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THE DIFFERENTIAL GEOMETRY
OF MODULAR SURFACES

by

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INTRODUCTION

The present paper is concerned with a study of the properties of the modular surfaces of meromorphic functions from the standpoint of differential geometry. This work was undertaken with the idea of possible use of such results for further interpretation of the properties of meromorphic functions.

The initial impetus was the work of Heimrdes, in which an attempt was made to prove the Picard Theorem by a detailed study of certain properties of the modular surface of an analytic function. Landau pointed out that if Picard's Theorem could be proved for polynomials, it could be proved (quite simply) for entire functions \(1\). Consequently, Heimrdes attempted to establish the Picard Theorem in the following form: \( F(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n (a_0, a_1 \neq 0) \) takes on the value zero or one in a disc \(|z| < \rho(a_0, a_1)\) where \( \rho(a_0, a_1) \) depends only upon \( a_0 \) and \( a_1 \). The proof of this form of Picard's Theorem depends, for the most part, upon the properties of the level lines and fold lines of the modular surface; and Heimrdes thus confined his attention to these curves. However, Heimrdes was able to prove the

*Numbers in parentheses refer to the bibliography at the end of the paper.
Theorem only for two special cases, both cases requiring
special conditions on the concavity of the fall lines.

In 1944, Zaat considered the properties of modular
surfaces, not restricting his attention to level and fall
lines [7]. In addition to work similar to that of Heiderode
he investigated in some detail the Gaussian and mean curvature
at various points on the modular surface.

In this paper the essential results of Heiderode and
Zaat are presented in slightly different form, and a detailed
application of these results is made to polynomials. The
Gaussian curvature relationship between the modular surfaces
of a function and the function squared is also considered.

The first section is devoted to some general facts
pertaining to modular surfaces. In the second section the
properties of fall and level lines are discussed. The third
section deals with curvature.
1. GENERAL REMARKS

The modular surface or relief (Beträgflächen [7]) of $F(z)$ is $\gamma = |F(z)|$. In what follows we shall use the notation:

$$F = u + iv = f^2 = (\alpha + i\beta)^2$$

$$\gamma = \gamma(x,y) = \gamma(z) = |F(z)| = |F|; \gamma' = |F'|; \gamma'' = |F''|$$

(1.1)

$$P = \frac{\partial \gamma}{\partial x} = \gamma_x; q = \gamma_y; r = \gamma_{xx}; s = \gamma_{xy}; t = \gamma_{yy}.$$  

**Theorem (1.2):** In any region where $F$ is holomorphic,

(1.2.1)  

$$\gamma(r + t) = p^2 + q^2$$

and

$$\frac{F'}{F} = \frac{\gamma_x - iv\gamma_y}{\gamma}$$

(1.2.2)

**Proof:** $\gamma = \sqrt{u^2 + v^2}$. Since $F$ is holomorphic, and hence $u$ and $v$ satisfy the Cauchy–Riemann equations and Laplace's equation,

$$p = \frac{uu_x + vuv_x}{\sqrt{u^2 + v^2}}$$

$$q = \frac{vu_x + uu_x}{\sqrt{u^2 + v^2}}$$

(1.2.3)

$$r = \frac{uu_x + vuv_x + u_x^2 + u_{xx}^2}{\sqrt{u^2 + v^2}} - \frac{(uu_x + vuv_x)^2}{(u^2 + v^2)^{3/2}}$$

$$s = \frac{vu_x + uu_x}{\sqrt{u^2 + v^2}} - \frac{(uu_x + vuv_x)(vuv_x - uu_x)}{(u^2 + v^2)^{3/2}}$$

$$t = \frac{-uu_x - vuv_x + u_x^2 + u_{xx}^2}{\sqrt{u^2 + v^2}} - \frac{(vuv_x - uu_x)^2}{(u^2 + v^2)^{3/2}}$$
Then
\[ \gamma (x + t) = 2(u_x^2 + v_x^2) = \left( \frac{\mu^2 - \mu v^2 + v^2 \mu_x^2 + \mu^2 v_x^2}{\mu^2 + v^2} \right) \]
Simplifying, we have
\[ \gamma (x + t) = u_x^2 + v_x^2 = \gamma^2. \]

Now
\[ p^2 + q^2 = \frac{\mu^2 u_x^2 + 2 \mu v u_x v_x + \mu^2 v^2}{\mu^2 + v^2} + \frac{\mu^2 u_x^2 - 2 \mu v u_x v_x + u^2 v_x^2}{\mu^2 + v^2} \]
or
\[ (1.2.1) \quad p^2 + q^2 = u_x^2 + v_x^2 = \gamma^2. \]

Hence
\[ \gamma (x + t) = p^2 + q^2. \]

Also
\[ \frac{F'}{F} = \frac{\mu_x + i v_x}{\mu + i v} = \frac{(\mu_x + i v_x) (\mu - i v)}{\mu^2 + v^2} = \frac{(\mu u + v v_x)}{\gamma \sqrt{\mu^2 + v^2}} - i \frac{(v u_x - \mu v_x)}{\gamma \sqrt{\mu^2 + v^2}}. \]

Hence
\[ \frac{F'}{F} = x_x - i \frac{\gamma v_x}{\gamma}. \]

**Theorem (1.5):** If at any point in a closed region the modular surface of a non-constant meromorphic function is lower (or higher) than at each point of the boundary, then there exists a zero (or pole) of the surface in this region. That is, if \( F(z) \) is a non-constant function meromorphic in a domain \( D \) and on its boundary \( B \), and if there exists a \( z_0 \in D \) such that \( |F(z_0)| < |F(z)| \) (or \( |F(z_0)| > |F(z)| \)) for all \( z \in B \), then there exists a \( z_1 \in D \) such that \( |F(z_1)| = 0 \) (or \( |F(z_1)| = \infty \)).

**Proof:** Suppose \( |F(z_0)| < |F(z)| \) for all \( z \in B \). Since \( F(z) \) is meromorphic in \( \overline{D} \) (\( \overline{D} = D \cup B \)) there exists at most a
finite number of poles $\alpha_i \in \mathbb{D}$. Associate with each $\alpha_i$ a $C_i$: $|z - \alpha_i| \leq r_i > 0$ such that $|F(z)| > |F(z_0)|$ for all $z \in C_i$. Let $R = \mathbb{D} - \{C_i\}$. $F(z)$ is holomorphic in $R$ and hence $|F(z)|$ takes on a minimum value, say $|F(z')|$, there.

Suppose $z'$ is on the boundary of $R$. Then $z' \in B$ or $z' \in \{z| |z - \alpha_i| = r_i\}$. But from the above this implies that $|F(z')| > |F(z_0)|$, which is a contradiction. Hence, $z' \in R$.

Then $|F(z')| = 0$. For suppose not. From a theorem of Cauchy [4], there exists an $h$ such that $z' + h \in R$ and $|F(z' + h)| < |F(z')|$, which is a contradiction since $|F(z')|$ is a minimum. Hence, if $|F(z_0)| < |F(z)|$ for all $z \in B$, there exists a $z_1$ such that $|F(z_1)| = 0$.

