A CLASS OF BORDERED AXISYMMETRIC DETERMINANTS

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1. Introduction.

1(1). The purpose of this paper is to establish an analytic proof of several theorems on a class of bordered axisymmetric determinants, which we will denote by

\[ B_n(1, 2, \ldots, n) = \begin{vmatrix} 
0 & 1 & 1 & 1 & \cdots & 1 \\
1 & 0 & \sqrt{3} & \sqrt{4} & \cdots & \sqrt{n} \\
1 & \sqrt{2} & 0 & \sqrt{3} & \sqrt{4} & \cdots & \sqrt{n} \\
1 & \sqrt{3} & \sqrt{2} & 0 & \sqrt{3} & \sqrt{4} & \cdots & \sqrt{n} \\
1 & \sqrt{4} & \sqrt{2} & \sqrt{3} & 0 & \sqrt{4} & \cdots & \sqrt{n} \\
1 & \sqrt{n} & \sqrt{n-1} & \sqrt{n-2} & \sqrt{n-3} & 0 & \cdots & 0 
\end{vmatrix} \]

where the elements, \( i_j \), satisfy the following conditions:

- \( i_j = i \overline{j} \),
- \( i_j > 0 \), if \( i \neq j \),
- \( i_j = 0 \), if \( i = j \).

1(2) In an early paper, Cayley derives the determinant form \( D_3(1, 2, 3) \) which expresses the relation between the distances determined by three points 1, 2, 3 on a Euclidean line:

\[ D_3(1, 2, 3) = \begin{vmatrix} 
0 & 1 & 1 & 1 \\
1 & 0 & \sqrt{2} & \sqrt{3} \\
1 & \sqrt{2} & 0 & \sqrt{3} \\
1 & \sqrt{3} & \sqrt{2} & 0 
\end{vmatrix} = 0. \]

* Formulated by Karl Menger.
Analogous vanishing determinant forms exist, expressing the relations between the distances determined by four points in a Euclidean plane, or five points in a Euclidean three-space, respectively, $D_4(1,2,3,4) = 0$ and $D_5(1,2,3,4,5) = 0$. *

1(3). The general determinant form, $D_2(1,2,..., n)$, has been used by Karl Menger † in the characterisation of the n-dimensional Euclidean space among the general semi-metric spaces by means of relations between the distances of its points.

1(4). We make the following definitions, after Menger:

(a) A set of elements, called points, is called a semi-metric space if to each couple of points, $i,j$, there corresponds a real number, $ij$, called the distance between them, such that $ij = ji$; $ij > 0$, if $i \neq j$; $ij = 0$ if $i = j$.

(b) Two semi-metric spaces, $S$ and $S'$, are congruent if there exists a one to one correspondence between their elements which preserves distances.

1(5). Menger has proved the following theorem ‡ on the congruence of a general semi-metric space, $S$, to a subset of the n-dimensional Euclidean space, $R_n$.

Theorem:

The necessary and sufficient condition that a semi-metric space

* Cayley, loc. cit.


S be congruent to a subset of the $R_n$ is that if $S$ contains more than $n+3$ points, for each integer $k \leq n+1$, and for each $k$-tuple of points $1, 2, \ldots, k$, of $S$, sign $D_k(1, 2, \ldots, k) = (-1)^k$ or $0$, and that $D_{n+3}(1, 2, \ldots, n+2) = 0$, for each $n+2$ points of $S$.

If $S$ contains exactly $n+3$ points, $1, 2, \ldots, n+3$, it is necessary and sufficient for the congruence of $S$ with a subset of the $R_n$ that besides these conditions, $D_{n+3}(1, 2, \ldots, n+3) = 0$.

1(6). Definition. ¶

A semi-metric space, $R$, is called "pseudo-euclidean," $(R_n)$, if each $n+2$ points of $R$ is congruent to $n+2$ pts. of the $R_n$, while $R$ is not congruent to a subset of the $R_n$.

1(7). Menger has shown that if $R$ is pseudo-euclidean $(R_n)$, that it contains exactly $n+3$ points and that the sign of $D_{n+3}(1, 2, \ldots, n+3) = (-1)^n$. ¶

A pseudo-euclidean

1(8). If $R_2$ contains exactly four points $1, 2, 3, 4$, it is called a "pseudo-linear quadruple." ‡‡ It has been shown ‡‡ that these quadruples are characterized among the general semi-metric quadruples by the relations

$12 = 34; \quad 23 = 14; \quad 13 = 24; \quad 12 + 23 = 13$.

