MICROWAVE CAVITY RESONATORS FOR
A PARAMAGNETIC-RESONANCE SPECTROMETER

by

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1. INTRODUCTION

1.1 Purpose of This Investigation

The purpose of this investigation is to study, design, and build microwave cavity resonators for use at both room and low temperatures with an electron paramagnetic resonance (EPR) spectrometer. The resonant cavity provides the high frequency magnetic field necessary for resonance absorption in paramagnetic solids; and it must be carefully constructed such that it will have a high Q and low noise. A high Q is necessary because the spectrometer detector signal is directly proportional to Q and the minimum detectable signal is inversely proportional to Q. In order to reduce the noise due to interactions between the DC magnetic field and currents induced in the walls by the modulation field, the cavity must be constructed of a non-conducting material coated with a thin layer of silver. The problems connected with the design and construction of such a cavity are discussed in detail, and characteristics are given for the resulting resonators.

1.2 EPR Experiments

Electronic paramagnetism occurs in a system of charges when its electronic angular momentum, spin plus orbital, is non-vanishing. Such paramagnetism is found in several forms of matter. The most important of these for EPR spectroscopy are substances containing ions of the transition groups
with partly filled inner electron shells. Single crystals of these solids containing paramagnetic ions provide the most useful information. When paramagnetic-resonance measurements are performed with these crystals and the results are interpreted, an accurate description of the ground state and the effect of the crystalline fields on the energy levels of the paramagnetic ion is obtained. The results permit the calculation of such quantities as the specific heat and magnetic susceptibility at low temperatures, and parameters which give information about the fine structure of optical absorption spectra in solids.

If a paramagnetic substance is placed in a constant magnetic field, and subjected to a high frequency magnetic field perpendicular to the steady field, energy will be absorbed near the frequency for which the relation

\[ f_0 = \frac{geB_0}{4\pi m} \]

is satisfied. Here, \( B_0 \) is the value of the constant magnetic induction; \( e \) and \( m \) are the charge and mass of the electron, respectively; and \( g \) is the spectroscopic splitting factor. (For a single electron whose orbital angular momentum is zero, \( g \) is approximately two.) The frequency \( f_0 \), may be of the order of 10 kmc when \( B_0 \) is of the order of 0.3 webers/m².

Considerations of maximum sensitivity² and resolution³
reveal that one should operate at the highest possible frequency. However, as can be seen from equation (1), this would lead to impractical values of magnetic field. Also, there is a practical upper limit on the operating frequencies of microwave components. A closer look at all the issues then shows that the best compromise is in the region of X or K-Band frequencies. The spectrometer shown in Figure 1 is designed to operate at 9.3 kmc.

1.3 The Cavity's Role in the Spectrometer

A block diagram of the EPR spectrometer with which the cavity is to be used is shown in Figure 1. The klystron frequency is locked on to the resonant frequency of the cavity (in which is placed the paramagnetic sample) by means of the frequency stabilizer. The external magnetic field is slowly varied and superimposed upon this large field is a small modulation field varying at an audio rate. As the field is swept through a resonance line of the paramagnetic sample, energy is absorbed from the high-frequency field in the cavity. The signal appears at the crystal detector as a change in the reflected power from the cavity, varying in magnitude at an audio rate due to the modulation field. The output from the crystal is fed through the narrow band amplifier and phase-sensitive detector to the recorder. The signal there appears as the derivative of the resonance line.

The sensitivity and proper operation of this spectrometer
impose several requirements upon the cavity. These requirements are considered in detail in Part II.
II. REQUIREMENTS OF APPROPRIATE CAVITIES

2.1 Quality Factor - Q

The ratio of the energy stored in the cavity to the average power loss is an indication of the "quality" of the cavity. The quality factor, or Q, is defined as:

\[
Q \equiv \frac{\omega (\text{energy stored in cavity})}{\text{(average power loss)}} \equiv \frac{\omega U}{P_L}
\]

where \(\omega\) is the resonant angular frequency of the cavity. Thus, the Q of a cavity is defined in the same manner as the Q of a low-frequency, lumped-constant circuit. As in the case of a lumped-constant circuit, a high value of Q is usually desirable.

A high value of Q is desirable for cavities used in EPR experiments for three major reasons:

1) The detector signal is directly proportional to Q.
2) The minimum detectable signal is inversely proportional to Q.
3) The discriminator output to the klystron frequency stabilizer is directly proportional to Q.

2.2 Electromagnetic Field Near Sample

The high-frequency magnetic field necessary for the resonance absorption should be perpendicular to the large, steady magnetic field. The electric field at the position of the sample should be small so that dielectric losses in the sample and sample holder will not reduce the Q of the
cavity. Some means for placing the sample in the proper region of the cavity field pattern must be available. The holder should have low loss and low dielectric constant, and it should be small enough so that the fields are not significantly disturbed.

2.3 Unwanted Cavity Modes

Modes of oscillation other than the one with the appropriate field configuration should be eliminated near the frequency of operation. Not only does the possibility exist of confusing some other mode with the desired one, but a reduction in Q can result from coupling between modes when their resonance curves overlap.

2.4 Noise

It has been found that metallic cavities are sources of noise. This is due to interactions between the DC magnetic field and eddy currents induced in the walls by the modulation field (the so called "cavity effect"). Vibrations in the cavity walls produce noise at the modulation frequency which can be quite large. Various non-conducting materials coated with silver have been used to reduce this effect.\textsuperscript{7,8,9} The silver coating is made thick compared with a skin-depth at 9.3 kmc but thin compared with a skin-depth at the modulation frequency. (For example, for silver, a skin-depth at 9.3 kmc is approximately $6.64 \times 10^{-5}$ cm, while at the modu-
ation frequency of 10 kc, it is $6.4 \times 10^{-2}$ cm, or 25.2 thousandths of an inch. A silver coating one or two thousandths of an inch thick should then be adequate to reduce the cavity effect.)

Also, the surfaces of the cavity walls should be smooth and free from small flakes of metal that might produce noise.

2.5 Coupling of Waveguide to Cavity

An iris, or coupling hole, must be provided so that electromagnetic energy can be coupled from the waveguide into the cavity. The amount of energy coupled determines the load impedance as seen by the generator. There is an optimum value for the coupled energy which depends upon the type of detection system used. Thus, it is desirable that the coupling be adjustable. This can be accomplished by placing a capacitive tuning screw in the waveguide connected to the cavity.

2.6 Cavities at Various Temperatures

Ideally, one would construct a cavity that could be used at all temperatures. Of course, this may not be possible, and even if possible, it may not be practical. The features necessary at low temperatures that are unnecessary at room temperature might well be discarded in a room temperature cavity for the sake of convenience of assembly. Also, because of thermal expansion, a given cavity will not have the same resonant frequency at both room and low temperature.
At room temperature it is convenient to have a cavity that is easy to assemble and disassemble. Since there is no coolant with which to contend, the cavity does not have to be sealed. A hole in the end-plate can be provided through which the sample holder can be inserted without disconnecting the cavity.

When designing a cavity for use at liquid nitrogen temperatures (77.4° K), one must consider several factors. Allowances must be made for differences in expansion of dissimilar materials which are joined together. The cavity must be sealed so that liquid nitrogen cannot penetrate it. Since liquid nitrogen has a dielectric constant of 1.454,\(^{10}\) it would strongly affect the resonant frequency of the cavity. Although the resonant frequency of a dielectric-filled cavity can be easily calculated, any disturbance in the liquid, such as bubbling, would produce a change in dielectric constant and, consequently, noise. Furthermore, liquid nitrogen is lossy and the Q would be reduced.

Difficulties also arise in connection with the tuning of the waveguide for the purpose of adjusting the coupling to the cavity. Since the nitrogen must be excluded from the cavity, tuning screws cannot conveniently be placed in the guide below the level of the liquid.

At liquid helium temperatures (4.2° K), the problems associated with the dielectric constant become unimportant as the dielectric constant at 4.19° K is 1.048.\(^{11}\) The cavity
does not have to be sealed, but it is best to maintain the liquid level well above the cavity to avoid bubbling. The thermal expansion problem is essentially the same as at liquid nitrogen temperatures since most of the thermal contraction occurs between room temperature and 77° K. A slide screw tuner can be placed near the cavity because the exclusion of helium from the cavity is not necessary.
III. THEORY

3.1 The TE$_{011}$ Cylindrical Cavity

The TE$_{011}$ right-circular cylindrical cavity fulfills the primary requirements and is easy to construct compared with other shapes. It is constructed as a tube with attached end-plates. A sketch of the basic cavity and its field configuration is shown in Figure 2.

The high-frequency magnetic field necessary for the resonance absorption is provided by $H_z$, the axial component of $H$. The amplitude of $H_z$ is a maximum at the geometric center of the cavity, while the other field components, $H_r$ and $E_\phi$, are zero all along the axis. As both $H_r$ and $E_\phi$ vary with $r$ as a first-order Bessel function, they have small amplitudes near the center of the cavity. All field components are axially symmetric, i.e., independent of the angle $\phi$.

The Q of the TE$_{011}$ cavity is high, its theoretical value being about 31,000 at room temperature (q.v. Section 3.4). Although in practice, the Q may be only one-third the above figure, that is sufficiently high for EPR experiments; and at low temperatures, the Q increases due to an increase in conductivity of the silvered walls.

The TE$_{011}$ and TM$_{111}$ modes of a cylindrical cavity are degenerate, that is, they have the same resonant frequency. If the cavity is perfectly symmetrical, both modes will be excited. Some means for eliminating the TM$_{111}$ mode must be
found; this problem is considered in Section 4.1.

Energy is coupled into the cavity from the rectangular waveguide via a slot (the iris) in the end plate. The transverse component of the magnetic field in the waveguide excites the radial component of the magnetic field in the cavity (q.v. Section 4.1). Tuning screws are placed in the waveguide at strategic points to adjust the coupling factor to the desired value.

The paramagnetic sample is placed in a sample holder which positions it in the center of the cavity. Since the electric field amplitude is small near the cavity axis, the sample holder is placed along it to hold dielectric losses to a minimum.

