( \overline{V}_x \) and the complement of \( U_x \) do not intersect.

(10.1.2) Regularity is a topological property. This follows immediately from the fact that inclusion in a set and the operation of closure are topologically invariant. (cf. (7.4)).

(10.2) If \( x \) and \( y \) are two distinct points in a regular space \( R \), there exist neighborhoods \( U_x \) and \( U_y \) that do not intersect(9). This is obvious from (iv) of \([6.1]\) and \([10.1]\).

(10.3) From example (7.5.2) it is clear that two topological spaces may be homeomorphic while considerable differences exist between their respective systems of neighborhoods. It is because certain conditions on the neighborhood system of the space \( R \) are sufficient, but not necessary for certain topological propositions to hold in \( R \), that we make the following definition:

Let \( R \) be any set of points with a certain system of neighborhoods, and let \( R' \) be the same set of points with the system of neighborhoods \( \Sigma' \); then if the map which associates with a point \( R \) the corresponding point in \( R' \) is homeomorphic, \( R \) is said to admit (or contain) the basis of neighborhoods \( \Sigma \). Also, if \( \Sigma' \) is the set

(9) In other words, Hausdorff's fourth axiom is satisfied in its stronger form (cf. footnote on [6.1]). That Vehors' regularity axiom is stronger than Hausdorff's fourth axiom is evident from example (10.9.1).
of neighborhoods in $\Sigma$ of a point in $R^*$, we say that the corresponding point in $R$ admits the basis $\Sigma'$.

Whenever a space $R$ admits a basis $\Sigma$ of neighborhoods, we shall assume, for brevity, that the neighborhoods in $R$ are the elements of the system $\Sigma$.

[10.4] $R$ satisfies Hausdorff's first axiom of countability if each point in $R$ admits a countable basis (we include this axiom for completeness only).

$R$ satisfies Hausdorff's second axiom of countability if it admits a basis containing not more than a countable set of neighborhoods.

(10.5) If $R$ is regular (see [10.1]), each of its subspaces $R^*$ is regular; if $R$ satisfies the second axiom of countability, so does each of its subspaces $R^*$.

Suppose that $R^*$ contains the point $x$, and that in $R$ the neighborhood $U_x$ contains a neighborhood $V_x$ such that each point of the complement of $U_x$ is interior to the complement of $V_x$.

$U_x^* = U_x \cap R^*$ and $V_x^* = V_x \cap R^*$ are non-empty neighborhoods of $x$ (they contain $x$). If $y$ is a point of the complement of $U_x^*$ in $R^*$, it is also a point of the complement of $V_x$ in $R$. It follows that $y$ as a point of $R$ has a neighborhood $U_y$ interior to the complement of $V_x$, and therefore $y$ as a point of $R^*$ has a neighborhood $U_y^* = U_y \cap R^*$ interior to the complement of $V_x^*$; i.e., $R^*$ is regular.

The second part of the theorem is even more obvious.

[10.6] We shall use the term $S$-space to denote a regular topological space satisfying the second axiom of countability.

(10.6.1) If $R$ is an $S$-space, each point $a$ in $R$ has a basis of neighborhoods $U_1, U_2, \ldots$ such that, for all positive $n$, $U_{n+1} \subseteq U_n$.

From the countable basis $\Sigma$ of $R$, we choose a neighborhood
U^{(1)}_x for each point x. From the countable neighborhoods
U^{(1)}_x, U^{(2)}_x, ..., containing the point x, we reject all except
those with the property that they either contain U^{(1)}_x or that
their closure is contained in U^{(1)}_x. The neighborhoods U^{(1)}_x,
V^{(2)}_x, V^{(3)}_x, V^{(4)}_x, ... we treat similarly with respect to V^{(2)}_x,
obtaining thus a sequence of neighborhoods
U^{(1)}_x, V^{(2)}_x, V^{(3)}_x, V^{(4)}_x, ...
Repeating this process indefinitely, we obtain the sequence of neighborhoods
U^{(1)}_x, V^{(2)}_x, U^{(3)}_x, V^{(4)}_x, ...
We see that if this is done for each point in R, the Hausdorff
axioms in [6.1] are satisfied by the newly obtained system \( \Sigma' \)
of neighborhoods:

1) holds by construction;
2) is trivially true;
3) and (iv) hold because they hold for the system
\( \Sigma \), and because \( \Sigma \) satisfies (ii) and R is regular.

It is readily seen from [7.1], [7.2], and [10.3] that \( \Sigma' \)
is a basis for R.

[10.7] A sequence of points \( a_1, a_2, ..., a_n, ... \) of an S-space R converges to
the point \( a \) of this space if for every neighborhood \( U_a \) there exists
an integer \( k \) such that \( U \) contains all the points \( a_n \) (\( n > k \)).

[10.7.1] A convergent sequence of points converges to one point
only; if more than a finite number of points of its set are distinct
from the point to which it converges, this point is the unique
limit point of the set. The proofs of these statements are too
familiar to need inclusion.

[10.7.2] If \( E \) is a set in an S-space R, the point \( a \) in R belongs
to \( E \) if and only if \( E \) contains a sequence converging to \( a \). We omit
(10.3) If \( \mathfrak{B} \) is a set in a topological space \( \mathfrak{B} \) satisfying the second axiom of countability, and if \( \mathfrak{A} \) is a set of open sets whose sum contains \( \mathfrak{P} \), then we can select from \( \mathfrak{A} \) a countable system \( \mathfrak{A}' \) of sets whose sum also contains \( \mathfrak{P} \). Let \( \Sigma \) be a countable system of neighborhoods in \( \mathfrak{B} \), and let \( \Sigma' = \{ U_1, U_2, \ldots \} \) be the set of all neighborhoods in \( \Sigma \) which are contained in one or more of the sets of \( \mathfrak{A} \). Let \( \{ G_1, G_2, \ldots \} \) be a system of sets from \( \mathfrak{A} \) such that \( G_i \supset U_i \) \((i = 1, 2, \ldots)\). It is clear that the system \( \{ G_1, G_2, \ldots \} \) is a countable system from \( \mathfrak{A} \), and that the sum of its sets contains \( \mathfrak{P} \).

(10.9) Examples

(10.9.1) Let \( \mathfrak{B} \) be the set of points of the Cartesian plane. For convenience we denote by \( U(x_0, y_0, r) \) the set of points \((x, y)\) for which \((x - x_0)^2 + (y - y_0)^2 < r^2\) and \( y \neq y_0 \) unless \( x = x_0 \). We shall topologize \( \mathfrak{B} \) in different manners and thus exhibit some of the intuitive ideas behind Victoria's axiom and Hausdorff's fourth axiom (cf. footnote on [6.1]).

1) If every set \( U(x_0, y_0, r) \) is a neighborhood of the point \((x_0, y_0)\), \( \mathfrak{B} \) is a non-regular space satisfying the Hausdorff axioms, even if the fourth axiom is given in its stronger form.

ii) If we require \( r \) to take on rational values only, we see that the space satisfies the first axiom of countability (cf. [10.4]). The topologizations (i) and (ii) are obviously equivalent.

iii) If we require \( x_0, y_0, \) and \( r \) to be rational and say that \( U(x_0, y_0, r) \) is a neighborhood of every point it contains, we
see that the space R satisfies the second axiom of countability.
Under the present topologization it is still not regular, since
the Victoria axiom is not satisfied for neighborhoods of \((x_0, y_0)\)
where \(x_0\) and \(y_0\) are rational. To see that the topologization under
(iii) is not equivalent with that under (i) and (ii), consider the
sequence of points \((x_n, y_n)\) \((n = 1, 2, \ldots; x = 1/n, y = \sqrt{2})\). Under
the topologization of (iii) this sequence has the limit point
\((0, \sqrt{2})\), and its points do not form a closed set. Under the
topologization of (i) and (ii), the sequence has no limit point,
and the set is therefore closed.

iv) If we require \(x_0, y_0, r\) to be rational, and say that
\(U(x_0, y_0, r)\) is a neighborhood of every point it contains except
\((x_0, y_0)\), a topological space is no longer defined. For if \(x_0\) and
\(y_0\) are rational numbers, the set \(U(x_0, y_0, r)\) is a neighborhood of
some point and contains the point \((x_0, y_0)\), but contains no neighbor-
hood of the point \((x_0, y_0)\). Therefore axiom (iii) of [6.1] is not
satisfied.

(10.9.2) Let \(R\) be the space of (6.6.4), in which every set \(U\) is a
neighborhood of \(x\) provided it contains \(x\) and all but a finite number
of points in \(R\). If \(R\) contains only a finite number of points, it is
a discrete space. If and only if \(R\) has not more than a denumerable
infinity of points, the second axiom of countability is satisfied.
If \(R\) has an infinity of points, the fourth Hausdorff axiom in its
stronger form is not satisfied, and therefore \(R\) is not a regular
space.

11. Compactness

[11.1] A set \(E\) in a topological space \(R\) is compact if every infinite
subset of \(E\) has at least one limit point in \(E\). \(R\) is a compact space
if the set \(R\) is compact. \(R\) is locally compact if each point in \(R\)
has a neighborhood the closure of which is compact.

(11.1.1) By (10.7.2), a necessary and sufficient condition for a set in a compact 3-space to be compact is that the set is closed.

(11.2) The intersection of a set \( \{F_1, F_2, \ldots\} \) \( (F_i \supseteq F_{i+1}, i = 1, 2, \ldots) \) of compact non-null sets in an 3-space is non-empty and compact.

If, for some \( i \), \( F_i = F_{i+1} = F_{i+2} = \ldots \), the theorem is obvious. Otherwise, we can select sets \( \{E_1, E_2, \ldots\} \) from the sets \( F_1, F_2, \ldots \) such that \( E_i \neq E_{i+1} \) and \( E_i \supseteq E_{i+1} \) for all \( i \). We choose \( a_i \) in \( E_i - E_{i+1} \), obtaining thus an infinite set of distinct points \( a_1, a_2, \ldots \) which must have at least one limit point. Since the sets \( F_i \) are closed, and since each of them contains all but a finite number of the set of points \( a_1, a_2, \ldots \), the limit points of the set \( \{a_1, a_2, \ldots\} \) are contained in the intersection of \( F_1, F_2, \ldots \), i.e. this intersection is not empty. Again, since the sets \( F_i \) are closed, their intersection, is closed by (6.4.3), and by (11.1.1) it is compact.

(11.2.1) If \( F \) is the intersection of the sets \( F_i \) in (11.2), and if \( G \) is an open set containing \( F \), then there exists a number \( k \) such that \( G \supseteq F_n \) (\( n > k \)). If we write \( E_n = F_n - G \), \( E_n \) is the intersection of the closed sets \( F_n \) and \( F_n \) is therefore closed and compact. Obviously \( E_n \supseteq E_{n+1} \) (\( n = 1, 2, \ldots\)), and by (11.2) the intersection \( E \) of the sets \( E_1, E_2, \ldots \) is non-null, unless one of the sets \( E_i \) is null. Since \( E = F - G = 0 \), this implies that \( E_i \) is a null set, for some \( i \), and the theorem is proved.

(11.3) If \( E \) is a compact set in an 3-space and \( \Omega \) a system of open sets whose sum contains \( E \), then the system \( \Omega \) contains a finite system of open sets whose sum contains \( E \).

By (10.8) \( \Omega \) contains a countable set of sets \( \{G_1, G_2, \ldots\} \) whose sum contains \( E \). If we write \( H_n = G_1 \cup G_2 \cup \ldots \cup G_n \) and \( F_n = E - H_n \), \( F_n \) is compact (\( E \) is closed and \( F_n \supseteq F_{n+1} \) for all \( n \)). The theorem is
proved if we show that, for some \( n \), \( F_n \) is empty. But if none of the sets \( F_n \) were empty, their intersection would by (11.2) not be empty, and the system \( \{G_1, G_2, \ldots \} \) would not cover the set \( E \), contrary to hypothesis.

(11.4) If \( f \) maps a compact space \( R \) on \( R' \) continuously, \( R' \) is compact.

Suppose that \( R' \) is not compact, so that \( R' \) contains an infinite subset \( N' \) without limit points in \( R' \). For each point \( x' \) of \( N' \) we select in \( R \) a point \( x \) such that \( f(x) = x' \). The set \( N \) of points thus obtained in \( R \) has at least one limit point \( a \). Let \( a' = f(a) \), and let \( U' \) be any neighborhood of \( a' \), and \( U \) a neighborhood of \( a \) such that \( f(U) \subseteq U' \). Since \( a \) is a limit point of \( N' \), \( U \) contains at least two distinct points \( p \) and \( q \) of \( N \). It follows from the selection of the points \( N' \) that \( f(p) \) and \( f(q) \) are distinct, so that \( U' \) contains at least one point of \( N' \) other than \( a' \). Therefore \( a' \) is a limit point of \( N' \) contrary to hypothesis.

(11.4.1) If the mapping \( f \) in (11.4) is also one-to-one and \( R' \) is an \( S \)-space, then \( f \) is a topological mapping.

We have to show that the inverse map \( f^{-1}(x') \) is also continuous, i.e., that the image by \( f(x) \) of every closed set \( F \) in \( R \) is closed (cf. (7.3.2)). But by (11.4.1) \( f \) is compact, and by (11.4) \( f(F) \) is compact and therefore closed.

(11.5) Every sequence of points in a compact \( S \)-space contains a convergent subsequence. This follows immediately from (10.7.2).

12. Continuous functions

[22.1] A real-valued function \( f(x) \) is defined on a topological space \( R \) provided to every point \( x \) in \( R \) is assigned exactly one real

(9) This section is not a necessary element in the logical structure of the present paper. We include it because the concept of continuous functions on a topological space is essential in the theory of representation of topological groups, a theory that we exclude here only because of the extreme limitations imposed by circumstances beyond our control.
number. The function is continuous at the point \( a \) if for every number \( \varepsilon > 0 \) there exists a neighborhood \( U_a \) such that for all \( x \) in \( U_a \), \( |f(x) - f(a)| < \varepsilon \); i.e., the function is continuous at all points of \( \mathbb{R} \) (we shall say continuous on \( \mathbb{R} \)) if it gives a continuous map (cf. [7.1]) of \( \mathbb{R} \) on the set of real numbers topologized as in (3.3.1).

(12.1.1) If \( f(x) \) is continuous on the connected space \( \mathbb{R} \), and if \( f(x) \) takes on the values \( a \) and \( b \), it takes on all intermediary values.

Otherwise, \( f(x) \) would give a continuous map of a connected space \( \mathbb{R} \) on a non-connected set (cf. [9.2]), contrary to (9.4).

(12.1.2) A real-valued function on a compact topological space \( \mathbb{R} \) is bounded and takes on its maximum and its minimum value.

For the set of values on which \( \mathbb{R} \) is mapped is by (11.4) compact, and therefore closed.