1(9). From these considerations we see that if a semi-metric space, $R$, consisting of the four points $1, 2, 3, 4$, has each three of its points congruent to three points of a straight line, $R_1$, then either

* $D_k(1, 2, \ldots, k)$, as defined in paragraph 1(1).
‡ See above.
(a) \( R \) is congruent to a subset of the \( D_4 \), and \( D_4(1,2,3,4) = 0 \),
or
(b) \( R \) is a pseudo-linear quadruple, the sign of \( D_4(1,2,3,4) \)
is negative, and \( 12 = 34; \ 23 = 14; \ 13 = 24 \).

2. The algebraic formulation of the above statement is as follows:

**Theorem 1.**

If the determinant,

\[
D_4(1,2,3,4) = \begin{vmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 12 & 13 \\
1 & 21 & 0 & 23 \\
1 & 31 & 32 & 0
\end{vmatrix}
\]

has its four principal bordered minors of order four vanishing, then the determinant \( D_4 \) vanishes, unless \( 12 = 34; \ 23 = 14; \ 13 = 24 \). Then \( D_4 \) has the value

\[-32(12, 23, 13)^2 \text{, where one of the numbers } 12, 23, 31 \text{, is the sum of the two others.}\]

2(1). We shall prove first the Lemma:

**Lemma:** If the principal bordered minor, \([5,5]\), obtained by striking out the fifth row and fifth column of \( D_4 \), vanishes, then \( D_4 \) has one of the three developments:

(a) \( D_4 = -2(12, 23, 13, 14 - 12, 34 - 23, 14 + 13, 24) \) if \( 12 + 33 = 13 \).

(b) \( D_4 = -2(12, 23, 13, 14 + 23, 14 - 13, 24) \) if \( 12 + 13 = 23 \).

(c) \( D_4 = -2(12, 23, 14 + 12, 34 - 23, 14 - 13, 24) \) if \( 23 + 13 = 12 \).

**Proof of the Lemma:**

\[
[5,5] = \begin{vmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 12 & 13 \\
1 & 21 & 0 & 23 \\
1 & 31 & 32 & 0
\end{vmatrix} = 0, \text{ by hypothesis.}
\]

* This Lemma is due to L. M. Blumenthal, The Rice Institute.*
Denote the signed cofactors of the elements in the first row of $[5, 5]$ by $(1, 1), (1, 2), (1, 3), (1, 4)$.

Perform the following operation on the determinant $D_4$:

$$\text{Col. } 1 \times (1, 1) + \text{Col. } 2 \times (1, 2) + \text{Col. } 3 \times (1, 3) + \text{Col. } 4 \times (1, 4).$$

We obtain:

$$\begin{vmatrix}
0 \cdot (1, 1) + 1 \cdot (1, 2) + 1 \cdot (1, 3) + 1 \cdot (1, 4) & 1 & 1 & 1 & 1 \\
1 \cdot (1, 1) + 0 \cdot (1, 2) + 12 \cdot (1, 3) + 13 \cdot (1, 4) & 0 & 12 & 13 & 14 \\
1 \cdot (1, 1) + 21 \cdot (1, 2) + 0 \cdot (1, 3) + 22 \cdot (1, 4) & 21 & 0 & 23 & 24 \\
1 \cdot (1, 1) + 51 \cdot (1, 2) + 52 \cdot (1, 3) + 0 \cdot (1, 4) & 51 & 52 & 0 & 34 \\
1 \cdot (1, 1) + 41 \cdot (1, 2) + 42 \cdot (1, 3) + 43 \cdot (1, 4) & 41 & 42 & 43 & 0 \\
\end{vmatrix}$$

$$\begin{vmatrix}
0 \cdot (1, 1) + 1 \cdot (1, 2) + 1 \cdot (1, 3) + 1 \cdot (1, 4) & 1 & 1 & 1 & 1 \\
1 \cdot (1, 1) + 0 \cdot (1, 2) + 12 \cdot (1, 3) + 13 \cdot (1, 4) & 0 & 12 & 13 & 14 \\
1 \cdot (1, 1) + 21 \cdot (1, 2) + 0 \cdot (1, 3) + 22 \cdot (1, 4) & 21 & 0 & 23 & 24 \\
1 \cdot (1, 1) + 51 \cdot (1, 2) + 52 \cdot (1, 3) + 0 \cdot (1, 4) & 51 & 52 & 0 & 34 \\
1 \cdot (1, 1) + 41 \cdot (1, 2) + 42 \cdot (1, 3) + 43 \cdot (1, 4) & 41 & 42 & 43 & 0 \\
\end{vmatrix}$$

Evidently, all the elements in the first column of this determinant are zero, except the last element, since the first element represents the value of $[5, 5]$, which is zero by hypothesis, and the other three elements are zero since they are the sums of the elements of a row of $[5, 5]$ multiplied by the cofactors of corresponding elements in another row.

