RICE UNIVERSITY

THE COUPLING THROUGH UNITARITY
OF THE NN AND NN* CHANNELS

by

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We present a formalism to calculate the nucleon-nucleon interaction with contributions from the NNπ channel in the isobar approximation for the Nπ system. In order to write dispersion relations for the $T = 1, J = 0$ amplitudes, we study the complex singularities that are present in the Born amplitude for the production process. We use the N/D coupled channels method for the helicity partial wave amplitudes and write the explicit equations for the $T = 1, J = 0$ amplitudes taking into account, through a simplified model, the contributions of the complex singularities.
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I. INTRODUCTION

In this thesis we will study the effect of an inelastic channel on the nucleon-nucleon interaction in the context of S-matrix theory.\(^1\), \(^2\) We have chosen the NN\(^*\) channel because the N\(^*\) production dominates the experimental data after the elastic region\(^3\) and thus we expect it to have greater influence on NN elastic scattering than the other inelastic channels. This N\(^*\) is the lowest of the N\(\pi\) resonances, with \(J^P = 3/2^+, I = 3/2\), mass of 1238 MeV and a width of about 125 MeV. In potential scattering by a trivial calculation, one can see that, below the inelastic threshold, an inelastic channel does influence the amplitude. This influence is given by an always attractive potential. We want to study the analogous situation in relativistic S-matrix theory.

There is little doubt that the inelastic channels can cause an enhancement in the amplitude (and cross section) in the elastic region near and below the inelastic threshold. There are several theoretical models that predict such enhancements and also experimentally several resonances are observed near the inelastic thresholds of a reaction.

The three most important enhancement causing mechanisms predicted by the theoretical models are the mechanism due to strong coupling between elastic and inelastic channels,
the threshold effects due to a rapidly rising inelastic cross section, and finally the mechanism by which a bound state in the inelastic channel decays into the elastic channel. The first mechanism is proposed by Cook and Lee (CL). It assumes a strong coupling between the elastic and inelastic channels due to unitarity and resonances are produced when the system has virtual oscillations between the two channels. In this way, the elastic channel can resonate (virtually) into the inelastic channel at energies much lower than the inelastic threshold, though it will be near to this threshold where the sharper resonances are possible. At energies above the inelastic threshold, unitarity also damps the elastic cross section. An interesting feature is that in the limit of very strong coupling between the elastic and inelastic channels, the amplitudes for two and three particle scattering actually become decoupled. This happens because the amplitude for transitions between the channels becomes zero as the coupling is too strong for a channel to make a permanent transition into the other channel.

In the second mechanism, one has that a rapidly rising production amplitude produces resonances near the production threshold. Ball and Frazer showed that the second and third \( \pi N \) resonances are explained by this mechanism. If the production amplitude led to the production of stable particles, then there would be very sharp cusps at threshold; but, due to the fact that one has the production of
unstable particles which have a mass distribution about a
certain value, the cusp is rounded and becomes what Nauem-
berg and Pais call wooly cusp. It turns out that the
prominence of the cusp depends on the strength of the elas-
tic inelastic coupling; and also, they are more prominent
when the production amplitude leads to particles in a low
orbital angular momentum state. As the coupling increases,
the cusp still remains but the appearance of the inelastic
channel is more gradual.

The third mechanism assumes that one can effective-
ly separate the channels and if the three particle system
is such that it can have a bound state, then as one "turns
on" the coupling between the elastic and inelastic channels
the virtual bound state decays and is seen as a resonance
in the elastic region, usually not very far from the inelas-
tic threshold. This mechanism can explain the resonance as a bound state in the $\bar{KN}$ channel. We do not expect
this mechanism in our problem because there are no bound
states in the $NN\pi$ system.

In coupled channels calculations, the system that has
been better studied is the $\pi N$, $\pi^* N$. In order to reduce the complexity of the calculations, only
the effects due to the three body system being in a two
particle ($pN$ or $N^*\pi$) state have been considered. The inter-
action leading to the $pN$ channel has a longer range than
than leading to a $\pi N^*$ channel since it can proceed by one
pion exchange while the latter cannot; and thus the more important inelastic channel is the $\rho N$. Ball, Frazer, and Nauenberg (BFN)\textsuperscript{14} and Cook and Lee\textsuperscript{5} derived the N/D equations for this two channel problem. Their solution would give relativistic amplitudes which include the effects of the inelastic channel. Later in this paper we will discuss this further.

Now we turn our attention to the NN interaction. We give a brief review on some of the more recent theoretical calculations.

Besides the short range effects of the NN interaction, the intermediate range, $\frac{1}{2\mu} < r < \frac{1}{\mu}$, ($\mu$ is the mass of the pion), is also heavily dependent on other than OPE interactions.\textsuperscript{16} The intermediate range properties of the NN interaction include the contributions from exchange of mesons heavier than the pion and two pion exchange (TPE). Scotti and Wong\textsuperscript{17} do one channel calculations using the N/D method with input consisting of the Born amplitudes corresponding to the exchange of the $\pi$, the pseudoscalar $\eta (I = 0)$, and the three vector mesons, $\rho (I = 1)$, $\omega (I = 0)$, $\phi (I = 0)$ with masses ranging from 140 MeV for the pion to about 1020 MeV for the $\phi$ meson. They consider these as stable particles. They also consider the exchange of the $\sigma$ meson ($I = 0$, scalar), but as this resonance has not been observed, they also calculate the exchange of an $S$-wave $\pi\pi$ system and observe that both contribute significantly to the intermediate range...
attractive force. They obtain a good fit with data up to 388 MeV, but the value they obtain for some of the parameters does not agree very well with experimental values, the main discrepancy probably being in the mass of the $\rho$ meson, for which they obtain 591 MeV while the experimental value is around 770 MeV.\textsuperscript{18}

Coulter, Scotti, and Shaw\textsuperscript{19} calculate the effect of the inelastic channels on NN scattering by doing a one channel N/D calculation with the amplitudes modified by an inelastic factor $\eta$ which is fixed by experimental data. Their results agree with the results of Scotti and Wong for $E_{\text{lab}} \lesssim 200$ MeV, but show appreciable deviation for higher energies and they conclude that for $E_{\text{lab}} \gtrsim 300$ MeV inelasticity should be taken into account.

A semi-phenomenological theory which can be fitted very well to the experimental data is Feshbach and Lomon's "Boundary Condition Model".\textsuperscript{20} In this potential model of the NN interaction, a fixed radius $r_0$ is found (at about $\frac{1}{2\mu}$) such that for distances greater than $r_0$ a local potential is calculated from the amplitudes for boson exchanges while the potential for $r < r_0$ is represented by an energy independent boundary condition on the wave function at $r_0$.\textsuperscript{21} This model is not entirely phenomenological and the energy independent boundary condition can be derived from potential scattering assuming inside $r_0$ maximum nonlocality and strength of the interaction compatible with causality, also
assuming a very general model for the core (that the potential does not vary very much with energy). This model is not relativistic and is intended to work up to about 350 MeV. Lomon and Feshbach include some two pion exchange contributions but their effect is ambiguous as several cancellations are present. They do not include any inelastic channels although the BCM can be applied to coupled channels calculations and actually some have been performed in KN scattering, including the NK* channel.

Sugawara and von Hippel (SvH) develop a simple idea to calculate the effects of the inelastic channels on elastic NN scattering in their paper where they develop zero parameter potentials for the NN interaction. Theirs is a non-relativistic theory so obviously it cannot be expected to be very realistic near the inelastic threshold. In their theory, they take into account the OPE amplitude, the exchange of vector mesons, and the contributions from modified OPE amplitudes corresponding to the NN ---\(\rightarrow\) NN* and NN ---\(\rightarrow\) N*N* processes. The modification of the OPE amplitudes for the production amplitudes consists in introducing a cutoff factor which is fixed by the N* production cross section. In their work, they fix no parameters with the NN scattering data but nevertheless they find good qualitative agreement with phenomenological potentials. SvH then calculate the potentials by taking the Fourier transforms (in momentum transfer variable) of the non-relativistic limit of the
amplitudes, and from the Schroedinger equation for coupled channels for definite partial waves, they obtain an effective potential from the potentials calculated from the amplitudes. They find that the inelastic channels contribute an attractive intermediate range \( \frac{1}{2\mu} < r < \frac{1}{\mu} \) potential which is of the same order of magnitude in that region as the potential due to vector meson exchange and, for very short ranges, \( r < \frac{1}{2\mu} \), the repulsive vector meson potential dominates. For long ranges, \( r > \frac{1}{\mu} \), the potential due to the inelastic channels is larger than the vector meson exchange potential.

The effect of the inelastic channel has also been studied in relativistic models. Shephard\(^{23}\) studies the effect of the NN* channel on the elastic region of NN scattering with the determinantal method, which is an approximation to the N/D method (it is the N/D method letting N equal the Born amplitude) and does not take into account the effects of the complex cuts which are present in the production amplitude. Shephard calculates the \( \mathcal{J} = 2^+ \) amplitudes for the NN \( \rightarrow \) NN, NN \( \rightarrow \) NN*, and NN* \( \rightarrow \) NN* processes with the only driving force being the OPE amplitude for NN \( \rightarrow \) NN*. He finds that a cusp near the inelastic threshold can occur depending on the values of the strength of the coupling between channels and on the width of the N but for the experimental value of \( \Gamma_{NN^*} \) he finds that the cusp is so wooly as to be virtually undetectable experimentally. In addition, he finds that the
real part of the phase shift is close to $90^\circ$, even for $\Gamma_{N^*}$ near the experimental value, indicating a resonance; but experimentally the value of the phase shift in that region is on the order of $10^\circ$, the large discrepancy probably being caused by the neglect of the $NN \rightarrow NN, NN^* \rightarrow NN^*$ forces and of other exchanges which contribute to $NN \rightarrow NN^*$. Coulter, Scotti, and Shaw$^{19}$, as a comparison, obtain that the real part of the phase shift does peak between 400 and 500 MeV and its value there is about $16^\circ$.

Leung$^4$ also does a study of the $NN^*$ effect with the N/D method. He also neglects the contribution of the anomalous cut but he estimates it, by replacing the cut by two poles, to be on the order of $3\%$. Shephard$^{23}$ also gives a rough estimate, but his value is very large, comparable in magnitude to the contributions of the cuts on the real axis.