If $|F(z_0)| > |F(z)|$ for all $z \in B$, then a proof similar to the above considering $\frac{1}{F(z)}$ and $\beta_i$, the zeros of $F(z)$, gives the required result.

**Theorem (1.4):** The tangent of the (acute) angle of inclination $\Theta$, between the plane tangent to the modular surface $\gamma = \gamma(z)$ at $z = \alpha$ and the $z$-plane, is equal to the value of the modulus of the derived function evaluated at $z = \alpha$. That is

$$
(1.4.1) \quad \tan \Theta = \gamma'(\alpha)
$$

**Proof:** $\Theta$ also equals the angle between the normal at the
point of tangency and the $\gamma$ axis. Then

$$\cos \theta = \frac{1}{\sqrt{p^2(\alpha) + q^2(\alpha) + 1}}$$

and

$$\tan \theta = \sqrt{p^2(\alpha) + q^2(\alpha)}.$$

From (1.2.4),

$$\tan \theta = \gamma'(\alpha).$$
The intersection of the modular surface with a plane parallel to the xy plane is called a contour line, level curve, or level line (Niveau linie [4]).

The equations of the projections of the level lines on the xy plane are found by setting \( z = c \) in the equation of the surface. Then the differential equation of the family of such projections is \( dY = 0 \) or

\[
(2.1.1) \quad \gamma_x dx + \gamma_y dy = 0.
\]

If at every point on a curve on a modular surface the tangent to the curve is perpendicular to the intersection of the tangent plane and the xy plane, then the curve is called a slope line or fall line (Fallinie [4]). This definition lends itself readily to establishing the essential properties of slope lines.

The intersection of the tangent plane at \((x_0, y_0)\) and the xy plane is the line

\[
\gamma = 0
\]  

\[
\gamma_x(x_0, y_0)(x - x_0) + \gamma_y(x_0, y_0)(y - y_0) + [\gamma - \gamma(x_0, y_0)] = 0.
\]

Since the tangent to the slope line is perpendicular to this line, so too is the tangent to the projection of the slope line on the xy plane. Then the family of projections of the slope lines satisfies the differential equation:
Comparing (2.1.1) and (2.1.2) we see that the projections of the level lines are orthogonal trajectories of the projections of the slope lines. Hence, the level lines are the orthogonal trajectories of the slope lines.

Furthermore, from (1.2.1), \( \gamma_x = -\chi'(f') \) and \( \gamma_y = -\gamma'(f') \). Substituting in (2.1.2) and simplifying:

\[
\int \left( \frac{f'}{f} \right) \, dx + \mathcal{R} \left( \frac{f'}{f} \right) \, dy = 0 \quad \text{or} \quad \int \left( \frac{f'}{f} \right) \, dz = 0
\]

Integrating, we have

\[
\int (\log F) = \text{constant} \quad \text{or} \quad \arg(F) = C_1.
\]

Hence the projections of the slope lines are the curves \( \arg(F) = C_1 \), and the projections of the level lines are the curves \( |F| = C_2 \).

Theorem (1.3) can now be restated in terms of level lines. If \( F \) is not identically a constant, then inside any closed level line there is at least one zero or one pole. Furthermore, the only minimum is zero and the only maximum is infinity.

For every \( z \), not a zero of \( F \) or \( f' \) and not a pole of \( F \), there is one slope line and one level line through \( z \). The projection of each slope line ends at a zero or a pole of \( F \).
through every point \((z_0, |F(z_0)|)\) such that
\[
F(z_0) \neq 0, \quad F'(z_0) = F''(z_0) = \ldots = F^{(m-1)}(z_0) = 0, \quad \text{and}
F^{(m)}(z_0) \neq 0,
\]
there pass \(m\) slope lines and \(m\) level lines whose tangents intersect in \(2m\) equal angles [6].

**Proof:** Consider the function \(W = F(z)\). From the hypotheses above, \(W' = F'(z)\) has an \((m-1)\)-fold zero at \(z = z_0\) and
\[
W_0 = F(z_0) \neq 0. \quad \text{Without loss of generality we can let } z_0 = 0.
\]
We then have the development
\[
W - W_0 = z^m(a_0 + a_1 z \ldots) = z^m \phi(z),
\]
where \(a_0\) and hence also \(\phi(0)\) are not zero. Let \(\psi(z) = \sqrt[m]{\phi(z)}\) be some one \(m\)-th root of \(\phi(z)\), and let \(t = z \psi(z)\). Then we have that
\[
W - W_0 = z^m \psi(z) \psi = t^m \psi \quad \text{and} \quad \frac{\partial t}{\partial z} = \psi(z) + z \psi'(z)
\]
or
\[
\frac{\partial t}{\partial z} = \psi(z) + z \psi'(z) \quad \text{is not zero at } z = 0, \quad \text{since } \psi(0) \neq 0.
\]
Therefore, there exists a function \(z = H(t)\), the single valued inverse of \(t = z \psi(z)\). Then any curve through \(W = W_0\) is mapped conformally onto \(m\) equally spaced curves at \(t = 0\), and these \(m\) curves are mapped conformally onto \(m\) equally spaced curves at \(z = 0\). Hence, any curve through \(W = W_0\) is mapped conformally onto \(m\) equally spaced curves at \(z = 0\).

Now the locus \(|F(z)| = |F(0)|\) in the neighborhood of the origin is the image in the \(z\)-plane of the circle (degenerate when \(F(0) = 0\) \(|W| = |F(0)|\), and the locus \(\arg [F(z)] = \arg [F(0)]\) is the image in the \(z\)-plane of the line \(\arg W = \arg [F(0)]\). If \(F(0), F'(0) \neq 0\), then in a
neighborhood of \( z = 0 \) the loci \(|F(z)| = |F(0)|\) and 
\[ \arg [F(z)] = \arg [F(0)] \]
consist of two Jordan arcs through the origin, and the two loci are orthogonal there. If \( F'(z) \) 
has an \((n-1)\)-fold zero at \( z = 0 \) but \( F(0) \neq 0 \), each of the loci consists in the 
neighborhood of the origin of \( m \) Jordan arcs through the origin making successive angles \( \frac{2\pi}{m} \) there.

The arcs of \(|F(z)| = |F(0)|\) bisect the angles between successive 
arc of \( \arg [F(z)] = \arg [F(0)] \). That is, the projections 
and therefore the tangents to the slope and level lines 
intersect in \( 2m \) equal angles.

Note that if \( F(z) \) has a \( m \)-fold zero at \( z = z_0^* \), then in a 
neighborhood of \( z = z_0^* \) the level curve is a point, and through 
\((z_0^*, 0)\) go \( m \) slope lines whose tangents intersect in \( m \) equal angles.