(12.2) **Urysohn's lemma**

If \( E \) and \( F \) are two closed non-intersecting sets in a compact \( T \)-space \( \mathbb{R} \), there exists a continuous function \( f(x) \) on \( \mathbb{R} \) such that \( f(x) = 0 \) in \( E \), \( f(x) = 1 \) in \( F \), and \( 0 \leq f(x) \leq 1 \) for all \( x \) in \( \mathbb{R} \).

We shall give the proof in three steps.

(12.2.1) If \( A \) and \( B \) are two closed non-intersecting sets in a compact \( T \)-space \( \mathbb{R} \), there exists an open set \( G \) in \( \mathbb{R} \) such that \( G \supset A \) and \( \overline{G} \cap \overline{B} = \emptyset \).

Since \( \mathbb{R} \) is regular, every point \( x \) in \( A \) has a neighborhood \( \overline{U} \) the closure of which does not intersect \( B \). The closed set \( A \) is compact, and by (11.3) we can select from the neighborhoods of points in \( A \) a finite set \( U_1, U_2, \ldots, U_k \) such that \( U_1 \cup U_2 \cup \ldots \cup U_k = \overline{G} \supset A \) and \( \overline{G} \) does not intersect \( B \).

(12.2.2) For every \( n \), we construct a system \( \sum \) of open sets
$G_n (r = \frac{1}{2}, \frac{2}{2^n}, \ldots \frac{2^n-1}{2^n})$ such that

1) $G_n \cup E$ and $\overline{G_n \cap F} = \emptyset$;

2) if $r^* < r$, $\overline{G_n} \subset G_n$.

The system $\Sigma_1$ contains only the set $G_2$. In (12.2.1) we constructed such a set. Suppose now that $\Sigma_n$ is constructed; we establish $\Sigma_{n+1}$ as follows: for $p = 2, 4, \ldots, 2^{n+1}-2$

we take for the set $G_\eta (r = \frac{p}{2^{n+1}})$ the set $G_\eta$ in $\Sigma_n$; for $p = 1, 3, \ldots, 2^{n+1}-1$, we construct, by the method of (12.2.1), the set $G_\eta (r = \frac{p-1}{2^{n+1}})$ from the two “neighboring” sets $\overline{G_\eta}$ and $R - G_\eta$ where $r_1 = \frac{p-1}{2^{n+1}}$, $r_2 = \frac{p+1}{2^{n+1}}$, and $G_0 = E$, $G_1 = R - F$.

(12.2.3) We are now ready to construct the function. Let $\Sigma$

be the system of all the sets $G$ established in the construction of the systems $\Sigma_1, \Sigma_2, \ldots$, together with the set $R$ which we denote by $G_1$ (12.2.2) notwithstanding.

For any point $x$ in $R$ we denote by $f(x)$ the greatest lower bound of all values of $r$ for which $x$ is contained in the set $G_r$ of the system $\Sigma$. It is easy to see that $f(x)$ satisfies the conditions imposed on it in (12.2).

13. Topological products

In this section, we show a development analogous to our treatment of the direct product of groups in section 5. Whererever we omit the proof of a proposition, a moment’s reflection will show that all is obvious.

[13.1] From two topological spaces $R$ and $S$ we construct a new topological space $T$, the **topological product** $T = R \times S$. Every pair of points $(x, y)$ ($x$ in $R$, $y$ in $S$) is a point of $T$. If $E$ and $F$ are sets in $R$ and $S$, respectively, we denote by $E \times F$ the set of all points $(x, y)$ ($x$ in $E$, $y$ in $F$). If $\Sigma$ and $\Sigma'$ are bases of neighborhoods in $R$ and $S$, respectively, we say that the set $W_{x,y} = U_x \times V_y$ is a neighborhood of $(x, y)$, where $U_x$ and $V_y$ are
neighborhoods of \( x \) and \( y \) in \( R \) and \( S \). It is easy to see that the system \( \Sigma'' \) of neighborhoods thus obtained satisfies the Hausdorff axioms of \( (5.1) \).

(13.2) If \( \Sigma \) and \( \Sigma' \) are replaced by topologically equivalent bases, \( \Sigma'' \) is replaced by a topologically equivalent basis.

For if \( \Sigma \) is replaced by an equivalent basis, the same is true for \( \Sigma'' \). A repetition of the argument with \( \Sigma' \) completes the proof.

(13.2.1) If \( G \) and \( H \) are open sets in \( R \) and \( S \), respectively, \( G \& H \) is an open set in \( T = R \& S \).

(13.2.2) If \( S \) and \( F \) are closed sets in \( R \) and \( S \), respectively, \( S \& F \) is a closed set in \( T \).

For if we write \( G = R - E, H = S - F \), then

\[
E \& F = (R - G) \& (S - H) = R \& S - (G \cup R \& H),
\]

and since \( G \cup R \& H \) is an open set by (13.2.1), the theorem is proved.

(13.2.3) If the second axiom of countability holds in \( R \) and \( S \), it holds in \( T \).

(13.2.4) If \( R \) and \( S \) are regular, \( T \) is regular. For if \( U_x \cdot U_y \) is any neighborhood of \( (x, y) \), \( x \) and \( y \) have neighborhoods \( V_x \) and \( V_y \), respectively, such that \( V_x \subset U_x \) and \( V_y \subset U_y \). If we write

\[
W = V_x \cdot V_y, \quad \overline{V_x} \cdot \overline{V_y}
\]

is a closed set and therefore \( \overline{W} \subset \overline{V_x} \cdot \overline{V_y} \); but

\[
\overline{V_x} \cdot \overline{V_y} \subset U_x \cdot U_y,
\]

and therefore \( \overline{W} \subset U_x \cdot U_y \), and therefore \( T \) is a regular space.

(13.3) If \( R \) and \( S \) are compact 3-spaces, \( T \) is a compact 3-space.

In (13.2.3) and (13.2.4) we have shown that \( T \) is an S-space. Now, if \( E \) is any infinite set in \( T \), we choose in \( E \) an infinite sequence of points \( e_1, e_2, \ldots \). If \( e_n = (a_n, b_n) \), it follows from the compactness of \( R \) and \( S \) that for some sequence \( n_1, n_2, \ldots \) each of the sequences \( a_1, a_2, \ldots \) and \( b_1, b_2, \ldots \) converges, say to...
points a and b, respectively. It is easy to see that \((a, b)\) is a
limit point of \(E\), and we have proved that \(T\) is compact.

(13.4) If \(A\) and \(B\) are locally compact \(S\)-spaces, \(T\) is a locally compact
\(S\)-space.

If \(a\) and \(b\) are points in \(A\) and \(B\), they have neighborhoods
\(U_a\) and \(U_b\) whose closures are compact. The method of (13.5) can
be applied to show that the closure of the neighborhood \(U_a \cdot U_b\) is
compact.

(13.5) If \(R_1, R_2, \ldots\) is a countable set of compact \(S\)-spaces, we
define their topological product \(T\): Any sequence \(x = \{x_1, x_2, \ldots\}\)
\((x_i\text{ in } R_i)\) is a point in \(T\). If \(U_i\) is a neighborhood of \(x_i\) in \(R_i\)
\((i = 1, 2, \ldots, n)\), we say that the set \(U(n) = U_1 \cdot U_2 \cdot \ldots \cdot U_n \cdot R_{n+1} \cdot R_{n+2} \ldots\)
in \(T\) is a neighborhood of \(x\).

It can be shown that the space \(T\) is an \(S\)-space. To demonstrate
that \(T\) is also compact, we can show, by the well-known "diagonal
process", that every infinite set \(B\) in \(T\) has at least one limit
point (cf. (19.3.4)).

(13.6) Example

If \(R^m\) and \(R^n\) are euclidean spaces of dimension \(m\) and \(n,
respectively, \(R^m \cdot R^n\) is homeomorphic with \(R^{m+n}\).
III. Topological Groups

We shall now combine the methods and results of the first two chapters by considering topologized sets of elements with a binary operation that satisfies the requirements for a group. From the fact that the object of our attention is a group, we shall deduce some of its topological properties; and the fact that it is a topological space will in turn shed light in its behavior as a group.

14. The concept of a topological group.

[14.1] A set $G$ of elements with a binary operation and a neighborhood system is a topological group provided

i) $G$ is an abstract group as defined in [1.1];

ii) $G$ is a topological space as defined in [6.1];

iii) the binary operation is continuous in the topological space; i.e., if $a$ and $b$ are any two elements of the set $G$, then for every neighborhood $W$ of the element $ab^{-1}$ there exist neighborhoods $U$ and $V$ of the elements $a$ and $b$ such that $UV^{-1} \subseteq W$ (notation as in [2.1]).

Condition iii) may be replaced by iii'):

a) If $W$ is any neighborhood of the element $ab$, there exist neighborhoods $U_a$ and $V_b$ such that $U_aV_b \subseteq W$.

b) If $V$ is any neighborhood of $a^{-1}$, $a$ has a neighborhood $U$ such that $U^{-1} \subseteq V$.

For suppose that iii) is satisfied; then, for $a = e$, there exist for every neighborhood $W$ of $b^{-1}$ two neighborhoods $U_e$ and
\(V_e\) such that \(U_eV_e^{-1} \subseteq W\); and since \(U_e\) contains the element \(e\),\n
\(V_e^{-1} \subseteq W\), i.e. (iii' b) is satisfied.

If now \(W\) is any neighborhood of \(ab\), there exist neighborhoods \(U_a\) and \(V_a^{-1}\) such that \(U_aV_a^{-1} \subseteq W\). But we have already shown that condition b) is satisfied; in other words that there exists a neighborhood \(V_e\) such that \(V_e^{-1} \subseteq V_e^{-1}\), i.e. \(V_e \subseteq V_e^{-1}\). It follows now that \(U_aV_e \subseteq W\), and therefore (iii' a) is satisfied.

Quite similarly we can show that if (iii'a) and (iii'b) hold, then (iii) holds.

(14.1.1) If (iii) is satisfied and the neighborhood system is replaced by another that is topologically equivalent, (iii) is still satisfied.

(14.2) If \(W\) is any neighborhood of \(c = a_1^{r_1} a_2^{r_2} \ldots a_n^{r_n}\) (\(r_i\) positive or negative integers), there exist neighborhoods

\(U_1, U_2, \ldots, U_n\) of the elements \(a_1, a_2, \ldots, a_n\) such that \(U_1U_2^{r_2} \ldots U_n^{r_n} \subseteq W\), where \(U_i = U_{i'}\) if \(a_i = a_{i'}\).

For simplicity in the proof, we assume that \(r_i = 1\) (\(i = 1, 2, \ldots, n\)).

By (iii'a) there exist neighborhoods \(U_1\) and \(U_2, U_3, \ldots, U_n\) of \(a_1\) and \(a_2, a_3, \ldots, a_n\) such that \(U_1U_2^{r_2} \ldots U_n^{r_n} \subseteq W\). Again there are neighborhoods

\(U_1, U_2, U_3, \ldots, U_n\) of \(a_1\) and \(a_2, a_3, \ldots, a_n\) such that

\(U_1U_2^{r_2} \ldots U_n^{r_n} \subseteq W\).

A finite number of repetitions of this argument prove the proposition. It is clear that if \(r_i\) is allowed to take on values other than unity, the same argument applies, except that (iii'b) must be invoked if \(r_i\) is negative.

(14.2.1) If \(a\) is any fixed element in the topological group \(G\), the one-to-one mappings \(f(x) = xa, f'(x) = ax,\) and \(f''(x) = x^{-1}\) of \(G\) on itself are topological.

In the case of \(f(x)\), it is clear how the continuity of \(f(x)\)

\(xa\) and its inverse \(f^{-1}(x) = xa^{-1}\) follows at once from (iii'a) of
[14.1] Similar proofs hold for $f'(x)$ and $f''(x)$.

(14.2.2) If $F$ is a closed set and $a$ any element in $G$, the sets $Fa$, $aF$, and $F^{-1}$ are closed, by (14.2.1).

(14.2.3) If $F$ is an open set and $K$ any arbitrary set in $G$, $FK$, $KF$, and $F^{-1}$ are open sets. For by (14.2.1) any set of the form $Fa$ is open, so that $FK$, $KF$, and $F^{-1}$ are sums of open sets, is also open.

Similar arguments apply to $KF$ and $F^{-1}$.

(14.3) If $p$ and $q$ are any two elements in $G$, there exists a homomorphic map (e.g., $f(x) \mapsto p^{-1}q$) of $G$ on itself which maps $p$ on $q$. We express this by saying that a topological group is a homogeneous space.

(14.3.1) It follows from the homogeneity of $G$ that if one element of $G$ has a neighborhood for which a condition $P$ is satisfied, then condition $P$ is locally satisfied throughout $G$. For example, if some neighborhood of the identity $e$ has a compact closure, then $G$ is locally compact.

(14.3.2) A topological group $G$ is discrete, (i.e. it contains no limit elements) if and only if the identity element of $G$ is a neighborhood of itself.

(14.4) The topological space of a topological group is regular it will follow that if the topological space of a topological group satisfies the second axiom of countability, it is an $S$-space.

If $U$ is any neighborhood of the identity $e$, it follows from (14.2) and from the relation $e^{-1} = e$ that $e$ has a neighborhood $V$ such that $V^{-1} \subseteq U$. We shall show that $V \subseteq U$, and by (10.1.1) and (14.3.1) the theorem will be proved.

If $p$ is a point of $V$, every neighborhood of $p$ contains points of $V$. By (14.2.3) the set $pV$ containing $p$ is open and therefore contains a neighborhood of $p$, so that $pV$ contains a
point $a$ of $V$; i.e., $V$ contains a point $b$ such that $pb = a$.

We have then $p = ab^{-1}$, and therefore $p$ is contained in $W^{-1} \subset U$.

(14.5) If the topological group $G$ satisfies the second axiom of countability, and if $P$ and $Q$ are compact sets in $G$, then $PQ$ is compact.

By (14.4), $G$ is an $S$-space. If $a_n = a b_n$ ($n = 1, 2, \ldots$), we can find a sequence $n_1, n_2, \ldots$ such that $a_{n_1}, a_{n_2}, \ldots$ converges to some point $a$ (by (11.5)). The sequence $b_{n_1}, b_{n_2}, \ldots$ has a limit point $b$. Therefore $a_{n_1}, a_{n_2}, \ldots$ has the limit point $ab$, by (iii'b) of (14.1).

15. The topologization of groups

In this section we shall construct, for a topological group $G$, a basis of neighborhoods for $G$ from the basis of the identity $e$. Conversely, we shall show how an abstract group $G$ can be topologized if it is possible to find a system of neighborhoods of $e$ that satisfies certain conditions which every basis of $e$ in a topologized group must satisfy.