In our work we use the N/D$^{1,24}$ method to obtain relativistic helicity partial wave amplitudes$^{25,26}$ which satisfy unitarity. As we have said above, we will only consider one inelastic channel, the $NN^*$, because the interactions leading to it are the longest ranged (OPE) and also because the $NN^*$ production is the most important experimentally. The effect of this inelastic channel will therefore be felt in the $T = 1$ NN scattering amplitudes (since the $NN^*$ can only be coupled to $T = 1$ due to conservation of isospin). The input to the N/D equations will be the OPE amplitudes, i.e., the (antisymmetrized) amplitudes corresponding to the fol-
owing diagrams:

\[
\begin{array}{c}
N \\
N \\
N \\
N \\
\end{array}
\quad
\begin{array}{c}
N \\
N \\
N \\
N \\
\end{array}
\quad
\begin{array}{c}
N \\
N \\
N \\
N \\
\end{array}
\quad
\begin{array}{c}
N \\
N \\
N \\
N \\
\end{array}
\]

Thus, the resultant amplitude for the process \( NN^* \rightarrow NN^* \) will be just the part of it due to the coupling between channels. This simplifies the problem to be solved.

By taking only the OPE pole as driving force in the NN interaction, we are neglecting the other singularities in the crossed channels such as other boson exchange poles and also the multipion logarithmic singularities (which, if done exactly, include the exchange of the pion resonances). Since all of these neglected singularities lie further away from the physical region, we expect that their contributions to the amplitudes will be smaller than the OPE contributions, especially for higher partial waves; this is a large approximation since we have seen that, e.g., vector meson exchanges are important, but we are mainly trying to see the effect of the inelastic channel and not calculate realistic amplitudes. Actually, in lowest order the calculated amplitudes will give the contribution of the following diagram

\[
\begin{array}{c}
N \\
N \\
N \\
N \\
\end{array}
\quad
\begin{array}{c}
N \\
N \\
N \\
N \\
\end{array}
\quad
\begin{array}{c}
N \\
N \\
N \\
N \\
\end{array}
\quad
\begin{array}{c}
N \\
N \\
N \\
N \\
\end{array}
\]

so part of the effect of the two pion exchange is included.
Furthermore, we will subsequently add directly the contribution of heavier meson exchange amplitudes which do not have complex singularities.

We take into account the three body aspect of the problem only in the fact that we take the $N^*$ to be unstable and this enters only in some kinematical factors (see Sec. V) when we assume that the width of the resonance is small although it is not. By varying the width we should get an idea of the validity of the approximation. The dynamical cuts will be those with the $N^*$ mass fixed at 1238 MeV.

In Section II we present a brief description of the N/D method for a one-channel partial wave amplitude. Here we consider the case where the left and right hand cut do not overlap and lie on the real $s$ axis (where $s$ is the square of the energy in the CM system). For the production amplitudes the cuts on the physical sheet are not as simple. There we have overlapping cuts and one must be very careful in setting up the N/D equations. This is discussed further in Section V.

In Section III we study and define the helicity states and amplitudes for the processes $NN \rightarrow NN$ and $NN \rightarrow NN^*$ and we also state our kinematical conventions. With these definitions and conventions, we proceed into Section IV where we calculate the OPE $J^P = 0^\pm$ partial wave helicity amplitudes for the above processes.

In Section IV we also study the analytic structure of
these Born amplitudes, which we need since in the N/D method one takes the dynamical cuts of the exact partial wave amplitude to be all those of the corresponding partial wave projection of the Born amplitude.

In Section V we treat the N/D formalism for the coupled channels. There we also discuss briefly the complications due to the anomalous cuts in the production amplitudes and we present a model which approximates the effect of the anomalous cuts and will simplify the numerical work. Within this model, we obtain the integral equations for the $J = 0^+$ amplitudes but we do not solve them here. We expect that the solutions of this model will give an estimate of the effects of the NN* channel.

The thesis includes several appendices where we do some of the explicit calculations and include some transformation tables for amplitudes between LS states and helicity amplitudes. Also in Appendix E we present an example of a function to illustrate some of the properties of the OPE production amplitude.

This thesis intends to be a basis for our later work, which will incorporate fully the effect of the NN* production on NN scatterings. Also we will include the other boson exchanges to this problem and this can be done easily within the formalism in Section V.
II. THE N/D METHOD

In this section we derive from Cauchy's integral formula the N/D equations for a partial wave helicity amplitude. Let $F(s)$ be a partial wave helicity amplitude where $s$ is a real analytic function on the $s$ plane except for branch cuts for real $s$. $F(s)$ satisfies elastic unitarity and this gives it a kinematical cut on the right hand side, beginning at $s^t$ which is the threshold for the reaction. This is not the complete unitarity cut. This cut is for unitarity in the $s$ channel, and unitarity in the crossed channels will be neglected except for the one pion exchange pole terms which are the nearest singularities to the physical region for NN scattering. $F$ also has dynamical cuts for $s$ less than a real constant $s_a$ which are due to unitarity in the crossed channels. Now if we assume $F(s)$ goes to zero as $s \rightarrow \infty$, we can write an unsubtracted dispersion relation for $F$ if we assume that the above two cuts are the only regions where $F$ is not analytic. If $F$ does not go to zero at $\infty$, we can still write substracted dispersion relations if we have that $F$ diverges at most as a polynomial in $s$.

For the discontinuity in $F$ across the dynamical cut, one takes generally the discontinuity in the first Born approximation due to a one particle exchange, and the unitarity relation gives us the discontinuity across the right hand cut. Then the dispersion relation gives us an amplitude
with the correct analytical structure but incorrect threshold behavior. This is easily seen since the integral over the dynamical cuts is the Born amplitude which does have the correct behavior at threshold and it cannot cancel the nonvanishing contribution at threshold of the integral over the kinematical cut. In fact, it seems that even if we had as input the exact dynamical discontinuity, the resulting amplitude would not have the correct threshold behavior.\textsuperscript{29} This becomes apparent when we consider the fact that the dispersion integral equations generally do not give a unique solution so the threshold behavior is a property which must be built into the amplitude.

Generally, the desired amplitude must have zeroes of a given order at threshold and then we introduce the function $\rho(s)$ with zeroes of order $m$ at threshold and such that it goes to a constant as $s \rightarrow \infty$. We write dispersion relations for $M(s) = F(s)\rho^{-1}(s)$ and this assures that $F(s)$ has at least the zeroes of $\rho(s)$ at threshold. In our case, where we consider $J = 0^+$ amplitudes, we do not need to fix the threshold properties but we still need a $\rho$-type factor to remove certain kinematical singularities which are present in the helicity amplitudes. This is considered in the next sections.

The integral equation for $M(s)$ provided by the dispersion relation is not linear, but we can obtain instead of a nonlinear equation a pair of linear integral equations.

We do this by expressing $M(s)$ as a quotient of two
functions\textsuperscript{1, 24, 30}, $M(s) = N(s)/D(s)$ such that $D(s)$ has the unitarity cut and is analytic elsewhere, and $N(s)$ is analytic except where $M(s)$ has dynamical cuts. Both $D(s)$ and $N(s)$ can be taken to be real analytic functions. $M$ can always be expressed in this manner for we can let

$$D(s) = \exp \left[ -\frac{1}{\pi} \int_{s_t}^{s_\infty} \frac{ds'}{s'-s} \delta(s') \right]$$

where $\delta$ is the phase shift and thus $D(s)$ is real analytic with branch points at $s_t$ and $s_\infty$ only, and in the physical region it has the phase $\exp(-i\delta)$. This is easily seen since

$$D(s+i\epsilon) = \exp \left[ -\frac{1}{\pi} \int_{s_t}^{s_\infty} \frac{ds'}{s'-s-i\epsilon} \delta(s') \right] = \exp \left[ -\frac{1}{\pi} \int_{s_t}^{s_\infty} ds' \frac{\delta(s')}{s'-s} -i\delta(s) \right] = |D(s+i\epsilon)|e^{-i\delta(s)}$$

and

$$D(s-i\epsilon) = |D(s-i\epsilon)|e^{i\delta(s)} = |D(s+i\epsilon)|e^{i\delta(s)}$$

for $s_t < s < s_\infty$, therefore, $D(s)$ has the unitarity cut of $M$ for $s_t < s < s_\infty$. Now with $D(s)$ given by (1), we define $N(s) = M(s)D(s)$ and it has the dynamical cut and is real for $s \in \mathbb{R}$.

The equations we obtain for $N$ and $D$ are linear integral equations, usually of the Fredholm type, which are easier to solve than the equation for $M$.

So now let us suppose $M$ is the (partial wave helicity) amplitude for a one channel process and that its only cuts as a function of $s$ are a right hand cut due to unitarity and a left hand cut due to the two particle interaction. The
unitarity cut is that one due to two particle intermediate states only. Let the right cut be for \( s < s_t \), and the left hand cut be for \( s > s_a \), then

\[
\text{disc}_R D(s) = \frac{1}{2\pi i} [D(s+i\epsilon) - D(s-i\epsilon)]
\]

\[
= N(s) \text{disc}_R [M(s)]^{-1} \quad s > s_t
\]

(2)

since \( N(s) \) is continuous for \( s > s_t \). Analogously for \( N(s) \),

\[
\text{disc}_L N(s) = \text{disc}_L M(s) D(s) \quad s < s_a
\]

(3)

Using (2) and (3), we can write dispersion equations for \( N \) and \( D \) if we assume that they are analytic everywhere, except for the above cuts and that they go to zero for large \( s \). We use Cauchy's integral formula,

\[
D(s) = \frac{1}{2\pi i} \int \frac{ds'}{s'-s} [D(s'+i\epsilon) - D(s'-i\epsilon)]
\]

\[
= \frac{1}{\pi} \int \frac{ds'}{s'-s} N(s') \text{disc}_R M^{-1}(s)
\]

(4)

and analogously,

\[
N(s) = \frac{1}{\pi} \int \frac{ds'}{s'-s} [\text{disc}_L M(s')] D(s')
\]

(5)

In general, we cannot assume that \( \lim_{s \to \infty} D(s) = 0 \), so the above equations are not correct. However, we can assume at least initially, that \( \lim_{s \to \infty} (s-s_0)^{-1} D(s) = 0 \) where \( s_0 \) is a constant, \( s_a < s_0 < s_t \), and then we have

\[
\frac{D(s)}{s-s_0} + \frac{1}{2\pi i} \int \frac{ds'}{s'-s} \frac{D(s')}{s'-s_0} = \frac{1}{\pi} \int \frac{ds'}{s'-s_0} \text{disc} \frac{D(s')}{s'-s_0}
\]