(2.5) For the special case when \( F(z) \) is a polynomial of 
the form \( F(z) = a_0(z-z_1)(z-z_2)\cdots(z-z_n) \) the projections of the 
level curves on the \( xy \)-plane, \(|F(z)| = C = |a_0| \), the distances from \( z \) to the \( z_j \)'s are called lemniscates and the \( z_j \)'s are called poles [3]. In 
this case \(|F(z)| = C \) is the locus of all points \( z \) such that the product of the distances from \( z \) to the \( z_j \)'s is a constant \( \frac{C}{|a_0|} \).

Now \(|F(z_j)| = 0\); and \(|F(z)|\) becomes infinite as \( z \) becomes 
infinite. Then \(|F(z)| = C \) separates the points \( z_j \) from the 
point at infinity. For any \( C \) the lemniscate consists of 
\( r \) closed Jordan curves \( J_1, \ldots, J_r \), \( 1 \leq r \leq n \).
These Jordan curves are mutually exterior (except that a point \( z \) such that \( F'(z) = 0 \) may belong to several such curves).

**Proof:** Assume that this is not true. Then there exists \( J_1 \) and \( J_2 \), curves of a lemniscate \( |F(z)| = C \), and such that \( J_1 \) is strictly interior to \( J_2 \). Call the annular region between \( J_1 \) and \( J_2 \), \( D \). Then from (1.3), \( |F(z)| < C \) for all \( z \) in \( D \). Again from (1.3) this implies that the region inside \( J_1 \) contains a pole, which is a contradiction. Hence, for any \( C \) the Jordan curves are mutually exterior except possibly at points \( z \) such that \( F'(z) = 0 \).

Let \( F'(z) \) be of the form:

\[
F'(z) = n a_0 (z - z_1')^{p_1} (z - z_2')^{p_2} \cdots (z - z_q')^{p_q}
\]

the \( z_j' \)'s distinct, \( \sum_{j=1}^{q} p_j = n - 1 \). Arrange the \( z_j' \)'s so that

\[
|F(z_1')| < |F(z_2')| < \cdots < |F(z_q')|.
\]

From (2.h) the lemniscate \( |F(z)| = |F(z_j')| \neq 0 \) is such that through \( z_j' \) go \( p_j + 1 \) Jordan curves whose tangents at \( z_j' \) are equally spaced.

We can now describe the change in the lemniscate

\[
|F(z)| = C \text{ as } C \text{ increases from zero to infinity. Assume the } z_j' \text{'s are distinct. For } C < |F(z_1')| \text{ the lemniscate consists of } n \text{ small ovals each containing a } z_j. \text{ When } C = |F(z_1')| \text{ the lemniscate consists of } p_1 + 1 \text{ intersecting}
Jordan curves and $n - p_l - 1$ mutually exterior ones. For $C$ such that $|F(z_1')| < C < |F(z_2')|$, $|F(z)| = C$ consists of $n - p_l$ mutually exterior Jordan curves. For $C$ such that $|F(z_k')| < C < |F(z_{k+1}')|$, $k < q$, $|F(z)| = C$ consists of $n - (p_k + p_{k-1} + \ldots + p_1)$ mutually exterior Jordan curves. Finally, for all $C$ such that $C > |F(z_q')|$, $|F(z)| = C$ consists of one Jordan curve.
3. **GAUSSIAN AND MEAN CURVATURE**

From a well known result of differential geometry [2], the Gaussian curvature $K_{\gamma}$ and the mean curvature $k_{\gamma}$ at a point $P$ of a surface $\gamma = \gamma(x,y)$ are given by the formulas

\[(3.1.1) \quad K_{\gamma} = \frac{r \frac{\partial^2}{\partial s^2} - s^2}{(1 + p^2 + q^2)^2}\]

\[(3.1.2) \quad k_{\gamma} = \frac{(1 + q^2) r - 2 p \frac{\partial}{\partial s} s + (1 + q^2) \frac{\partial}{\partial s}}{2(1 + p^2 + q^2)^{3/2}}\]

where $p$, $q$, $r$, $s$, and $t$ are the partial derivatives of $\gamma = |F(z)|$ as defined in (1.1).

If we write $F$ in terms of $f = \alpha + i\beta$ then $\gamma = \alpha^2 + \beta^2 = 5^2$.

And similar to (1.2.3)

\[p = 2(\alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial x})\]

\[q = 2(\beta \frac{\partial}{\partial x} - \alpha \frac{\partial}{\partial x})\]

\[r = 2[(\alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial x}) + (\alpha^2_x + \beta^2_x)]\]

\[s = 2[\beta \frac{\partial}{\partial x} - \alpha \frac{\partial}{\partial x}]\]

\[t = 2[(\alpha^2_x + \beta^2_x) - (\alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial x})]\]

Then

\[1 + p^2 + q^2 = 1 + 4(\alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial x})^2 + 4(\beta \frac{\partial}{\partial x} - \alpha \frac{\partial}{\partial x})^2\]

Simplifying:

\[1 + p^2 + q^2 = 1 + 4(\alpha^2 + \beta^2)(\alpha^2_x + \beta^2_x)\]
Then

\[(3.1.3) \quad 1 + p^2 + q^2 = 1 + 4s^2 \tilde{s}'^2\]

Also

\[rt - s^2 = 4 \left[ (\alpha \beta_{xx} + \beta \beta_{xx}) + (\alpha_x^2 + \beta_x^2) \right] \left[ (\alpha_x^2 + \beta_x^2) - (\alpha \beta_{xx} + \beta \beta_{xx}) \right] - \tilde{t}_t (\beta \alpha_{xx} - \alpha \beta_{xx})^2.\]

Simplifying:

\[rt - s^2 = 4 \left[ (\alpha_x^2 + \beta_x^2) - (\alpha_x^2 + \beta_x^2) (\alpha_{xx} - \alpha \beta_{xx}) \right] \]

Then

\[(3.1.4) \quad rt - s^2 = 4 \left( \frac{s}{t} \frac{\dot{t}}{\dot{s}} - \frac{\dot{s}}{s} s'^2 \right)\]

Substituting (3.1.3) and (3.1.4) into (3.1.1) we have

\[(3.2.1) \quad K \gamma = \frac{4 \left( \frac{s}{t} - \frac{s'}{s} s'^2 \right)}{(1 + 4 \frac{s}{t} s'^2)^2} \]

In terms of \( F(z) \) \( s^2 = |F|, \frac{s'}{s} = \frac{F'}{\sqrt{F}}, \) and \( \frac{s''}{s} = \frac{|2 F F'' - F'|^2}{4 F^{3/2}}. \)