[15.1] A subset $E$ of a topologized space $R$ is everywhere dense in $R$ if $\overline{E} = R$, i.e. if every neighborhood in $R$ (cf. (6.2.2)), contains points of $E$.

(15.2) If $G$ is a topological group and $\Sigma^*$ a basis of neighborhoods of its identity $e$ (cf. [10.3]), then

i) the intersection of all neighborhoods in $\Sigma^*$ contains only the element $e$;

ii) if $U$ and $V$ are two neighborhoods in $\Sigma^*$, $U \cap V$ contains a third neighborhood in $\Sigma^*$;

iii) for every neighborhood $U$ in $\Sigma^*$ there exists a neighborhood $V$ in $\Sigma^*$ such that $VV^{-1} \subset U$. 
iv) for every neighborhood $U$ in $\Sigma^*$ and every element $a$ in $U$, $\Sigma^*$ contains a neighborhood $V$ such that $Va \subseteq U$.

v) for every neighborhood $U$ in $\Sigma^*$ and every element $a$ in $G$, $\Sigma^*$ contains a neighborhood $V$ such that $a^{-1}Va \subseteq U$.

We see immediately that (i) is a consequence of the Hausdorff axioms (i) and (iv) in [6.1]. (ii) is identical with (ii) of [6.1]. (iii) follows from (14.2), as in (14.4). (iv) follows from (iii+a) of [14.1]. (v) can be deduced from (14.2) by taking $a_1 = a^{-1}$, $a_2 = e$, $a_3 = a$.

Q5.2.1 If $\Sigma^*$ is a basis of the element $e$ in the topological group $G$, and the set $M$ is everywhere dense in $G$; if is the system of sets of the form $Ux$ ($U$ in $\Sigma^*$, $x$ in $M$); and if each set $Ux$ is assigned as a neighborhood to every point it contains, then $\Sigma$ is a basis for the topological group $G$.

We have to show that the system $\Sigma$ satisfies the conditions of [6.1], and that every set which is open under the topologization $\Sigma'$ is an open set in $G$, and conversely.

Suppose that $a$ is a point in an arbitrary open set $\mathcal{E}$ in $G$. Then $Ea^{-1}$ is an open set containing $a$, and by (14.2) the set $\Sigma^*$ contains a neighborhood $U$ such that $UU^{-1} \subseteq Ea^{-1}$. It is easily seen that since $E$ is everywhere dense, $aE^{-1}$ is everywhere dense and therefore contains an element $d$ in $U$.

d$^{-1}a$ is contained in $M$, and therefore the set $Ud^{-1}a$ is one of the sets in $\Sigma$. But also $UU^{-1} \subseteq Ea^{-1}$, and since $U$ contains the element $d$, this implies $Ud^{-1}a \subseteq Ea^{-1}$, i.e. $Ud^{-1}a \subseteq E$. Finally, $Ud^{-1}$ contains $e$, since $d$ is in $U$, and therefore $a$ is in $Ud^{-1}a$; i.e., the point $a$ has a neighborhood $Ud^{-1}a$ contained in $E$. We have thus
shown that $E$ is open under the topologization $\Sigma$.

Conversely, it follows from (14.2.3) that all sets of the system $\Sigma$ are open sets in the topological space $G$. It follows that if $E$ is an open set under the topologization $\Sigma$, it is an open set of the space $G$.

Moreover, since $M$ is everywhere dense, the first of the Hausdorff axioms in (6.1) is satisfied by the system $\Sigma$. Since the remaining three axioms are statements requiring the existence of certain neighborhoods relative to given neighborhoods, and since every open set $E$ of $G$ is an open set under the topologization $\Sigma$ and all sets of the system $\Sigma$ are open sets in $G$, the proof is complete.

(15.3) Let $G$ be an abstract group and $\Sigma^*$ a system of sets of $G$ satisfying the five conditions in (15.2); then $G$ is a topological group if $G$ assigns to each point $x$ as neighborhoods the sets $U_x = U_x (U \text{ in } \Sigma^*)$. Moreover, the system $\Sigma$ of neighborhoods thus obtained gives the only topologization of $G$ under which $\Sigma^*$ is a basis of $G$.

We have to prove that the system $\Sigma$ satisfies the Hausdorff axioms of (6.1).

(i) is obviously satisfied. (ii) follows from (ii) of (15.2). (iii) follows from (iv) of (15.2). (iv) can be deduced from (i) of (15.2); for if $y$ is any point other than $x$ in a neighborhood $U_x$, there exists in $\Sigma^*$ a set $V$ that does not contain the point $yx^{-1}$. The neighborhood $V_x$ of $x$ therefore does not contain the point $yx^{-1}z$, and (iv) is satisfied. We have therefore proved that under the system $\Sigma$ $G$ is a topological space.

We show next that condition (iii) of (14.1) is satisfied in the space $G$: i.e. that if $a$ and $b$ are any two elements of $G$ and $W$ is any neighborhood of $ab^{-1}$ there exist neighborhoods $U$ and $V$
of a and b such that \( UV^{-1} \subset W \).

Let \( c = ab^{-1} \) and let \( Wc \) be a neighborhood of \( ab^{-1} \). By

(iii) of (15.2) there exists a set \( U \) in \( Z^* \) such that \( UW^{-1} \subset W \),
and by (v) of (15.2) there exists a set \( V \) in \( Z^* \) such that
\( ab^{-1}Vba^{-1} \subset U \). This implies \( ab^{-1}V^{-1}U^{-1}ab^{-1} \) and therefore
\( Ua(Vb)^{-1} = Uab^{-1}V^{-1}C \subset ab^{-1}W \).

It remains to show that \( \Sigma \) gives the only topologization
under which \( \Sigma * \) is a basis of the identity element \( e \). Suppose
that \( S \) is any system of neighborhoods in which the neighborhoods
of \( e \) can be replaced by those of the system \( \Sigma * \). Then, if \( S' \) is
the system of sets of the form \( Ux (U \in \Sigma^*, x \in G) \) and if every
set in \( S' \) is assigned as neighborhood to each point it contains, the
topological space thus obtained is not distinct from that under \( S \),
by (15.2.1); but by (7.2.2) it is not distinct from the space
obtained under the system \( \Sigma \), and the theorem is proved.

(15.4) Examples

(15.4.1) Let \( G^n \) be the additive group of vectors in the n-
dimensional euclidean vector space. Clearly the null vector is the
identity element. If its neighborhoods are the sets \( U_x \) of all
non-null vectors whose magnitude is less than \( r \), \( G^n \) is topologized.

(15.4.2) The set of non-singular square matrices of order \( n \) can be
topologized: If \( \Phi_k \) is the set of all matrices whose elements do not
exceed \( \frac{1}{k} \) in absolute value, and if \( U_k \) is the set of matrices of
the form \( e + a \), where \( a \) is in \( \Phi_k \) and \( e \) is the identity matrix, then the
system of sets \( U_k \) satisfies all conditions of (15.3).


[16.1] If \( G \) is a topological group and \( H \) is a set of elements of \( G \)
which is a subgroup of the abstract group \( G \) and a closed set in the
topological space \( G \), then \( H \) is a (normal) subgroup of the topological
group $G$. In other words, a subgroup of the abstract group $G$ is a subgroup of the topological group provided it is a closed set in the topological space $G$.

(16.2) If $H$ is a subset of the topological group $G$ and a subgroup of the abstract group $G$, and if $H$ is topologized as a subspace of the space $G$, then $H$ is a topological group.

We need only prove that the group operations in $H$ are continuous in the topological space $H$, i.e. that condition (iii) of [14.1] is satisfied.

If $a$ and $b$ are any two elements in $H$ and $ab^{-1} = c$, and if $W$ is a neighborhood of $c$ in the space $H$, there exists a neighborhood $W$ of $c$ in the space $G$ such that $W = H \cap W$. $a$ and $b$ have neighborhoods $U$ and $V$ in $G$ such that $UV^{-1} \subseteq W$. If we write $U' = H \cap U$, $V' = H \cap V$, we have $U'V'^{-1} \subseteq W'$, and the theorem is proved.

(16.3) If $H$ is a subgroup of a compact (locally compact) topological group $G$, $H$ is compact (locally compact).

If $G$ is compact, $H$ is compact because it is closed (11.1.1). If $G$ is locally compact and $a$ is an element of $H$, then $a$ has a neighborhood $U$ in $G$ such that $U$ is compact. If we write $U' = U \cap H$, then $U' \subseteq H$, since $H$ is closed relative to $G$. The set $U'$ is a neighborhood of $a$ in the space $H$, and the set $U'$ is compact, since $U' \subseteq U$. Therefore $H$ is locally compact.

(16.4) If $H$ is a subgroup of a topological group $G$, and $G/H$ the totality of all right cosets of $H$ in $G$, and if $\Sigma$ is a basis of neighborhoods $U$ for $G$, we construct a system $\Sigma'$ of neighborhoods for the set $G/H$ by choosing as neighborhoods $U'$ the sets of all cosets of the form $Hx$, where $x$ is any element in one of the neighborhoods $U$ in the system $\Sigma$. The topological space $G/H$ thus obtained we call the space of right cosets of $H$ in $G$. Analogously we define the space of left cosets $G/H$. It is easy to see that the
neighborhood system $\Sigma$ satisfied the Hausdorff axioms of \([6.1]\).

\((16.4.1)\) If $H$ is a subgroup of a compact (locally compact) topological group $G$, then the space of cosets $G/H$ is compact (locally compact).

For let $A_1$, $A_2$, ... be any sequence of elements in $G/H$. Then $A_n$ is of the form $Ha_n$. If $G$ is compact, the sequence $a_1$, $a_2$, ... has a subsequence $a_{n_1}$, $a_{n_2}$, ... converging to a limit point $a$.

The subsequence $A_{n_1}$, $A_{n_2}$, ... converges to $A=Ha$, since the neighborhoods of $A$ are sets of cosets of the form $Hx$ ($x$ in some neighborhood $U$ of $a$).

The proof for the case of local compactness is similar.

\((16.5)\) A mapping $f$ of a topological space $R$ on a topological space $R'$ is open if, for every open set $E$ in $R$, $f(E)$ is an open set in $R'$. Clearly, a mapping $f$ is open if and only if for every point $a$ and every neighborhood $U$ of $a$ in $R$ the point $a'=f(a)$ has a neighborhood $V'$ such that $V' \subseteq f(U)$. The necessity of the condition is immediate. To see the sufficiency, suppose that the condition is satisfied, and that $E$ is an open set in $R$. If $a$ is a point in $E$, $E$ contains a neighborhood $U$ of $a$, and by hypothesis $a'=f(a)$ has a neighborhood $V'$ such that $V' \subseteq f(U) \subseteq f(E)$. It follows that $f(E)$ is an open set.

It will be evident in section 17 that the concept of open mappings is very important in the topology of groups. A homomorphism cannot be a topological mapping, except in the special case where it is an isomorphism. In an open continuous homomorphism we shall have a mapping which "would be topological if it were one-to-one."

\((16.5.1)\) If $H$ is a subgroup of a topological group $G$ and $G/H$ the space of cosets of $H$, we call the natural mapping of $G$ on $G/H$ the mapping which associates with each element $x$ in $G$ the element
X \in \mathfrak{f}(x) \text{ which is the coset containing the element } x. \text{ We show that this mapping is continuous and open.}

Suppose that G/H is the space of right cosets. If a is an element of G and we write A = Na, then \( f(a) = A \). If \( U^* \) is a neighborhood of A, \( U^* \) is composed of all cosets of the form \( Hx \), where \( x \) belongs to some neighborhood \( U \) of a. By (14.2.3) \( HU \) is an open set in \( G \), and therefore it contains some neighborhood \( V \) of a. We have then \( f(V) \subset U \), and therefore \( f \) is continuous. That \( f \) is an open mapping is obvious from the choice of neighborhoods in the space \( G/H \).

(16.6) If \( N \) is a normal subgroup of a topological group \( G \), then the set of cosets \( G/N \) with the topology assigned in [16.4] and the binary operation defined in [2.7] is called the \textbf{factor group} of the topological group \( G \) by the normal subgroup \( N \). We prove that the factor group is a topological group.

If \( A \) and \( B \) are two elements of \( G/N \), and if \( C = AB^{-1} \) and \( \mathbb{W}^* \) is a neighborhood of \( G \), \( \mathbb{W}^* \) is a set of cosets of the form \( Hx \), where \( x \) is any element in a neighborhood \( \mathbb{W} \) of some element \( c \). Now if \( b \) is any element in \( B \) and \( a \) is the element \( cb \), then \( a \) is in \( A \). There exist neighborhoods \( U \) and \( V \) of \( a \) and \( b \) such that \( UV^{-1} \subset \mathbb{W} \). If \( U^* \) and \( V^* \) are the corresponding neighborhoods of \( A \) and \( B \), with elements of the form \( Hx, Ny \) (\( x \) in \( U \), \( y \) in \( V \)), we have

\[
Nx(Ny)^{-1} = Nxy^{-1}N^{-1} = NN^{-1}xy^{-1} = Nxxy^{-1} \subset \mathbb{W}.
\]

Therefore \( U^*V^*-1 \subset \mathbb{W}^* \), and condition (iii) of [14.4] is satisfied.

(15.6.1) If \( G \) is a topological group satisfying the second axiom of countability, then every factor group \( G/N \) satisfies this axiom.

(15.6.2) If a topological group \( G \) is compact (locally compact), each of its factor groups \( G/N \) is compact (locally compact).

If \( G \) is compact, it follows from the fact that \( G \) can be mapped
If $G$ is compact, its identity $e$ has a neighborhood $U$ such that $U$ is compact. We apply the natural mapping $f$ of $(16.5.1)$; since this is open, $f(U) = U^*$ is an open set in $G/N$, and since $f$ is continuous, $f(U)$ is compact. Since $U$ is a neighborhood of $e$, $N$ is an element of $U^*$, and since by $(14.4)$ $G/N$ is a regular space, the element $N$ has a neighborhood $V^*$ such that $V^* \subseteq U^* \subseteq f(U)$. $V^*$ is closed and $f(U)$ is compact. Therefore $V^*$ is compact, and since $G/N$ is a homogeneous space, $G/N$ is locally compact.

(16.6.3) If $N$ is a normal subgroup of a topological group $G$, and if $N$ and $G/N$ are both compact and $G$ satisfies the second axiom of countability, then $G$ is compact.

We denote by $G^*$ the factor group $G/N$ and by $f$ the natural mapping of $G$ on $G^*$ as described in $(16.5.1)$, and we write $a^*_n = f(a_n)$, where $a_1, a_2, \ldots$ is an infinite sequence of points in $G$. Since $a_1^*, a_2^*, \ldots$ has convergent subsequence, we shall suppose for simplicity that the sequence itself converges to a point $a^*$. 