Thus we have:

$$\begin{vmatrix}
0 & 1 & 1 & 1 & 1 \\
0 & 12 & 13 & 14 & 0 \\
0 & 21 & 0 & 23 & 24 \\
0 & 51 & 52 & 0 & 34 \\
0 & 41 & 42 & 43 & 0 \\
\end{vmatrix}$$

then,

$$(1, 1).D_4 = \begin{vmatrix}
(1, 1) + 41 \cdot (1, 2) + 42 \cdot (1, 3) + 43 \cdot (1, 4) \\
\end{vmatrix}$$

and finally,

$$(A) \begin{vmatrix}
(1, 1) + 41 \cdot (1, 2) + 42 \cdot (1, 3) + 43 \cdot (1, 4) \\
\end{vmatrix}^2$$

since, developing in terms of the last column,
Developing the cofactors appearing in (A), we obtain:

\[ (B) \quad 2 \begin{pmatrix} 12 & 23 & 13 \end{pmatrix}, \quad D_4 = -\left\{ 2 \left( 12^2, 23^2, 13^2 \right) - \frac{14}{13} \left( 12^2, 23^2 + 23^2 - 13^2 \right) \right\}^2 \]

Now, since \([5, 5] = 0\), we have:

\[ (12 + 23 - 13)(12 - 23 + 13)(12 + 23 + 13) = 0 \]

Case I.

Substituting \((12 + 23 - 13) = 0\) in (B), we have:

(a) \[ D_4 = -2 \left( 12^2, 23^2, 13^2 \right) \]

Case II.

Similarly, if \((12 - 23 + 13) = 0\), we have:

(b) \[ D_4 = -2 \left( 12^2, 23^2, 13^2 \right) \]

Case III.

Lastly, if \((-12 + 23 + 13) = 0\), we obtain:

(c) \[ D_4 = -2 \left( 12^2, 23^2, 13^2 \right) \]

This completes the proof of the Lemma.

2(2). We now show if all the fourth order principal bordered minors of \(D_4\) vanish, \(D_4\) vanishes, unless \(12 = 34;\ 23 = 14;\ 13 = 24\), in which case \(D_4 = -32(12^2, 23, 13^2)\).

The vanishing of all the fourth order bordered principal minors of \(D_4\) implies:

\[ \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 12 & 13 & 14 \\ 1 & 1 & 23 & 24 \\ 1 & 1 & 31 & 32 \end{pmatrix} = -\frac{1}{13}(12 + 23 + 13)(12 + 23 - 13)(12 - 23 + 13)(-12 + 23 + 13) \]

Cayley, loc. cit.
A priori there are $3^6$, or 81, possible combinations to be considered. However, it is easily seen that only 15 of these combinations are consistent.

e.g. Suppose, if possible, the combination:

(a) \( \sqrt{2} - \sqrt{4} + \sqrt{7} = 0 \)

(b) \( \sqrt{2} - \sqrt{4} + \sqrt{7} = 0 \)

(c) \( \sqrt{2} - \sqrt{4} + \sqrt{7} = 0 \)

(d) \( \sqrt{2} - \sqrt{4} + \sqrt{7} = 0 \)

is consistent. We shall show this is impossible.

Subtracting (b) from (a), (d) from (c), and adding reminders, we have:

\[
- \sqrt{3} + \sqrt{3} + \sqrt{4} - \sqrt{4} + \sqrt{7} - \sqrt{7} - \sqrt{2} = 0; \\
\text{c.e.} \quad 2(\sqrt{3}) - 2(\sqrt{3}) = 0, \\
\sqrt{3} - \sqrt{3} = 0.
\]

But, from (a) \( \sqrt{2} = \sqrt{3} - \sqrt{3} \), hence \( \sqrt{2} = 0 \), which contradicts the hyp.

that in \( D_4 \), \( \sqrt{i} > 0 \) if \( i \neq j \).

Twelve of the possible combinations are:

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* Cf. footnote of the preceding page.