3.2 Field Configurations

The electric and magnetic field configurations are given by the following equations, and are illustrated in Figure 2.12

\[
(1)\quad H_z = H_0 J_0 \left( \frac{S_{01} \Gamma}{a} \right) \sin \frac{\pi z}{d},
\]

\[
(2)\quad H_r = -\frac{\pi a}{S_{01} d} H_0 J_1 \left( \frac{S_{01} \Gamma}{a} \right) \cos \frac{\pi z}{d},
\]

\[
(3)\quad E_\phi = -\frac{J \omega \mu a}{S_{01}} H_0 J_1 \left( \frac{S_{01} \Gamma}{a} \right) \sin \frac{\pi z}{d},
\]

\[
(4)\quad E_z = E_r = H_\phi = 0,
\]
Figure 2

(A) COORDINATE SYSTEM

Z = d

RADIUS = \alpha

X Y

\phi

(B) TE_{011} FIELD CONFIGURATION

--- E

--- H
where,

\[ a = \text{radius of cavity,} \]
\[ d = \text{length of cavity,} \]
\[ J_0 = \text{zero-order Bessel function,} \]
\[ J_1 = \text{first-order Bessel function,} \]
\[ S_{01} = 3.832 = \text{the value of the argument for which} \]
\[ \text{the derivative of the zero-order Bessel function has its first zero,} \]
\[ H_0 = \text{amplitude factor.} \]

These equations are derived in the Appendix. The time dependence, which is understood, is given by the factor \( e^{j\omega t} \). The units are MKS throughout this thesis.

### 3.3 Resonant Frequency and Dimensions

The resonant frequency of a cavity resonator is a function of its dimensions and the mode of oscillation. The equation relating these quantities for the \( \text{TE}_{011} \) cavity is found from equation (57), Section 8.4, to be:

\[
(1) \quad \nu^2 = c^2 \left[ \left( \frac{S_{01}}{\pi D} \right)^2 + \frac{1}{4d^2} \right],
\]

where \( D = 2a = \text{diameter of the cavity} \). It can be seen from this equation that one is free to choose either the diameter or the length for a given resonant frequency, the unchosen quantity then being determined. The shape selected for the cavity is usually determined after considering the \( Q \),
interfering modes, and spatial limitation. The effect of the shape, or D/d ratio, on the Q is discussed in the following section.

Again, from equation (57), Section 8.1, the general expression for the resonant frequency of a cylindrical cavity can be written:

\[ (f_{nmpD})^2 = \left( \frac{u_{nmpc}}{\pi} \right)^2 + \left( \frac{pc}{2} \right)^2 \left( \frac{D}{d} \right)^2 \]

where \( u_{nmp} \) refers to the arguments for which \( J_n (u) = 0 \) for TE modes, or the arguments for which \( J_n (u) = 0 \) for TM modes. The graph of this equation, called the "mode chart", relates the resonant frequency of each mode to the length and diameter of the cavity. With its use, one can select the D/d ratio such that there are no unwanted modes near the resonant frequency of the desired mode. This chart is most useful in the design of cavity wave-meters.

3.4 Q

The theoretical Q for the TE\(_{011}\) cylindrical resonator is calculated in Section 8.3 to be:

\[ Q = \frac{\varepsilon \mu^2 \omega^3}{4R_s} \left[ \frac{\pi^2}{d^3} + \frac{s_{01}^2}{2a^3} \right]^{-1} \]

It is possible to write this equation in a more useful form. From equation (1), Section 3.3, the resonant frequency is
where \( \delta = \left[ \frac{\pi f \mu \sigma}{2} \right]^{-\frac{1}{2}} \) is the skin-depth. By substituting equations (2) and (3) into equation (1) and simplifying, one obtains

\[
Q = \frac{\lambda_0 S_{01}}{2 \pi \delta} \left[ \frac{1 + \left( \frac{\pi}{2 S_{01}} \right)^2 \cdot (D/d)^2}{1 + \left( \frac{\pi}{2 S_{01}} \right)^2 (D/d)^3} \right]^{3/2}
\]

where \( \lambda_0 \) is the free space wavelength. Since \( S_{01} = 3.832 \), equation (4) further reduces to

\[
Q = 0.610 \left( \frac{\lambda_0}{\delta} \right) \left[ \frac{1 + 0.1681X^2}{1 + 0.1681X^3} \right]^{3/2}
\]

where \( X = D/d \). Equation (5) shows that the \( Q \) of the TE\(_{011} \) cavity can be expressed as a function of two variables, frequency, and diameter-to-length ratio. However, it must be remembered that the frequency, diameter, and length are related by equation (2). If the frequency is chosen, then \( Q \) is a function of only \( D/d \). One can then maximize \( Q \) with respect to \( D/d \). If one selects this maximizing value of \( D/d \), then equation (2) fixes both \( d \) and \( D \), for a given frequency. By differentiating \( Q \) with respect to \( X = D/d \), one finds by the usual method that \( D/d = 1 \) gives \( Q \) a maximum value.

One would then suppose that it is most desirable to construct a cavity with \( D/d = 1 \). However, a closer inspection
of equation (5) reveals that \( Q \) is a slowly varying function of \( X \); hence the shape is not particularly important from the standpoint of \( Q \) value (For example, \( Q \) is down by only 3.6\% of its maximum for \( S = 0.5 \) and 4.6\% at \( X = 1.5 \)). The shape can thus be selected to satisfy limitations on cavity size or to eliminate mode interference.

The \( D/d \) ratio for the dimensions selected in Section 4.1 is 0.637. The \( Q \) is calculated in Section 8.3 to be 31,400, as compared with a maximum value of 32,000 for \( D/d = 1 \).

### 3.5 Influence of Cavity Parameters on Detector Signal

The equivalent circuit for a resonant cavity and its coupling system is shown in Figure 8, and the equations describing it are derived in Section 8.4. The equivalent circuit is most useful in providing a description involving salient cavity parameters which can be experimentally determined. These parameters describe the behavior of a cavity in a microwave circuit. In particular, the parameters \( Q_0 \) and \( \beta \) are useful in determining the detector signal and minimum detectable signal. One is referred to the article by Feher\(^2\) for a thorough discussion of spectrometer sensitivity.

The average power absorbed per unit volume of a paramagnetic sample is

\[
(1) \quad P_p = \frac{1}{2} \omega \mu \chi'' H^2 ,
\]

where \( \chi'' \) is the imaginary part of the RF susceptibility and
\( H \) is the high-frequency magnetic field. The \( Q \) of a cavity containing a paramagnetic sample is

\[
Q = \frac{\omega U}{P_L + P_p} = \frac{\omega U}{P_L (1 + P_p/P_L)}
\]

where \( P_L \) is the power loss in the cavity without the sample and \( U \) is the energy stored. Assuming that \( P_p \) is small compared to \( P_L \), and noting that \( (\omega U)/(P_L) \equiv Q_0 \),

\[
Q = \frac{\omega U}{P_L} (1 - P_p/P_L) = Q_0 \left[ 1 - \left( \frac{P_p}{\omega U} \right) \left( \frac{\omega U}{P_L} \right) \right]
\]

\[
= Q_0 \left[ 1 - \frac{Q_0 P_p}{\omega U} \right].
\]

Now, defining the "filling factor",

\[
\eta = \frac{\int_{V_s} H^2 dV_s}{\int_{V_c} H^2 dV_c}
\]

where \( V_s \) is the sample volume and \( V_c \) the cavity volume, one obtains

\[
Q = Q_0 \left( 1 - Q_0 \chi^\eta \eta \right),
\]

\[
\frac{\Delta Q}{Q_0} = Q_0 \chi^\eta \eta.
\]

Consider now the case where the detector signal is proportional to the voltage in the waveguide due to the reflected
wave. Assume that the incident wave to the cavity is negligibly affected by the absorption of the sample. Then, the reflected voltage is

\[ V_r = \Gamma V, \]  

where \( \Gamma \) is the magnitude of the reflection coefficient and \( V \) is the magnitude of the incident voltage. Referring to Section 8.5, one sees that the relations between \( \Gamma \), \( \beta \), and the VSWR allow one to write, for an undercoupled cavity,

\[ V_r = \left[ \frac{S - 1}{S + 1} \right] V = \left[ \frac{1 - \beta}{1 + \beta} \right] V. \]

The change in \( V_r \), which represents the absorption signal, is given by

\[ \Delta V_r = \frac{\partial V}{\partial \beta} \frac{\partial \beta}{\partial R_s} \Delta R_s, \]

where \( R_s \) is the equivalent resistance of the cavity (which changes when the sample absorbs power). From the defining equation for \( \beta \), equation (6) in Section 8.4, one obtains

\[ \frac{\partial \beta}{\partial R_s} = -\frac{\beta}{R_s}. \]

Then,

\[ \Delta V_r = \frac{2V\beta}{(1 + \beta)^2} \frac{\Delta R_s}{R_s}. \]
But,

\[ \Delta R_s = \frac{\Delta Q}{Q_o} \tag{12} \]

Thus,

\[ V_r = \frac{2V\beta}{(1 + \beta)^2} \frac{\Delta Q_o}{Q_o} = \frac{2V\beta}{(1 + \beta)^2} Q_o X'' \eta \tag{13} \]

This expression can be optimized with respect to \( \beta \). In so doing, one finds that \( \beta = 1 \) gives the maximum change in \( V_r \). However, since \( \beta = 1 \) corresponds to a matched load (q.v. equation (1), Section 8.5), one should not operate the spectrometer near the critically coupled condition. The reason is that the absorption signal would pass the cavity through the matched condition when the external field is swept through resonance, resulting in a sign reversal of the reflection and distortion of the line.