Substituting in (3.2.1) we have

\[K \gamma = \frac{|F'|^4 - 4 FF'' - F'^2|^2}{4 F^{3/2} (1 + 1F')^2}, \]

or in terms of \( F \) and its conjugate \( \bar{F} \)

\[(3.2.2) \quad K \gamma = \frac{-4 FF' \bar{F} F'' + 2 F'^2 \bar{F} F'' + 2 F''^2 F F'}{4 F^{3/2} (1 + 1F')^2} \]

or

\[K \gamma = \frac{-|F''|^2 + \frac{1}{2} \left( \frac{F'^2 \bar{F}''}{F} + \frac{F''^2 F}{\bar{F}} \right)}{(1 + 1F'^2)^2}. \]
Factoring $|F^n|^2$ and simplifying:

$$(3.2.3) \quad k_\gamma = \frac{|F''|^2 \left[ \mathcal{R} \left( \frac{F'}{F''} \right) - 1 \right]}{(1 + |F'|^2)^2}.$$  

If we multiply and divide (3.2.2) by $|F'|^4$ and simplify we have

$$(3.2.4) \quad k_\gamma = \frac{|F|^2 \left[ \mathcal{R} \left( \frac{F F''}{F'^2} \right) - |F F''|^2 \right]}{(1 + |F'|^2)^2}.$$  

From (1.2.1)

$$p^2 + q^2 = \gamma(r + t).$$

Substituting in (3.1.2) we have

$$(3.3.1) \quad k_\gamma = \frac{p^2 + q^2}{\gamma} \left[ 1 + \frac{r q - \gamma t - \gamma p^2 - 2 p q \gamma s}{\gamma p^2 + q^2} \right] \frac{\gamma p^2 + q^2}{2 \left(1 + |F'|^2\right)^{3/2}}.$$  

Consider

$$N = r \gamma q^2 + t \gamma p^2 - 2 p q \gamma s.$$  

From (1.2) substituting for $p, q, r, t$ and $\gamma'^2$ we have

$$N = q^2 [u_{xx} + v v_{xx} + \gamma'^2 - p^2] + p^2 [-(u_{xx} + v v_{xx}) + \gamma'^2 - q^2] - 2 p (u_{xx} + v v_{xx} - q) .$$

Simplifying

$$N = \gamma'^2 + (u_{xx} + v v_{xx})(q^2 - p^2) + 2 p (u v_{xx} - v u_{xx}) .$$
Substituting for \( p \) and \( \dot{q} \) and simplifying:

\[
(3.5.2) \quad \kappa = |F'|^2 + u_{xx}[-u(u_x^2 - v_x^2) - 2v_uxv_x] + v_{xx}[v(u_x^2 - v_x^2) - 2u_uxv_x].
\]

Substituting for \((3.5.2)\) in \((3.5.1)\) we have

\[
(3.5.3) \quad \kappa = \frac{1}{2} \frac{|F'|^2}{|F|} \left[ 1 + \frac{1 + |F'|^2}{1 + |F'|^2} \right] \frac{1 + \frac{A u_{xx} + B v_{xx}}{|F'|^2}}{1 + |F'|^2}.
\]

where \( A = -u(u_x^2 - v_x^2) - 2u_uxv_x \)

\[B = v(u_x^2 - v_x^2) - 2u_uxv_x.\]

Now consider \( R\left( \frac{F F''}{F'^2} \right) = \frac{1}{|F'|^2} R\left( \frac{F F''}{F'^2} \right) \)

\[= \frac{1}{|F'|^2} R\left( \frac{(\alpha + i \beta)(\alpha_x + i \beta_x)(\alpha_x - i \beta_x)^2}{|F'|^2} \right) \]

\[= \frac{1}{|F'|^2} (-A u_x v_x - B v_x^2).\]

Substituting in \((3.5.3)\) we finally get

\[
(3.5.4) \quad \kappa = \frac{1}{2} \frac{|F'|^2}{|F|} \left[ 1 + \frac{1 + |F'|^2}{1 + |F'|^2} \right] \frac{1 + \frac{1 - R\left( \frac{F F''}{F'^2} \right)}{|F'|^2}}{1 + |F'|^2}.
\]

**Theorem (3.4):** If \( w = F(z) \) is an analytic function such that at every point \((x, y, z)\) on its modular surface \( Y(x, y) = |F(z)| \)

the Gaussian curvature \( \kappa \) \((x, y) = 0, \) then

\[W = C, \quad W = e^{\beta z} , \quad \text{or} \quad W = C(z + b).\]

**Proof:** Let \( w = \left| F(z) \right|^2, \) then from \((3.2.1)\)
If the absolute value of two analytic functions are identical, then their logarithms differ by an imaginary constant. Hence, one function is equal to a constant times the other function, where the modulus of the constant is one. Applying this result to (3.4.1), we have

\[(3.4.2) \quad \xi'' = c_1 \xi'' \cdot \]

Let \( p = \frac{\partial f}{\partial z} \), then \( \frac{\partial^2 f}{\partial z^2} = \frac{\partial^2 f}{\partial \xi^2} \). Substituting in (3.4.2) we have \( p^2 = c_1 f p \frac{\partial f}{\partial \xi} \) or \( p(p - c_1 f \frac{\partial f}{\partial \xi}) = 0 \).

Now \( p = 0 \) implies \( f = A_1 \) is a solution of (3.4.2), and \( \frac{\partial f}{\partial \xi} = c_2 \frac{\partial f}{\partial \xi} \) implies \( p = A_2 \xi^3 \), implying \( \frac{\partial f}{\partial \xi} = A_2 \xi \).

Hence \( f = A_2 (z + b)^4 \), \( f = A_1 e^{A_2 z} \), and \( f = A_1 \) are the only solutions of (3.4.2).

If \( f = A_1 \) (or \( \xi = 0 \)), then \( \xi' = 0 \), \( \xi'' = 0 \), and therefore \( \xi'' = 0 \).

If \( f = A_1 e^{A_2 z} \) (or \( \xi = 0 \)), then \( \xi' = |A_1 A_2 e^{A_2 z}| \).
\[ \xi'' = |A_4 A_2^2 e^{A_2^2 z}|, \text{ and} \]

\[ \xi'^2 - \xi \xi'' = |A_4 A_2^2 e^{2A_2 z}| - |A_4 A_2^2 e^{A_2 z}| |A_4 A_2^2 e^{A_2 z}| = 0. \]

Therefore \( K \xi \equiv 0 \).

If \( f = A_3^2 (z + b)^4 \), then \( \xi' = |C_4 A_3^2 (z + b)^4 - 1|, \)

\[ \xi'' = |C_4 (C_4 - 1) A_3^2 (z + b)^4 - 2|, \text{ and} \]

\[ \xi'^2 - \xi \xi'' = |C_4^2 |A_2^2| (z + b)^4 - 2|C_4| |C_4 - 1||A_2^2|^2 (z + b)^4 - 2 = 0. \]

Now \( |C_4^2| - |C_4| |C_4 - 1| = 0 \text{ if and only if } |C_4| = 0 \text{ or} \)

\[ |C_4| - |C_4 - 1| = 0. \text{ If } |C_4| = 0, \text{ then } f \equiv A. \text{ If} \]

\[ |C_4| - |C_4 - 1| = 0, \text{ then } C_4^2 = C_4^2 - 2C_4 + 1 \text{ or } C_4 = \frac{1}{2}. \text{ That} \]

is, when \( f = A_2^2 (z + b)^1 \) or \( w = C(z + b), \text{ then } K \xi \equiv 0 \)

(except at \( z = -b \) where \( F(-b) = 0 \) and \( K \xi \) is undefined).