† We shall denote the relation \( pq + qr = pr \) by the symbol \( pqr \).
It may be verified that for each of these combinations, $D_4 = 0$, and the relations $12 = 34$, $23 = 14$, $13 = 24$ do not subsist.

E.g. Suppose the combination (1) subsists.

(1):

$12 + 23 = 13$
$12 + 34 = 14$
$13 + 34 = 14$
$23 + 34 = 24$

From the Lemma, Case I, since $12 + 23 = 13$,

$$D_4 = -2 \left\{ \frac{12}{13} \cdot \frac{13}{12} \cdot \frac{34}{14} - \frac{12}{14} \cdot \frac{34}{12} - \frac{12}{12} \cdot \frac{34}{12} + \frac{12}{12} \cdot \frac{14}{12} \right\}$$

whence:

$$D_4 = -2 \left\{ \frac{12}{13} \cdot \frac{13}{12} \cdot \frac{13}{13} - \frac{12}{12} \cdot \frac{14}{12} - \frac{12}{12} \cdot \frac{14}{12} + \frac{12}{12} \cdot \frac{14}{12} \right\}$$

expanding:

$$D_4 = -2 \left\{ \frac{12}{13} \cdot \frac{13}{12} - \frac{12}{12} \cdot \frac{14}{12} + \frac{12}{12} \cdot \frac{14}{12} + \frac{12}{12} \cdot \frac{14}{12} \right\}$$

Hence $D_4 = 0$, if (1) subsists.

Suppose $12 = 34$, $23 = 14$, $13 = 24$, and (1) subsist. We will show this is impossible.

Applying $12 = 34$, $23 = 14$, $13 = 24$ to (1) we get:

(a) $34 + 14 = 24$
(b) $34 + 24 = 14$
(c) $24 + 34 = 14$
(d) $14 + 34 = 24$

Adding (a) and (b) we obtain $34 = 0$ which is contrary to hypothesis as before.

We show, finally, that if any one of the three remaining possible combinations

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subsists, then \( \overline{12} = \overline{34}, \overline{23} = \overline{14}, \overline{13} = \overline{24}, \) and \( D_4 = -32 \left( \overline{12} \cdot \overline{23} \cdot \overline{34} \right) \).

e.g. Suppose that (14) subsists.

(14): (a) \( \overline{12} + \overline{23} = \overline{13} \)
(b) \( \overline{12} + \overline{14} = \overline{24} \)
(c) \( \overline{14} + \overline{34} = \overline{13} \)
(d) \( \overline{23} + \overline{34} = \overline{24} \)

The operation (a) + (b) - (c) - (d) yields \( \overline{12} = \overline{34} \).

The operation (a) - (b) - (c) + (d) yields \( \overline{23} = \overline{14} \).

The operation (a) - (b) + (c) - (d) yields \( \overline{13} = \overline{24} \).

Hence the first condition is satisfied.

By the Lemma, Case I, we have from

(14), (a): \( \overline{12} + \overline{23} = \overline{13} \),

\[ D_4 = -2 \left( \overline{12} \cdot \overline{13} \cdot \overline{34} - \overline{12} \cdot \overline{34} \cdot \overline{13} \right) \]

Upon the substitution of (14); (b), (c), (d), we obtain

\[ D_4 = -32 \left( \overline{12} \cdot \overline{23} \cdot \overline{34} \right) \]

These results are readily obtained for the other two combinations, (13), (15).

This completes the proof of Theorem 1.

3. Theorem 2:

If the determinant \( D_5(1,2,3,4,5) \),

\[
D_5 = \begin{vmatrix}
0 & 1 & 1 & 1 & 1 \\
1 & 0 & \overline{12} & \overline{13} & \overline{14} \\
1 & \overline{12} & 0 & \overline{23} & \overline{24} \\
1 & \overline{13} & \overline{23} & 0 & \overline{34} \\
1 & \overline{14} & \overline{24} & \overline{34} & 0 \\
\end{vmatrix}
\]

has all of its bordered fourth order principal minors equal to zero, then

it has all of its bordered fifth order principal minors equal to zero.
Proof of Theorem 2:

3(1). Suppose, if possible, none of the bordered fifth ordered principal minors of $D_5$ is zero, i.e., each $D_4(p_i, p_i', p_i'', p_i''') = 0$, where $p_i, p_i', p_i'', p_i'''$ is some quadruple selected from $1, 2, 3, 4, 5$. Now, each $D_3(t_i, t_i', t_i'') = 0$ by hypothesis, where $t_i, t_i', t_i''$ is some triple selected from $1, 2, 3, 4, 5$. By Theorem 1, $D_4(p_1, p_2, p_3, p_4)$ is different from zero if and only if $\frac{p_1p_2}{p_3p_4} = \frac{p_2p_3}{p_1p_4} = \frac{p_1p_3}{p_2p_4}$. Accordingly, we have: $12 = 13 = 14 = 15 = 23 = 24 = 34 = 35 = 45$, that is, all ten distinct elements in the determinant are equal.