The minimum detectable signal will now be calculated assuming that the limiting factor is random thermal noise. The voltage due to noise can be calculated by the usual method on the assumption that the noise of a system can be represented by a voltage generator in series with a noiseless resistor equal to the internal resistance of the system. Assume that the equivalent circuit consists of a noise generator in series with a resistor \( Z_o \) (which represents the cavity and waveguide) connected to a matched load (the crystal
detector). However, the crystal also contributes a noise voltage of the same magnitude. The circuit is shown below.

\[
E = \sqrt{4Z_o kTb}
\]

The open circuit noise voltage due to a system with resistance \(Z_o\), bandwidth \(b\), and operating at a temperature \(T\) is

\[
(14) \quad E = \sqrt{4Z_o kTb}
\]

The total RMS noise voltage appearing at the terminals shown is then

\[
(15) \quad V_N = \left[ Z_o kTb + Z_o kTb \right]^{1/2} = \sqrt{2 Z_o kTb}
\]

The noise voltage \(V_N\) must be set equal to \(\Delta V_r\) to find the minimum detectable signal;

\[
(16) \quad \sqrt{2 Z_o kTb} = \frac{2V \beta}{\sqrt{2(1 + \beta)^2}} Q_o
\]

where the \(\sqrt{2}\) factor converts \(\Delta V_r\) to RMS voltage. Solving for \(X''\),

\[
(17) \quad X''_{\text{min}} = \frac{(1 + \beta)^2}{\beta V Q_o \eta} \sqrt{2 Z_o kTb} = \frac{(1 + \beta)^2}{\beta Q_o \eta} \left[ \frac{kTb}{2P} \right]^{1/2}
\]

where the incident power, \(P = \frac{1}{2}V^2/Z_o\).
The minimum detectable number of spins, \( N_{\text{min}} \), can be calculated with the use of equation (17). For a Lorentzian line, the susceptibility can be represented at resonance by:

\[
\chi'' = \frac{X_0 \omega_0}{3 \Delta \omega} = \frac{X_0 B_0}{3 \Delta B},
\]

where \( \Delta B \) is the half-width of the line, and \( X_0 \) is given by \(^{15}\)

\[
X_0 = \frac{N}{V_s} \frac{\mu_0 \mu_B^2 g^2 S(S+1)}{3kT}.
\]

Here, the previously undefined symbols are:

- \( \mu_0 \) = permeability of free space = \( 4\pi \times 10^{-7} \) henrys/meter,
- \( \mu_B \) = Bohr magneton = \( 9.273 \times 10^{-24} \) joules/(weber/meter\(^2\)),
- \( S \) = spin of electron = \( \frac{1}{2} \),
- \( N \) = number of spins in volume \( V_s \).

Substituting the values of \( X_0 \) and \( B_0 = (hf_0)/(g \mu_B) \), one obtains

\[
\chi_{\text{min}}' V_s = \frac{N \mu_0 \mu_B hf_0 S(S+1)g}{3 \sqrt{3} \Delta B kT}
\]

Now, the filling factor, \( \eta \), can be calculated from equation (4) assuming that the magnetic field does not vary over the sample. After evaluating the integrals in equation (4) for the TE\(_{011}\) cavity, one obtains
(21) \[ \eta = \frac{2V_s}{V_c J_0^2 (S_{01})} \left[ 1 + \left( \frac{a \pi}{S_{01d}} \right)^2 \right]^{-1} \]

where \( V \) is the cavity volume. For a cavity with dimensions \( d = 2.511 \) inches and \( a = 0.800 \) inches,

(22) \[ \eta = (0.14 \times 10^6) V_s \]

where \( V_s \) is measured in cubic meters. If the value for \( \eta \) is substituted into equation (17) and the resulting value of \( X_{\text{min}} \), \( V_s \) is equated to that given by equation (20), the value of \( N_{\text{min}} \) is found to be:

(23) \[ N_{\text{min}} = \frac{3 \sqrt{3} \Delta B (1 + \beta)^2 (kT)^{3/2} h^1}{(1.4 \times 10^5) \beta Q_o \mu_o \mu_b h f_0 g S(S + 1) (2P)^{1/2}} \]
IV. DESIGN

4.1 Cavity for Use at Room Temperature

The desirable features for the cavity are several and have been discussed in the preceding sections. The discussion that follows concerns the design of a $TE_{011}$ cavity including as many of these features as possible with a minimum of compromise.

The resonant frequency of the cavity was chosen to be $9.3 \text{ kmc}$ on the basis of considering the optimum operating frequencies of the associated spectrometer equipment. Since there is no dewar with which to contend, the outer diameter of the cavity is restricted only to 2 inches (to provide clearance for the modulation coils). However, since the diameter of the liquid helium cavity is more restricted, it was decided to make all cavities the same inside diameter, namely 1.600 inches. The difference in $Q$ and the other properties is not significant for a slightly larger diameter, and the 1.600 inch dimension allows a sturdy wall of 0.200 inches in thickness for the room temperature and liquid nitrogen cavities. The length of the cavity then follows from equation (1), Section 3.3, to be 2.511 inches.

In order to reduce the cavity effect (Section 2.5), it was decided that a plastic body, coated on the inner surfaces with a thin layer of silver, would be used. The selected plastic, Hysol 6000 C-8 cast epoxy resin, has been used
The plastic was obtained in the form of a 2.5 inch diameter rod from the Hysol Corporation, Olean, New York. It was chosen over other non-conducting materials such as glass in the belief that it would not become brittle at liquid helium temperatures, and that it would be easier to machine.

The fields in the cavity are excited through an iris in one end plate, hereafter referred to as the "top". The location of the iris was chosen to couple the transverse magnetic field in the waveguide to the radial magnetic field in the cavity at their maxima. The transverse magnetic field in the waveguide is a maximum at the center of the guide. The maximum for \( H_r \) in the cavity occurs at the first maximum of \( J_1 \left( \frac{S_{01} r}{a} \right) \). This point is given by

\[
\frac{S_{01} r_m}{a} = 1.84
\]

(1) \hspace{1cm} \frac{S_{01} r_m}{a} = 1.84

\[
r_m = \frac{1.84 \, a}{S_{01}} = \frac{(1.84)(0.8)}{3.832} = 0.364 \text{ inches}
\]

(2) \hspace{1cm} r_m = \frac{1.84 \, a}{S_{01}} = \frac{(1.84)(0.8)}{3.832} = 0.364 \text{ inches}

The size of the iris, or coupling hole, was decided upon by experiment. It was found that a slot 3/32 by 5/16 inch, and 0.030 inch thick is satisfactory. The thickness of the wall surrounding the coupling hole is not critical as long as it does not become appreciable with a wavelength. The amount of coupling can, of course, be adjusted with tuning screws.

The top plate and iris were constructed of silver-plated
brass for ease of construction and rigidity. Since the currents due to the modulation field are primarily induced in the barrel, the brass top plate should produce little noise.

Above, the diameter of the cavity was chosen to be 1.600 inches, the length then following at 2.511 inches, giving D/d = 0.637. It has been said that this value of D/d has negligible effect on the Q of the cavity. The question of mode interference will now be considered.

As mentioned in Section 3.3, the mode-chart can be used to quickly determine if other modes have resonant frequencies near that of the TE$_{011}$ mode for a given value of D/d. By an inspection of the mode-chart, one finds that there are other modes with resonant frequencies in the neighborhood of 9.3 kmc for D/d = 0.637. The mode nearest in frequency to the TE$_{011}$, other than the degenerate TM$_{11}$ mode, is the TM$_{013}$. The resonant frequency of the TM$_{013}$ mode is

$$f = c \left[ \left( \frac{u_{01}}{\pi D} \right)^2 + \left( \frac{3}{2d} \right)^2 \right]^{1/2} = 9.04 \text{ kmc}.$$  

Thus, the TM$_{013}$ mode is not near enough to the TE$_{011}$ mode to cause confusion. Since the TM$_{013}$ mode is the nearest in frequency to the TE$_{011}$, the other modes are not excited either.

The TM$_{11}$ mode has the same resonant frequency as the TE$_{011}$ mode. Thus, it will be excited simultaneously with the TE$_{011}$ unless steps are taken to prevent it. There are several ways to suppress an unwanted mode. These will be dis-
cussed below.

One method of suppressing an unwanted mode is to locate the iris such that the desired mode is preferentially excited. For example, if magnetic field coupling is to be used, the iris would be placed at a point where the unwanted mode’s magnetic field is zero, but the desired mode’s magnetic field is in the right direction and of non-zero magnitude. An inspection of the field configurations for the TM_{11} and TE_{011} modes reveals that it is not possible to suppress the TM_{11} mode by this method. (It would be possible if the iris could be placed on a side wall, since H_z = 0 for TM modes.)

Another method consists of placing obstacles, such as gaps or "ruts", at points of high current density of the unwanted mode. This attenuates the unwanted mode and also tends to remove the degeneracy. This method is possible for suppressing the TM_{11} mode. A gap at the bottom plate, or a rut in the barrel wall serves to attenuate the TM_{11} currents. The TE_{011} currents are all circumferential and thus are much less affected. The practice of leaving a gap at the bottom plate is common in TE_{011} wavemeters.\(^{17}\)

A final method of mode suppression is the introduction of some asymmetry into the cavity such that the TM_{11} and TE_{011} are no longer degenerate. A technique that has been used with success consists of making the bottom plate concave. Since the Q of the TM_{11} is low, the Q of a TE_{011}
cavity is lowered when the TM_{111} is excited. Bussy and Birnbaum\textsuperscript{18} report that the Q of a TE_{011} cavity increased by a factor of 2.2 after the concave plate was used to remove the degeneracy.

It was found that the introduction of a gap in the conducting surface at the bottom end plate had more than a negligible effect on the TE_{011} mode. While the gap effectively suppressed the extraneous modes, it also affected the coupling to the TE_{011} mode. Some of the cavities were slightly asymmetric because of the difficulty in machining the plastic. These unavoidable asymmetries sufficiently split the degeneracy of the TE_{011} and TM_{111} modes such that no further mode suppression was necessary in some cases (such as the cavity described in Section 6.1).

For ease of assembly, the cavity is joined together with \#4-40 brass machine screws. A drawing of the cavity is shown in Figure 3.

4.2 Low Temperature Cavities

The cavity to be used at liquid nitrogen temperature is somewhat different in structure since it must be sealed to liquid nitrogen. The top of the cavity is closed (except for the iris) with only the bottom plate being removable. A brass plate through which the stainless steel waveguide is passed is permanently attached to the top. The waveguide slips into a channel in the silvered plastic which forms a waveguide leading to the iris. The sample holder is screwed
into a tapped hole in the inside of the bottom plate. Allow-
ance is made for differences in thermal expansion of the var-
ious materials, of course. The dimensions were chosen so
that the cavity would be resonant at 9.3 kmc at the low temp-
eratures. The thermal expansion data was taken from tables
in Scott's *Cryogenic Engineering*. A drawing of the cavity
is shown in Figure 4.

At liquid helium temperatures, the above cavity can be
used, or a model similar to the room temperature cavity,
since it does not have to be sealed to liquid helium.
DRILL # 43 X 1/4 DEEP.
TAP # 4-40, 3 PLACES ON EACH END.