By hypothesis, the identity $e$ in $G$ has a countable basis of neighborhoods, $U_1, U_2, \ldots$. By $(16.6.1)$, we may suppose that this sequence of neighborhoods is monotonic decreasing, i.e. $U_n \supseteq U_{n+1}$ for all $n$. If we write $U_n^* = f(U_n)$, the sequence $U_1^*, U_2^*, \ldots$ is monotonic decreasing and forms a basis of neighborhoods for $a^*$ in $G^*$. Since the sequence $a_1^*, a_2^*, \ldots$ converges to $a^*$, we can replace it by a subsequence so that $a_1^* a_n a_n^{-1}$ is in $U_n^*$. Now let $a^*$ be an inverse image in $G$ of $a^*$, and $b_n$ an inverse image in $U_n^*$ of $a^* a_n a_n^{-1}$; and write $a_n' = b_n a_n$. Then $f(a_n') = f(b_n) f(a_n') = a_n^* a_n^{-1} a_n^* = f(a_n)$. Moreover, since by $(16.5.1)$ $f$ is a continuous mapping, $a_1', a_2', \ldots$ converges to $a'$. 

If we now write $c_n = a_n a_n^{-1}$, $c_n$ belongs to $N$, since $f(a_n') = f(a_n)$, i.e. since $a_n'$ and $a_n$ belong to the same coset. Since $N$
is compact, we can select from \( c_1, c_2, \ldots \) a convergent
subsequence \( c_{n_1}, c_{n_2}, \ldots \). Since \( a_{n_1}', a_{n_2}', \ldots \) also converges,
the sequence \( c_{n_1}a_{n_1}', c_{n_2}a_{n_2}', \ldots \) converges; but this sequence
can be written \( a_{n_1}, a_{n_2}, \ldots \), and the theorem is proved.

\[ \text{(16.7)} \] A topological group is **simple** if each of its normal subgroup
is either discrete or coincides with \( G \).

\[ \text{(16.8)} \] If \( H \) is a (normal) subgroup of the abstract group \( G \), then
\( \bar{H} \) is a (normal) subgroup of the topological group \( G \).

By (2.2.1), \( \bar{H} \) is a group provided we can show that if \( a \) and \( b \)
are in \( \bar{H} \), then \( ab^{-1} \) is in \( \bar{H} \). If \( W \) is a neighborhood of \( ab^{-1} \), then
there exist neighborhoods \( U \) and \( V \) of \( a \) and \( b \) such that \( U^{-1}W \subset W \). But there
exist elements \( x \) and \( y \) in \( H \) belonging to \( U \) and \( V \), respectively, and
therefore \( xy^{-1} \) belongs to \( H \) and to \( W \). This shows that \( ab^{-1} \) belongs
to \( \bar{H} \). Since \( \bar{H} \) is closed in \( G \), it follows that \( \bar{H} \) is a subgroup of
the topological group \( G \).

Suppose now that \( H \) is normal, that \( a \) is an element of \( \bar{H} \) and
c an element of \( G \). If \( V \) is a neighborhood of \( c^{-1}ac \), a has by
(14.2) a neighborhood \( U \) such that \( c^{-1}Uc \subset V \). Since \( H \) is normal,
c^{-1}xc is in \( H \). Since \( V \) is an arbitrary neighborhood of \( c^{-1}ac \),
and \( c^{-1}xc \) is in \( V \), \( c^{-1}ac \) is in \( \bar{H} \), and \( \bar{H} \) is a normal subgroup of
the topological group \( G \).

\[ \text{(16.9)} \] Example

If \( G \) is the additive group of real numbers, topologized in the
obvious manner, \( G \) is a simple group. For either \( H \) contains
elements arbitrarily near the element \( 0 \), and is therefore everywhere
dense in \( G \); by definition of subgroups, \( \bar{H} = H \), and therefore \( H = G \).
Or else \( H \) contains a least positive number \( h \), and it is obvious that
\( H \) is composed of the elements \( nh \) and none others, where \( n \) is an
integer; clearly \( H \) is then discrete.
17. Isomorphism, Automorphism, Homomorphism.

Just as in the theory of abstract groups we do not distinguish between two groups with the same algebraic structure, and as in topology we do not consider differences between two topological spaces that can be mapped on each other homeomorphically, we establish classes of topological groups the members of which we regard as equivalent.

[17.1] A mapping $f$ of a topological group $G'$ is isomorphic if $f$ maps the abstract group $G$ on the abstract group $G'$ isomorphically and the topological space $G$ on the topological space $G'$ homeomorphically. If an isomorphism between the topological groups $G$ and $G'$ exists, we say that the two topological groups are isomorphic. An isomorphic mapping $f$ of a topological group on itself is an automorphism.

For brevity, we shall sometimes refer to a relation which maps an abstract group isomorphically on another as an algebraic isomorphism, in distinction to a topological isomorphism; the latter expression shall always denote a relation satisfying the conditions of the present paragraph.

[17.2] A mapping $g$ of a topological group $G$ into a topological group $G'$ is homomorphic if $g$ is an homomorphic mapping of the abstract group $G$ into the abstract group $G'$ and a continuous mapping of the topological space $G$ into the topological space $G'$. A homomorphic mapping $g$ of a topological group $G$ into a topological group $G'$ is open if it is an open mapping of the topological space $G$ into the topological space $G'$.

[17.3] If $G$ and $G'$ are two topological groups and $g$ is an homomorphic mapping of the abstract group $G$ into the abstract group $G'$, then

1) a sufficient condition for $g$ to be a continuous
mapping of the topological space $G$ into the topological space $G^*$ is that for every neighborhood $U^*$ of $e^*$ in $G^*$ the identity $e$ in $G$ have a neighborhood $U$ such that $g(U) \subseteq U^*$.

ii) a sufficient condition for the mapping of the topological space $G$ into the topological space $G^*$ to be open is that for every neighborhood $V$ of $e$ in $G$ the identity $e^*$ in $G^*$ have a neighborhood $V^*$ such that $g(V) \subseteq V^*$.

Suppose the condition in (i) is satisfied. If $a$ is an element in $G$, and if $g(a) = a^*$ and $U^*$ is a neighborhood of $a^*$, then $U^*a^{-1}$ contains a neighborhood of the identity $e^*$, and therefore there exists a neighborhood $U$ of the identity $e$ in $G$ such that $g(U) \subseteq U^*a^{-1}$. $U'a$ contains a neighborhood $U'$ of $a$, and $g(U) \subseteq g(U')g(a)U^*a^{-1}a^* = U'$, and $g$ is a continuous mapping.

The proof for (ii) is analogous.

[17.4] If $G$ is a topological group, $N$ one of its normal subgroups and $G/N$ the corresponding factor group, we define the natural homomorphic mapping $g$ of the topological group $G$ on its factor group $G/N$ by associating with every element $x$ in $G$ that coset $X$ of $N$ which contains $x$, i.e. by writing $g(x) = xN$.

In (3.4) we showed that the mapping just described is an homomorphism of the abstract group $G$ on the abstract group $G/N$, and in (16.5.1) it was shown that the mapping is continuous and open.

(17.4.1) If $g$ is an open homomorphic mapping of a topological group $G$ on a topological group $G^*$, and if $K$ is kernel of the homomorphism, then $K$ is a normal subgroup of the topological group $G$, and the topological group $G^*$ is isomorphic with $G/K$.

In (3.3) we proved that $N$ is a normal subgroup of the
abstract group $G$. Since $g$ is a continuous mapping and $N$ is composed of all elements which are mapped on the single element $e$, $N$ is closed in $G$, and therefore it is a normal subgroup of the topological group $G$.

Now let $x^r$ be an element of $G^r$, and $X$ the totality of all inverse images in $G$ of $x^r$. In (3.3) it was proved that $X$ is a coset of $N$ in the group $G$, and that the mapping determined by $f(x^r) = x$ is an isomorphic mapping of the abstract group $G/N$. Since $f$ is a one-to-one mapping of the space $G^r$ on the space $G/N$, the theorem will be proved if we show that $f$ is bicontinuous.

Suppose that $a^r$ is an element of $G^r$, $f(a^r) = A$, and $U^r$ is a neighborhood of $A$ in $G/N$. By [16.4], $U^r$ is the set of all cosets of the form $Hx$ where $x$ belongs to some fixed neighborhood $U$ in $G$, and where $U$ contains an element $a$ such that $A = Ha$. Since the mapping $g$ is open and $g(a) = a^r$, $a^r$ has a neighborhood $V^r$ which is contained in $g(U)$. Now let $x^r$ be any point in $V^r$; then $U$ contains an element $x$ such that $g(x) = x^r$. We have then $f(x^r) = Nx$; and since $Nx$ is an element in $U^r$, $f$ is a continuous mapping.

It remains to be proved that $f^{-1}$ is also a continuous mapping. Let $A = Ha$ be an element in $G/N$, and let $U^r$ be a neighborhood of $a^r = f^{-1}(A)$. Since the mapping $g$ is continuous and $g(a) = a^r$, the element $a$ has a neighborhood $V$ such that $g(V) \subseteq U^r$. If $V$ is the neighborhood of $A$ which is composed of all cosets of the form $Hx$ ($x$ in $V$), then $f^{-1}(V) \subseteq U^r$.

17.5) If the two topological groups $G$ and $G^r$ are locally compact and satisfy the second axiom of countability, then every homo-
morphic mapping $g$ of the topological group $G$ on the topological group $G^*$ is open. It will follow that the result of (17.4.1) applies to every homomorphism between the two topological groups.

We prove first that if $W$ is a neighborhood in $G$, $g(W)$ in $G^*$ contains a neighborhood $W^*$.

Since the topological space $G$ is locally compact and regular (cf. (14.4)), there exists a neighborhood $V$ such that $V$ is compact and $V \subseteq W$. Since $G$ satisfies the second axiom of countability, it follows from (10.8) that from the system of sets of the form $V x (x \in G)$ we can select a countable system covering $G$; in other words, there exists a sequence of points $a_1, a_2, \ldots$ such that the system of open sets $V a_1, V a_2, \ldots$ covers $G$. If we write $g(V a_n) = F_n$, we can apply the proof in (11.4) to show that $F_n$ is also compact. Since $G^*$ is an $S$-space, it follows from (11.1.1) that $F_n$ is closed. Obviously, the system $F_1, F_2, \ldots$ covers the space $G^*$.

Suppose now, if possible, that none of the sets $F_n$ contains a neighborhood. Since $G^*$ is locally compact, it contains at least one neighborhood $V^*$ with a compact closure. Since $F_1$ is closed and contains no neighborhood, $V^*$ contains a point $b_1$ which together with a neighborhood $V_1$ does not belong to $F_1$; since $G^*$ is regular, we may by (10.6.1) suppose that $V_1 \subseteq V^*$ and $V_1$ does not belong to $F_1$. Similarly, $V_1$ contains a point $b_2$ with a neighborhood $V_2$ such that $V_2 \subseteq V_1$ and $V_2$ does not intersect $F_2$. Continuing this process, we construct a sequence $V_1, V_2, \ldots$ with $\overline{V_n} \subseteq V_{n+1}$ such that each of the sets $\overline{V_n}$ is compact and has no points in $F_n$. By (11.2), the intersection of these sets $\overline{V_n}$ is not empty, and the sequence of sets $F_1, F_2, \ldots$ does not cover $G^*$.
It follows that at least one set $F_k$ contains a neighborhood. By (14.2.3), the set $g(V) = F_k G(a_k^{-1})$ also contains a neighborhood. But $V \subseteq U$, and therefore $g(U)$ contains a neighborhood $W^a$.

If $U$ is a neighborhood of $a$ in $G$, $a$ has a neighborhood $W$ such that $W^{a^{-1}} \subseteq U$ (cf. (14.2)) and observe that $a = a a^{-1}$. We have proved that $g(W)$ contains a neighborhood $W^a$. If $q$ is a point of $W^a$ and $p$ a point of $W$ such that $g(p) = q$, then $W q^{-1}$ is a neighborhood of $a$, and clearly $W q^{-1}$ is contained in $U$. Moreover, $W q^{-1}$ is a neighborhood of $a^*$ in $G^*$, and we have now $g(U) = g(W q^{-1}) = W^a q^{-1}$, and $g$ is an open mapping. q.e.d.

(27.6) If $G$ and $G^*$ are topological groups, and if $f$ is an open homomorphism mapping of the topological group $G$ on $G^*$ with the kernel $K^*$, there exists a one-to-one correspondence between the subgroups of $G^*$ and those subgroups of $G$ that contain $K^*$. Moreover, this one-to-one correspondence can be chosen so that normal subgroups correspond to each other and their respective factor groups $G/K$ and $G^*/K^*$ are isomorphic.

If $H^*$ is a subgroup of the topological group $G^*$, we associate with it the subgroup $H$ of elements in $G$ which are inverse images of elements in $H^*$. If $H$ is a subgroup of $G$, we associate with it the set of elements $H^* = f(H)$.

Let $H^*$ be any subgroup of the topological group $G^*$. It is obvious that $H$ is a subgroup of the abstract group $G$. Since $H^*$ is a closed set and $f$ is a continuous mapping, $H$ is a closed set and therefore a subgroup of the topological group $G$. 
If $N^*$ is a normal subgroup of the topological group $G^*$ and $N$ the subgroup of elements in $G$ which are mapped onto $N^*$, let $g$ be the natural homomorphism of $G^*$ on $G^*/N^* = G^*/N$. Then $h(x) = g(f(x))$ is by (16.5.1) an open homomorphism of $G$ on $G^*$, and its kernel is $N$. By (17.4.1) $N$ is a normal subgroup of the topological group $G$, and the topological groups $G/N$ and $G^*/N^*$ are isomorphic.

Conversely, let $H$ be a subgroup of the topological group $G$, and write $H^* = f(H)$. Suppose, moreover, that $H$ contains the kernel $N'$ of $f$; then every element in $G$ which belongs to the inverse image of $H^*$ also belongs to $N$. For if $f(a)$ is in $H^*$, $H$ contains an element $b$ such that $f(b) = f(a)$;

$$f(ab^{-1}) = f(a)f(b^{-1}) = f(a)[f(b)]^{-1} = e^*,$$

i.e. $ab^{-1}$ is in $N'$ and $e$ in $H$. Therefore $f(G - H) = G^* - H^*$, and since $G - H$ is open and $f$ is an open mapping, $H^*$ is closed in $G^*$. By (3.8.2) $H^*$ is a subgroup of the abstract group $G^*$, and $H^*$ is normal if $H$ is normal. This concludes the proof.

17.7) Example.

Let $G$ be the additive group of real numbers with the discrete topology, and $G^*$ the additive group of real numbers with the obvious "natural" topology. If we define the mapping $g$ of $G$ on $G^*$ by associating with the real number $x$ in $G$ the same real number $x^*$ in $G^*$, it is clear that $g$ is an algebraic isomorphism of $G$ on $G^*$. Topologically, $g$ is an isomorphism, since it is continuous, but not an isomorphism, since its inverse is not continuous.