But from Theorem 1, 2(2), it is readily seen that none of the $D_3(t_1, t_2, t_3)$ can vanish if $\frac{t_1t_2}{t_1t_3} = \frac{t_1t_3}{t_2t_3}$, which contradicts the hypothesis. Hence at least one bordered fifth order principal minor must vanish.

3(2). Suppose, to fix the ideas, that $D_4(2, 3, 4, 5)$ is the only vanishing fifth order bordered principal minor of $D_5$.

Then, by Theorem 1, we have $12 = 13 = 14 = 15 = 23 = 24 = 34 = 35 = 45$, which is impossible, as before. Hence at least two fifth order principal bordered minors must vanish.

3(3). Suppose now, $D_4(2, 3, 4, 5) = D(1, 3, 4, 5) = 0$, and

(1) $D(1, 2, 3, 4) = 0$
(2) $D(1, 2, 3, 5) = 0$
(3) $D(1, 2, 4, 5) = 0$.

Then from Theorem 1 and from (1), (2), (3)

$12 = 34; \quad 12 = 35; \quad 12 = 45$.

Hence $D_3(3, 4, 5) = 0$, since $34 = 35 = 45$. This is impossible, by hypothesis, and similarly for all choices of the two vanishing determinants. Hence at
least three bordered fifth order principal minors must vanish.

3(4). Suppose besides \( D_4(2,3,4,5) = 0 \) and \( D_4(1,3,4,5) = 0 \), that also \( D_4(1,2,4,5) = 0 \).

We have then:

(1) \( D_4(1,2,3,4) \neq 0 \)

(2) \( D_4(1,2,3,5) \neq 0 \).

From (1): \( 12 = 34, 23 = 14, 13 = 24 \).

From (2): \( 12 = 35, 23 = 15, 13 = 25 \).

From \( D_3(3,4,5) = 0 \), we obtain:

\[
(34 + 45 - 35)(34 - 45 + 35)(-34 + 45 + 35) = 0.
\]

Since \( 34 = 35 \) this becomes:

\[
45 \cdot [2(34) - 45] \cdot 45 = 0, \text{ or }
\]

(A) \( 2(34) = 45 \).

From \( D_3(1,4,5) = 0 \) and \( 14 = 15 \),

(B) \( 2(14) = 45 \).

From \( D_3(2,4,5) = 0 \), and \( 25 = 24 \),

(C) \( 2(24) = 45 \).

Hence, from (A), (B), (C),

\( 34 = 14 = 24 \).

But \( 12 = 34 \), hence

\( 12 = 14 = 24 \),

and \( D_3(1,2,4) \neq 0 \), as before, which is contrary to hypothesis. It may be verified that similar contradictions are obtained for all the \( \binom{5}{2} \) choices of the non-vanishing bordered fifth order determinants. Hence at least four bordered fifth order principal minors vanish.

3(5). Suppose now, to fix the ideas,
\[
D(1,2,3,5) = 0 \\
D(1,2,4,5) = 0 \\
D(1,3,4,5) = 0 \\
D(2,3,4,5) = 0,
\]

and the remaining minor \( D(1,2,3,4) \neq 0 \).

Then, if \( D(1,2,3,4) \neq 0 \) one of the three following cases exists,

from Theorem 1: \( (\cdot, \cdot) \)

\[
(1) \quad \begin{array}{c}
213 \\
124 \\
134 \\
243
\end{array}
\]

\[
(2) \quad \begin{array}{c}
123 \\
214 \\
143 \\
234
\end{array}
\]

\[
(3) \quad \begin{array}{c}
132 \\
142 \\
314 \\
324
\end{array}
\]

and each of these cases implies the relation \( 12 = 34, 23 = 14, 13 = 24 \), by Theorem 1.

Consider (1). Then, since \( 213 \), or \( 12 + 13 = 23 \) exists, we have from the Lemma, 2(1), (b):

\( (A) \): \( D(1,2,3,5) = -2(12.23.13 - 12.35 + 23.15 - 13.25)^2 = 0 \).