Hence we have Theorem (3.4).

(3.5) Curvature at finite zeros of \( F, F', \) and \( F''. \)

Without loss of generality we shall consider the
value of the Gaussian curvature \( K \xi \) and the mean curvature
\( K \gamma \) at \( z = 0 \) for different forms of \( F(z) \). We shall exclude
from consideration \( F(z) = A \) and \( F(z) = C(z + b) \) in what
follows. (See Theorem (3.4)).
If \( F, F', \) and \( F'' \) are not zero at \( z = 0 \), then

\[ K_\chi(0) \equiv 0 \text{ if and only if } R \left( \frac{F'}{FF'} \right) \equiv 1 \text{ (from (3.2.3))}. \]

Note that \( R \left( \frac{F'}{FF'} \right) = 0 \) is a level curve of a harmonic function.
And \( k_\chi(0) \equiv 0 \) if and only if \( R \left( \frac{F'}{FF'} \right) \equiv \frac{1}{|F'/F|^2} \equiv 1 \text{ (from (3.3.1)).} \]

If \( P(0), P'(0), \) and \( P^{(m)}(0) \not\equiv 0 \) and

\[ P''(0) = \ldots = P^{(m-1)}(0) = 0, \]

then \( K_\chi(0) = 0 \text{ (from (3.2.4))} \) and \( k_\chi(0) > 0 \text{ (from (3.3.4)).} \) In a deleted neighborhood of \( z = 0 \) there are \( m - 2 \) regions of positive Gaussian curvature, \( m - 2 \) regions of negative Gaussian curvature, and \( 2m - 4 \) lines of zero Gaussian curvature.

**Proof:** In a neighborhood of \( z = 0 \), \( P(z) \) is of the form:

\[ P = a_0 + a_1 z + a_m z^m + a_{m+1} z^{m+1} + \ldots, \quad a_0, a_1, a_m \not\equiv 0 \text{ and } m > 2. \]

Then

\[ P' = a_1 + ma_m z^{m-1} + \ldots, \]

\[ P'' = m(m-1)a_m z^{m-2} + \ldots. \]

And

\[ \left( \frac{F'^2}{FF''} \right) = \frac{a_1^2 + \lambda m a_1 a_m z^{m-1} + \ldots}{a_0 m(m-1) a_m z^{m-2} + \ldots} \]

\[ = \frac{1}{m(m-1)} \cdot \frac{a_1^2}{a_0 a_m} z^{-(m-2)} + O(z). \]
Then the function \( W = \frac{F'(z)^2}{F(z) F''(z)} \) has a pole of order \( m - 2 \) at the origin. If we locally uniformize \( W = W(z) \) in a neighborhood of \( z = 0 \) and \( W = \infty \), in a manner similar to that of (2.h), we see that the line \( G(W) = 1 \) is mapped on \( m - 2 \) equally spaced lines through \( z = 0 \). That is, there are \( 2m - 4 \) equally spaced lines emanating from the origin along which \( G \left( \frac{F'}{F} \right) = 1 \). The regions \( G \left( \frac{F'}{F} \right) > 1 \) and \( G \left( \frac{F'}{F} \right) < 1 \) are each mapped on \( m - 2 \) alternate regions bounded by the \( 2m - 4 \) lines along which \( G \left( \frac{F'}{F} \right) = 1 \).

From (3.2.3), this implies that in a deleted neighborhood of \( z = 0 \) there are \( m - 2 \) regions of positive curvature, \( m - 2 \) regions of negative curvature, and \( 2m - 4 \) lines of zero curvature.

\[ z\text{-plane} \]
(3.5.3) If \( P(0), P'(0) \neq 0 \) and \( P''(0) = 0 \), then
\[ k_\gamma(0) = 0 \] (from (3.3.4)) and \( \mathcal{R} \left( \frac{F'^2}{FF''} \right) = 0 \). Hence \( k_\gamma(0) < 0 \) (from (3.2.3)) and also \( k_\gamma(z) < 0 \) for \( z \) sufficiently small, since \( k(z) \) is a continuous function of \( z \) in this region.

(3.5.4) If \( P(0) = 0, P'(0), P''(0) \neq 0 \), then \( k_\gamma(0) \) and \( k_\gamma(0) \) are undefined (from (3.2.3) and (3.5.3)). However, in a deleted neighborhood of \( z = 0 \) there is one region of positive Gaussian curvature, one of negative Gaussian curvature, and two lines of zero Gaussian curvature.

Proof: In a neighborhood of \( z = 0 \), \( P(z) \) has the form:
\[ P = a_1 z + a_2 z^2 + \ldots, \quad a_1, a_2 \neq 0. \]
Then
\[ P' = a_1 + 2a_2 z \ldots, \]
\[ P'' = 2a_2 + \ldots, \]
and
\[ \frac{F'^2}{FF''} = \frac{a_1^2 + 4 a a_1 a_2 z + \ldots}{2 a, a_2 z + \ldots} = \frac{a_1}{2 a_2} \frac{z^{-1}}{z}. \]
Then \( \frac{F'(z)^2}{F(z)F''(z)} \) has a pole of order one at \( z = 0 \).
Therefore, as in (3.5.2), there is one region of positive curvature, one region of negative curvature, and two lines of zero curvature.
(3.5.5) If $F(0) = F^{(m)}(0) \neq 0$, $m > 2$, and $F^{(m)}(0) = F^{(n)}(0) = 0$, then $K_{y}(0) = k_{y}(0) = 0$ (from (3.2.3) and (3.3.4)). However, in a deleted neighborhood of $z = 0$, $K_{y}(z) < 0$.

Proof: In a neighborhood of $z = 0$, $F(z)$ has the form:

$$F = a_{0} + a_{m}z^{m} + a_{m+1}z^{m+1} + \cdots , \quad a_{0}, a_{m} \neq 0, m > 2.$$ 

Then

$$\frac{F^{,1}}{F^{,m}} = \frac{m^{2}a_{m} Z^{m-2} + \cdots}{m (m-1) a_{0} A_{m} Z^{m-2} + \cdots}.$$ 

This implies that

$$\left|\frac{F^{,1}}{F^{,m}}\right| < 1$$

and therefore $R \left( \frac{F^{,1}}{F^{,m}} \right) < 1$ for $z$ sufficiently small.

Hence $K_{y}(z) < 0$ (from (3.2.3)) in a deleted neighborhood of $z = 0$.