MAT'L: HYDOL 6000 C-B EPOXY RESIN

FIGURE 3(A)
ROOM TEMP CAVITY BARREL
**Figure 3**

**Room Temp Cavity**

**B) Bottom Plate**

- **Material:** Hysol 6000 C-B Epoxy Resin

**C) Top Plate**

- **Material:** Yellow Brass

- **Drill #32 Thru, 3 Places**

- **Mill 3/32 x 5/16 Slot Thru**

- **Mill 1/4 x 1 x .345 Deep to Fit Waveguide**

**Typical**

- **.030**

- **.375**

- **2.000**
ROOM TEMP CAVITY
**FIGURE 4(B)**

**LIQUID NITROGEN TEMP. CAVITY END PLATE**
V. CONSTRUCTION

5.1 Machining

The cavity parts were machined on a lathe at a low feed speed using a steel cutting tool with a radius of about 1/32 inch, except for the milled waveguide slot and iris. The plastic is highly abrasive, and it proved to be difficult to maintain the proper edge and radius on the cutting tool. This is important, as the tool marks in the barrel of the cavity are difficult to polish out; and a proper tool minimizes tool marks.

Since the low temperature cavities undergo a large thermal contraction it is necessary to remove stresses that might lead to distortion in the parts. The low temperature cavities were thus annealed between rough-cutting and final cutting in an attempt to remove any stresses that may have been introduced by machining. The rough-cut parts were placed in an oven at 75° C overnight, and then allowed to cool slowly to room temperature.

5.2 Polishing

The cavity barrels were polished by placing them in a lathe chuck and holding a cylindrical polishing tool against them. The polishing tool consisted of a 1 1/8 inch diameter lucite rod covered with metallurgical polishing cloth - "kitten-ear" broadcloth. Successively finer grades of
abrasive were used, finishing with an abrasive with particle size of about 0.3 micron (Linde type 5175-A). One problem connected with the polishing of the barrel was avoiding the introduction of asymmetries. It was decided that this danger would be lessened if only the finer abrasives were employed.

The end plates were easily polished by grinding them on successively finer grades of emery paper, and then polishing them on metallurgical polishing wheels. The final polish was again attained by the above-mentioned cloth and abrasive. The surfaces obtained by this method were of course superior to that obtained for the barrels.

5.3 Silvering

The conducting layer for the cavity was obtained by chemical silvering. The method used is a modification of the Brashear process for glass. It was suggested in the report by Chester, et al.\textsuperscript{20}

A disadvantage of the chemical silvering process is that the part requires further polishing after silvering in order that a highly reflecting surface be obtained. Furthermore, the silver coating is not highly adherent, so that it wears off easily when polished. Thus, it was found that a better surface was obtained by polishing the plastic well before silvering and omitting subsequent polishing.

Since silver coats that have been deposited by evaporation of silver metal in a vacuum need no subsequent pol-
ishing, this method suggested itself as a possible remedy. Silvering by evaporation was attempted on one end plate as an experiment. The part had to be placed sufficiently far from the source (a molybdenum boat full of silver metal) such that a uniform deposit would be formed, and also such that the plastic would not be deformed by the high temperature boat. This decreased the efficiency of the process.

The results obtained by evaporation were not conclusive, but were discouraging. The surface obtained was mirror-like; but the coating was apparently too thin, although it was calculated to be greater than a skin-depth. Comparison of the end plate with one that had been chemically silvered showed that the Q of the cavity was lower with the evaporation-coated part. Also, the plastic had "sweated" somewhat due to the heat, and the adherence of the coating was poor. In order to increase the thickness of the coating, one would have to place the part nearer the source at the expense of uniformity. But even at the far distance, the part "sweated", and since in the case of the barrel a wire filament would have to be placed along its axis, one would fear that the plastic would deform due to the proximity of the heat source. The poor adherence of the coating could possibly have been due to an improperly cleaned surface, but according to L. Holland, poor adherence is apparently a general difficulty with plastics.

It should not be inferred that the selection of chemical silvering over vacuum deposition was based upon exhaus-
tive tests; however, the choice did seem to be the better on the basis of the above findings.

The brass parts were easily silvered by electroplating. The coatings obtained were both highly conducting and adherent.
6.1 Q Measurement

A. Experimental Technique

The quality factors of the cavities were measured with the use of the impedance method described by Ginzton. The apparatus is shown in Figure 5.

Before using the somewhat longer impedance method, one can determine an approximate value of the loaded Q with the oscilloscope. By observing the oscilloscope pattern, which is essentially a plot of reflected voltage versus frequency, one can measure the half-power points with the wave-meter. The loaded Q is then calculated by dividing the readily determined resonant frequency by the half-power bandwidth. The coupling factor can also be approximately determined by observing the ratio of reflected signal to incident signal. The relation between \( \beta \) and \( \Gamma \) follows from equations (4), (5), and (12) in Section 8.5. However, this method will not reveal whether the cavity is undercoupled or overcoupled.

The first step of the impedance method is to determine the coupling coefficient, \( \beta \). To do this, one must find the detuned short position (q.v. Section 8.4). Since the cavity is not tunable, the frequency of the klystron is varied, and the detuned short position is found as a function of frequency using the standing wave detector (The detuned short position at each frequency occurs at a minimum in the standing wave
FIGURE 5

Q MEASUREMENT APPARATUS
pattern). The plot of the probe carriage position versus frequency for frequencies off resonance allows one to interpolate and find the detuned short position for the resonant frequency. The standing wave detector probe carriage is then set at the detuned short position and the klystron is tuned to the cavity resonance. If a minimum in the voltage standing wave pattern occurs at the detuned short position, the cavity is undercoupled. If a voltage maximum occurs there, the cavity is overcoupled. The coupling coefficient is calculated by

\[
(1) \quad \beta = \frac{1}{S_0} , \quad \text{or}
\]

\[
(2) \quad \beta = S_0 ,
\]

for the undercoupled and overcoupled cases, respectively, where \( S_0 \) is the VSWR at resonance (q.v. Section 8.5).

After \( \beta \) has been determined, the VSWR is measured as a function of frequency. The bandwidth defined by the frequencies at which the VSWR is equal to the value given by equation (13), Section 8.5, is used to calculate \( Q \).

\[
(3) \quad Q_0 = \frac{f_0}{\Delta f}
\]

A sample plot of this data is shown in Figure 6.

The loaded \( Q \) can then be calculated by

\[
(4) \quad Q_L = \frac{Q_0}{1 + \beta} .
\]
VSWR VS FREQUENCY FOR ROOM TEMR CAVITY WITH SAMPLE HOLDER

\[ Q_o = \frac{f_0}{\Delta f} \]

FREQUENCY IN MC + 9.268 KMC

FIGURE 6
B. Results

The Q values for each cavity were used as a basis of selection. The best room temperature cavity obtained was then used with several end plates and sample holders to determine their effect on the Q.

The highest Q was of course obtained with a closed end plate and no sample holder \( (Q_0 = 12,000) \). An end plate with a 0.200 inch hole was next used. The data for it is shown in Figure 6. The \( Q_0 \) obtained in this case was 11,200. Thus, as was expected, a hole in the center of the bottom plate has a small effect on the Q. Next, a sample holder, 0.200 inch in diameter and made of plastic foam, was inserted through the hole such that one end was in the center of the cavity. The Q was lowered to 10,300. Finally, a polystyrene sample holder containing a sample of diphenyl-picryl-hydrazyl(D.P.H.) and a beeswax plug was placed through the hole. The Q was lowered to about 9500, as the polystyrene is more lossy than the plastic foam.

It was found that the Q values and coupling coefficient varied from day to day, sometimes radically. Since the values of Q and \( \beta \) were lowest when a window was left open during humid weather, it is believed that the humidity affects the conducting surface. As long as the cavity is closed and kept from a damp atmosphere, the coupling and Q is fairly constant.

The \( Q_0 \) data are subject to some error other than the ordinary errors associated with instrument accuracy. Since
the resonant frequency varies with cavity dimensions, and these depend upon temperature, the resonant frequency changed slightly during each data run. The resonant frequency was thus recorded before and after each run, and only data involving a small change were used.

6.2 Adjustment of $\beta$

The coupling coefficient can be adjusted by tuning screws placed in the waveguide leading to the cavity. A machine screw placed in the center of the waveguide acts as a variable shunt capacitance. The impedance as seen by the transmission line can be tuned to any desired value with a variable susceptance screw that can be placed anywhere along the line (a slide-screw tuner). However, to a limited extent, a screw placed at a predetermined distance from the cavity can change the coupled energy to the desired amount.

To use this single, fixed position screw, one must measure the cavity impedance at resonance and calculate the position at which the screw must be placed for the desired value of $\beta$. The disadvantage of the single screw is that it is applicable only near the frequency for which the calculation is made.

Since the resonant frequency varies somewhat with different end plates, sample holders, temperature, etc., it is more useful to have a tuner that has a wider range of applicability. The double-screw tuner, two screws placed $3/8\lambda$ apart, is an improvement. It has a wider range of operation.
and may be sufficient if the frequency range and range of
desired coupling are small.\textsuperscript{25}

At low temperatures, the cavity will be in the dewar when the coupling factor must be adjusted. For best operation, the tuner must be placed near the cavity since a change in frequency of the generator would change the guide wavelength; and if the tuner is several wavelengths from the cavity, even a small frequency change would drastically change the impedance to the tuner. Thus, the only practical solution here is a slide-screw tuner which can be adjusted from outside the dewar.

6.3 Noise Measurements

Spectrometer noise which can be attributed to the cavity is the result of two major effects (q.v. Section 2.5). The most important is the noise caused by currents induced in the cavity walls by the modulation field. The other effect is somewhat nebulous, but can be thought of as being caused by small, vibrating, silver particles on the surface of the walls, or any similar effect associated with a silver layer that adheres poorly to the plastic. The former effect (which is most pronounced in metal cavities) should be essentially absent from the plastic cavity. In a metal cavity, the conducting surface is usually much smoother and definitely more adherent than that of the plastic cavity; thus, a metal cavity should have less noise produced by the latter effect than a plastic cavity.
The noise characteristics of the plastic cavity were compared to those of a silvered-brass cavity of the same dimensions at room temperature. Two methods were used to determine the relative merits of the two cavities.

The first method consisted of placing the cavities in the modulation field and measuring the field inside each cavity with a search coil. The results, shown in Table I, simply, but clearly show that the brass cavity strongly reduces the field inside it at the higher modulation frequencies as a result of the currents induced in the walls. The plastic cavity, however, has negligible effect.