Although $G$ and $G^*$ are both locally compact, theorem (17.5)
does not apply, since $G$ fails to satisfy the second axiom of
countability.


(13.1) If $M$ is a set of (normal) subgroups of a topological space
$G$, and $D$ is the intersection of the subgroups in $M$, then $D$ is a
(normal) subgroup of the topological group $G$. This follows at once
from (5.1) and (6.4.3).

(13.2) If $A$ is a set of elements of a topological group $G$, there
exists a unique minimal subgroup of $G$ which contains $A$, also a
unique minimal normal subgroup which contains $A$. For by (13.1) the
intersection of all (normal) subgroups containing $A$ is a (normal)
subgroup of the topological group $G$.

(13.3) If $H$ is a subgroup and $N$ a normal subgroup of the topolog-
cal group $G$, then the intersection $D = H \cap N$ is a normal subgroup
of the topological group $H$.

For by (5.3) $D$ is a normal subgroup of the abstract group $H$, and by (6.4.3) and (6.2) $D$ is a closed set relative to $H$.

(13.4) If $H$ is a subgroup and $N$ a normal subgroup of the topological
group $G$, and if the product $HN$ is a closed set in $G$, then
$HN = NH$ is a subgroup of the topological group $G$; and if $H$ is a
normal subgroup of $G$, $HN$ is also normal. This follows from (5.4).

(18.4.1) If the topological group $G$ is compact and satisfies the
second axiom of countability, the product $HN$ in (13.4) is a closed
set in $G$.

For $H$ and $N$ are in this case both compact subspaces of $G$. 
and since $G$ is an $S$-space, $H$ and $N$ as well as their direct product are compact $S$-spaces (cf. (13.3)). By (iii'a) of [14.1] the direct product $H \times N$ can be mapped continuously on the set $HN$ in $G$; by (11.4) $HN$ is compact, and since $G$ is an $S$-space it follows that $HN$ is closed.

(18.4.2) The conditions of (18.4.1) can be relaxed. We prove that if $G$ satisfies the second axiom of countability and one of the sets $H$ and $N$, say $H$, is compact, then $HN$ is a closed set.

Let the sequence $c_1, c_2, \ldots$ in $HN$ converge to $c$ in $G$. We can write $c_n = a_n b_n$ ($a_n$ in $H$, $b_n$ in $N$), and from $a_1, a_2, \ldots$ we can select a subsequence $a_{n_1}, c_{n_1}, \ldots$ converging to $a$ in $H$. Now $b_n = a_{n_1}^{-1} c_{n_1}$, and therefore $b_{n_1}, b_{n_2}, \ldots$ converges to $a^{-1}c$. But $N$ is closed, and therefore $a^{-1}c$ is in $N$, and $c = a(a^{-1}c)$ is in $HN$.

Therefore $HN$ is closed.

(18.4.3) If $H_1, H_2, \ldots, H_k$ are normal subgroups of a topological group $G$, and if the product $P = H_1 H_2 \cdots H_k$ is closed in $G$, then $P$ is a normal subgroup of the topological group $G$.

(18.5) If $H$ is a subgroup and $N$ a normal subgroup of a locally compact topological group $G$ satisfying the second axiom of countability, and if the product $P = HN$ is a closed subset of the topological space $G$, and $D$ denotes the intersection $H \cap N$, then the factor groups $H/D$ and $P/N$ are isomorphic.

It follows from (18.3) and (18.4) that $D$ and $F$ are topological groups. In proving (5.5) we showed that every element $X$ of the group $H/D$ is contained in exactly one element $X'$ of the group $P/N$. If we write $X' = f(X)$, we have, for $X = D x$ ($x$ in $H$), $X' = Nx$. 
We also proved, in (5.5), that the mapping $f$ is an algebraic isomorphism. It remains to prove that the isomorphism is topological.

If $U^*$ is a neighborhood of an element $A^*$ in the space $F/N$, it is composed of all cosets of the form $N x$ ($x$ in some neighborhood $U$ in the space $F$). Since $N U$ is an open set in $F$ and contains $A^*$, $N \cap (N U) = U$ is an open set in $H$ (cf. (6.2.1)) and contains $A = f^{-1}(A^*)$ ($A$ is contained in $A^*$). Let $U^*$ be the open set of all cosets of the form $D x$ ($x$ in $U$). Then, if $X$ is in $U^*$, $f(X)$ is in $U^*$, and $f$ is a continuous mapping.

We have proved so far that $f$ is an algebraically isomorphic and topologically homomorphic mapping of the topological group $H/D$ on $F/N$. Since the topological spaces $R/D$ and $F/N$ satisfy the second axiom of countability and are locally compact (cf. (16.3)), (16.4.1), (17.5) applies and $f$ is an open mapping.

[18.6] If $K$ and $N$ are two normal subgroups of the topological group $G$, we say that $G$ is decomposed into the direct product of its subgroups $K$ and $N$ if $K N = G$ and $K \cap N = \{e\}$.

[18.7] If $K$ and $N$ are two topological groups and $G$ is the set of all pairs $(x, y)$ ($x$ in $K$, $y$ in $N$) and if the algebraic operations and the topological relations in $G$ are defined as in (5.7) and (13.1), respectively, we say that $G$ is the direct product of the topological groups $K$ and $N$.

[18.7.1] The product $G$ defined in (18.7) is a topological group. That $G$ is an abstract group follows from (5.7), that $G$ is a topological space from (13.1). It remains to show that the group operations are continuous.

Suppose $a (a^*, a^\alpha)$ and $b (b^*, b^\beta)$ are two elements of $G$. Write $c = ab^{-1} (a^\alpha b^{-1}, a^\beta b^{-1}) (c^*, c^\alpha)$, and let $W$ be a neighborhood of $c$. By 13.1 $W$ is composed of all points $(z^*, z^\alpha)$ in
G where $z'$ and $z''$ are elements in neighborhoods $W'$ and $W''$ of
the points $c'$ and $c''$ in $K$ and $N$, respectively. Since the group
operations are continuous in $K$ and $N$, the elements $a'$, $a''$, $b'$
and $b''$ have neighborhoods $U'$, $V'$, $U''$, $V''$ such that $U'V' = 1$ $\subseteq W'$ and
$U''V'' = 1 \subseteq W''$. If $U$ is the set of all pairs $(x', x'')$ such that $x'$
is in $U'$ and $x''$ is in $U''$; and if $V$ is the set of all pairs $(y', y'')$
such that $y'$ is in $V'$ and $y''$ is in $V''$, then obviously $UV = 1 \subseteq W$,
and $G$ is a topological group.

(38.7.2) The extension of (18.7) and (18.7.1) to any finite number
of topological groups is obvious.

(33.7.5) If the topological group $G$ is decomposed into the direct
product of $K$ and $\Pi$, every element of $K$ commutes with every element
of $\Pi$, and every element of $G$ can be represented uniquely in the
form $kn$, where $k$ is in $K$ and $n$ in $\Pi$. The proof in (5.6.1) is
valid here.

(33.7.4) If the topological group $G$ is the direct product of two
groups $K$ and $\Pi$, and if $K'$ and $\Pi'$ are the sets of elements in $G$
which are of the form $(k, a')$ and $(n, n')$, respectively, then

i) $K'$ and $\Pi'$ are normal subgroups of $G$,

ii) $G$ is decomposed into the direct product of $K'$ and $\Pi'$, and

iii) $K'$ and $\Pi'$ are isomorphic with $K$ and $\Pi$, respectively.

The proofs of the algebraic properties are as in (5.7.2).

The topological aspect of the isomorphism follows directly from
the construction of neighborhoods in the direct product $G$.

(28.7.5) If $G$ is a locally compact topological group satisfying
the second axiom of countability, and if $G$ is decomposed into the
direct product of $K$ and $N$; if moreover $K'$ and $N'$ are topological
groups isomorphic with $K$ and $N$, respectively, and $G'$ is the di-
rect product of $K'$ and $N'$, then the topological groups $G$ and $G'$
are isomorphic. In short: If the space of $G$ is a locally compact
$S$-space, [18.6] and [18.7] are equivalent in the sense in which
[5.6] and [5.7] are equivalent (cf. (5.7.1)).

Let $f$ and $g$ be isomorphic mappings of the topological group
$K'$ on $K$ and of the topological group $N'$ on $N$, respectively. To
every element $(x, y)$ in $G'$, corresponds an element $h((x, y)) = f(x)g(y)$
in the group $G$; in (5.7.1) we showed that $h$ is an algebraic isomor-
phism of $G'$ on $G$. We show next that $h$ is a continuous mapping.

If $W$ is a neighborhood of $c = ab$ in $G$, where $a$ is in $K$, $b$ in
$N$, there exist neighborhoods $U^a$ and $V^b$ of $a$ and $b$ in $G$ such that
$U^aV^b \subseteq W$. If we write $U = K \cap U^a$ and $V = N \cap V^b$, $U$ and $V$ are neighbor-
hoods in the subspaces $K$ and $N$ (cf. (3.1)). If we write $a' = f^{-1}(a)$
and $b' = g^{-1}(b)$, there exist neighborhoods $U'$ and $V'$ of $a'$ and $b'$ such
that $f(U') \subseteq U$ and $g(V') \subseteq V$. If $W'$ is the neighborhood of the ele-
ment $(a', b')$ in $G'$ composed of all pairs $(x, y)$ ($x$ in $U'$, $y$
in $V'$), it is obvious that $h(W') \subseteq W$. Therefore $h$ is continuous.

Since $h$ is algebraically isomorphic and topologically contin-
uous, it is open, by (17.5), and therefore bicontinuous ($G$ is
locally compact); the same is true of $K$ and $N$, since these spaces
are closed in $G$; similarly, $K$ and $N$ satisfy the second axiom of
countability. By the isomorphism between $K$ and $K'$ and between
$N$ and $N'$, $K'$ and $N'$ and consequently $G'$ are locally compact (cf.
(13.4)) and satisfy the second axiom of countability; therefore
(17.5) applies. The proof is complete.

(18.8) If a locally compact topological group $G$ satisfying the second axiom of countability is decomposed into the direct product of its subgroups $K$ and $N$, then $K$ is isomorphic with the factor group $G/N$.

Since $K \cap N = \{e\}$, this theorem follows from (18.5).

(18.9) Example

The points (or vectors) of the Cartesian plane form an additive topological group $G$; clearly, $G$ is compact and satisfies the second axiom of countability. Let $H$ be a straight line through the origin, of slope $\alpha$, and $N$ the set of all points whose coordinates are integers. $K$ and $N$ are normal subgroups of the topological group $G$. If $P$ denotes the product $HN$ composed of all elements of the form $h+n$ ($h \in H, n \in N$), $P$ is closed in $G$ if $\alpha$ is a rational number. It follows that $P$ is then locally compact.

If however $\alpha$ is irrational, $P$ is neither closed nor locally compact. Here the intersection $D = H \cap N$ contains the identity element only. It is obvious that the result of (18.5) does not hold, since $H/D$ is a discrete topological space while $P/N$ is connected (the elements of $H/D$ are the sets containing exactly one of the points with integral coordinates; the space $P/N$ is an aggregate of straight lines everywhere dense in the plane).

19. The infinite direct product

[19.1] If $M$ is a set $\{G_1, G_2, \ldots\}$ of normal subgroups of a compact topological group $G$ satisfying the second axiom of countability, we say that $G$ decomposes into the direct product of the set $M$ of its subgroups provided
1) If \( N \) is the minimal normal subgroup of the topological group \( G \) which contains all the subgroups of \( H \), then \( N = G \);

ii) If \( H_n \) is the minimal normal subgroup of \( G \) containing the subgroups \( G_1, G_2, \ldots, G_{n-1}, G_{n+1}, G_{n+2}, \ldots \), then the intersection of all the subgroups \( H_1, H_2, \ldots \) contains only the identity \( e \) of \( G \).

It should be observed that while apparently no difference exists between the statement in the present paragraph and that in [5.3], the two definitions are not equivalent; for in the decomposition of a topological group, all the subgroups in question must be topologically closed.

\[(19.1.1) \text{If } G \text{ decomposes into the direct product of the subgroups } \{G_1, G_2, \ldots \}, \text{ then } G \text{ can be decomposed into the product of two of its subgroups } G_n \text{ and } H_n \text{ (notation and conditions as in (19.1)).}

By (13.4) \( G_n H_n \) is compact, and since \( G \) is an 3-space, it is therefore closed and constitutes a normal subgroup of \( G \) by (13.4). 2). Since \( G_n H_n \) contains all the groups \( G_1, G_2, \ldots \), we have \( G_n H_n = G \) by (i) of (19.1). If \( n' \) is the intersection of the subgroups \( H_1, H_2, \ldots, H_{n-1}, H_{n+1}, H_{n+2}, \ldots \), then \( G_n \cap G_n' \). It is obvious from (ii) of (19.1) that \( G_n \cap H_n = \{e\} \). Therefore \( G_n H_n = \{e\} \), and \( G \) decomposes into the direct product of \( G_n \) and \( H_n \).

19.2) If the compact topological group \( G \) satisfies the second axiom of countability and decomposes into the direct product of the set \( G_1, G_2, \ldots \) of normal subgroups, then every element \( G_1 \) commutes with every element of \( G_j \) \((i \neq j)\); and if \( x_1, x_2, \ldots \) is a sequence of elements in \( G \) such that \( x_i \) is in \( G_i \) for each \( i \), then the infinite product \( x_1 x_2 \ldots \) converges, and every element in
G has a unique representation as such a product.

The commutativity was shown in (5.8.2).

To prove that \( x_1x_2\ldots \) converges, write \( y_m = x_1x_2\ldots x_m \). Since the intersection of all the sets \( E_n \) contains only the identity \( e \), there exists for every neighborhood \( V \) of \( e \) an integer \( t \) such that \( H_1 \cap H_2 \cap \ldots \cap H_t \subset V \) (cf. (11.2.1)). If \( p > t \) and \( q > t \), \( y_qy_p^{-1} \) is in \( V \); for \( y_qy_p^{-1} \) can be written
\[
x_1x_2x_3\ldots x_qx_p^{-1}x_{p-1}^{-1}\ldots x_2^{-1}x_1^{-1} = x_1(x_2x_3\ldots x_qx_p^{-1}x_{p-1}^{-1}\ldots x_2^{-1})x_1^{-1} = x_1x_2(x_3\ldots x_qx_p^{-1}x_{p-1}^{-1}\ldots x_2^{-1})x_1^{-1},
\]
etc., and since \( H_1 \) is normal and contains all the \( x_1 \) except \( x_1 \), \( y_qy_p^{-1} \) is in \( H_1 \); since \( H_2 \) is normal and contains \( x_3, x_4, \ldots \), the element \( y_qy_p^{-1} \) is in \( H_2 \), etc. It follows that \( y_qy_p^{-1} \) is in \( H_1 \cap H_2 \cap \ldots \cap H_t \subset V \).