(3.5.6) If $F(0) = F^{n}(0) = F^{(m)}(0) = \cdots = F^{(m-1)}(0) = 0$ and $F^{(m)}(0), F^{(m)}(0) \neq 0$, $m > 2$, then $K_{y}(0)$ and $k_{y}(0)$ are undefined (from (3.2.3) and (3.3.4)). However, in a deleted neighborhood of $z = 0$ there are $m - 1$ regions of positive Gaussian curvature, $m - 1$ regions of negative Gaussian curvature, and $2m - 2$ lines of zero Gaussian curvature.

Proof: In a neighborhood of $z = 0$,
\[ F = a_1 z + a_m z^m + a_{m+1} z^{m+1} + \ldots, \quad a_1, a_m \neq 0, \ m > 2. \]

Then
\[
\frac{F'}{F''} = \frac{a_1^2 + 2 m a_1 a_m z^{m-1} + \ldots}{m (m-1) a_1 a_m z^{m-1} + \ldots}.
\]

or
\[
\frac{F'}{F''} = \frac{1}{m (m-1)} \frac{a_1}{a_m} z^{-(m-1)} + O(1).
\]

Then (as in (3.5.2)) there are \( m - 1 \) regions of positive curvature, \( m - 1 \) regions of negative curvature, and \( 2m - 2 \) lines of zero curvature in the deleted neighborhood.

(3.5.7) If \( F(0) = F'(0) = 0 \) and \( F''(0) \neq 0 \) then
\[ K_\gamma(0) = 4 \lambda_2^2 ( > 0) \] and \[ K_\delta(0) = 2 \lambda_2 ( > 0). \]

**Proof:** In a neighborhood of \( z = 0 \),
\[ F = a_2 z^2 + a_m z^m + \ldots, \quad a_2, a_m \neq 0, \ m > 2. \]

Then
\[ F' = 2a_2 z + m a_m z^m - 1 + \ldots, \]
\[ F'' = 2a_2 + m(m-1) a_m z^m + \ldots, \]
and
\[
\frac{F'}{F''} = \frac{4 a_2^2 Z^2 + O(z^m)}{2 a_2^2 Z^2 + O(z^m)}.
\]

Also
\[
\frac{F'}{F''} = 2 + O(z^{-2}), \quad m > 2.
\]
\[ F'_{12} = h a_{12} z^2 + O(z^3), \]
\[ F'_{12} = h a_{12}^2 + O(z^{m-2}). \]

Then
\[ R \left( \frac{F'(0)^2}{F(0) F''(0)} \right) = 2 \quad \text{and} \quad |F''(0)|^2 = h a_{12}^2. \]

Hence \( k_\gamma(0) = h a_{12}^2 \) and \( k_\delta(0) = 2 a_{12} \) (from (3.2.5) and (3.3.1)).

(3.5.8) If \( F(0) = F'(0) = \ldots = F^{(m-1)}(0) = 0 \)
and \( F^{(m)}(0) \neq 0, \) \( m > 2, \) then \( k_\gamma(0) = k_\delta(0) = 0. \) In a deleted neighborhood of \( z = 0, \) \( k_\gamma(z) > 0. \)

**Proof:** In a neighborhood of \( z = 0, \)
\[ F = a_m z^m + a_{m+1} z^{m+1} + \ldots, \quad m > 2. \]

Then
\[ F' = m a_m z^{m-1} + \ldots, \]
\[ F'' = m(m-1) a_m z^{m-2} + \ldots. \]

and
\[ \frac{F'_{12}^2}{F F''} = \frac{m^2 a_m^2 z^{2m-2} + O(z^{2m-1})}{m (m-1) a_m z^{2m-2} + \ldots} = \frac{m}{m-1} \left[ 1 + O(z) \right] \]
From (3.2.3) and (3.3.4), $K_\gamma(0) = k_\gamma(0) = 0$. Also, in a deleted neighborhood of $z = 0$, for $z$ sufficiently small, 

$$R \left( \frac{F''}{F'} \right) = \frac{m}{m-1} [1 + \rho] > 1.$$ 

Hence (from (3.2.3)) $K_\gamma(z) > 0$ in a deleted neighborhood of $z = 0$.

The results of (3.5) are shown in Table 3.5.
<table>
<thead>
<tr>
<th>$F(z)$</th>
<th>$F'(z)$</th>
<th>$F''(z)$</th>
<th><strong>GAUSSIAN CURVATURE</strong></th>
<th><strong>MEAN CURVATURE</strong></th>
<th><strong>GAUSSIAN CURVATURE $K_2(z)$ IN A DELETED NEIGHBORHOOD OF $z = z_0$</strong></th>
<th><strong>SECT.</strong></th>
</tr>
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<td>*</td>
<td>*</td>
<td>$K_2(z_{o}) \geq 0 \Leftrightarrow k_2(z_{o}) \geq 0 \Leftrightarrow$</td>
<td>$\mathcal{R}(\frac{F'^2}{F'}) \geq 1$ at $z = z_0$</td>
<td>$\mathcal{R}(\frac{F'^2}{F''}) - \frac{1}{</td>
<td>F'</td>
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<td>*</td>
<td>0</td>
<td>$K_2(z_{o}) = 0$</td>
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<td>$K_2(z_{o}) &lt; 0$</td>
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<td>0</td>
<td>*</td>
<td>$K_2(z_{o}) &lt; 0$</td>
<td>$k_2(z_{o}) = 0$</td>
<td>$K_2(z_{o}) &lt; 0$</td>
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</tr>
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<td>*</td>
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<td>undefined</td>
<td>2 lines of zero curvature</td>
<td>.4</td>
</tr>
<tr>
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<td>0</td>
<td>0</td>
<td>$K_2(z_{o}) = 0$</td>
<td>$k_2(z_{o}) = 0$</td>
<td>$K_2(z_{o}) &lt; 0$</td>
<td>.5</td>
</tr>
<tr>
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<td>undefined</td>
<td>2 lines of zero curvature</td>
<td>.6</td>
</tr>
<tr>
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<td>0</td>
<td>*</td>
<td>$K_2(z_{o}) = 4A_2^2 &gt; 0$</td>
<td>$k_2(z_{o}) = 2A_2 &gt; 0$</td>
<td>$K_2(z_{o}) &gt; 0$</td>
<td>.7</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$K_2(z_{o}) = 0$</td>
<td>$k_2(z_{o}) = 0$</td>
<td>$K_2(z_{o}) &gt; 0$</td>
<td>.8</td>
</tr>
</tbody>
</table>

* means not zero

**TABLE 3.5**
(3.6) Gaussian curvature in a neighborhood of a finite pole and the point at infinity (again excluding \( F(z) = C \), \( F(z) = Ce^{b^2} \), and \( F(z) = C(z + b) \) which were considered in (3.4)).

(3.6.1) \( K_\gamma(z) < 0 \) in a deleted neighborhood of a finite pole.