(6)
D(s) = D(s_o) + \frac{s-s_o}{\pi} \int_{s_o}^{\infty} \frac{ds'}{(s'-s)(s'-s_o)} \text{disc } D(s') \hspace{1cm} (6)

Now let

\[ A_L(s) = \text{disc}_L M(s) \]
\[ A_R(s) = \text{disc}_R M(s) \]

and then

\[ D(s) = D(s_o) + \frac{1}{2} (s-s_o) \int_{s_o}^{\infty} ds' \int_{s_o}^{s'} ds'' A_L(s'') D(s'') A_R(s') \hspace{1cm} (7) \]

and

\[ N(s) = \frac{1}{\pi} \int_{s_o}^{\infty} \frac{ds'}{s'-s} A_L(s') \times \left[ D(s_o) + \frac{s'-s_o}{\pi} \int_{s_o}^{\infty} \frac{ds''}{s''-s'} \frac{N(s'') A_R(s'')}{s''-s_o} \right] \]

Let \( B(s) = \frac{1}{\pi} \int_{s_o}^{\infty} \frac{ds'}{s'-s} A_L(s') \) then

\[ N(s) = D(s_o)B(s) + \frac{1}{2} \int_{s_o}^{\infty} ds' \int_{s_o}^{s'} ds'' \frac{(s'-s_o)}{(s'-s)(s''-s')} \times \frac{A_L(s') N(s'') A_R(s'')}{s''-s_o} \]

\[ = D(s_o)B(s) \]
\[ - \frac{1}{\pi} \int_{s_o}^{\infty} \frac{ds'}{s'-s} \left[ B(s') - \frac{s'-s_o}{s'-s_o} B(s) \right] N(s') A_R(s') \hspace{1cm} (8) \]

These are the N/D integral equations for the numerator and denominator functions, N and D. They possess a number of desirable properties; for example, the solutions are independent of the substraction point \( s_o \) and independent of the constant \( D(s_o) \); the integral equation for N has a
non-singular kernel and for reasonable choice of $B$ and $A$ is Fredholm. The many channel generalization we will eventually use provides us with a symmetric (time reversal invariant) scattering matrix. These are the features which make the N/D equations with an approximate left hand discontinuity a desirable dynamical model and we will find other advantages as we go on.
III. THE HELICITY AMPLITUDES

In this section we first construct the helicity states for the N and N* and then define the amplitudes between these states. The helicity, being the component of the total angular momentum in the direction of motion is invariant under proper Lorentz transformations and changes sign under coordinate inversions and thus we will have relativistic amplitudes.

A. HELICITY STATES

The N* is a resonance with $J = 3/2$, $I = 3/2$ and in this section we will study the treatment of the field operator representing such a particle.

The simplest way to represent a spin 3/2 field is by coupling a vector and a spinor field and requiring it to satisfy certain subsidiary conditions that will cancel the $J = 1/2$ part.\textsuperscript{31, 32} We will proceed analogously for the isospin part, but we will do it separately. Thus, we write the field for the N* as

$$\psi_{N^*}(x) = [\psi_0(x), \psi_1(x), \psi_2(x), \psi_3(x)] = \psi_{\alpha\mu}(x)$$

where each $\psi_{\mu}$ is a Dirac spin 1/2 spinor and $\mu$ is a vector index = 0, 1, 2, 3 and $\alpha$ is a spinor index = 1, 2, 3, 4; and we require the field to satisfy

$$(i\gamma^\mu + M)_{\alpha\beta} \psi_{\beta\mu}(x) = 0 \quad \mu = 0, 1, 2, 3 \quad (1)$$
and \((\gamma^\mu)_{\alpha\beta} \phi_{\beta\mu} = 0\) \(\alpha = 1, 2, 3, 4\) \(\text{(2)}\)

These are the Rarita-Schwinger equations for a spin \(3/2\) field.

From (1) and (2) we see that \(\psi_{N^*}\) satisfies the Klein-Gordon equation and also we see that

\[
(i\gamma_\mu \partial_\mu)\psi_\nu = -M\psi_\nu
\]

\[
\therefore 0 = \gamma_\nu \gamma_\mu \partial_\mu \phi_\nu = 2\partial_\nu \phi_\nu \gamma_\nu \gamma_\mu \partial_\mu \phi_\nu = 2\partial_\nu \phi_\nu
\]

\[
\therefore \partial_\nu \phi_\nu (x) = 0 \quad \text{(3)}
\]

The subsidiary condition in this form will be very useful in our calculations. The subsidiary conditions insure that in the non-relativistic limit \(\psi_{\alpha\mu}\) is just the spin \(3/2\) part of the coupling of a vector and a spin \(1/2\) field.

To get the configuration space representation of the spin \(3/2\) field, we then write

\[
\psi_\mu (x) = \int \frac{d^3p \sqrt{\frac{M}{p_0}}}{3/2} \sum_{\lambda = -3/2} \{ a(\vec{p},\lambda) \omega_\mu (\vec{p},\lambda) e^{-i\vec{p}\cdot\vec{x}} + b^+(\vec{p},\lambda) v_\mu (\vec{p},\lambda) e^{i\vec{p}\cdot\vec{x}} \}
\]

where \(p_0 = \sqrt{m^2 + p^2}\)

and where \(a(\vec{p},\lambda)\) destroys an \(N^*\) in a helicity state \((\vec{p},\lambda)\) and \(b^+(\vec{p},\lambda)\) creates an \(\bar{N}^*\) in a helicity state \((\vec{p},\lambda)\) and \(\omega_\mu (\vec{p},\lambda)\), \(v_\mu (\vec{p},\lambda)\) are the solutions to the Rarita-Schwinger equations, written in the helicity representation.
Then we can write

$$\omega_\mu(\vec{p}, \lambda) = \sum_{\lambda'=-1/2}^{1/2} \langle l, 1/2, 3/2; \lambda-\lambda', \lambda', \lambda \rangle \epsilon_\mu(\vec{p}, \lambda-\lambda') u(\vec{p}, \lambda')$$

(5)

where $u(\vec{p}, \lambda)$ are the usual Dirac equation solutions in the helicity representation, and $j_1, j_2, j; m_1, m_2, m$ are Clebsch-Gordan coefficients.

The $\epsilon_\mu(\vec{p}, \lambda)$ are the spin 1 vectors and they must satisfy, because of (3),

$$p^\mu \epsilon_\mu(\vec{p}, \lambda) = 0$$

(6)

$$\epsilon^*(\vec{p}, \lambda)\epsilon_\mu(\vec{p}, \lambda) = -1$$

(7)

For a particle at rest we have

$$\epsilon_\mu(0, \pm 1) = \pm \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix} \quad \epsilon_\mu(0, 0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and to obtain $\epsilon_\mu(\vec{p}, \lambda)$ we must apply first a Lorentz transformation to $\epsilon_\mu(0, \lambda)$ so as to get the vectors for a particle moving with momentum $p = (E^2 - M^2)^{1/2}$ along the z axis and then we rotate them so as to give them a direction parallel to $\vec{p}$. We will work in a coordinate system where $\phi = 0$, i.e., $\vec{p}$ lies in the x-z plane. Then

$$\epsilon_\mu(\vec{p}, \lambda) = R(0, \theta, 0) L^\nu_\mu(p\hat{z}) e_\nu(0, \lambda)$$

where $[L^\nu_\mu(p\hat{z})] = \begin{pmatrix} E/M & 0 & 0 & P/M \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ P/M & 0 & 0 & E/M \end{pmatrix}$.
and \( R(0, \theta, 0) = \exp(-i\theta J_y) =
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos\theta & 0 & \sin\theta \\
0 & 0 & 1 & 0 \\
0 & -\sin\theta & 0 & \cos\theta
\end{pmatrix}
\]
and we get
\[
e^p_{\mu}(p, \pm l) = \pm \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \cos\theta \\ i \sin\theta \\ -\sin\theta \end{pmatrix} \quad e^p_{\mu}(p, 0) = \begin{pmatrix} p/M \\ E/M \sin\theta \\ 0 \\ E/M \cos\theta \end{pmatrix} \quad (8)
\]

The helicity spinors \( u(p^\dagger, \lambda) \), \( \lambda = \pm 1/2 \), \( \hat{p} \) in the x-z plane making an angle \( \theta \) with the z axis are obtained in the same way.

\[
u(p^\dagger, \lambda_1) = \sqrt{\frac{E+M}{2M}} \begin{pmatrix} \chi_1 \\ \frac{p\cdot\xi}{E+M} \\ \chi_1 \end{pmatrix}
\]

where \( \chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) for \( \lambda_1 = 1/2 \)

and \( \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) for \( \lambda_2 = -1/2 \)

and then, \( u(p^\dagger, \lambda_1) = \exp(-i\theta J_y^*) u(p^\dagger, \lambda_1) \)

\[
u(p, \lambda_1) = \sqrt{\frac{E+M}{2M}} \begin{pmatrix} \chi_1(\theta) \\ \frac{2p\cdot\lambda_1}{E+M} \\ \chi_1(\theta) \end{pmatrix}
\]
where

\[
\chi_1(\theta) = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix} \quad \text{and} \quad \chi_2(\theta) = \begin{pmatrix} -\sin\theta/2 \\ \cos\theta/2 \end{pmatrix}
\]

The isospin part of the \( N^* \) field is treated similarly. We use the three isospin = 1 polarization vectors in isospace, \( \hat{\epsilon}(1), \hat{\epsilon}(0), \) and \( \hat{\epsilon}(-1), \) where
\[ \hat{\varepsilon} (\pm 1) = \pm \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ \pm i \\ 0 \end{pmatrix} \quad \hat{\varepsilon} (0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]

and we write

\[
N^{*+}_i = \varepsilon_i (1) \chi_{1/2} \\
N^+_i = \frac{2}{\sqrt{3}} \varepsilon_i (0) \chi_{1/2} + \frac{1}{\sqrt{3}} \varepsilon_i (1) \chi_{-1/2} \\
N^0_i = \frac{1}{\sqrt{3}} \varepsilon_i (-1) \chi_{1/2} + \frac{2}{\sqrt{3}} \varepsilon_i (0) \chi_{-1/2} \\
N^-_i = \varepsilon_i (-1) \chi_{-1/2}
\]

where \( \chi_{1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) a \( I_3 = 1/2 \) isospinor