\( G \) is compact, and therefore the sequence \( y_1, y_2, \ldots \) has at least one limit point \( x \). To show that this limit point is unique, we suppose that \( x' \) \( (x \neq x') \) is also a limit point of \( y_1, y_2, \ldots \) and that \( U \) and \( U' \) are neighborhoods of \( x \) and \( x' \) whose closures do not intersect (cf. (10.2)). Then \( \overline{U}^{-1} \) is a compact set (and therefore a closed set, since \( G \) is an \( S \)-space) not containing the identity \( e \). Therefore \( e \) has a neighborhood \( V \) such that \( V \) does not intersect \( \overline{U}^{-1} \). Since \( x \) and \( x' \) are limit points of \( y_1, y_2, \ldots \), there exist integers \( p > t \) and \( q > t \) such that \( y_p \) is in \( U \) and \( y_q \) is in \( U' \); it follows that \( y_qy_p^{-1} \) is not contained in \( V \), contrary to what was proved above. Therefore every infinite product \( x_1x_2\ldots \) \( (x_1 \text{ in } G_1) \) converges.

Below, we shall make use of the fact that if the product
x_1x_2... converges to x, then the product x_2x_3... converges to 
x_1^{-1}x: if U is any neighborhood of x_1^{-1}x, x has a neighborhood V
such that x_1^{-1}V \subseteq U; and if every neighborhood V of x contains
all but a finite number of the elements y_m, it follows that every
neighborhood U of x_1^{-1}x contains all but a finite number of the
elements x_1^{-1}y_m.

It remains to show that every element x in G has a unique
representation x = x_1x_2... (x in G; i = 1, 2, ...). Since G de-
composes into the direct product of its subgroups G_n and H_n, x
has by (5.6.1) a unique representation x = x_nz_n (x_n in G_n, z_n in
H_n). We determine x' by writing x = x_1z_1 = x_2z_2 = x_3z_3 = ... and
x' = x_1x_2x_3... since by (5.6.1) and (19.1.1) x_nz_n = z_nx_n, this
gives us x'x^{-1} = (x_1x_2...x_n...x_n^{-1}z_n^{-1}. Since the product
x_n+1x_{n+2}... converges to an element y in H_n (H_n is a normal sub-
group of G, contains the elements x_{n+1}, x_n+1x_{n+2}... and is closed),
x_n(x_n+1x_{n+2}...x_n^{-1} is in H_n, and since x_1, x_2, ..., x_{n-1}, and
z_n^{-1} are in H_n, x'x^{-1} is in H_n. This holds for all n, and therefore
x'x^{-1} = e, by (ii) of [19.1]; in other words x = x' = x_1x_2... 

Suppose now that x_1x_2... = x'_1x_2'... = x; then x_1^{-1}x_1x_2...
= x_1^{-1}x_1'x_2'..., Since G decomposes into the direct product of
H_1 and G_1, and x can be expressed uniquely in the form yz (y in
H_1, z in G_1), x_1 = x_1'. It follows now that x_2x_3... = x_2'x_3'..., and we argue as before that x_2 = x_2', etc.

[19.3] From a countable set M of compact topological groups
G_1, G_2, ... satisfying the second axiom of countability, we con-
struct a compact topological group satisfying the second axiom
of countability. The elements of G, the direct product of the
groups in \( \mathbb{N} \), are all the sequences \( x = \{ x_1, x_2, \ldots \} \) \( \in \mathbb{N} \),
n = 1, 2, \ldots \). If \( x = \{ x_1, x_2, \ldots \} \) and \( y = \{ y_1, y_2, \ldots \} \)
are two elements in \( \mathbb{N} \), we define their product to be
\[ xy = \{ x_1y_1, x_2y_2, \ldots \} \]. We say that the set of sequences of the form
\( x' = \{ x'_1, x'_2, \ldots \} \) is a neighborhood \( U = \mathcal{U}(r) \) of the point
\( x = \{ x_1, x_2, \ldots \} \) where for \( i = 1, 2, \ldots \), \( r \) the element \( x_i' \) takes
on all values in a neighborhood \( U_1 \) of the point \( x_1 \) in the group
\( G_1 \), while for \( i = r+1, r+2, \ldots \) the element \( x_i' \) takes on all values
in \( G_i \).

We shall now show that the structure \( \mathbb{N} \) has the promised pro-
properties.

(19.3.1) If \( e_i \) is the identity in the group \( G_i \), then for all
\( x = \{ x_1, x_2, \ldots \} \) in \( \mathbb{N} \) we have
\[ \{ x_1, x_2, \ldots \} \{ e_1, e_2, \ldots \} = \{ x_1, x_2, \ldots \} = x \text{, and} \]
\[ \{ x_1, x_2, \ldots \} \{ x_1^{-1}, x_2^{-1}, \ldots \} = \{ e_1, e_2, \ldots \} \].
Therefore \( \mathbb{N} \) contains an identity, and every element in \( \mathbb{N} \) has an
inverse. It is clear that the first two requirements in \( 1.1 \) are
satisfied. It follows that \( \mathbb{N} \) is an abstract group.

(19.3.2) We examine next the topological properties of \( \mathbb{N} \).

i) Clearly, each point of \( \mathbb{N} \) has a neighborhood containing
the point.

ii) Let \( \mathcal{U}(p) \) and \( \mathcal{V}(q) \) be any two neighborhoods of \( x \), and
suppose, for definiteness, that \( p > q \). \( \mathcal{U}(p) \) is composed of all
points \( x' = \{ x'_1, x'_2, \ldots \} \) where \( x'_1 \) lies in some neighborhood
\( U_1 \) of \( x_1 \) in \( G_1 \), \( x'_2 \) in some neighborhood \( U_2 \) of \( x_2 \) in \( G_2 \), etc.;
\( x_{p-1}', x_{p-2}', \ldots \) are any points in \( G_{p-1}, G_{p-2}, \ldots \). A similar
statement holds for \( \mathcal{V}(q) \). But \( x_i \) in \( G_i \) has a neighborhood \( \mathcal{W}_i \)
contained in \( U_1 \cap V_1 \) \( (i = 1, 2, \ldots, q) \). If we define \( \mathcal{W} \) to be the set
of points \( x'' = \{ x''_1, x''_2, \ldots \} \) where \( x''_1 \) is in \( \mathcal{W}_1 \) \( (i = 1, 2, \ldots, q) \).
$x_i$ is in $U_i$ ($i = q+1, q+2, \ldots, p$), and $x_i$ is in $G_i$ ($i = p+1, p+2, \ldots$), then $u \subseteq U(p) \cup U(q)$.

iii) If $y = \{y_1, y_2, \ldots\}$ is any point in the neighborhood $U(r)$ of $x$, then the neighborhood $V(r)$ of $y$ composed of the points $y' = \{y_1', y_2', \ldots\}$ is contained in $U(r)$, provided that for $i = 1, 2, \ldots, r$ the point $y_i'$ in $G_i$ is restricted to lie in some neighborhood $V_i$ of $y_i$ where $V_i \subseteq U_i$.

iv) If $y = \{y_1, y_2, \ldots\} \neq x = \{x_1, x_2, \ldots\}$, there exists an integer $r$ such that $x_r \neq y_r$. It follows that $x$ has a neighborhood $U(r)$ which does not contain the point $y$.

(19.3.3) We observe that the neighborhoods of the form $U(1)$ constitute a countable set, since $G_1$ satisfies the second axiom of countability. Since $G_2$ also satisfies the second axiom of countability, the neighborhoods of the form $U(2)$ can be arranged in a countable set of countable sets, and therefore they form a countable set. The cases of $U(3), U(4), \ldots$ can be handled similarly.

Since all the neighborhoods of the form $U(r)$ are countable for each value of $r$, they are countable if $r$ is allowed to take on all values $1, 2, \ldots$. Therefore the topological space $G$ satisfies the second axiom of countability.

(19.3.4) We show next that the topological space $G$ is compact. Let $x(1), x(2), \ldots$ be any sequence of points $(x(1) = \{x_1(1), x_2(1), \ldots\}$, $i = 1, 2, \ldots)$. Since $G_1$ is compact, we can select a subsequence $x(1), x(n_1), x(n_2), \ldots$ such that the sequence $x_1(1), x_1(n_1), x_1(n_2), \ldots$ in $G_1$ converges to a point $x_1$. For simplicity in the notation we shall write $x_1(1), y(2), y(3), y(4), \ldots$ in place of $x_1(1), x(n_2), x(n_3), \ldots$, and $y_r(i)$ for $x_r(n_i)$.
Similarly, the sequence \( x_1(1), y(2), y(3), \ldots \), contains a subsequence \( x(1), y(2), y(n_1), y(n_2), \ldots \) with the property that the sequence \( x_2(1), y_2(2), y_2(n_1), y_2(n_2), \ldots \) in \( G_2 \) converges to a point \( x_2 \). We write \( x(1), y(2), z(3), z(4), \ldots \) for this subsequence, and \( x_r(n) \) for \( y_r(n) \) \((r = 3, 4, \ldots, i = 3, 4, \ldots)\). Continuing this process indefinitely, we obtain the sequence \( x(1), y(2), z(3), u(4), \ldots \), converging, as we shall show, to the point \( x = \{x_1, x_2, \ldots\} \).

Now every neighborhood of the point \( x \) is of the form \( U(p) \), containing all points \( x' = \{x_1', x_2', \ldots\} \) where \( x_i' \) is any point in \( U_i \) for \( i = 1, 2, \ldots, p \) and \( x_1 \) is any point in \( G_1 \) for \( i = p+1, p+2, \ldots \) \((U_i \) a neighborhood of \( x_1 \) in \( G_1 \)). For any fixed neighborhood \( U(p) \) of \( x \) and any integer \( i \) \((i = 1, 2, \ldots, p)\), \( U_i \) contains all but a finite number of the points \( x_1(1), y_1(2), \ldots, x_1(i), x_1(i+1), x_1(i+2), \ldots \), and therefore contains all but a finite number of the points \( x_1(1), y_1(2), \ldots, x_1(i), s_1(i+1), t_1(i+2), \ldots \). Therefore all but a finite number of the points \( x, y, \ldots \) are contained in \( U(p) \), and the sequence \( x(1), y(2), \ldots \) converges to \( x \). But the sequence is extracted from \( x(1), x(2), x(3), \ldots \), and we have therefore shown that \( G \) is a compact space.

(19,3,5) It remains to show that the group operation in \( G \) is continuous.

If \( x \) and \( y \) are any two points in \( G \), and if \( W(p) \) is any neighborhood of \( xy^{-1} \), constructed from neighborhoods

\[ \{x, y, \ldots\} \]
of the points \( x_1 y_1^{-1} , x_2 y_2^{-1} , \ldots , x_p y_p^{-1} \), there exist neighborhoods \( U_1 , U_2 , \ldots , U_p \) and \( V_1 , V_2 , \ldots , V_p \) of \( x_1 , x_2 , \ldots , x_p \) and \( y_1 , y_2 , \ldots , y_p \) such that \( U_1 y_1^{-1} \subseteq W \).

(i 1, 2, \ldots). From these neighborhoods we can construct neighborhoods \( U(p) \) and \( V(p) \) of \( x \) and \( y \), respectively, so that \( U(p) V(p)^{-1} \subseteq W(p) \), and the continuity of the group operation is established.

(19.4) If \( G \) is the direct product of the topological groups \( G_1 , G_2 , \ldots \), where \( G_1 \) is compact and satisfies the second axiom of countability \((i=1, 2, \ldots)\); and if \( G_k' \) is the set of elements in \( G \) of the form \( x = \{ x_1 , x_2 , \ldots \} \) with \( x_i = e_i \) for \( i \neq k \), then every set \( G_k' \) is a normal subgroup of the topological group \( G \), and \( G \) is decomposed into the direct product of the subgroups \( G_k' \) \((k=1, 2, \ldots)\).

That \( G_k' \) is a subgroup of the abstract group \( G \) follows at once from its definition. Since \( G \) is an \( S \)-space and \( G_k' \) is obviously compact in the topological group \( G \), \( G_k' \) is closed in \( G \); i.e., \( G_k' \) is a subgroup of the topological group \( G \). Since all the elements of \( G_k' \) commute with all elements of \( G_j' \) \((j \neq k)\), the subgroups \( G_k' \) are normal.

It remains to show that conditions (i) and (ii) of [19.1] are satisfied. Suppose that \( N \) is the minimal normal subgroup of \( G \) containing all groups \( G_k' \), and that \( x = \{ x_1 , x_2 , \ldots \} \) is any element in \( G \). Then the sequence of elements \( y_1 , y_2 , \ldots \)

\((y_n = \{ x_1 , x_2 , \ldots , x_n , e_{n+1} , e_{n+2} , \ldots \} \) is in \( N \), and since \( N \) is closed and every neighborhood \( U(p) \) of \( x \) in \( G \) contains all the elements \( y_n \) for \( n \geq p \), \( N \) contains \( x \). It follows that \( N = G \).
If \( H_n \) is the minimal normal subgroup of \( G \) containing all the subgroups \( G_k \), all elements of \( H_n \) are of the form \( x = \{ x_1, x_2, \ldots, x_{n-1}, x_n, x_{n+1}, \ldots \} \). It is now obvious that the only element of \( G \) contained in all the subgroups \( H_n \) is the element \( e = \{ e_1, e_2, \ldots \} \). This completes the proof.

(19.5) If \( G \) is a compact topological group satisfying the second axiom of countability; if \( G \) is decomposed into the direct product of a countable set \( \{ G_1, G_2, \ldots \} \) of its subgroups; and if \( G_k' \) is a topological group isomorphic with \( G_k \) \((k=1, 2, \ldots)\) and \( G' \) is the direct product of the groups \( G_1', G_2', \ldots \), then the groups \( G \) and \( G' \) are isomorphic.

Let \( f_1 \) be an isomorphic mapping of the topological group \( G_1' \) on \( G_1 \) \((i=1, 2, \ldots)\). To every element \( x' = \{ x_1', x_2', \ldots \} \) in \( G' \) there corresponds an element \( x = f(\{ x_1, x_2, \ldots \}) = x_1 x_2 \ldots \), where \( x_1 = f(x_1') \). Since the representation of points in \( G \) as infinite products is unique (cf. (19.2)), the mapping \( f \) of \( G' \) on \( G \) is one-to-one.