<table>
<thead>
<tr>
<th></th>
<th>No Cavity</th>
<th>Plastic Cavity</th>
<th>Brass Cavity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_m$</td>
<td>Brms in webers/m$^2$ x 10$^4$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100 cps</td>
<td>2.3</td>
<td>2.3</td>
<td>2.3</td>
</tr>
<tr>
<td>1 kc</td>
<td>2.6</td>
<td>2.6</td>
<td>0.68</td>
</tr>
<tr>
<td>10 kc</td>
<td>2.7</td>
<td>2.7</td>
<td>0.34</td>
</tr>
</tbody>
</table>

Table I. Effect of Plastic and Silvered-Brass Cavities on Modulation Field

In order that the noise caused by surface effects in the plastic cavity could be determined, the peak-to-peak signal-to-noise ratio (S/N) of the spectrometer was measured using both cavities under similar operating conditions. The same end plates, sample holder, and sample were used with both cavities. A known amount of paramagnetic sample, 0.9
milligram of diphenyl-picryl-hydrazyl (D.P.H.), was placed in the sample holder. The experimental value of the minimum detectable number of spins was calculated from the number of spins of the sample and the signal-to-noise ratio. The theoretical value of \( N_{\text{min}} \) was calculated using equation (23), Section 3.5. The ratios of \( (N_{\text{min}})_{\text{exp}} \) to \( (N_{\text{min}})_{\text{th}} \) for the plastic and brass cavities were then compared for various values of modulation frequency and detector bandwidth. The results are shown in Table II. A sample calculation illustrating the method follows.

An advantage of this method is that the "coherent noise", that is, noise which is in phase with the modulation signal, can be cancelled out to a certain extent in the detector. In this way, the noise due to the induced wall currents can be partially separated from that due to surface effects. The coherent noise signal becomes quite large for the brass cavity as the modulation frequency is increased. The effect of this appears in the data of Table II. Even though the coherent noise could be cancelled effectively at 100 cps, the effectiveness decreased at higher frequencies, and at 10 kc, the coherent noise was so large that the amplifiers saturated, and no data could be taken for \( b = 0.11 \) cps. That the coherent noise is much smaller for the plastic cavity is further verification of the conclusions derived from the results shown in Table I.

For D.P.H., \( B = 2 \times 10^{-14} \) w/m², \( g = 2 \), and \( S = \frac{1}{2} \). For the plastic cavity, \( Q_0 = 9500 \) and \( f_0 = 9.269 \) kmc. If these...
values are substituted into equation (23), Section 3.5, the theoretical minimum detectable number of spins becomes

\[ (N_{\text{min}})_{\text{th}} = \left( \frac{1 + \beta}{\beta} \right)^2 \left( \frac{1}{2P} \right)^{\frac{1}{2}} (1.93 \times 10^9) \]

For an incident power of \( P = 1.44 \text{ mw} \), \( \beta = 0.447 \), and a detector bandwidth of 0.11 cps,

\[ (N_{\text{min}})_{\text{th}} = 5.48 \times 10^{10} \]

At a modulation frequency of 1 kc and detector bandwidth of 0.11 cps, \( S/N \) was measured to be \( 1.1 \times 10^{14} \). The number of spins in 0.9 mg of D.P.H. is \( 1.4 \times 10^{18} \), therefore, the experimental minimum detectable number of spins is

\[ (N_{\text{min}})_{\text{exp}} = \frac{1.4 \times 10^{18}}{1.1 \times 10^4} = 1.27 \times 10^{14} \]

\[ A_p = \frac{(N_{\text{min}})_{\text{exp}}}{(N_{\text{min}})_{\text{th}}} = 2.2 \times 10^3 \]

At first glance, there appears to be a large discrepancy between the calculated and theoretical values of \( N_{\text{min}} \). However, so far, no consideration has been given to the noise introduced by the detection system. If this source of noise is taken into account, the experimental and theoretical values agree much more closely. The purpose of the calculation here is not to determine an absolute value for \( N_{\text{min}} \), but rather to compare the relative merits of the plastic and
brass cavities. Since the same detector was used for both cavities, the ratio, $A_p/A_b$ gives the desired information, since the effect of the detector is cancelled out.

The parameters for the brass cavity are: $Q_0 = 6800$, $f_0 = 9.262$ kmc, $P = 1.31$ mw, and $\beta = 0.643$. Thus, for a bandwidth of 0.11 cps,

\begin{equation}
(N_{\text{min}})_{\text{th}} = 7.24 \times 10^{10}.
\end{equation}

The signal-to-noise ratio using the brass cavity was found to be $1.1 \times 10^4$ at $f_m = 1$ kc, therefore,

\begin{equation}
(N_{\text{min}})_{\text{exp}} = \frac{1.4 \times 10^{18}}{1.1 \times 10^4} = 1.27 \times 10^{14}.
\end{equation}

\begin{equation}
A_b = \frac{(N_{\text{min}})_{\text{exp}}}{(N_{\text{min}})_{\text{th}}} = 1.8 \times 10^3.
\end{equation}

The ratio of $A_p$ to $A_b$ is found using the values from equations (4) and (7):

\begin{equation}
\frac{A_p}{A_b} = \frac{2.2 \times 10^3}{1.8 \times 10^3} = 1.2
\end{equation}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
\textbf{f}_m & \textbf{b} = 2.1 \text{ cps} & \textbf{b} = 0.11 \text{ cps} \\
\hline
100 \text{ cps} & 2.1 & 5.9 \\
1 \text{ kc} & 1.9 & 1.2 \\
10 \text{ kc} & 0.56 & --- \\
\hline
\end{tabular}
\caption{\textit{Ap/A_b for Various Values of Modulation Frequency and Detector Bandwidth}}
\end{table}
The results in Table II. show that the plastic cavity produces slightly more noise which might be attributed to surface effects than the brass cavity. However, the noise caused by induced currents in the brass cavity far overshadows similar noise in the plastic cavity. It must be concluded, therefore, that the plastic cavity is effective in reducing cavity noise.
VII. CONCLUSIONS

7.1 Critique

A major factor in obtaining a high $Q$ plastic cavity is the quality of the surface obtained by the machining of the barrel. The difficulties associated with removing tool marks by polishing have already been discussed. If an improvement in the machined surface could be made such that little or no polishing is necessary, much of the difficulty would be removed. A possible solution is the use of a tool harder than steel, such as diamond or ceramic. A diamond cutting tool has been used on silvered metal surfaces to obtain a final polish, with $Q$'s resulting of the order of $85\%$ theoretical values.\textsuperscript{27} Ceramic tools are less expensive and would probably be nearly as effective as diamond. A properly shaped ceramic tool would hold its edge against the abrasive action of the plastic much better than ordinary tool steel.

A possible method of increasing the conductivity of the silvered surface is electrolytic polishing.\textsuperscript{28} This process is used mainly for polishing small metallurgical samples and is rather tedious to set up; however, once the technique is mastered, it would provide a means of controlled final polishing.

7.2 Discussion of Results

The results of this investigation indicate that a $T_{E_{011}}$
cylindrical cavity is satisfactory as a means of obtaining the high-frequency magnetic field necessary for paramagnetic-resonance absorption. Its high Q and field configuration are quite desirable for the EPR spectrometer with which it is to be used.

The use of the epoxy resin as a base material was not without disadvantages, as has previously been pointed out. However, the results of the noise measurements show that the silvered-plastic cavity is most effective in reducing cavity noise. Although the Q's obtained were not near the theoretical maximum, values of the order of 10,000 are adequate, and higher Q's could undoubtedly be obtained with the employment of more refined construction techniques.

7.3 Ceramic Cavities

Ceramic has been successfully used as a cavity material.\textsuperscript{9,29} It can be accurately machined, metallized with silver, and highly polished. The silver coating obtainable is much more adherent than that on the epoxy resin. Since ceramic is much stronger than plastic, a ceramic cavity can be constructed with thinner walls and thus an increased resonator volume for a given external diameter. Various metals can be brazed to ceramic. If then, a metal with thermal expansion properties similar to those of the ceramic can be found, end plates can be soldered to the cavity barrel. This provides a convenient method of sealing the cavity without barring disassembly.
A suggested design for a $\text{TE}_{011}$ ceramic cavity to be used at low temperatures is shown in Figure 7. The dimensions are selected so that the resonant frequency will be near 9.3 kmc at liquid helium temperatures. The material is Coors Type AD-94 alumina ceramic (Coors Porcelain Co., Golden, Colorado). The thermal expansion of molybdenum is similar to that of the AD-94 ceramic, thus it is employed as shown.

Several other types of ceramics are suitable for cavity materials, and the manufacturers in each case are equipped to construct the cavities to specifications.
CERAMIC CAVITY

FIGURE 7(A)

CERAMIC CAVITY

NOTES:
1. BRAZE Nb PARTS TO BARREL.
2. SILVER ALL CONDUCTING SURFACES.

CERAMIC: COORS TYPE AD-94
NOTES:
1. BRAZE M6 RING TO CERAMIC.
2. SILVER INNER FACE OF CERAMIC PLATE.

CERAMIC: COORS TYPE AD-94

FIGURE 7 (B)
CERAMIC CAVITY END PLATE
VIII. APPENDIX

8.1 General Theory of Resonant Cavities

The theory of resonant cavities involves the solution of Maxwell's equations subject to the boundary conditions at the cavity walls. For an air filled cavity, the curl equations are:

\[ \nabla \times \mathbf{E} = -\mu \dfrac{\partial \mathbf{H}}{\partial t}, \]

\[ \nabla \times \mathbf{H} = \varepsilon \dfrac{\partial \mathbf{E}}{\partial t}. \]

Taking the curl of equation (1) and using equation (2),

\[ \nabla \times \nabla \times \mathbf{E} = -\mu \dfrac{\partial}{\partial t} (\nabla \times \mathbf{H}) = -\mu \varepsilon \dfrac{\partial^2 \mathbf{E}}{\partial t^2}. \]

By a vector identity,

\[ \nabla \times \nabla \times \mathbf{E} = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E}, \]

since \( \nabla \cdot \mathbf{E} = 0 \) in a medium containing no free charge. The wave equation for \( \mathbf{E} \) follows,

\[ \nabla^2 \mathbf{E} = \mu \varepsilon \dfrac{\partial^2 \mathbf{E}}{\partial t^2}. \]

A similar development for \( \mathbf{H} \) yields
Since the source of the excitation of the cavity fields is a generator with angular frequency $\omega$, the time dependence for the fields is given by the real part of the factor $e^{j\omega t}$, written with the "real part" understood. Defining $\kappa^2 = \omega^2 \mu \varepsilon$, the wave equations become:

\begin{align*}
\nabla^2 \mathbf{E} &= -\kappa^2 \mathbf{E} , \\
\nabla^2 \mathbf{H} &= -\kappa^2 \mathbf{H} .
\end{align*}

These wave equations are then to be solved subject to the boundary conditions. However, it must be remembered that $\mathbf{E}$ and $\mathbf{H}$ are not independent but are related by equations (1) and (2). For purposes of obtaining a solution to the wave equations, the assumption is made that the walls of the cavity are perfectly conducting. The electric and magnetic fields in the walls are zero, then, and the boundary conditions are:

\begin{align*}
\hat{n} \times \mathbf{E} &= 0 , \\
\hat{n} \cdot \mathbf{H} &= 0 ,
\end{align*}

where $\hat{n}$ is a unit vector normal to the surface at the cavity walls. That is, the normal component of $\mathbf{H}$ and the tangential component of $\mathbf{E}$ are zero at the boundaries. Another useful condition which follows from the assumption of perfect conductivity is that the normal derivative of the tangential component of $\mathbf{H}$ must vanish at the boundaries.
Now, assume that the cavity is cylindrical in shape and that the z-axis is located along its symmetry axis. Write the Laplacian operator as the sum of its transverse and z parts. Let the vector \( \mathbf{E} \) be redefined as \( \mathbf{E} = \mathbf{E}(u,v)F(z) \), where \( u \) and \( v \) are any general transverse coordinates, and \( F(z) \) gives the z dependence of the electric field vector.