\( \chi_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) a \( I_3 = -1/2 \) isospinor.
B. HELICITY AMPLITUDES FOR NN \rightarrow NN AND NN \rightarrow NN^*

We have the following process which will be described in the center of momentum system

\[ 1 + 2 \rightarrow 3 + 4 \]

and particle \( i \) has momentum \( p_i \). We will use the Mandelstam variables, which are

\[ s = (p_1 + p_2)^2 = (p_3 + p_4)^2 = W^2, \text{ the square of the energy} \]

\[ t = (p_1 - p_3)^2 = (p_2 - p_4)^2, \text{ the negative of the momentum transfer} \]

\[ u = (p_1 - p_4)^2 = (p_2 - p_3)^2 \]

Now, in the CM system we have that

\[ p_i = (E_i, \hat{p}_i) \]

and \( \hat{p}_1 = \hat{p} = -\hat{p}_2 \) and \( \hat{p}_4 = \hat{p}' = -\hat{p}_3 \)

where we let \( \hat{p}' \) lie in the x-z plane. We then have

\[ t = m_1^2 + m_3^2 - 2E_1E_3 - 2pp'\cos \theta \]

\[ u = m_1^2 + m_4^2 - 2E_1E_4 + 2pp'\cos \theta \]

and

\[ p = \frac{1}{2W} A(W, m_1, m_2) \quad p' = \frac{1}{2W} A(W, m_3, m_4) \]

where \( A(W, a, b) = \left[ (W+a+b)(W+a-b)(W-a+b)(W-a-b) \right]^{1/2} \)

Throughout the thesis we let

\( m \equiv \text{mass of nucleon} \)
$M \equiv $ mass of N$^*$

$\mu \equiv $ mass of pion

Let particle $i$ be in a helicity state with helicity $\lambda_i$. Then the helicity amplitude is given by the T-matrix element

$$T(\lambda_4, \lambda_3, \lambda_2, \lambda_1) = \langle \lambda_4 \lambda_3 | T | \lambda_1 \lambda_2 \rangle$$

and we define the partial wave helicity amplitudes for a given total angular momentum $J$ by$^{25, 26}$

$$T(\lambda_4, \lambda_3, \lambda_2, \lambda_1) = \frac{1}{\mathbf{p} \mathbf{p}'} \sum_J (2J+1) \langle \lambda_4 \lambda_3 | T^J(s) | \lambda_1 \lambda_2 \rangle d^J_{\lambda \mu}(\theta)$$

where $\lambda = \lambda_1 - \lambda_2$, $\mu = \lambda_4 - \lambda_3$ and normalized such that

$$\frac{d\sigma}{d\Omega} = |T(\lambda_4, \lambda_3, \lambda_2, \lambda_1)|^2$$

where we defined $T^J(\lambda_4, \lambda_3, \lambda_2, \lambda_1) = \langle \lambda_4 \lambda_3 | T^J(s) | \lambda_1 \lambda_2 \rangle$

$$= \frac{(\mathbf{p} \mathbf{p}')^{\frac{3}{2}}}{2} \int_{-1}^{1} d(cos \theta) T(\lambda_4, \lambda_3, \lambda_2, \lambda_1) d^J_{\lambda \mu}(\omega; \theta)$$

In terms of this partial wave helicity amplitudes, the S-matrix element is given by

$$\langle \lambda_4 \lambda_3 | S^J(s) | \lambda_1 \lambda_2 \rangle = \delta_{1,4} \delta_{2,3} + 2i \sqrt{\frac{p \cdot p'}{p}}$$

$$\cdot \langle \lambda_4 \lambda_3 | T^J(s) | \lambda_1 \lambda_2 \rangle$$

where

$$\langle E_{\text{final}} | f^M \lambda_4 \lambda_3 | S | \lambda_1 \lambda_2 E_{\text{initial}} \rangle = \delta(E_f - E) \delta J J_f \delta M M_f$$

$$\cdot \langle \lambda_4 \lambda_3 | S^J | \lambda_1 \lambda_2 \rangle$$
In the helicity formalism, we have simple relations between the amplitudes due to the space-time symmetries.

Let P be the parity operator, T be the Wigner time reversal operator, we then have that:

\[ P^{-1}SP = S \quad \text{and} \quad T^{-1}ST = S^{-1} \]

and as

\[ P|JM\lambda_1 \lambda_2> = \eta_1 \eta_2 (-1)^{S_1-S_2} |JM, -\lambda_1, -\lambda_2> \quad (4) \]

and

\[ T|JM\lambda_1 \lambda_2> = (-1)^{J-M} |J, -M, \lambda_1 \lambda_2> \quad (5) \]

where \( \eta_i, s_i \) are intrinsic parity and spin of particle i, then one has

\[ <\lambda_4 -\lambda_3|T^J| -\lambda_1 -\lambda_2> = \eta_g <\lambda_4 \lambda_3|T^J|\lambda_1 \lambda_2> \quad (6) \]

\[ <\lambda_1 \lambda_2|T^J|\lambda_4 \lambda_3> = <\lambda_4 \lambda_3|T^J|\lambda_1 \lambda_2> \quad \text{(in the inverse process)} \quad (7) \]

where

\[ \eta_g = \frac{\eta_3 \eta_4}{\eta_1 \eta_2} (-1)^{S_3+S_4-S_1-S_2} \]

We will use definite parity amplitudes, so we have to express the definite parity states in terms of the \( |JM\lambda_1 \lambda_2> \). From eq. (4) we see that these are given by linear combinations of \( |JM\lambda_1 \lambda_2> \) and \( |JM-\lambda_1-\lambda_2> \).

States with correct symmetry under particle exchange are constructed by using the particle exchange operator \( P_{12} \), then:

\[ P_{12} \]
\[ P_{12} |JM\lambda_1\lambda_2> = (-1)^{J-2S} |JM\lambda_2\lambda_1> \]

for particles 1 and 2 identical and the state of correct symmetry is

\[ \frac{1}{\sqrt{2}}(1 + (-1)^{2S} P_{12}) |JM\lambda_1\lambda_2> = \frac{1}{\sqrt{2}}(|JM\lambda_1\lambda_2> + (-1)^J |JM\lambda_2\lambda_1>) \]

For NN scattering there is a maximum of \((2 \times 1/2 + 1)^4 = 16\) helicity amplitudes (for a given I). Using (6) and (7), and also the fact that they are amplitudes between states of identical particles, we find that there are only 5 independent ones (for a given I); let us choose

\[ T_1 = <1/2, 1/2 |T|1/2, 1/2> \]
\[ T_2 = <1/2, 1/2 |T|-1/2, -1/2> \]
\[ T_3 = <1/2, -1/2 |T|1/2, -1/2> \]
\[ T_4 = <1/2, -1/2 |T|-1/2, 1/2> \]
\[ T_5 = <1/2, 1/2 |T|1/2, -1/2> \]

and then the definite parity partial wave helicity amplitudes are

\[ T_{JP}^1 = \frac{1}{2} p \int dx \, dJ_{\lambda\mu}(x) [T_{1\mu} T_{2\mu}] \]
\[ T_{JP}^3 = \frac{1}{2} p \int dx \, dJ_{\lambda\mu}(x) [T_{3\mu} T_{4\mu}] \]
\[ T_{JP}^5 = \frac{1}{2} p \int dx \, dJ_{\lambda\mu}(x) [2T_{5\mu}] \]

\[ P = \pm (-1)^J \]  \hspace{1cm} (9)

and, due to the different conventions, we have that

\[ T(\lambda_4 \lambda_3 \lambda_2 \lambda_1) = \frac{-iE^2}{(2\pi)^5} W F(\lambda_4 \lambda_3 \lambda_2 \lambda_1) = \frac{-i}{2(2\pi)^5} EF(\lambda_4 \lambda_3 \lambda_2 \lambda_1) \]
where $F$ is the amplitude as calculated from the Feynman diagrams, and this $T^J$ satisfies $|T^J| \leq 1$ and thus admit a parametrization $T^J \sim \exp(i\delta_J) \sin\delta_J$, with $\delta_J$ real. Moreover $\sqrt{s} T^J_1, \sqrt{s} T^J_3$, and $T^J_5$ are regular at $s = 0$. 

In Appendix C we give tables for conversion between helicity and LS partial waves for $J = 0, 1, 2$.

NN $\rightarrow$ NN*

For the NN $\rightarrow$ NN* process we can have a maximum of $(2 \times 1/2 + 1)^3 \times (2 \times 3/2 + 1) = 32$ amplitudes for a given $I$, but using parity conservation, eq. (6), we see that there are only 16 independent ones. Let,

$$t^J_i = \frac{1}{2}(pp')^{3/2} \int dx t_i(\lambda_4 \lambda_3 \lambda_2 \lambda_1) \, d^J_{\lambda \mu}(x)$$

and then

<table>
<thead>
<tr>
<th>NN</th>
<th>1JM</th>
<th>1JM</th>
<th>1JM</th>
<th>1JM</th>
<th>1JM</th>
<th>1JM</th>
<th>1JM</th>
<th>1JM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1JM &amp; 1/2</td>
<td>$t^J_1$</td>
<td>$t^J_2$</td>
<td>$t^J_3$</td>
<td>$t^J_4$</td>
<td>$t^J_5$</td>
<td>$t^J_6$</td>
<td>$t^J_7$</td>
<td>$t^J_8$</td>
</tr>
<tr>
<td>1JM &amp; 1/2</td>
<td>$t^J_9$</td>
<td>$t^J_10$</td>
<td>$t^J_11$</td>
<td>$t^J_12$</td>
<td>$t^J_13$</td>
<td>$t^J_14$</td>
<td>$t^J_15$</td>
<td>$t^J_16$</td>
</tr>
<tr>
<td>1JM &amp; 3/2</td>
<td>$t^J_17$</td>
<td>$t^J_18$</td>
<td>$t^J_19$</td>
<td>$t^J_20$</td>
<td>$t^J_21$</td>
<td>$t^J_22$</td>
<td>$t^J_23$</td>
<td>$t^J_24$</td>
</tr>
<tr>
<td>1JM &amp; 3/2</td>
<td>$t^J_25$</td>
<td>$t^J_26$</td>
<td>$t^J_27$</td>
<td>$t^J_28$</td>
<td>$t^J_29$</td>
<td>$t^J_30$</td>
<td>$t^J_31$</td>
<td>$t^J_32$</td>
</tr>
</tbody>
</table>

**TABLE I**

As the definite parity helicity states for NN are given by
\[
\frac{1}{\sqrt{2}}(|J\lambda_1 \lambda_2\rangle \pm |J-\lambda_1-\lambda_2\rangle) \quad P = \mp (-1)^J
\]

and those for NN* by

\[
\frac{1}{\sqrt{2}}(|J\lambda_1 \lambda_2\rangle \pm |J-\lambda_1-\lambda_2\rangle) \quad P = \pm (-1)^J
\]

we have that

\[
t^J_{1,4+}, t^J_{5,8+}, t^J_{9,12+}, t^J_{13,16+} \text{ are for parity } \pm (-1)^J
\]

and

\[
t^J_{2,3+}, t^J_{6,7+}, t^J_{10,11+}, t^J_{14,15+} \text{ are for parity } -(\pm 1)^J
\]

where we have defined \( t^J_{i,j} = t^J_i \pm t^J_j \); and note that there are only 12 independent amplitudes because for \( T = 1 \), which is the only state in which NN and NN* can couple, we have that \( |J^{3/2}_{-3/2}\rangle - |J^{-1/2}_{-1/2}\rangle \) for NN is not an allowed state because of the generalized Pauli principle. Also, for the same reason, for \( J = \text{odd} \) the only allowed states for NN will be with \( \lambda_1 \neq \lambda_2 \).