Suppose \( x = f(\{ x_1', x_2', \ldots \}) = x_1 x_2 \ldots \) and \( y = f(\{ y_1', y_2', \ldots \}) = y_1 y_2 \ldots \). Since \( x_1 \) and \( y_1 \) commute when \( i \neq j \), \((x_1 x_2 \ldots x_m)(y_1 y_2 \ldots y_m) = x_1 y_1 x_2 y_2 \ldots x_m y_m \). Since for every neighborhood \( W \) of the point \( xy \) there exist neighborhoods \( U \) and \( V \) of the points \( x \) and \( y \) such that \( UV \subseteq W \), and since every neighborhood of \( x \) (of \( y \)) contains all but a finite number of the points \( x_1 x_2 \ldots x_m \) \((y_1 y_2 \ldots y_m)\), where \( m=1, 2, \ldots \), it follows that the infinite product \( x_1 x_2 y_2 \ldots \) converges to \( xy \). Therefore \( f(x_1', x_2', \ldots)f(y_1', y_2', \ldots) = f(x_1 y_1, x_2 y_2, \ldots) \), and we have established that the mapping \( f \) is algebraically isomorphic.
We show next that $f$ is a continuous mapping of $G'$ on $G$.

By (17.3) it is sufficient to prove that for every neighborhood $U$ of the identity $e = e_1 e_2 \ldots$ in $G$ there exists a neighborhood $U(p)$ of $e' = \{e_1', e_2', \ldots\}$ in $G'$ such that $f(U(p)) = U$.

Suppose that infinitely many of the subgroups $G_i$ of $G$ have points not contained in the neighborhood $U$ of $e$ in $G$. Then there exists a sequence of points $x_{n_1}, x_{n_2}, \ldots$ (in $G_i'$) converging to a point $x$ not in $U$. Each set $H_n$ contains all of these points $x_{n_i}$, except possibly one, and since $H_n$ is closed, $H_n$ contains $x$. Condition (ii) of [13.1] is thus violated, and it follows that if $U$ is any neighborhood of $x$, then $f_i'(G_i') \subset U$ provided $i$ exceeds a certain constant $p$.

But by hypothesis, the identity $e_i'$ of $G_i'$ has a neighborhood $U_i'$ such that $f_i'(U_i') \subset U \cap G_i = U_i$. If we construct the neighborhood $U_i'(p)$ of $e' = \{e_1', e_2', \ldots\}$ in $G'$ from these neighborhoods $U_i'$ ($i = 1, 2, \ldots, p$), it is now clear that $f(U_i'(p)) \subset U$, and $f$ is proved to be a continuous mapping.

Since both $G$ and $G'$ are compact topological groups satisfying the second axiom of countability, we can apply (17.5), and $f$ is a continuous open one-to-one mapping of $G'$ on $G$; i.e. $G'$ and $G$ are isomorphic topological groups, q.e.d.

(19.6) Example

Let $G_1, G_2, \ldots$ be an infinite set of finite abstract groups. If we topologize these groups by assigning to each element the element itself as neighborhood, we obtain compact topological groups satisfying the second axiom of countability. The direct product $G$ of these topological groups is compact and satisfies
the second axiom of countability (cf. (19.3)). It is however not a discrete group.

As a special case we take as $G_n$ the additive group of integers modulo the $n$th prime integer. In the product $G$ of the discretely topologized groups, the points are of the form $x = \{x_1, x_2, x_3, \ldots \}$, where $x_1 = 0, 1; x_2 = 0, 1, 2, 3; x_3 = 0, 1, 2, 3, 4; x_4 = 0, 1, \ldots, 6; \ldots$. The element $x = \{1, 0, 4, 3, 6, 12, 2, \ldots \}$ for example, has the following neighborhoods:

- $U(0)$: the set of all points of $G$;
- $U(1)$: all points of the form $x = \{1, x_2, x_3, x_4, \ldots \}$
- $U(2)$: all points of the form $x = \{1, 0, x_3, x_4, \ldots \}$

$\ldots$

- $U(5)$: $x = \{1, 0, 4, 3, 6, x_5, x_7, \ldots \}$

$\ldots$

We note that in any infinite direct product of topological groups $G_1, G_2, \ldots$ there exists exactly one neighborhood of the form $U(p)$ if and only if $G_1, G_2, \ldots, G_p$ are discrete groups.

20. Connected and zero-dimensional groups

(20.1) If $G$ is a topological group and $N$ is the component of the point $e$ in the topological space $G$ (cf. [9.3]), then $N$ is a normal subgroup of the topological group $G$.

Let $a$ and $b$ be two elements in $N$. By (14.2.1) and (9.4.1) $aN^{-1}$ is a connected set containing $e$ ($N^{-1}$ contains $a^{-1}$). Therefore $aN^{-1} \subseteq N$; in particular $ab^{-1}$ is in $N$, and by (2.2.1) $N$ is a subgroup of the abstract group $G$. By (9.5.1) $N$ is a closed set in $G$, and therefore $N$ is a subgroup of the topological group $G$. 
If \( x \) is any element in \( G \), \( x^{-1}Nx \) is a connected set containing the identity, so that \( x^{-1}Nx \subseteq N \); i.e. \( N \) is a normal subgroup of the topological group \( G \).

[20.2] If the topological space of a topological group \( G \) is connected, i.e. if the component of \( e \) coincides with \( G \), \( G \) is said to be a connected group.

If the component of the identity of a group \( G \) contains only the identity, \( G \) is a 0-dimensional or totally disconnected group.

(20.2.1) If \( G \) is a topological group and \( N \) the component of its identity, then the factor group \( G^* = G/N \) is a 0-dimensional group.

By [17.4] the natural homomorphic mapping \( f \) of \( G \) on \( G^* \) is open. We show first that if \( P^* \) is the component of the identity of \( G^* \) and \( P \) the set of points in \( G \) which are mapped into \( P^* \), then the mapping \( f \) of the subspace \( P \) on the subspace \( P^* \) is open. If \( U \) is any neighborhood in \( P \), there exists a neighborhood \( V \) in \( G \) such that \( U = P \cap V \) (cf. (8.2.2)). Since \( f(U) = P^* \cap f(V) \) and \( f \) is an open mapping of \( G \) on \( G^* \), \( f(V) \) is an open set in \( G^* \), and therefore \( f(U) \) is an open set in the space \( P^* \).

Suppose now that the group \( G^* \) is not 0-dimensional, so that \( P^* \) contains elements other than the identity of \( G^* \). The identity of \( G^* \) is the coset \( Ne = N \), and our supposition implies that \( P \) contains elements not in \( N \). Since \( N \) is a maximal connected set, this implies that \( P = A \cup B \) where \( A \) and \( B \) are non-empty non-intersecting open sets in the space \( P \) (cf. (9.1.1)). Now the sets \( f(A) \) and \( f(B) \) do not intersect; for if \( a \) is an element in \( A \), then \( Na \subseteq A \), since \( Na \) is a connected set and therefore can not have points in \( B \). But \( f(A) \) and \( f(B) \) are open sets in \( P^* \), and therefore
P* is not connected, which is contrary to the definition that P is the component of the identity of G*.

(20.3) If U is a neighborhood of a connected topological group G, every element of G can be represented as a finite product of elements in U.

Let V be the sum of all sets of the form U^n. By (6.2.4) and (14.2.3) V is an open set. If we can show that V is also a closed set, it will follow from the supposed connectedness of G that the complement of V is empty, i.e. V = G.

Let a be a point of V. Since aU^{-1} is open, it contains a neighborhood of a, and therefore a point b of V. By hypothesis

\[ b = u_1 u_2 \ldots u_m \quad (u_i \text{ in } U, i = 1, 2, \ldots, m). \]

But since b is in aU^{-1},

\[ b = au_{m+1}^{-1} \quad (u_{m+1} \text{ in } U). \]

Therefore a = u_1 u_2 \ldots u_{m+1} and V is closed.

(30.4) If G is a topological group and the element z commutes with every element in G, z is a central element of G; the set Z of all central elements is the center of the topological group G.

(30.4.1) It follows from (4.2) and (4.2.1) that the center Z of the topological group G coincides with the center of the abstract group G and is therefore a normal subgroup of the latter. We shall show that Z is a closed set, i.e. Z is a normal subgroup of the topological group G. It will follow from (4.2.2) that every subgroup of the topological group Z is a normal subgroup of the topological group G. Such subgroups are called central normal subgroups of G.

Suppose that Z is not closed. Then Z contains an element a and G an element x such that a = x^{-1}ax = a. Since G is a regular space, there exist neighborhoods U and U' of a and a' such that
U and U' do not intersect (cf. (10.2) and (14.4)). If V = Z \cap U, a
is in V, since for every neighborhood \( W \) of a the intersection
U \cap W contains a neighborhood of a which contains points of Z,
(a is contained in Z). But \( a' = x^{-1}ax \) is contained in \( x^{-1}Vx = x^{-1}Vx \cdot V \)
(cf. (14.2.1)). But U' and V do not intersect, since \( V \subseteq U \). Therefore
\( a' \) cannot be in V, and the contradiction proves that \( Z \) is closed.

(20.5) Every discrete normal subgroup \( N \) (cf. [6.5.4]) of a con-
nected topological group \( G \) is a central normal subgroup of \( G \).

If \( N \) is a discrete normal subgroup of \( G \), each element \( a \) of
\( N \) has a neighborhood \( V \) of \( N \) containing no element of \( N \) other than
the element \( a \). By (14.2) it follows from the relation \( e^{-1}ae = a \)
that the identity \( e \) has a neighborhood \( U \) such that \( U^{-1}aU \subseteq V \). If
\( u \) is any element in \( U \), we have then \( u^{-1}au \subseteq V \); but since \( N \) is normal,
we have also \( u^{-1}au \subseteq N \), and it follows that \( u^{-1}au = a \), since \( a \) is the
only element of \( N \) in \( V \).

Now let \( x \) be any element in \( G \); by (20.3), we may write
\( x = u_1u_2 \ldots u_m \) (\( u_i \) in \( V \), \( i = 1, 2, \ldots, m \)). Therefore \( x^{-1}ax \)
\( = u_m^{-1}u_{m-1}^{-1} \ldots u_1^{-1}au_1^{-1} \ldots u_m^{-1}a = u_m^{-1}u_{m-1}^{-1} \ldots u_1^{-1}u_1^{-1}u_2^{-1} \ldots u_m^{-1} \)
\( = a \). Therefore every element in \( N \) belongs to \( Z \); i.e. \( N \) is a central
normal subgroup of \( G \).

(20.6) If \( G \) is a locally compact topological 0-dimensional group
satisfying the second axiom of countability, every neighborhood
\( U \) of the identity of \( G \) contains a set \( H \) which is open in \( G \) and
which is a subgroup of the topological group \( G \). Since \( H \) is also
a closed set, the space \( G/H \) is by [17.4] discrete.

The proof is in three steps:

(20.6.1) By (10.6.1), the identity \( e \) of \( G \) has a basis of neighbor-
hoods \( V_1, V_2, \ldots \) such that \( V_n \supseteq V_{n+1} \) (\( n = 1, 2, \ldots \)). By hypothesis there exists in \( G \) a compact set \( E \) containing \( e \). We say that a point \( a \) in \( E \) can be connected to \( e \) over the set \( E \) by a chain of order \( n \) provided there exists a set of points \( a_1, a_2, \ldots, a_k \) (\( a_1 = e, a_k = a \)) of \( E \) such that \( a_i^{-1} a_{i+1} \) is in \( V_n \) (\( i = 1, 2, \ldots, k-1 \)).

We define the set \( E_n \) as the set of points which can be connected to \( e \) by chains of order \( n \) over the set \( E \). Every point of \( E_n \) can be connected to \( e \) by a chain of order \( n \) over the set \( E_n \); for if the point \( a \) in \( E_n \) is connected to \( e \) by the chain \( a_1, a_2, \ldots, a_k \), the same is true of \( a_1, a_2, \ldots, a_k^{-1} \). Obviously \( E_n \supseteq E_n+1 \).

The set \( E_n \) is open relative to \( E \); for if the point \( a \) in \( E \) can be connected to \( e \) by a chain of order \( n \) composed of \( k \) points, then every point of \( V_n \cap E \) can be connected to \( e \) by a chain of order \( n \) composed of \( k+1 \) points.

The set \( E_n \) is a closed and therefore compact set in the space \( G \); for the set \( E \) is chosen to be compact, so that any point not in \( E \) cannot belong to \( \overline{E_n} \). But if the point \( a \) in \( E \) does not belong to \( E_n \), the set \( aV_n^{-1} \) cannot intersect \( E_n \), and again \( a \) is not in \( \overline{E_n} \). Therefore \( E_n = \overline{E_n} \).

(20.6.2) We show next that the intersection \( E^* \) of the sets \( E_1, E_2, \ldots \) is connected. Since \( G \) is a 0-dimensional group, it will follow that \( E^* = \{e\} \).

Suppose that \( E^* \) is the sum of two closed non-intersecting non-null sets \( A \) and \( B \) (\( e \) in \( A \)). By (14.2.2) \( A^{-1} \) is also closed, and therefore \( A^{-1} \) and \( B \) are both compact. If \( c_1, c_2, \ldots \) is any sequence in \( A^{-1}B \), we can write \( c_n = a_n^{-1} b_n \) (\( a_n \) in \( A \), \( b_n \) in \( B \)), and from the sequence \( a_1^{-1}, a_2^{-1}, \ldots \) we can extract a sequence \( a_{n_1}^{-1}, a_{n_2}^{-1}, \ldots \) converging to some element \( a^{-1} \) in \( A^{-1} \). The set of points \( b_{n_1}, b_{n_2}, \ldots \) has at least one limit point \( b \), and by
(iii'a) of [6.1] the set of points $c_1, c_2, \ldots$ has the limit point $c = a^{-1}b$. In other words, $A^{-1}B$ is compact and therefore closed.

Since $A$ and $B$ do not intersect, $A^{-1}B$ does not contain the identity, and therefore $e$ has a neighborhood $V_r$ such that $V_r e^{-1}$ does not intersect $A^{-1}B$ (cf. (14.2)). It follows that the sets $AV_r$ and $B V_r$ do not intersect (if $av_1 = bv_2$ (notation obvious), then $v_1 v_2^{-1} = a^{-1}b$).

If $b$ is any point in $B$, $b$ cannot be connected to $e$ by a chain of order $r$. For since $AV_r$ and $B V_r$ do not intersect, such a chain would have two "adjacent" points $a_i$ and $a_{i+1}$ contained in the sets $AV_r$ and $B V_r$, respectively. The point $a_i^{-1} a_{i+1}$ would then be contained in both of the sets $V_r$ and $V_r^{-1} A^{-1} B V_r$, and therefore $V_r e^{-1}$ and $A^{-1} B$ would intersect. It is therefore impossible to connect $b$ to $e$ by a chain of order $s$ ($s > r$). But by (11.2.1), there exists an integer $s$ ($s > r$) such that $E_s \subset a^s V_r$. Now any element $b$ in $B$ belongs to $E_s$ and can therefore be connected to $e$ by a chain of order $s$. The contradiction proves that $E_s$ is a connected set.