Since \( 12 + 24 = 14 \), we have by 2(1), (a):

\( (B) \): \( D(1,2,4,5) = -2(12.24.14 - 12.45 - 24.15 + 14.25)^2 = 0 \).

Since \( 13 + 34 = 14 \), we have by 2(1), (a):

\( (C) \): \( D(1,3,4,5) = -2(13.34.14 - 13.45 - 34.15 + 14.35)^2 = 0 \).

Since \( 24 + 34 = 23 \), we have by 2(1), (c):

\( (D) \): \( D(2,3,4,5) = -2(23.34.24 + 23.45 + 34.25 - 24.35)^2 = 0 \).

From this relation \( (A), (B), (C), (D) \), we get by applying \( 12 = 34, 23 = 14, 13 = 24 \):

\( 15 = 24 \):

From \( (A) + (D) \): \( 2(12.13) - 35 + 15 + 45 - 25 = 0 \).

From \( (B) + (C) \): \( 2(12.13) + 35 - 15 - 45 + 25 = 0 \).

Adding the above: \( 4(12.13) = 0 \).

This implies that either \( 12 = 0 \) or \( 13 = 0 \) which is contrary to the hypothesis
that \( i_j > 0 \) if \( i \neq j \).

By an examination of the remaining cases it can be shown in an entirely analogous manner that the same contradiction is obtained. This completes the proof of Theorem 2.

4. Theorem 3.

If the determinant \( D_5(1,2,3,4,5) \) has all its bordered fourth order principal minors equal to zero then it has all of its unbordered principal minors of orders four and five equal to zero.

4(1). The 5 principal unbordered minors of order four of \( D_5 \) vanish, since each is the principal unbordered minor of one of the 5 bordered principal minors of \( D_5 \), which were shown to vanish by Theorem 2.

4(2). The unbordered principal minor of order five of \( D_5 \), \( E_5 \),

\[
E_5 = \begin{vmatrix}
0 & 12^1 & 13^1 & 14^1 & 15^1 \\
21^1 & 0 & 23^1 & 24^1 & 25^1 \\
31^1 & 32^1 & 0 & 34^1 & 35^1 \\
41^1 & 42^1 & 43^1 & 0 & 45^1 \\
51^1 & 52^1 & 53^1 & 54^1 & 0
\end{vmatrix}
\]

vanishes.

From 4(1), all its principal minors of order four vanish.

Perform the following operation upon \( E_5 \):

Divide

row 1, by 1 
row 2, by \( 12^2 \)

col. 1, by 1 
col. 2, by \( 12^2 \)

\[\&\text{ Cf. L. M. Blumenthal, Bull. Amer. Math. Soc., Oct. 1931. The unbordered principal minor, } E_4, \text{ of order four, of a vanishing fifth order bordered minor, must vanish with the determinant.}\]
all positive, non-zero quantities.

Thus, we have:

\[
\begin{pmatrix}
12 & 13 & 14 & 15 \\
0 & 1 & 1 & 1 \\
1 & 0 & \frac{x_3^2}{13^2 14^2} & \frac{x_4^2}{14^2 15^2} \\
1 & \frac{x_3^2}{13^2 14^2} & 0 & \frac{x_5^2}{15^2 16^2} \\
1 & \frac{x_4^2}{14^2 15^2} & \frac{x_5^2}{15^2 16^2} & 0
\end{pmatrix}
\]

Now all the principal minors of order four vanish in the above determinant.

Hence it must vanish and from this, \( E_5 \) vanishes.

This completes the proof of Theorem 3.

5. Theorem 4.

If the determinant \( D_5(1,2,3,4,5) \) has all its bordered fourth order principal minors equal to zero, then \( D_5 \) vanishes.

This theorem is an immediate consequence of Theorems 2 and 3. All the principal minors of \( D_5 \) of orders four and five have been shown to vanish, whence \( D_5 \) must vanish.

6. Theorem 5.

If the determinant \( D_n(1,2,...,n) \) has all its bordered fourth order principal minors equal to zero, then \( D_n \) vanishes, \( n \geq 4 \).


\[ \text{Cf. Bocher, Introduction to Higher Algebra, page 57.} \]
Each bordered principal minor of $D_n$ of order six has all its principal minors of orders four and five vanishing, by Theorems 2 and 3. Hence all the principal minors of $D_n$ of orders four and five vanish, and accordingly $D_n$ vanishes.

FINIS