For the non-relativistic limit where we can use the usual LS states, we include in Appendix C the transformation tables for \( J = 0, 1, 2 \) between the LS partial wave amplitudes and the helicity partial wave amplitudes.
IV. THE BORN AMPLITUDES

In this section we proceed to calculate the Born amplitudes for the two processes, \( NN \to NN \) and \( NN \to NN^* \). These amplitudes are given by the one pion exchange diagrams which we calculate using the Feynman rules. They are calculated for initial and final helicity states.

A. CALCULATION

1. \( NN \to NN \)

The interaction Lagrangian we use for the \( NN\pi \) vertex is

\[
L(x) = ig\bar{\psi}(x)\gamma_5 \gamma^\tau \phi(x)\psi(x), \quad \frac{g^2}{4\pi} \sim 15
\]

where \( \psi(x) \) is the (eight component) nucleon field and \( \phi(x) \) is the pseudoscalar isovector pion field.

The Born amplitude is then given by

\[
F_{13}(\lambda_4\lambda_3\lambda_2\lambda_1) = ig^2(2\pi)^4 \frac{m^2}{E^2} [A_D(\lambda_4\lambda_3\lambda_2\lambda_1) - A_E(\lambda_4\lambda_3\lambda_2\lambda_1)] \delta(\lambda_1) (r_4 r_3 r_2 r_1)
\]

(1)

where particles 1, 2 (3, 4) are incident (final) with momentum \( p_1 \) and helicity \( \lambda_1 \), and with isospin \( r_i \). \( \lambda_1^{(a)} \) is the isospin amplitude which will be calculated in Appendix B, and \( A_D \) and \( A_E \) are given by

\[
A_D(\lambda_4\lambda_3\lambda_2\lambda_1) = [\bar{u}(p',\lambda_4)\gamma_5 u(-p,\lambda_2)][t-\mu^2]^{-1}[\bar{u}(-p',\lambda_3)\gamma_5 u(p,\lambda_1)]
\]

\[
A_E(\lambda_4\lambda_3\lambda_2\lambda_1) = [\bar{u}(p',\lambda_4)\gamma_5 u(p,\lambda_1)][u-\mu^2]^{-1}[\bar{u}(-p',\lambda_3)\gamma_5 u(-p,\lambda_2)]
\]
We only want to calculate the $J=0^{\pm}$, $T=1$ amplitudes due to OPE, and from Section III, we have that they are given by

$$T_{1,B}^{0^{\pm}} = \frac{p}{2} \int_{\omega} dx \, P_{\omega}(x) \frac{1}{2} \{ [\langle z_{1/2} \rangle^{+} \langle -z_{1/2} \rangle^{-}] T_{\omega} \}$$

$$= \frac{p}{2} \int_{\omega} dx \, P_{\omega}(x) \{ [A_{D}(\omega_{1/2}, \omega_{1/2}, \omega_{1/2}) - A_{E}(\omega_{1/2}, \omega_{1/2}, \omega_{1/2})] + [A_{D}(\omega_{1/2}, \omega_{1/2}, -\omega_{1/2}) - A_{E}(\omega_{1/2}, \omega_{1/2}, -\omega_{1/2})] \}$$

and $A_{D}$ and $A_{E}$ are calculated in Appendix D to be

$$A_{D}(1/2, 1/2, 1/2, 1/2) = A_{E}(1/2, 1/2, 1/2, 1/2) = 0$$

$$A_{D}(1/2, 1/2, -1/2, -1/2) = (4m^{2})^{-1} \frac{t}{t-\mu^{2}}$$

$$A_{E}(1/2, 1/2, -1/2, -1/2) = (4m^{2})^{-1} \frac{u}{u-\mu^{2}}$$

and then we obtain for the helicity amplitudes

$$T_{1,B} = 0$$

$$T_{2,B} = ig^{2} (2\pi)^{4} \frac{1}{4E^{2}} \left[ \frac{t}{t-\mu^{2}} + \frac{u}{u-\mu^{2}} \right]$$

$$T_{3,B}, T_{4,B}^{'}$$ and $T_{5,B}$ do not contribute to $J=0$ scattering but they do contribute to scattering in higher angular momentum states (except for $T_{5,B}$ which is equal to 0).\(^{33}\)

Finally, the OPE partial wave helicity amplitudes for $J=0^{\pm}$, $I=1$ are given by

$$T_{1,B}^{0^{\pm}} = \pm \frac{g^{2}}{16 \pi} \frac{p}{W} \int_{\omega} \frac{t}{t-\mu^{2}} + \frac{u}{u-\mu^{2}} \, dx$$

$$= \pm \frac{g^{2}}{8 \pi} \sqrt{\frac{s-4m^{2}}{s}} \left\{ 1-(1+z)Q_{\omega}(z) \right\}$$

where $z = -1 - \frac{\mu}{2p^{2}}$
$T_{1,B}^{0\pm}$ have kinematical singularities at $s=0, s=4m^2$. In order to remove this we introduce

$$\rho_1 = \frac{1}{2} \sqrt{\frac{s-4m^2}{s}}$$

and in the dispersion equations we will use $T_{1,B}^{0\pm}^{-1}$ which is regular at $s=0, 4m^2$.

2. $NN \rightarrow NN^*$

The interaction Lagrangian for the $N^*N\pi$ interaction will be assumed to be

$$L_{N^*N\pi}(x) = \frac{G}{m} \{ \bar{\psi}(x) \gamma^\mu \phi(x) \gamma^\mu \phi(x) + h.c. \} \equiv L_I$$

where we have omitted the isospin part (the isospin amplitudes are given in Appendix B). Note that although $\phi(x)$ is a pseudoscalar we have that $L_I$ is CP invariant since

$$U_{CP} (\bar{\psi}_\mu \psi) U_{CP}^{-1} = -g_{\mu\mu} \bar{\psi}_\mu \psi \quad \text{(no summation!)}$$

and also

$$U_{CP} (\bar{\phi} \gamma^\mu \phi) U_{CP}^{-1} = -g_{\mu\mu} \bar{\phi} \gamma^\mu \phi$$

$$\therefore \quad U_{CP} L_I U_{CP}^{-1} = L_I$$

To determine the effective coupling constant $G$, we calculate the $N^*$ decay rate $\Gamma$ and equate it to the experimental value (which is the full width at half maximum of the resonance curve). From Appendix A we have

$$\Gamma = \frac{1}{3} \frac{G}{2m} \frac{q^3}{M} [m+E(q)]$$
where \( q \) is the momentum of the nucleon and \( E(q) \) its energy, evaluated in the CM system.

We want to calculate the amplitudes corresponding to the following diagrams

\[
\begin{array}{c}
\text{N}_3 \quad \text{N}^*_4 \\
\text{N}_1 \quad \text{N}_2 \\
\end{array}
\]

The minus sign is due to the fact that we must antisymmetrize the NN state. Using the Feynman rules we obtain

\[
f_{1, B} (\lambda_4 \lambda_3 \lambda_2 \lambda_1) = \frac{(-i)^2}{(2\pi)^4} \frac{gG}{\sqrt{E_1 E_2 E_3 E_4}} \frac{m^3}{m}
\]

\[
\{ M_{B, D} (\lambda_4 \lambda_3 \lambda_2 \lambda_1) - M_{B, E} (\lambda_4 \lambda_3 \lambda_2 \lambda_1) \lambda^{(b)}_I \}
\]

where \( \lambda^{(b)}_I \) is the isospin amplitude, and

\[
M_{B, D} (\lambda_4 \lambda_3 \lambda_2 \lambda_1) = \bar{u} \gamma \mu (\lambda_4, \lambda_3) u(-\lambda_2, \lambda_1) \left( \frac{k}{t-\mu} \right) \bar{u}(-\lambda_3, \lambda_2) \gamma_5 u(\lambda_4)
\]

\[
M_{B, E} (\lambda_4 \lambda_3 \lambda_2 \lambda_1) = \bar{u} \gamma \mu (\lambda_4, \lambda_3) u(+\lambda_2, \lambda_1) \left( \frac{k'}{u-\mu} \right) \bar{u}(-\lambda_3, \lambda_2) \gamma_5 u(-\lambda_4)
\]

where \( k_\mu, k'_\mu \) are the momentum transfers.