From equation (7);

\[
\left[\nabla_t^2 + \nabla_z^2\right] \mathbf{E}(u,v) F(z) = -K^2 \mathbf{E}(u,v) F(z) ,
\]

where the subscript "t" designates "transverse".

\[
F \nabla_t^2 \mathbf{E} + \mathbf{E} \nabla_z^2 F = -K^2 \mathbf{E} \mathbf{F} .
\]

Separating the variables,

\[
\nabla_t^2 \mathbf{E} + (K^2 - \beta^2) \mathbf{E} = 0 ,
\]

\[
\frac{d^2 F}{dz^2} + \beta^2 F = 0 ,
\]

where \( \beta^2 \) is some separation constant. The solution of (14) gives:

\[
F = A \begin{cases} 
\sin \beta z \\
\cos \beta z 
\end{cases} .
\]

Each component of \( \mathbf{E} \) satisfies equation (12), so the proper combination of sine and cosine solutions for \( F \) depends upon the particular boundary conditions for that component. Also, the value of \( \beta \) is fixed by the application of the boundary conditions.
A similar argument can be made for $\mathbf{H}$, and since $\mathbf{E}$ and $\mathbf{H}$ are related by Maxwell's curl equations, the factor $\beta$ is the same for $\mathbf{H}$. Thus, the vectors $\mathbf{E}$ and $\mathbf{H}$ both satisfy each of the following equations:

\begin{align}
(16) \quad \left[ \frac{\partial^2}{\partial z^2} + \beta^2 \right] \begin{cases} \mathbf{E} \\ \mathbf{H} \end{cases} &= 0, \\
(17) \quad \left( \nabla^2 + k^2 \right) \begin{cases} \mathbf{E} \\ \mathbf{H} \end{cases} &= 0.
\end{align}

It will now be shown that the transverse components of electric and magnetic field can be expressed in terms of the $z$-components, $E_z$ and $H_z$. First, rewrite equation (1), separating each vector and each operator into transverse and $z$ components.

\begin{align}
(18) \quad \left( \nabla_t + \nabla_z \right) \times (\mathbf{E}_t + \mathbf{E}_z) &= -j \omega \mu (\mathbf{H}_t + \mathbf{H}_z)
\end{align}

Expanding the left side of (18):

\begin{align}
(19) \quad \left( \nabla_t + \nabla_z \right) \times (\mathbf{E}_t + \mathbf{E}_z) &= \nabla_t \times \mathbf{E}_t + \nabla_t \times \mathbf{E}_z \\
&\quad + \nabla_z \times \mathbf{E}_t + \nabla_z \times \mathbf{E}_z.
\end{align}

Since $\nabla_z \times \mathbf{E}_z = 0$, and $\nabla_t \times \mathbf{E}_t = -j \omega \mu \mathbf{H}_z$, equation (18) becomes:

\begin{align}
(20) \quad -j \omega \mu \mathbf{H}_t &= \nabla_t \times \mathbf{E}_z + \nabla_z \times \mathbf{E}_t.
\end{align}

A similar development from equation (2) yields:

\begin{align}
(21) \quad j \omega \epsilon \mathbf{E}_t &= \nabla_t \times \mathbf{H}_z + \nabla_z \times \mathbf{H}_t.
\end{align}
Now, substituting $\mathbf{H}_t$ from (20) into (21):

\begin{align}
(22) \quad j \omega \varepsilon \mathbf{E}_t &= \nabla_t \times \mathbf{H}_z + j/(\omega \mu) \left[ \nabla_z \times \nabla_t \times \mathbf{E}_z + \nabla_z \times \nabla_z \times \nabla_z \mathbf{E}_t \right], \\
(23) \quad \nabla_z \times \nabla_t \times \mathbf{E}_z &= \nabla_t (\nabla_z \cdot \mathbf{E}_z) - (\nabla_z \cdot \nabla_t) \mathbf{E}_z \\
&= \nabla_t \left( \frac{\partial \mathbf{E}_z}{\partial z} \right), \\
(24) \quad \nabla_z \times \nabla_z \times \mathbf{E}_t &= \nabla_z (\nabla_z \cdot \mathbf{E}_t) - (\nabla_z \cdot \nabla_z) \mathbf{E}_t \\
&= -\nabla_z^2 \mathbf{E}_t = \beta^2 \mathbf{E}_t,
\end{align}

making use of equation (16). Thus, (22) becomes:

\begin{align}
(25) \quad j \omega \varepsilon \mathbf{E}_t &= \nabla_t \times \mathbf{H}_z + \frac{J}{\omega \mu} \left[ \nabla_t \left( \frac{\partial \mathbf{E}_z}{\partial z} \right) + \beta^2 \mathbf{E}_t \right].
\end{align}

Solving for $\mathbf{E}_t$,

\begin{align}
(26) \quad \mathbf{E}_t \left[ j \omega \varepsilon - \frac{J \beta^2}{\omega \mu} \right] &= \nabla_t \times \mathbf{H}_z + \frac{J}{\omega \mu} \nabla_t \left( \frac{\partial \mathbf{E}_z}{\partial z} \right).
\end{align}

Multiplying through by $-j \omega \mu$, remembering that $K^2 \equiv \omega^2 \mu \varepsilon$, and solving for $\mathbf{E}_t$ gives:

\begin{align}
(27) \quad \mathbf{E}_t &= \frac{\nabla_t \left( \frac{\partial \mathbf{E}_z}{\partial z} \right)}{K^2 - \beta^2} - \frac{j \omega \mu}{K^2 - \beta^2} \nabla_t \times \mathbf{H}_z.
\end{align}

Now, define $K^2 - \beta^2 \equiv k^2$. 

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Solving similarly for $\vec{H}_t$:

\begin{align*}
(29) \quad \vec{H}_t &= (1/k^2) \nabla_t \left( \frac{\partial \vec{E}_z}{\partial z} \right) + \frac{j \omega \varepsilon}{k^2} \nabla_t \times \vec{E}_z.
\end{align*}

Thus, the transverse fields can be found once the $z$-components are known. The $z$-components of the fields satisfy equations of the form of equation (13):

\begin{align*}
(30) \quad (\nabla_t^2 + k^2) \begin{cases} 
\vec{E}_z = 0 \\
\vec{H}_z = 0 
\end{cases}.
\end{align*}

Consider now the boundary conditions on $\vec{E}_z$ and $\vec{H}_z$ at the boundary of a cross-section of the cavity. Here $\vec{E}_z$ and the normal derivative of $\vec{H}_z$ must vanish. If one solved equation (30) for $\vec{E}_z$ and $\vec{H}_z$ subject to these boundary conditions, one would find, in general, that each condition leads to a different set of eigenvalues, i.e., different values for $k$. Thus, except in special cases of degeneracy, the boundary conditions correspond to two different resonant frequencies: one for which $\vec{E}_z$ is permitted, but $\vec{H}_z$ is zero, and the other for which $\vec{H}_z$ is permitted, but $\vec{E}_z$ is zero. These two mutually exclusive modes of oscillation are designated as transverse magnetic (TM) and transverse electric (TE), respectively. There is a permitted mode of oscillation for each value of the eigenvalue $k$. Thus, the double
infinity of modes, TM and TE, form a complete set from which any arbitrary field configuration can be constructed.

If the conductivity of the walls is finite, the boundary conditions are such that a superposition of TM and TE waves is necessary to satisfy them at a single frequency. Refer to p. 526 of reference no. 31 for a discussion of this point.