**Proof:** Without loss of generality assume that the pole is at \( z = 0 \). Then in a neighborhood of the pole, \( F(z) \) has the form:

\[
F = a_m z^{-m} + \ldots, \quad a_m \neq 0.
\]

Then

\[
F' = -ma_m z^{-m-1} + \ldots, \quad F'' = m(m + 1)a_m z^{-m-2} + \ldots
\]

and

\[
\frac{F'^2}{F'F''} = \frac{m^2 a_m^2 z^{-2m-2} + \ldots}{m(m+1)a_m^2 z^{-2m-2} + \ldots}
\]

or

\[
\frac{F'^2}{F'F''} = \frac{m}{m+1} [1 + O(z)]
\]

Hence \( |\frac{F'^2}{F'F''}| < 1 \) implying \( R (\frac{F'^2}{F'F''}) < 1 \) in a neighborhood of \( z = 0 \). Therefore \( K_\gamma(z) < 0 \) in a deleted neighborhood of \( z = 0 \) (from (3.2.5)).

(3.6.2) \( K_\gamma(z) < 0 \) in some neighborhood of \( z = \infty \) if
$z = \infty$ is a regular point for $F(z)$.

**Proof:** For all $z$ sufficiently large $F(z)$ has the form:

$$F = a_0 + a_m z^{-m} + \ldots, \quad a_m \neq 0.$$  

Then

$$F' = -ma_m z^{-m-1} + \ldots$$

$$F'' = m(m + 1)a_m z^{-m-2} + \ldots$$

and

$$\frac{F'}{F} \sim \frac{m^2 a_m z^{-2m-2} + \ldots}{a_0 m (m+1)a_m z^{-m-2} + \ldots}$$

1) If $a_0 \neq 0$, then for $z$ sufficiently large $|\frac{F'}{F''}| < 1$, implying $R(\frac{F'}{F''}) < 1$.

2) If $a_0 = 0$, then for $z$ sufficiently large $\left|\frac{F'}{F''}\right| = \frac{m^n}{m+n+1} [1 + \rho] < 1$, implying $R(\frac{F'}{F''}) < 1$.

Hence $K_\gamma(z) < 0$ (from (3.2.3)).

(3.6.3) $K_\gamma > 0$ in some neighborhood of a repeated pole at infinity.

**Proof:** For all $z$ sufficiently large $F(z)$ has the form:

$$F = a_m z^m + \ldots, \quad a_m \neq 0, m > 1.$$  

Then

$$F' = ma_m z^{m-1}, \quad F'' = m(m - 1)a_m z^{m-2} + \ldots$$

and

$$\frac{F'}{F} = \frac{m^2 a_m z^{2m-2} + \ldots}{m (m-1) a_m z^{2m-2} + \ldots} = \frac{m}{m-1} + O\left(\frac{1}{z}\right)$$
Then for \( z \) sufficiently large, \( \Re \left( \frac{F'}{F} \right) > 1 \). Hence, from (3.2.3), \( \kappa_\gamma(z) > 0 \) for all \( z \) sufficiently large.

(3.6.4) If \( F(z) \) has a simple pole at infinity, then for all \( z \) sufficiently large there are \( m + 1 \) regions of positive Gaussian curvature, \( m + 1 \) regions of negative Gaussian curvature, and \( 2m + 2 \) lines of zero Gaussian curvature, where \( m \) is such that

\[
F(z) = a_1 z + a_0 + a_{-m} z^{-m} + \ldots; \quad a_1, a_{-m} \neq 0, \; m > 0.
\]

Proof:

\[
F' = a_{-1} z^{-m} + \ldots,
\]

and

\[
\frac{F'}{F} = \frac{F''}{F} = \frac{a_i^2 - 2m a_i a_{-m} z^{-m} + \ldots}{a_i a_m z^{-m} + \ldots} = \frac{1}{m(m+1)} \cdot \frac{a_i}{a_m} z^{m+1} + O(1).
\]

Then \( \frac{F'(z)^2}{F(z) F''(z)} \) has a pole of order \( m + 1 \) at \( z = \infty \). Hence, as in (3.5.2), there are \( m + 1 \) regions of positive curvature, \( m + 1 \) regions of negative curvature, and \( 2m + 2 \) lines of zero curvature for \( z \) sufficiently large.

(3.7) Let us consider the special case when \( F(z) \) is a polynomial, say \( F(z) = C_0 z^m + C_1 z^{m-1} + \ldots + C_{m-1} z + C_m \).
\( c_0 \neq 0, n > 2 \), and describe the various regions of Gaussian curvature.

For \( z \) sufficiently large \( K_\gamma(z) > 0 \) (from (3.6.5)).

Hence the regions where \( K_\gamma(z) < 0 \) are bounded. If we exclude all isolated points such that \( K_\gamma(z) = 0 \) (from Table 3.5 this means exclude all \( z_o \) such that both \( F'(z_o) \) and \( F''(z_o) = 0 \)), then the locus of all points such that \( K_\gamma(z) = 0 \) is the locus of all points such that \( \mathcal{R}(\frac{F'^2}{F F''}) = 1 \), along with the points where \( F'' = 0 \) and \( F, F' \neq 0 \). This is a curve in the \( xy \) plane.

If crosses itself at points \( z_o \) where \( F, F' \neq 0 \) and \( F'' = 0 \) (from (3.5.2)); and it abuts on singularities of \( K_\gamma \) where \( F = 0 \) and \( F' \neq 0 \) (see (3.5.4) and (3.5.6)).

Note that any point where the line of zero curvature crosses itself is a pole of \( \mathcal{R}(\frac{F'^2}{F F''}) \). If it were not, then the harmonic function \( \mathcal{R}(\frac{F'^2}{F F''}) - 1 \) would be zero along a closed curve. This implies \( \mathcal{R}(\frac{F'^2}{F F''}) = 1 \) or \( F'^2 = CFF'' \)

which is considered in (3.4) and which we are now excluding.

\[ (3.8) \quad \text{The relationship between } K_\gamma \text{ and } K_\gamma^2. \]

From (3.2.1):

\[ (3.8.1) \quad K_\gamma = \frac{4(z'^4 - z^2 z''^2)}{(1 + 4z^2 z'^2)^2} \quad \text{and} \quad K_\gamma^2 = \frac{4(z'^4 - z^2 z''^2 z'^2)}{(1 + 4z^2 z'^2)^2} \]
Now \( \gamma = 25^2, \quad \gamma' = 255', \) and \( \gamma'' = |251' + 25f''|, \) implying

\[
K \gamma^2 = \frac{45^2 (45^2 - \gamma') (45^2 + \gamma'')}{(1 + 16 \gamma') \gamma''^2}
\]

Let \( 45^2 + \gamma'' = D. \) Then

\[
(3.8.2) \quad 25^2 = 255'' \leq D \leq 45^2 + |251' - 255''|
\]

(A) Suppose \( \gamma' \neq 0 \) which implies \( \gamma, \gamma' \neq 0. \) Then