We are only interested here in the \( J=0^\pm, I=1 \) and therefore the only cases we need consider* are

a. \( \lambda_4=\lambda_3=\lambda_2=\lambda_1=\frac{1}{2} \)

b. \( \lambda_4=\lambda_3=\frac{1}{2} \quad \text{and} \quad \lambda_2=\lambda_1=-\frac{1}{2} \)

*See Appendix C and Table I in Section III.
Actually, we want definite parity amplitudes, and from Section III we see that they are given by

\[ M_{1,B}^{0\pm} = \frac{1}{2} \int_0^\infty \{ \mathcal{M}_{B,D}^{(1 \pm \frac{3}{2} - \frac{1}{2})} - \mathcal{M}_{B,E}^{(1 \pm \frac{3}{2} + \frac{1}{2})} \} P_0(x) dx \]  

These are calculated in Appendix D, where we get

\[ M_{1,B}^{0+} = \frac{2}{3} \frac{A}{2 \rho \rho' \gamma} \{ Q_0(z) [z^2 (\frac{p E_4^{+}}{M \Delta_1^+} + \frac{p E_4^-}{M \Delta_1^-}) + z (\frac{p E_4^{+}}{M \Delta_2^+} + \frac{p E_4^-}{M \Delta_2^-}) + \frac{p E_4^{+}}{M \Delta_1^+} - \frac{p E_4^-}{M \Delta_1^-}] + z (\frac{p E_4^{+}}{M \Delta_2^+} - \frac{p E_4^-}{M \Delta_2^-}) + \frac{p E_4^{+}}{M \Delta_1^+} - \frac{p E_4^-}{M \Delta_1^-}] \]  

and

\[ M_{1,B}^{0-} = \frac{2}{3} \frac{A}{2 \rho \rho' \gamma} \{ Q_0(z) [z^2 (\frac{p E_4^{+}}{M \Delta_1^+} + \frac{p E_4^-}{M \Delta_1^-}) + z (\frac{p E_4^{+}}{M \Delta_2^+} + \frac{p E_4^-}{M \Delta_2^-}) + \frac{p E_4^{+}}{M \Delta_1^+} - \frac{p E_4^-}{M \Delta_1^-}] + z (\frac{p E_4^{+}}{M \Delta_2^+} - \frac{p E_4^-}{M \Delta_2^-}) + \frac{p E_4^{+}}{M \Delta_1^+} - \frac{p E_4^-}{M \Delta_1^-}] \]  

where we have defined

\[ A = i \frac{E+m}{2m} \sqrt{E_3+m} (E_4+m) \]

\[ z = \frac{1}{4 \rho \rho' \gamma} \{ 3m^2 + M^2 - 2 \mu^2 - s \} \]

\[ \Delta_1^+ = \frac{2p}{E+m} \left[ \frac{p'}{1 - (E_3+m)(E_4+m)} \right] \]

and

\[ \Delta_2^+ = 2p \left[ \frac{p'}{(E+m)(E_4+m)} + \frac{1}{E_3+m} \right] \]

Finally, we obtain for the normalized partial wave helicity amplitudes for \( J=0^\pm \), \( I=1 \)
\[
\begin{aligned}
t_{1,B}^{0^\pm} &= \frac{(2\pi)^4}{(2\pi)^5} (-i)^2 \frac{m^3}{E} \frac{G}{M} \frac{\sqrt{PP'}}{M} \frac{\sqrt{E_1 E_2 E_3 E_4}}{W} \\
\lambda_{I=1}^{(b)} &= \frac{0^\pm}{M_{1,B}}
\end{aligned}
\]

where \( \lambda_{I=1}^{(b)} = -2 \sqrt{\frac{2}{3}} \)

as calculated in Appendix B.

The \( t_{1,B}^{0^\pm} \) have kinematical singularities at both thresholds, \( p \to 0, p' \to 0, \) and at \( s=0. \) We remove these singularities by defining

\[
\begin{aligned}
\rho_1 &= \frac{p}{W} = \frac{1}{2} \sqrt{\frac{s-4m^2}{s}} \\
\rho_2 &= \frac{p'}{W} = \frac{1}{2s} \sqrt{[s-(M+m)^2][s-(M+m)^2]} 
\end{aligned}
\]

and then we have that \( t_{1,B}^{0^\pm} (\rho_1 \rho_2)^{-\frac{1}{2}} \) do not have kinematical singularities in the physical region. They do have some kinematical singularities at \( s = 0 \) but they will not be considered here. They can be taken into account by working in the \( \sqrt{s} \) plane but that would introduce other difficulties.
B. ANALYTIC STRUCTURE OF THE BORN AMPLITUDES

1. \( T_{1,B} \)

First define

\[
B_{11}^{\pm}(s) = \frac{1}{\rho_1(s)} T_{1,B}^{0\pm}(s) = 2 \sqrt{\frac{s}{s-4m^2}} T_{1,B}^{0\pm}
\]

This will be the input amplitude to the N/D equations. Also, the dynamical cuts of the amplitude for the process \( NN \rightarrow NN \) will be the same as those of \( B_{11}^{\pm}(s) \).

According to our conventions, for this process we have

\[
s = 4(p^2+m^2)
\]
\[
t = -\beta + \frac{2}{\mu} \frac{\beta - \alpha \cos \theta}{\gamma}
\]
\[
u = -\beta + \frac{2}{\mu} \frac{\beta + \alpha \cos \theta}{\gamma}
\]

where \( \alpha = 2p^2 \) and \( \beta = 2p^2+\mu^2 \).

The helicity amplitudes, as we have seen, contain terms proportional to \( Q_j(-\beta/\alpha) \) where

\[
\frac{1}{\alpha} Q_j(-\beta/\alpha) = \frac{1}{2\alpha} \int_0^\infty \frac{P_j(\cos \theta)}{t-\mu^2} \frac{d(\cos \theta)}{d\theta} = \left(\frac{-1}{\alpha}\right)^{J+1} Q_j(\beta/\alpha)
\]

Then, for \( B_{11} \) we will have a term proportional to

\[
\frac{1}{2\alpha} \ln \frac{\beta + \alpha}{\beta - \alpha}
\]

The analytic structure of this function is well known, it is analytic everywhere except on the logarithmic cut.

Let \( w = \frac{\beta + \alpha}{\beta - \alpha} \) and define the principal branch of the log for \(-\pi < \arg w < \pi \) and \( \log w = |w| + \pi i \arg w \)
so the Riemann surface of $B_{12}$ as a function of $s$ will be an infinite number of copies of the $s$ plane, each with a cut which goes from $s = -\infty$ to $s = 4m^2 - \mu^2$, and the sheets are joined in the usual way. The physical sheet is the one for which arg $s \in (-\pi, \pi)$. As $s = 4m^2 - \mu^2$ is a zero of $w(s)$, we have that

$$\text{disc} \left\{ \frac{1}{2a} \ln \frac{\beta + \alpha}{\beta - \alpha} \right\} = \frac{\pi i}{\alpha(s)}$$

for $s < 4m^2 - \mu^2$.

2. $t_{12}^{0\pm}$

Here we define

$$B_{12}^\pm(s) = B_{21}^\pm(s) = [\rho_1(s)\rho_2(s)]^{-\frac{1}{2}} t_{12}^{0\pm}$$

This will be the other input of the N/D equations. Its cuts are those of the amplitude for $NN \rightarrow NN^*$. For this process we have,

$$s = 4(p^2 + m^2)$$
$$t = -\beta + \mu^2 - \alpha \cos \theta$$
$$u = -\beta + \mu^2 + \alpha \cos \theta$$

where $\alpha(s) = 2pp' = \frac{1}{2} \left[ \frac{1}{s}(s-a_1)(s-a_2)(s-a_3) \right]^{\frac{1}{2}}$

and $\beta(s) = \frac{1}{2} [3m^2 + M^2 - 2\mu^2 - s] = \frac{1}{2} (s-a)$

and we have defined

$$a = 3m^2 + M^2 - 2\mu^2 - s$$
\( a_1 \equiv 4m^2 \) = the elastic threshold
\( a_2 \equiv (M+m)^2 \) = the elastic threshold
\( a_3 \equiv (M-m)^2 \)

The analytic structure of \( B_{12}(s) \) is that of
\[
\frac{1}{\alpha Q_0}(-\beta/a) = \frac{1}{\alpha Q_0}(\beta/a) = \frac{1}{\alpha} \ln \frac{\beta+a}{\beta-a}
\]

and we will study the structure of the latter function.

Let
\[
w(s) = \frac{\beta(s)+\alpha(s)}{\beta(s)-\alpha(s)}
\]

The log \( w \) has branch points at \( w=0 \) and at \( w=-\infty \), i.e., where
\[
\alpha(s) = \pm \beta(s)
\]

As \( \lim_{s \to 0} w(s) = -1 \), we have that \( s=0 \) is not a logarithmic branch point, and then (11) gives
\[
s^2(2a-a_1-a_2-a_3)+s(-a^2+a_1a_2+a_1a_3+a_2a_3)-a_1a_2a_3 = 0
\]
whose roots are
\[
s_{\pm} = \frac{1}{2}[3m^2+M^2-m^2]^{\pm 1} \sqrt{(4m^2-m^2)[(m+\mu)^2-M^2][M^2-(m-\mu)^2]}
\]

Following Mandelstam\(^{35}\), we know that the position of the singularities is obtained by letting the mass of an external particle (in our case the \( N^* \)) become complex by adding \(+i\epsilon \) and then taking the limit as the variable mass approaches its value. Also, our amplitudes should not have a cut as a function of the external mass\(^{14}\), so we could add \(-i\epsilon \) and
obtain the same result. Thus, what we want to see is the path of \( s_+ \) as we vary the external mass from the value at which the particle would be stable to its observed value. To do this, call the mass of the external particle \( \omega \) and also define \( x \),

\[
\omega^2 = (m+\mu)^2 + x + i\epsilon \quad x \in \mathbb{R} \quad \epsilon > 0
\]

Note that if \((m-\mu)^2 < \omega^2 < (m+\mu)^2\), i.e., if \( N \) and \( N^* \) are stable then the branch points are on the real line. We then have

\[
s_\pm(x) = 2m^2 + m \mu + x \left( \frac{(4m^2 - \mu^2)}{2\mu} \right)^{1/2} \sqrt{-x(x + 4m \mu) + i\epsilon(2x + 4m \mu)} + \frac{i\epsilon}{2}
\]

Let us examine the radical as a function of \( x \),

- \( x < 0 \)
  - i) \( x < -4m \mu \) large imaginary part \( > 0 \).
  - ii) \( -4m \mu < x < -2m \mu \) small positive imaginary part (i.e., goes to zero with \( \epsilon \));
  - iii) \( -2m \mu < x < 0 \) small imaginary part \( < 0 \).
    - Note that here, as we vary \( x \) continuously, we have entered the second sheet of the square root.
  - iv) \( x > 0 \) large negative imaginary part.