Consider the case of TE waves. The solution for $H_z$ is first determined; then, the other field components are obtained from equations (28) and (29). A specific cavity will be considered - a right circular cylinder with length "$d$" and radius "$a$". The $z$ dependence of $H_z$ is given by equation (15). Since the normal component of $H_z$ is zero at $z = 0$, the sine solution must be selected. Also, since $H_z = 0$ at $z = d$: (28) $\sin \beta d = 0 \Rightarrow \beta d = p\pi, p = 1, 2, \ldots$. Thus, the eigenvalue $\beta$ is determined:

$$\beta = \frac{(p\pi)}{d}, p = 1, 2, \ldots$$

To complete the solution for $H_z$, equation (30) must be solved. In cylindrical coordinates, equation (30) becomes:

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial H_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 H_z}{\partial \phi^2} + k^2 H_z = 0\right).$$

Solving by separation of variables, assume:

$$H_z = R(r) \Phi(\phi).$$
Substitute in (32):

\[ (34) \quad \Phi \frac{\partial^2 R}{\partial r^2} + \frac{\Phi}{r} \frac{\partial R}{\partial r} + \frac{R}{r^2} \frac{\partial^2 \Phi}{\partial \phi^2} + k^2 R = 0, \]

\[ (35) \quad \frac{r^2}{R} \frac{\partial^2 R}{\partial r^2} + \frac{r}{R} \frac{\partial R}{\partial r} + k^2 r^2 + \frac{1}{\Phi} \frac{\partial^2 \Phi}{\partial \phi^2} = 0. \]

Separating variables:

\[ (36) \quad \frac{1}{\Phi} \frac{d^2 \Phi}{d \phi^2} = -n^2, \]

where \( n \) is some constant to be determined. The solution of (36) is:

\[ (37) \quad \Phi = A_1 \cos n\phi + A_2 \sin n\phi = A \cos (n\phi + \delta). \]

Now since \( \Phi \) must be single-valued, \( n \) must be equal to an integer, \( n = 0, 1, 2, \cdots \).

\[ (38) \quad r^2 \frac{d^2 R}{dr^2} + r \frac{dR}{dr} + (k^2 r^2 - n^2)R = 0. \]

This is the form of Bessel's equation, and its solution is:

\[ (39) \quad R = C_1 J_n (kr) + C_2 N_n (kr), \]

where \( J_n \) is the \( n \)th order Bessel function. The Neumann functions, \( N_n \), are improper for this problem since they have an infinity at \( r = 0 \). Thus, the total solution of \( H_z \) becomes:
\[ H_z = H_0 J_n (kr) \cos (n\phi + 8) \sin \beta z. \]

However, the boundary condition has not yet been applied at the side walls of the cavity. This condition is that the normal derivative of \( H_z \) must be zero at \( r = a \) and its application determines the eigenvalues, \( k \).

\[ \frac{\partial H_z}{\partial r} \bigg|_{r=a} = 0, \Rightarrow \frac{\partial J_n(ka)}{\partial r} = 0. \]

Thus:

\[ ka = S_{nm}, \]

where \( S_{nm} \) is the value of the argument for which the first derivative of the nth order Bessel function has its mth zero. That is, \( S_{nm} \) is the mth solution of equation (39), \( m = 1, 2, \ldots \).

Then, setting the arbitrary phase angle, \( 8 = 0 \),

\[ H_z = H_0 J_n \left( \frac{S_{nm} r}{a} \right) \cos n\phi \sin \frac{p\pi z}{d} \]

The other field components follow from equations (28) and (29) \((E_z = 0)\):

\[ H_t = (1/k^2) \nabla_t \left( \frac{\partial H_z}{\partial z} \right) = (1/k^2) \beta \cos \beta z \quad H_0 \nabla_t \left[ J_n(kr) \cos n\phi \right] \]

\[ = \frac{\beta \cos \beta z}{k^2} H_0 \left[ \frac{\partial J_n(kr)}{\partial r} \cos n\phi - \phi \frac{n}{r} J_n(kr) \sin n\phi \right], \]

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The result of the magnetic field 

\[ \mathbf{H} = \frac{j \omega \mu_0}{k^2} \nabla \times \mathbf{E} \]

can be expressed in terms of Bessel functions as:

\[ L = -\frac{J_0}{k^2} \left[ \hat{\phi} \frac{\partial J_n}{\partial r} \cos n \phi - \hat{r} \frac{J_n}{r} \sin n \phi \right] \sin \beta z. \]

Thus:

\[ H_r = \frac{j \omega \mu a^2 \mathbf{H}_0}{S_{nm}^2} \frac{\partial J_n}{\partial r} \left( \frac{S_{nm} r}{a} \right) \cos n \phi \cos \frac{p \pi z}{d}, \]

\[ H_\phi = -\frac{j \omega \mu a^2 \mathbf{H}_0}{S_{nm}^2} r J_n \left( \frac{S_{nm} r}{a} \right) \sin n \phi \cos \frac{p \pi z}{d}, \]

\[ E_r = -\frac{j \omega \mu a^2 \mathbf{H}_0}{u_{nm}^2} \frac{\partial}{\partial r} J_n \left( \frac{u_{nm} r}{a} \right) \sin n \phi \sin \frac{p \pi z}{d}, \]

\[ E_\phi = -\frac{j \omega \mu a^2 \mathbf{H}_0}{u_{nm}^2} \frac{\partial}{\partial r} J_n \left( \frac{u_{nm} r}{a} \right) \cos n \phi \sin \frac{p \pi z}{d}. \]

The equations for TM waves are similarly derived. The results are:

\[ E_z = E_0 J_n \left( \frac{u_{nm} r}{a} \right) \cos n \phi \cos \frac{p \pi z}{d}, \]

\[ E_r = -\frac{p \pi a^2 E_0}{u_{nm}^2} \frac{\partial}{\partial r} J_n \left( \frac{u_{nm} r}{a} \right) \cos n \phi \sin \frac{p \pi z}{d}. \]
The quantity \( u_{nm} \) is the solution of the equation \( J_n(u) = 0 \); that is, it is the value of the argument for which the \( n \)th order Bessel function has its \( m \)th zero.

The resonant frequency of the cavity is a function of the mode type and the dimensions. The equation relating these quantities comes from the defining equation for \( k \).

\[
(55) \quad k^2 = K^2 - \beta^2.
\]

But, since \( K^2 = \omega_\mu \varepsilon \), \( \beta^2 = \left( \frac{p\pi}{d} \right)^2 \), and (for TE waves)

\[
(56) \quad \omega_\mu \varepsilon = \left( \frac{S_{nm}}{a} \right)^2 + \left( \frac{p\pi}{d} \right)^2
\]

If the cavity medium is air; \( \mu \varepsilon = 1/c^2 \). Then,

\[
(57) \quad f_{nm}^2 = \frac{c^2}{4\pi^2} \left[ \left( \frac{S_{nm}}{a} \right)^2 + \left( \frac{p\pi}{d} \right)^2 \right]
\]

The subscripts on \( f \) denote the particular TE mode of oscilla-
tion. Also, it is common practice to speak of a mode defined by \( n, m, \) and \( p \) as a \( TE_{nmp} \) mode.

For TM modes, one would only replace \( S_{nm} \) by \( u_{nm} \) in equation (57) for the resonant frequency of a \( TM_{nmp} \) mode.

8.2 Independence of Cavity Modes

It will be shown that, in general, the modes of oscillation in a resonant cavity are orthogonal. The conditions for which there is energy transfer between modes will be discussed. The method of proof follows that of Smythe.\textsuperscript{32}

Consider the vector analog of Green's theorem which states:

\begin{equation}
\int_S [\bar{A} \times (\nabla \times \bar{B}) - \bar{B} \times (\nabla \times \bar{A})] \cdot d\bar{S} = \int_V [\bar{E} \cdot (\nabla \times (\nabla \times \bar{A})) - \bar{A} \cdot (\nabla \times (\nabla \times \bar{E}))] \, dV.
\end{equation}

Now, for the general vectors \( \bar{A} \) and \( \bar{B} \), write \( \bar{E}_i \) and \( \bar{E}_j \), respectively, where \( i \) and \( j \) represent different modes. Making use of equation (7) in Section 8.1,

\begin{equation}
\nabla \times \nabla \times \bar{E}_i = -\nabla^2 \bar{E}_i = k^2 \bar{E}_i = \omega_i^2 \mu \varepsilon \bar{E}_i.
\end{equation}

Substituting in equation (1):

\begin{equation}
\int_S [\bar{E}_i \times (\nabla \times \bar{E}_j) - \bar{E}_j \times (\nabla \times \bar{E}_i)] \cdot d\bar{S} = \mu \varepsilon (\omega_i^2 - \omega_j^2) \int_V \bar{E}_i \cdot \bar{E}_j \, dV
\end{equation}
The vectors $\mathbf{E}_i$ and $\mathbf{E}_j$ are normal to the boundary. Thus, both terms in the bracket are vectors tangential to the surface, and their scalar products with $d\mathbf{S}$ are zero. It follows that:

$$\int_V \mathbf{E}_i \cdot \mathbf{E}_j \, dV = 0, \quad \omega_i \neq \omega_j$$

and also,

$$\int_V \mathbf{B}_i \cdot \mathbf{B}_j \, dV = 0, \quad \omega_i \neq \omega_j$$

The electric and magnetic fields for different modes are then indeed independent, but subject to two important restrictions. First, the limit of infinite conductivity of the cavity walls has been assumed; and secondly, there must be no degeneracy in resonant frequencies. In practice, of course, the conductivity is finite and there will be some coupling between modes (refer to Section 8.1). However, even if the conductivity is very high so that this effect is negligible, there is strong coupling between modes when their resonant frequencies coincide. An important example of this is the degeneracy of the $\text{TE}_{01m}$ and $\text{TM}_{1mp}$ modes.

8.3 Q of the $\text{TE}_{011}$ Mode

The $Q$ of the $\text{TE}_{011}$ mode of a resonant cavity is calculated according to the definition
as defined in Section 2.1. The calculation is based upon the assumption that the electric and magnetic fields have the same form as calculated in Section 8.2, and losses due to the finite conductivity of the walls are the result of currents caused by these fields.

Since \( \mathbf{E} \) and \( \mathbf{H} \) are 90° out of time phase, the total energy stored can be found from \( \mathbf{E} \) when it is a maximum,

\[
U = \varepsilon / 2 \int |\mathbf{E}|^2 \, dv.
\]

The power loss can be calculated from

\[
P_L = 1/2 \int_{s} R_s |\mathbf{J}|^2 \, ds,
\]

where

\[
\mathbf{J} = \text{current density} = n \times \mathbf{H}
\]

\[
R_s = \text{surface resistivity} = \left[ \frac{\pi f \mu}{\sigma} \right]^{1/2}
\]

\( \sigma = \text{conductivity of cavity walls in mhos/meter} \)

\( \hat{n} = \text{unit vector normal to surface} \)

For the \( \text{TE}_{011} \) mode, the electric field is

\[
\mathbf{E} = \mathbf{E}_\phi = \hat{\phi} \left( \frac{j \omega \mu a^2}{S^2_{011}} \right) \frac{\partial}{\partial r} J_0 \left( \frac{S_{01} r}{a} \right) \sin \frac{n \pi}{d}
\]