(20.6.3) We can now proceed to the construction of the subgroup $H$ of (20.6). If $U'$ is any neighborhood of $e$, it contains a neighborhood $U$ such that $\overline{U}$ is compact. We apply the construction of (20.6.1) to the set $E = \overline{U}$. By (iii'a) of [14.1] the identity has a neighborhood $V$ such that $V e^{-1}$ contains $u$, and by (11.2.1) and the fact that the sets $E_1, E_2, \ldots$ are closed and compact, there exists an integer $s$ such that $E_s \subset V$. But in (20.6.1) we showed that $E_s$ is open relative to $\overline{U}$, and by (8.2.2) there exists an open set $U$ of the space
such that \( \mathbb{E}_t = \mathbb{U} \cap \mathbb{W} \). Since \( \mathbb{E}_t = \mathbb{V} \subset \mathbb{V}^2 \subset \mathbb{U} \), we have then
\[ \mathbb{E}_t = \mathbb{U} \cap \mathbb{V} \cap \mathbb{W} \subset \mathbb{W}, \]
i.e. \( \mathbb{U} \cap \mathbb{W} \subset \mathbb{U} \cap \mathbb{W} \), and therefore \( \mathbb{E}_t = \mathbb{U} \cap \mathbb{W} \); since \( \mathbb{E}_t \) is the intersection of two open sets in \( \mathbb{G} \), \( \mathbb{E}_t \) is open in \( \mathbb{G} \).

We conclude the proof by showing that the set \( \mathbb{E}_t \) which we have shown to be open and closed set contained in \( \mathbb{U} \) is a subgroup of the abstract group \( \mathbb{G} \).

Every point in \( \mathbb{E}_t^2 \) can be connected to \( \mathbb{e} \) by a chain of order \( n \) over \( \mathbb{E}_t^2 \); for if \( a \) and \( b \) are two elements in \( \mathbb{E}_t \), connected to \( \mathbb{e} \) by chains of order \( t \) of the form \( a_1, a_2, \ldots, a_t \) and \( b_1, b_2, \ldots, b_t \), respectively, then the chain \( a_1, a_2, \ldots, a_t, b_1, b_2, \ldots, b_t \) is a chain of order \( t \), since
\[ (a_1 b_1)^{-1} a_{i+1} b_{i+1} = a_i^{-1} b_i^{-1} a_i b_i \] \( \text{since \( \mathbb{E}_t \subset \mathbb{U} \) and \( \mathbb{V}^2 \subset \mathbb{U} \), it follows that} \)
\( \mathbb{E}_t \subset \mathbb{U} \) and therefore by definition of \( \mathbb{E}_t \) that \( \mathbb{E}_t^2 = \mathbb{E}_t \). By induction
\[ (\mathbb{E}_t^m)^2 = \mathbb{E}_t (m=1, 2, \ldots). \]

To show that \( \mathbb{E}_t^{-1} \subset \mathbb{E}_t \), we consider, for any element \( a \) in \( \mathbb{E}_t \), the sequence \( a^1, a^2, \ldots \), the points of which we have shown to be contained in \( \mathbb{E}_t \). Since \( \mathbb{E}_t \) is compact, it contains a limit point for this sequence. This implies that corresponding to any neighborhood \( \mathbb{V}_n \) of the identity there exist integers \( m \) and \( m' \) \( (m' > m) \) such that \( a^m (a^{m'})^{-1} = a^{m'-m} = a^{k_n} \) is in \( \mathbb{V}_n \) (cf. (iii) of [14.4]). Now let \( b \) be a limit point of the sequence \( a^{k_1}, a^{k_2}, \ldots \). Then every neighborhood of \( \mathbb{a b} \) contains points of the sequence \( a^{k_1}, a^{k_2}, \ldots \). Therefore \( \mathbb{a b} = \mathbb{e} \). Since \( \mathbb{E}_t \) is closed set, \( \mathbb{b} \) is in \( \mathbb{E}_t \), i.e. \( \mathbb{E}_t^{-1} \subset \mathbb{E}_t \). By (2.2.1 b) the proof is complete.

(3.7) If \( \mathbb{G} \) is a compact topological 0-dimensional group satisfying the second axiom of countability, every neighborhood \( \mathbb{U} \) of the
identity of $G$ contains a set $N$ which is open in $G$ and which is a
normal subgroup of the topological group $G$. Moreover, the space
$G/N$ is by [17.4] both discrete and compact and is therefore
finite.

In other words, if in (20.6) we strengthen the condition
of local compactness of $G$ to compactness of $G$, the subgroup in the
result becomes a normal subgroup, and the space of its factor
group a finite space.

We construct the subgroup $H$ as in (20.6) and denote by $H$
the intersection of all the sets $x^{-1}Hx$, where $x$ is an arbitrary
element of $G$. It is easily verified that the sets $x^{-1}Hx$ are subgroups
of $G$, and by (13.1) $N$ is a subgroup of $G$. If $n$ is any element in
$N$, $x^{-1}nx$ belongs to $N$ for all $x$ in $G$, and therefore $N$ is normal.

We show next that $N$ contains a neighborhood $V$ of the identity.
Since $N$ is a group, it will follow that for any point $a$ in $N$, $N$
contains a neighborhood $V_a$, i.e. $N$ is an open set, and the proof
of the theorem will be complete.

If no neighborhood $V$ of $e$ is contained in $N$, $G - N$ contains
a sequence of points $a_1, a_2, \ldots$ converging to $e$. Since not each
of the sets $x^{-1}Hx$ contains the point $a_1$, we can write
$a_i = x^{-1}_i b_i x_i$ (for $i = 1, 2, \ldots$). Since $G$ is compact we
may suppose that the sequence $a_1, a_2, \ldots$ and the elements
$x_1, x_2, \ldots$, $b_1, b_2, \ldots$ have been selected so that the sequences
$x_1, x_2, \ldots$ and $b_1, b_2, \ldots$ converge to $x$ and $b$, respectively.
Since $H$ is an open set, $b$ belongs to $G - H$. Moreover, since
$a_1, a_2, \ldots$ converges to $e$, $x^{-1}bx = e$, i.e. $b = x$. But this is
impossible, since $e$ belongs to $N$. The contradiction completes
(20.8) If G is a topological 0-dimensional group, it has no connected subset containing more than one element.

For if E were a connected subset of G containing two distinct elements a and b, the set $EA^{-1}$ would be a connected set containing the identity and the point $ba^{-1} = e$, contrary to the supposition that the component of $e$ is the set $\{e\}$.

(20.9) If every neighborhood $U$ of the identity of the topological group G contains an open subgroup $H$ of the group G, G is a 0-dimensional group.

We note that this theorem is not exactly a converse of (20.6). For nothing is said about compactness or local compactness of G.

If U is any neighborhood of $e$ in G, G decomposes into two non-intersecting non-null open sets $H$ and $G - H$ such that $H \subseteq U$. The component of the identity is connected and must therefore contain no elements other than $e$.

(20.10) Examples

(20.10.1) Let G be the direct product of the discretely topologized finite groups $G_1, G_2, \ldots$ (cf. (19.6)). If $C_k$ denotes the set of all elements of the form $x = \{x_1, x_2, \ldots\}$, $x_i = e_i$, $i = 1, 2, \ldots, k$, we see at once that $C_k$ is a normal subgroup of the topological group G. It is easy to see that if U is any neighborhood $U(m)$ of the identity of G, then $C_m \subseteq U$. G is a 0-dimensional group, by (20.9).

(20.10.2) Let G be the additive group of real numbers topologized in the usual way, and H the set of all rational numbers, H is a subgroup of the abstract group G, and therefore a topological group (cf. (16.2)). Since the component of zero in H contains
only zero, \( H \) is a 0-dimensional group. We note however that \( H \) can be generated by any neighborhood of its identity, so that (20.3) applies to some groups that are not connected. By (20.9) we deduce that some neighborhood \( U \) of the identity \( H \) contains no open subgroups of \( H \), and therefore that (20.6) does not apply to general 0-dimensional groups (\( H \) satisfies the second axiom of countability, but is not locally compact.

21. Local isomorphism

[21.1] Two topological groups \( G \) and \( G' \) are locally isomorphic if there exist neighborhoods \( U \) and \( U' \) of the identities \( e \) and \( e' \) and a homeomorphic mapping of \( U \) and \( U' \) such that

i) If the elements \( x, y, \) and \( xy \) belong to \( U \), then

\[
f(xy) = f(x)f(y);
\]

ii) If the elements \( x', y', \) and \( x'y' \) belong to \( U' \), then

\[
f^{-1}(x'y') = f^{-1}(x')f^{-1}(y').
\]

We include two relations that are immediate consequences of (i):

iii) \( f(e) = e' \);

iv) if \( x \) and \( x^{-1} \) are in \( U \), \( f(x^{-1}) = f(x)^{-1} \).

(iii) follows from the fact that \( e \) and \( ee \) are contained in \( U \), so that \( f(e) = f(e)f(e) \). (iv) follows from the fact that \( f(x)f(x^{-1}) = f(xx^{-1}) = e' \).

We can show, moreover, that (ii) is a consequence of (i):

if (i) is satisfied for a neighborhood \( U \) of the identity we choose a neighborhood \( V \) such that \( V^2 \subset U \), and we write \( f(V) = V' \).

Clearly condition (i) is satisfied for \( V \). Now let \( x', y' \), and \( x'y' \) belong to \( V' \), and write \( x = f^{-1}(x') \), \( y = f^{-1}(y') \). Then \( xy \) is in \( U \), and therefore \( f(x)f(y) = x'y' \). i.e. \( f^{-1}(x'y') = f^{-1}(x')f^{-1}(y') \), and (ii) is satisfied for \( V \).
(21.2) If \( H \) is a discrete normal subgroup of the topological group \( G \), then the groups \( G \) and \( G' = G/N \) are locally compact.

Let \( f \) be the natural homomorphic mapping of \( G \) on \( G' \) (cf. (17.4)) and let \( W \) be a neighborhood of the identity of \( G \) containing no element of \( N \) other than \( e \). If \( U \) is a neighborhood of \( e \) in \( G \) such that \( uu^{-1} \subset W \), and if we write \( f(U) = U' \), the mapping \( f \) is one-to-one between \( U \) and \( U' \). For if the two elements \( x \) and \( y \) in \( U \) have the same image, i.e. belong to the same coset of \( N \), then \( xy^{-1} \) is in \( N \cap W \) and therefore \( xy^{-1} = e \), i.e. \( x = y \). By (17.4) \( f \) is an open and continuous mapping, and since it is one-to-one it is homeomorphic.

(21.3) If \( G \) and \( G' \) are two connected locally isomorphic topological groups, there exists a group \( H \) with two discrete normal subgroups \( N \) and \( N' \) such that \( G \) and \( G' \) are isomorphic to the factor groups \( H/N \) and \( H/N' \), respectively.

There exist neighborhoods \( U \) and \( U' \) of the identities of \( G \) and \( G' \) and a mapping \( f \) of \( U \) on \( U' \) so that the conditions of [21.1] are satisfied. For simplicity in the discussion we assume that \( U^{-1} = U \). This is not an essential restriction; for if \( \Sigma \) is a system of neighborhoods \( U \) of the identity, we can replace the system \( \Sigma \) by \( \Sigma' \), a system of neighborhoods \( U^* = U \cup U^{-1} \) (\( U \) in \( \Sigma \)), without altering the topological properties of the space. Every set \( U \) contains a set \( U^* \). \( U^* \) is an open set, and by (iii' b) of [14.1] \( U^* \) contains a neighborhood \( U_1 \) of the system \( \Sigma \) such that \( U_1 U^{-1} \subset U^* \).

By \( K \) we denote the direct product of the groups \( G \) and \( G' \) (cf. [18.7]), and by \( V \) the set of elements in \( K \) which can be represented in the form \( (x, f(x)) \), where \( x \) is in \( U \). The set \( H \) we
define to be the set of all elements in $K$ which can be represented as a finite product of elements in $V$; i.e.

$H = VUV^2UV^3U \ldots$. Because $U = U^{-1}$, $H$ is a subgroup of the abstract group $K$, and it is therefore a topological group, by (16.2).

However, we shall introduce a different topology into $H$. Let $\{ U_\alpha \}$ be a system of neighborhoods of the identity of the group $G$ (where the index $\alpha$, in general, takes on the values of a non-countable set), and let $\{ U_\alpha \}$ be the system of neighborhoods in $K$. 
of the form \( U_\alpha = f(U_\alpha) \). We denote by \( V_\alpha \) the set in \( K \) composed of all points of the form \((x, f(x))\) where \( x \) is in \( U_\alpha \), and we topologize \( H \) by choosing as system of neighborhoods for the identity the system of sets \( \{ V_\alpha \} \). By (15.2), (15.2.1), and (15.3) this establishes a topology for \( H \).

With every element \( z = (x, x') \) in \( K \) we associate the element \( x \) in \( G \), and we write \( g(z) = g(x, x') = x \). Obviously \( g \) is a homomorphic mapping of the abstract group \( K \) on the abstract group \( G \), and therefore \( g \) is also a homomorphic mapping of the abstract group \( H \) on some subgroup \( G^* \) of the abstract group \( G \). Now \( g(V) = U \), and therefore \( U \subseteq G^* \). But by (20.3) \( G \) is generated by \( U \), and since \( G^* \) is a subgroup of the abstract group \( G \), we have \( G^* = G \).

Moreover, since \( g(V_\alpha) = U_\alpha \), it follows by (17.3) that \( g \) is an open continuous homomorphism of the topological group \( H \) on the topological group \( G \).

By (17.4.1) the topological group \( G \) is isomorphic with the factor group \( H/N \), where \( N \) is the kernel of the homomorphism \( g \). The neighborhoods \( V_\alpha \) of the identity of \( H \) contain no elements of \( N \) other than the identity of the group \( N \), since \( N \) is composed of all elements of the form \((a, x')\), and \( V_\alpha \) of elements of the form \((x, f(x))\). Therefore \( N \) is a discrete normal subgroup of \( H \).

Since \( G \) and \( G' \) enter into the construction of the topological group \( K \) in a symmetrical manner, it is obvious how a discrete normal subgroup \( N' \) can be constructed so that \( G' \) is isomorphic to the factor group \( H/N' \).
Example

If $G$ is the additive group of real numbers and $N$ the subgroup of all integers, it follows from (21.2) that the groups $G$ and $G/N$ are locally isomorphic. But $G/N$ is compact, and $G$ is not compact; therefore the two groups are not isomorphic.
References


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