It is easily seen that to first order in \( \epsilon \) the point where \( s \) becomes real is \( s < 4m^2 \) and is independent of \( \epsilon \). Let

\[
x_0 = -2m \mu + 2\eta \quad \eta \text{ small and } > 0
\]

\[
s_+(x_0) = 2m^2 + \eta + \left( \frac{(4m^2 - \mu^2)}{2\mu} \right)^{1/2} \sqrt{4m^2 - 4\eta^2 - 4i\epsilon \eta} + i\epsilon/2
\]
\[ s^+ = 2m^2 + \eta \pm (4m^2 - \mu^2)^{1/2} \left[ 1 - \frac{\eta^2}{2m^2 \mu^2} \right]^{1/2} + i \frac{s}{2} \left[ 1 - (4m^2 - \mu^2)^{1/2} \right] \frac{n}{m \mu^2} \]

so \( s^+ \) becomes real when

\[ \eta \text{ is } \geq \frac{\eta}{m \mu^2} \left( 4m^2 - \mu^2 \right)^{1/2} = 1 \]

i.e., \( \eta = \frac{\mu^2}{2} \) which certainly is small. The exact value of \( s \) for which \( s^+ \) crosses the Re(s) axis is \( s = 4.13 \text{(BeV)}^2 \).

Finally, when \( \omega = M^2 \) we have that

\[ s^+ (M) = (2.08 \pm 3.85i) \text{(BeV)}^2 \]

Let \( s_1 = 2.08 + 3.85i \)

\[ s_2 = 2.08 - 3.85i \]

Let \( A \) be the sheet of \( a(s) \) for which for very large \( sa+s \), also let

\[ w(s) = \frac{\beta(s) + a(s)}{\beta(s) - a(s)} \]

then, on sheet \( A \), \( s_1 \) is a pole of \( w(s) \) and \( s_2 \) is a zero of \( w(s) \). (In the other sheet of \( a(s) \) it is the other way around.) Thus, we can draw a cut in the log surface from \( s_1 \) to \( s_2 \). This log cut is not the only one that log \( w(s) \) has. It is easy to see that \( s = -\infty \) is a pole of \( w(s) \) (for \( s \) on sheet \( A \)), so we need to study more carefully the structure of \( \frac{1}{a(s)} \ln w(s) \).*

*In Appendix E, we do the analysis of a simpler function which still has the same general properties of \( \frac{1}{a(s)} \log \frac{\beta + a}{\beta - a} \).
Let \( a_1(s) = s \alpha(s) \) and \( \beta_1(s) = s \beta(s) \), then

\[
f(s) = \frac{s}{a(s)} \ln \frac{\beta(s)+\alpha(s)}{\beta(s)-\alpha(s)} = \frac{1}{a_1(s)} \ln \frac{\beta_1(s)+\alpha_1(s)}{\beta_1(s)-\alpha_1(s)}
\]

First let us examine the cuts in \( a_1(s) \). \( a_1(s) \) is analytic on a two sheeted Riemann surface. The branch points are at \( s=0, s=a_1, s=a_2, \) and \( s=a_3 \). Let

\[
\theta = \tan^{-1} \left[ \frac{\Im(s)}{\Re(s)} \right]
\]

\[
\theta_i = \tan^{-1} \left[ \frac{\Im(s)}{\Re(s)-a_i} \right] \quad i = 1, 2, 3
\]

then \( a(s) = |a(s)| \exp \left[ \frac{1}{2}(\theta_1 + \theta_2 + \theta_3 + \theta) \right] \)

and the two sheets, A and B, will be defined by

- \( s \) on sheet A when \(-3\pi/2 < \theta < \pi/2\) \quad \( 0 < \theta_1 < 2\pi \)
  \(-\pi < \theta_2 < \pi \quad 0 < \theta_3 < 2\pi \)

- \( s \) on sheet B when \( \pi/2 < \theta < 5\pi/2 \) \quad \( 0 < \theta_1 < 2\pi \)
  \(-\pi < \theta_2 < \pi \quad 0 < \theta_3 < 2\pi \)

and so forth. Thus the sheets A and B will have the following cuts:

![Diagram](image)

The cut beginning at \( s=0 \) was chosen in the above manner for simplicity, for later the cut in \( \log w(s) \) is going to be taken along the negative real \( s \). We actually want the
cuts of \( a_1(s) \) to be on the Re(s) line and the cut as defined above on the positive Im(s) has to be rotated by \( \pi/2 \), but this will not alter our conclusions for the physical sheet; this will be shown later.

Now we need to examine the Riemann surface for \( \log w(s) \). Note that the surface for \( w \) is also a two sheeted surface because \( w \) satisfies

\[
w^2[\beta_1^2(s) - a_1^2(s)] - 2w[\beta_1^2(s) + a_1^2(s)] + [\beta_1^2(s) - a_1^2(s)] = 0 \quad (14)
\]

and \( a_1^2 \) and \( \beta_1^2 \) are polynomials in \( s \); also it is clear that \( a_1 \) and \( w(s) \) have the same branch cuts. We define the principal branch of the log when \( \arg(w) \in (-\pi, \pi) \). Now we have to see what surface we are to construct as domain for the log as a function of \( s \).

We have already seen that for \( s \) on sheet A we have a pole of \( w \) at \(-\infty\), another pole at \( s_1 \) and a zero of \( w \) at \( s_2 \). These are the only zeroes or poles of \( w(s) \) for \( s \) on sheet A. Now, to obtain the branch cut for the log \( w \), we can only join a zero and a pole of \( w \) (we cannot join two zeroes or two poles). So there is no problem with defining a cut between \( s_1 \) and \( s_2 \), this cut can be defined so that it does not cross any of the square root cuts due to \( a_1(s) \), and so there will be a cut between \( s_1 \) and \( s_2 \) on sheet A and a cut between \( s_2 \) and \( s_1 \) on sheet B. Now we must deal with the branch point at \( s=-\infty \). This is a pole type branch point (b.p.)
and it must be joined to a zero type b.p., but there are no more of the zero type b.p. This problem is solved by noticing that on sheet B there is an extra zero type b.p. at s=-∞, and it turns out that the log cut begins at s=-∞ on sheet A, goes along the negative real s up to s=0 (which is a square root b.p.) and goes into sheet B where it goes to s=-∞ (thus, note that the sense of the branch cut will be different on both sheets).

This is shown by mapping -∞ < \( w < 0 \) into the s-plane. This is done by solving (14) for s. We get

\[
s^3 + ps^2 + gs + r = 0
\]

where

\[
p = \frac{1}{4w}[w^2(-2a+a_1+a_2+a_3)+2w(2a+a_1+a_2+a_3)+(-2a+a_1+a_2+a_3)]
\]

\[
q = \frac{1}{4w}[w^2(a^2-a_1a_2-a_1a_3-a_2a_3)-2w(a^2+a_1a_2+a_1a_3+a_2a_3)
+ (a^2-a_1a_2-a_1a_3-a_2a_3)]
\]

\[
r = \frac{1}{4w}[w^2+2w+1]a_1a_2a_3
\]

When we let w vary between -∞ and 0, we obtain the cuts described above. In Fig. 1 we plot the log cuts for s on sheet A of \( \alpha_1(s) \), i.e., where \( \alpha_1 \sim s^2 \) for large s.

Now we want to find the surface over which \( f \) is analytic, except at some branch points.

We have already defined the surface for \( \alpha_1(s) \), it consists of two sheets A and B connected along the cuts shown
on page 40. Now let $s$ lie on

sheet $A_n$ when $a(s)$ is on $A$ and $\arg(w) \in ((2n-1)\pi, (2n+1)\pi)$

$A_{-n}$

$B_n$ $B$ $\arg(w) \in ((2n-1)\pi, (2n+1)\pi)$

$B_{-n}$ $\arg(w) \in ((2n+1)\pi, -(2n-1)\pi)$

For $s$ on these sheets,

$$\log w(s) = \text{log/w}(s) + 2\pi ij \arg w \text{ on any } A_j$$

$$\log w(s) = -\text{log/w}(s) - 2\pi ij \arg w \text{ on any } B_j, \ j \text{ any integer.}$$

In Fig. 3 we show the cuts on the $A_j$ and $B_j$ and how they are connected, i.e., the square root cuts connect $A_j$ with $B_{-j}$ and the log cuts connect $A_j$ with $A_{j+1}$ and $B_j$ with $B_{j+1}$ where the + sign stands when one goes down on the left hand log cut and from right to left on the log cut between $s_1$ and $s_2$.

From the above, we see that if for $s \in \mathbb{R}$ one has that on

$$A_j \quad \log w(s) = \text{log/w}(s) + 2\pi ij$$

$$B_j \quad \log w(s) = -\text{log/w}(s) - 2\pi ij$$

Now it only rests to find what happens when we have the full function

$$f(s) = \frac{1}{a_1(s)} \log \frac{\beta_1(s) + a_1(s)}{\beta_1(s) - a_1(s)}$$
CUTS FOR \( \text{LOG}_W(S) \)

- LOGARITHMIC BRANCH CUTS
- SQUARE ROOT BRANCH CUTS

FIGURE 3
Let

sheet $I^+ = I^- = I$ such that $\log w$ is over $A_0$ or $B_0$,

$$II^+$$

$$III^+$$

and $II^-$

$$III^-$$

and so forth,

and $II^-$

$$II^-$$

$$III^-$$

$$A_{-1} \text{ or } B_{-1}$$

$$A_{-2} \text{ or } B_{-2}$$

Now let $Q_i$ stand for the $i^{th}$ Roman numeral, and let $s \in \mathbb{R}$, then over

sheet I

$$f(s) = \frac{1}{|a|} \ln |w(s)|$$

$$II^+ f(s) = \frac{1}{|a|} \ln |w(s)| \pm \frac{2\pi i}{|a|}$$

and in general, over sheet $Q_{n+1}^+$

$$f(s) = \frac{1}{|a(s)|} \ln |w(s)| \pm \frac{2\pi \ln |a(s)|}{|a(s)|} \quad n = 0, 1, 2, ...$$

now let us see how they are connected.

Firstly, note that sheet I does not have any square root cut, it only has the logarithmic cuts. Going down on the left hand cut one gets to sheet $II^+$, and going up on it one gets to $II^-$. Going from right to left on the log cut on the right hand side one goes from I to $II^+$. Sheet I does not have any square root cuts because they connect $A_0$ to $B_0$ and they define sheet I (the square root cuts connect $I^+$ and $I^-$ but they are the same sheet). Sheet I is drawn of Fig. 1.

For $i > 1$, $Q_i^+$ is connected to the following sheets:
$Q_i^-$ through its square root cuts
$Q_i^{+}$ through its logarithmic cuts,

and analogously for $Q_i^{-}$, $i > 1$.