The energy stored is given by
The result of this calculation is

$$U = \frac{\epsilon}{2} \int_0^2 \int_0^{2\pi} \int_0^d \left| E_\phi \right|^2 r dr d\phi dz$$

$$= \frac{\epsilon}{2} \int_0^2 \int_0^{2\pi} \int_0^d \frac{\omega^2 \mu^2 a^4}{s_{01}^2} H_o^2 \left[ \frac{\partial}{\partial r} J_o \left( \frac{S_{01}r}{a} \right) \right]$$

$$\cdot \sin^2 \frac{\pi z}{d} r dr d\phi dz.$$

The integral for $P_L$ is taken over all walls. Thus,

$$P_L = \frac{R_s}{2} \left\{ \int_0^2 \int_0^{2\pi} \int_0^d \frac{\omega^2 \mu^2 a^4}{s_{01}^2} H_o^2 \left[ \frac{\partial}{\partial r} J_o \left( \frac{S_{01}r}{a} \right) \right] r dr d\phi \right.$$  

$$\left. + H_o^2 \int_0^2 \int_0^{2\pi} \int_0^d \left[ J_o \left( S_{01}r \right) \right]^2 \sin^2 \frac{\pi z}{d} ad \phi dz \right\},$$

where the integrals are over the end plates and barrel of the cavity, respectively. The result is

$$P_L = R_s H_o^2 \left\{ \frac{\pi^3 a^4}{s_{01}^2 d^2} \left[ \frac{\partial}{\partial r} J_o \left( S_{01}r \right) \right]^2 + (\pi/2) \left[ J_o \left( S_{01}r \right) \right]^2 \right\}$$

Substituting equations (6) and (7) into (1) and simplifying, one obtains

$$Q = \frac{\epsilon \mu^2 \omega^3}{4R_s \left[ \pi^2 + \frac{s_{01}^2}{2a^3} \right]}$$
The values of d and a selected in Section 4.1 are:

(10) \[ 2a = 1.600 \text{ inches}, \]

(11) \[ d = 2.511 \text{ inches}. \]

The resonant frequency is

(12) \[ f = 9.3 \text{ kmc}. \]

At this frequency, the surface resistance for silver is

(13) \[ R_s = (2.52 \times 10^{-7}) f^\frac{1}{2} = (2.52 \times 10^{-7})(9.3 \times 10^9)^\frac{1}{2} \]

\[ = 24.3 \text{ ohms}. \]

The values of \( \varepsilon \) and \( \mu \) are their free-space values, and

\[ S_{01} = 3.832. \]

Substituting these values into equation (9), one obtains for \( Q \),

(14) \[ Q = 31,400. \]

8.4 Equivalent Circuit and Impedance of Cavity

There exist equivalent circuits for resonant cavities and their coupling systems. This will not be proved here. One is referred to Chapter 7 in reference 23, Chapter 11 in reference 14, and the literature in general for a discussion of this subject.

An equivalent circuit for a resonant cavity, its coupling system, and the waveguide of impedance \( Z_0 \) leading to it
is shown in Figure 8a. Figures 8b and 8c show the equivalent circuit with the primary impedance referred to the secondary.

(1) \[ Z_s = R_s + j \left[ \omega L - 1/\omega C \right] \]

By ordinary lumped-circuit analysis,

(2) \[ E' = \frac{-jEX_m}{z_1} \]

(3) \[ Z = \frac{-(jX_m)^2}{z_1} = \frac{(X_m^2)}{z_o} \left[ \frac{1 - jX_1/z_o}{1 + (X_1/z_o)^2} \right] \]

where

(4) \[ z_1 = z_o + j\omega L_1 = z_o + jX_1 \]

(5) \[ X_m = \omega M \]

Now, define

(6) \[ \beta \equiv \frac{(\omega M)^2}{z_oR_s} \left[ \frac{1}{1 + (X_1/z_o)^2} \right] = \beta_1 \left[ \frac{1}{1 + (X_1/z_o)^2} \right] \]

where

(7) \[ \beta_1 \equiv \frac{(\omega M)^2}{z_oR_s} \]

When \( \beta_1 = 1 \), the cavity is said to be critically coupled; when \( \beta_1 < 1 \), the cavity is undercoupled; and when \( \beta_1 > 1 \), the cavity is overcoupled. In general, \((X_1/z_o)^2 \ll 1\), and
FIGURE 8 — EQUIVALENT CIRCUIT
\[ \beta = \beta_1. \]

The loaded Q of the circuit in Figure 3c is

\[ Q_L = \frac{\omega L - \beta R_s (X_1/Z_0)}{R_s (1 + \beta)} = \frac{\omega L}{R_s} \left[ \frac{1 - (\beta R_s/Z_0) (X_1/\omega L)}{1 + \beta} \right]. \]

The second term in the numerator of equation (8) is usually negligible, thus,

\[ Q_L = \frac{\omega L}{R_s} \left[ \frac{1}{1 + \beta} \right]. \]

But,

\[ \frac{\omega L}{R_s} \equiv Q_o, \]

where \( Q_o \) is the Q of the unloaded cavity. Therefore,

\[ Q_L = \frac{Q_o}{1 + \beta}. \]

Defining,

\[ Q_{\text{ext}} = \frac{Q_o}{\beta}, \]

one has three different types of quality factor. \( Q_o \) represents the losses of the unloaded cavity; \( Q_L \) represents the losses of the cavity plus its coupling system; and \( Q_{\text{ext}} \) represents the losses of the coupling system. The factor \( \beta \) is called the coupling coefficient.
(13) \[ 1/Q_L = 1/Q_o + 1/Q_{\text{ext}} \]

The input impedance looking into terminals a-a of the coupling network is

\[(16) \quad Z_{aa} = jX_1 + \frac{(\omega M)^2}{R_s + j(\omega L - 1/\omega C)} \]

Rearranging (16),

\[(17) \quad \frac{Z_{aa}}{Z_o} = \frac{jX_1}{Z_o} + \frac{\beta_1}{1 + jQ_o \left[ \frac{\omega}{\omega_o} - \frac{\omega_o}{\omega} \right]} \]

where

\[(18) \quad \omega_o^2 = 1/LC \]

If \( \omega = \omega_o \), \( \beta_1 = (\omega_o M)^2/Z_oR_s \) and

\[(19) \quad \frac{Z_{aa}}{Z_o} = \frac{jX_1}{Z_o} + \frac{\beta_1}{1 + j2Q_o \delta} \]

where \( \delta \) is the frequency tuning parameter,

\[(20) \quad \delta = \frac{\omega - \omega_o}{\omega} \]

Notice that if the cavity is detuned, the second term in equation (19) becomes small and

\[(21) \quad Z_{aa}/Z_o = jX_1/Z_o \]

Now, select terminals b-b in Figure 8a at a distance d
from a-a such that \( Z_{bb} = 0 \). From transmission line theory, the impedance at b-b with the cavity detuned is

\[
\frac{Z_{bb}}{Z_o} = \frac{Z_{bb} + jZ_o \tan \lambda d}{Z_o + jZ_{aa} \tan \lambda d} = \frac{jX_i + jZ_o \tan \lambda d}{Z_o - X_i \tan \lambda d} = 0 .
\]

Solving for \( \lambda d \),

\[
\lambda d = -\tan^{-1}\left(\frac{X_i}{Z_o}\right) .
\]

Thus, the impedance at b-b for any value of \( \delta \) is

\[
\frac{Z_{bb}}{Z_o} = \frac{Z_{aa}/Z_o + jZ_o(-X_i/Z_o)}{Z_o + jZ_{aa}(-X_i/Z_o)} .
\]

Upon the substitution of the value of \( Z_{aa} \) from equation (19), \( Z_{bb} \) becomes, after simplifying,

\[
\frac{Z_{bb}}{Z_o} = \frac{\beta}{1 + j2Q_o(\delta - \delta_o)} ,
\]

where

\[
\delta_o = \frac{\beta X_i}{2Q_o Z_o} .
\]

The b-b is called the detuned short position, since the impedance at that point is zero if the cavity is detuned. The detuned short position is important in the experimental determination of \( \beta \).

Equation (26) also shows that resonance occurs at a frequency slightly different from the natural resonant fre-
quency of the unloaded cavity. That is, the resonant frequency is a function of the amount of coupling to the cavity.

8.5 Theory of $Q_0$ Measurement Technique

The magnitude of the coupling coefficient, $\beta$, can be obtained by measuring the VSWR at resonance. From equation (25) of the preceding section, the impedance at the detuned short position at resonance ($\delta = \delta_0$) is

$$Z_{bb} = \beta Z_0$$

From transmission line theory, one knows that the impedance at a voltage maximum is

$$Z = SZ_0,$$

and at a voltage minimum is

$$Z = Z_0/S,$$

where $S$ is the voltage standing wave ratio. Thus if a voltage maximum occurs at the detuned short position, the coupling coefficient is

$$\beta = S,$$

and the cavity is overcoupled.

If a voltage minimum occurs at the detuned short position

$$\beta = 1/S,$$

and the cavity is undercoupled.
Now, referring again to equation (25), Section 8.4, one sees that at certain frequencies the impedance at the detuned short position becomes

\[(6) \quad \frac{Z_{bb}}{Z_o} = \frac{\beta}{1 \pm j}.\]

The locus of these points for all possible values of \(\beta\), plotted on a Smith chart, is a circle with the center on the periphery of the chart at 90° points which passes through the two endpoints of the resistive axis. The plot of the impedance as a function of frequency also is a circle. The intersection of these two circles determines the frequencies at which

\[(7) \quad 2Q_o (\delta - \delta_o) = \pm 1.\]

If these two values of \(\delta\) are designated as \(\delta_1\) and \(\delta_2\),

\[(8) \quad 2Q_o (\delta_1 - \delta) = 1,\]
\[(9) \quad 2Q_o (\delta_2 - \delta) = -1,\]

it follows that

\[(10) \quad Q_o = \frac{1}{\delta_1 - \delta_2} = \frac{f_o}{f_1 - f_2}.\]

Frequencies \(f_1\) and \(f_2\) are called the half-power points.

Now, consider some results from transmission line theory. The magnitude of the voltage reflection coefficient is given
(11) \[ \Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} \]

The standing wave ratio is related to \( \Gamma \) by

(12) \[ S = \frac{1 + \Gamma}{1 - \Gamma} = \frac{|Z_L + Z_0| + |Z_L - Z_0|}{|Z_L + Z_0| - |Z_L - Z_0|} \]

where \( Z_L \) is the load impedance.

At the half-power points, the input impedance at the detuned short position is given by equation (6). If this value of \( Z_{bb} \) is substituted into equation (12), there results the value of the VSWR at the frequencies \( f_1 \) and \( f_2 \),

(13) \[ (S_{\frac{1}{2}})_o = \frac{2 + \beta^2 + \sqrt{4 + \beta^4}}{2\beta} \]

If one measures the VSWR as a function of frequency, and determines the coupling coefficient by means of the VSWR at resonance, one can then calculate \( Q_o \) from the bandwidth defined by equation (13).
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