1. \( K \gamma = 0 \Rightarrow 25^2 = 255'' \Rightarrow 0 \leq D \leq 45^2 \Rightarrow D > 0 \)

\[ \Rightarrow K \gamma^2 > 0 \]

2. \( K \gamma < 0 \Rightarrow 25^2 < 255'' \)

\[ \Rightarrow K \gamma^2 \text{ may be } \leq 0 \text{ for } 25^2 < 255'' < 35^2 \]

\[ K \gamma^2 < 0 \text{ for } 255'' > 35^2 \]

3. \( K \gamma > 0 \Rightarrow 25^2 > 255'' \Rightarrow 0 < D \leq 45^2 \Rightarrow D > 0 \)

\[ \Rightarrow K \gamma^2 > 0 \]

(B) Suppose \( \gamma' = 0 \) which implies \( 255' = 0. \) Then either

\( \gamma = 0 \) or \( \gamma' = 0. \) If \( \gamma = 0, K \gamma = 45^2 \) and \( K \gamma^2 = 0. \) If \( \gamma' = 0, K \gamma = -45^2 \gamma''^2 \) and \( K \gamma^2 = -45^2 \gamma''^2. \)

Then

1. \( K \gamma > 0 \Rightarrow \exists \gamma = 0 \Rightarrow K \gamma^2 = 0 \)

2. \( K \gamma = 0 \Rightarrow \exists \gamma' = 0 \Rightarrow K \gamma^2 = 0 \)

or \( \exists \gamma = \gamma'' = 0 \Rightarrow \gamma'' = 0 \)

\[ \Rightarrow K \gamma^2 = 0 \]

3. \( K \gamma < 0 \Rightarrow K \gamma = -45^2 \gamma''^2 < 0 \Rightarrow \exists \gamma, \gamma'' \neq 0. \)
Also $K_{\gamma^2} = -\gamma^2 \gamma'^2$. But 

$$\gamma^2 = \gamma'^2 \neq 0$$

and $\gamma'' = |2\gamma' + 2\gamma''|$. 

Since $\gamma' = 0$, $\gamma'' = |2\gamma''| = 2\gamma'' \neq 0$ 

$$\Rightarrow K_{\gamma^2} < 0.$$ 

Now $\gamma = \sqrt{\gamma}$, $\gamma' = \frac{\gamma'}{2\gamma}$, and $\gamma'' = \frac{1}{4\gamma^{3/2}} |2\gamma'' - \gamma'^2|$. 

Let $\beta = |2\gamma'' - \gamma'^2|$. Then: 

$$|2\gamma'' - \gamma'^2| \leq \beta \leq 2\gamma'' + \gamma'^2.$$ 

Substituting in (3.8.1) and simplifying: 

$$K_{\gamma} = \frac{(\gamma'^2 - \beta)(\gamma'^2 + \beta)}{\gamma^2 (1 + \gamma'^2)^2}.$$ 

Let $E = \gamma'^2 - \beta$. Then from (3.8.3): 

$$-2\gamma'' \leq E \leq \gamma'^2 - |2\gamma'' - \gamma'^2|.$$

(A) Suppose $\gamma' \neq 0$. Then 

(1) $K_{\gamma^2} = 0 \Rightarrow \gamma'^2 = \gamma\gamma'' \Rightarrow -2\gamma'^2 \leq E \leq 0$ 

$$\Rightarrow K_{\gamma} \leq 0.$$ 

(2) $K_{\gamma^2} < 0 \Rightarrow \gamma'^2 < \gamma\gamma'' \Rightarrow -2\gamma'' \leq E \leq 2\gamma'^2 - 2\gamma'' < 0$ 

$$\Rightarrow K_{\gamma} < 0.$$ 

(3) $K_{\gamma^2} > 0 \Rightarrow \gamma'^2 > \gamma\gamma'' \Rightarrow -2\gamma'' \leq E \leq 2\gamma''$ 

$$\Rightarrow K_{\gamma} \text{ may be } > 0.$$ 

(B) Suppose $\gamma' = 0$. Then from (3.8.1) and (3.8.1), 

$$K_{\gamma^2} = 4\gamma^2 K_{\gamma}$$

and $K_{\gamma^2} \leq 0$. 

(1) $K_{\gamma^2} < 0$ implies $\gamma$, $\gamma'' \neq 0$ which implies $\gamma \neq 0$. 

Now $\gamma' = 0$ and $\gamma \neq 0$ implying $\gamma' = 0$. $\gamma' = 0$
and \( \gamma'' \neq 0 \) imply \( \delta'' \neq 0 \). Hence from (3.8.2) part (B), \( K \gamma < 0 \).

(2) \( K \gamma^2 = 0 \) implies \( \gamma \) or \( \gamma'' = 0 \). For \( \gamma = 0 \), \( \delta = 0 \)
implies from (3.8.2) part (B) that \( K \gamma \neq 0 \). When
\( \gamma' = 0 \), \( \delta \) or \( \delta' = 0 \). From (3.8.2) part (B) \( \delta = 0 \)
implies \( K \gamma = \frac{1}{4} \delta'^2 \). Now \( \gamma'' = 0 \) and \( \delta = 0 \) imply \( \delta' = 0 \)
which implies \( K \gamma = 0 \). And \( \gamma'' = 0 \) and \( \delta' = 0 \) imply
\( 2 \delta'' = 0 \) which implies that \( K \gamma = -\frac{1}{4} \delta^2 \delta''^2 = 0 \).

Therefore, from (3.8.2) and (3.8.4) parts (A) and (B),

we have

For \( \gamma' \neq 0 \)

(1) \( K \gamma = 0 \Rightarrow K \gamma^2 \geq 0 \) \( K \gamma^2 = 0 \) \( K \gamma \leq 0 \)
(2) \( K \gamma < 0 \Rightarrow K \gamma^2 \) may be \( \gamma' = 0 \) \( K \gamma^2 < 0 \) \( K \gamma < 0 \)
(3) \( K \gamma > 0 \Rightarrow K \gamma^2 > 0 \) \( K \gamma^2 > 0 \) \( K \gamma \) may be \( \gamma' = 0 \)

For \( \gamma' = 0 \)

(1) \( K \gamma = 0 \Rightarrow K \gamma^2 = 0 \) \( K \gamma^2 = 0 \) \( K \gamma \geq 0 \)
(2) \( K \gamma < 0 \Rightarrow K \gamma^2 < 0 \) \( K \gamma^2 < 0 \) \( K \gamma < 0 \)
(3) \( K \gamma > 0 \Rightarrow K \gamma^2 = 0 \)

Or in tabular form:

\[
\begin{array}{ccc}
\delta' \neq 0 & K \gamma & K \gamma^2 \\
> 0 & = 0 & < 0 \\
< 0 & > 0 & = 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
\delta' = 0 & K \gamma & K \gamma^2 \\
> 0 & = 0 & < 0 \\
< 0 & > 0 & = 0 \\
\end{array}
\]
BIBLIOGRAPHY


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