Finally, let us discuss briefly what happens when we rotate the cut in $\alpha(s)$ that lies on the $\text{Im}(s)$ axis by $\pi/2$ so that it lies on the negative real $s$ axis. As we have said above, sheet I is not altered (since it does not have the square root cuts) but the others are. One has to consider that when one crosses the negative real $s$ one is changing both the determination of $\alpha_1(s)$ and of $\log w(s)$, and one proceeds exactly as above.
In this section we deal with the coupled NN and NN* channels. The N/D method of Section II generalizes easily to take into account a two channel process and we obtain equations for the amplitudes. The only inputs that we use for the equations are the NN \(\rightarrow\) NN and NN \(\rightarrow\) NN* Born amplitudes. We do not include any input for the NN* \(\rightarrow\) NN* process, and thus the amplitude obtained for this process is just the part due to the coupling to the NN channel.

In our study, we are considering two coupled channels and we must generalize the N/D method for these processes. Call the NN channel 1 and the NN* channel 2, then we have that \(F_{ij}\) is the partial wave helicity amplitude for the process with initial channel \(j\) and final channel \(i\). We also define a kinematical factor \(\rho_i\) for the \(i^{th}\) channel. Then let

\[
M_{ij} = F_{ij} (\rho_i \rho_j)^{-\frac{1}{2}}
\]

no sum over \(i, j\) and then

\[
M = (M_{ij}) = N / D = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \begin{pmatrix} D_{22} & -D_{12} \\ -D_{21} & D_{11} \end{pmatrix} \frac{1}{||D||} \tag{1}
\]

Bjorken and Nauemberg\(^{36}\) have shown that if the input to the N/D equations is symmetric then the resulting \(M\) is symmetric, as it should be by time reversal invariance.

In this thesis we are going to consider an approximation
to the full complex cuts in $t^0_{1'i'B}$. It consists in taking
for the complex cut in $(M_{ij})$ just two separate log cuts,
one in the first quadrant and the other in the fourth quadrant
of sheet $I$, see Fig. 2. In this way, the contours in
the integral equations for $D_{ij}$ are not altered and the re-
sulting equations are much simpler than the equations that
take into full account the complex cut.* Only $N$ has these
cuts and one must take them such that $N$ is still a real
analytic function, except on the cuts for $B$. The other
parameters can be fixed by requiring that the approximate
function, $B^+_1(s)$, approximate as well as possible the actual
OPE input over some set in the $s$ plane.

We also take the $N^*$ as a zero width particle.

Then, for each parity, and omitting the $J=0$ superscript,
for $i,j = 1,2$

\[
\text{disc } N_{ij} = (\text{disc } M_{ik}) D_{kj} = \alpha^{-1}_{ik} D_{kj} \quad (2)
\]

\[
\text{disc } D_{ij} = (\text{disc } M^{-1})_{ik} N_{kj} = \rho_i(s) N_{ij} \quad \text{(no sum over $i$)} \quad (3)
\]

$s > a_i$

where $\alpha^{-1}_{ij}(s) = \text{disc}_L B_{ij}$, $\alpha^{-1}_{22} = 0$, and the $\rho_i$ are given in
page 35.

We then have

*See Ref. 14 for the integral equations with full complex cut
for the $\pi N \rightarrow \pi N$and $\pi N \rightarrow \rho N$ processes.
\[ N_{11}(s) = \frac{1}{\pi} \int_{-i\infty}^{i\infty} \frac{ds'}{s'-s} \alpha_{11}(s') D_{11}(s') \]
\[ + \frac{1}{\pi} \int_{-i\infty}^{i\infty} \frac{ds'}{s'-s} \alpha_{12}(s') D_{12}(s') + A_{12}(s) D_{21}(s) \]  
(4a)

\[ N_{12}(s) = \frac{1}{\pi} \int_{-i\infty}^{i\infty} \frac{ds'}{s'-s} \alpha_{12}(s') D_{12}(s') \]
\[ + \frac{1}{\pi} \int_{-i\infty}^{i\infty} \frac{ds'}{s'-s} \alpha_{12}(s') D_{22}(s') + A_{12}(s) D_{22}(s) \]  
(4b)

\[ N_{21}(s) = \frac{1}{\pi} \int_{-i\infty}^{i\infty} \frac{ds'}{s'-s} \alpha_{21}(s') D_{11}(s') + A_{21}(s) D_{11}(s) \]  
(4c)

\[ N_{22}(s) = \frac{1}{\pi} \int_{-i\infty}^{i\infty} \frac{ds'}{s'-s} \alpha_{21}(s') D_{12}(s') + A_{21}(s) D_{12}(s) \]  
(4d)

where

\[ A_{12}(s) = A_{21}(s) = \frac{\gamma_3}{\pi} \alpha_{21}(s) \ln(\frac{s-s_c-i\gamma_2}{s-s_c+i\gamma_2} \cdot \frac{s-s_c+i\gamma_1}{s-s_c-i\gamma_1}) \]

where

\[ s_c = m^2 + M^2 - \mu^2 = 4.13\text{ (BeV)}^2 \]

is the point where the exact cut crosses the Re(s) axis.

Thus, we are approximating \( B_{12}(s) \) by \( \hat{B}_{12}(s) \), where

\[ \hat{B}_{12}(s) = \frac{1}{\pi} \int_{-i\infty}^{i\infty} \frac{ds'}{s'-s} \alpha_{12}(s') D_{12}(s') + A_{12}(s) \]  
(5)

and \( \gamma_1 \gamma_2 \gamma_3 \) are parameters, see Fig. 2. \( \gamma_3 \) gives a value to the discontinuity across the cuts introduced and \( \gamma_2 \gamma_1 \gamma_1 \) describe the length of the cut and distance from the real s axis. They can be fixed by having

\[ \int_{a}^{b} |\hat{B}_{12}^{0\pm}(s) - B_{12}^{0\pm}(s)|^2 ds \]
minimized on a properly chosen interval \([a,b]\).

With the cuts for \(N\) as above, we have that the contours in the dispersion integrals for \(D_{ij}\) (from \(a_i\) to \(\infty\)) are not altered (the dynamical and kinematical cuts do not overlap), and the \(D_{ij}\) do not have the complex cut. We then get the following dispersion equations for the \(D_{ij}\), with one subtraction at \(s_o\) and normalized such that \(D_{ij}(s_o) = \delta_{ij}\).

\[
D_{11}(s) = 1 - \frac{(s-s_o)}{\pi} \int_{a_1}^{\infty} \frac{ds'}{(s'-s)(s'-s_o)} \rho_1(s') N_{11}(s') \quad (6a)
\]

\[
D_{12}(s) = -\frac{(s-s_o)}{\pi} \int_{a_1}^{\infty} \frac{ds'}{(s'-s)(s'-s_o)} \rho_1(s') N_{12}(s') \quad (6b)
\]

\[
D_{21}(s) = -\frac{(s-s_o)}{\pi} \int_{a_2}^{\infty} \frac{ds'}{(s'-s)(s'-s_o)} \rho_2(s') N_{21}(s') \quad (6c)
\]

\[
D_{22}(s) = 1 - \frac{(s-s_o)}{\pi} \int_{a_2}^{\infty} \frac{ds'}{(s'-s)(s'-s_o)} \rho_2(s') N_{22}(s') \quad (6d)
\]

Inserting these equations in the equations for \(N_{ij}\) (equations 4), we get after some algebraic manipulation

\[
N_{11}(s) = b_{11}(s) + \int_{a_1}^{\infty} ds' K_{11}(s,s';s_o) \rho_1(s') N_{11}(s') \quad (7a)
\]

\[
N_{12}(s) = b_{12}(s) + \int_{a_1}^{\infty} ds' K_{11}(s,s';s_o) \rho_1(s') N_{12}(s') \quad (7b)
\]

\[
N_{21}(s) = b_{21}(s) + \int_{a_2}^{\infty} ds' Q_{21}(s,s';s_o) \rho_1(s') N_{11}(s') \quad (7c)
\]

\[
N_{22}(s) = b_{22}(s) + \int_{a_2}^{\infty} ds' Q_{21}(s,s';s_o) \rho_1(s') N_{12}(s') \quad (7d)
\]
where
\[ b_{11}(s) = B_{11}(s) + \int_{\Delta_1} ds' Q_{12}(s, s', s_0) \rho_2(s') B_{21}(s') \]
\[ b_{12}(s) = b_{21}(s) = \tilde{B}_{12}(s) \]
\[ b_{22}(s) = 0 \]

and
\[ Q_{12}(s, s', s_0) = \frac{1}{\pi^2(s' - s_0)} \int_{-\infty}^{\infty} ds'' \frac{(s'' - s_0) \alpha_{12}(s'' - s) - \frac{1}{2}}{(s'' - s)(s'' - s') \alpha_{12}(s)} \]
\[ = Q_{21}(s, s', s_0) \]

and
\[ K_{11}(s, s', s_0) = \frac{1}{\pi^2(s' - s_0)} \int_{-\infty}^{\infty} ds'' \frac{(s'' - s_0) \alpha_{11}(s'' - s)}{(s'' - s)(s'' - s') \alpha_{11}(s)} \]
\[ + \int_{\Delta_1} ds'' Q_{12}(s, s'', s_0) \rho_2(s') Q_{21}(s'', s', s_0) \]

Thus the whole problem has been reduced to solving two integral equations (7a and 7b) with the same kernel \( K_{11} \), and then once one has solved these equations for \( N_{11}, N_{12} \) one can obtain \( N_{21}, N_{22} \) from 7c and 7d respectively and the \( D_{ij} \) from equations 6.

It is not difficult to include the width of the \( N^* \). To do this we need to change \( a_2 \) from the value for a zero width \( N^* \), \( a_2 = (m+M)^2 \) to \( a_2 = (2m+\mu)^2 \) which is the threshold for
the NN\pi process and one also needs to change\textsuperscript{14, 23} the
\[ p_2(s) = \frac{p'}{w} \]
by changing \( p' \) to
\[ p'_v(s) = \int_{-\infty}^{\infty} \frac{d\omega \theta[\omega-(m+\mu)^2]_{\theta}[(s^{1/2}-m)\omega]p'(\omega)|f(\omega)|^2}{(s^{1/2}-m)^2} \]
\[ = \int_{(m+\mu)^2}^{\infty} d\omega \ p'(\omega)|f(\omega)|^2 \]
where \( f(\omega) \) is the resonance form for the \( \pi N \) P-wave amplitude,
\[ |f(\omega)|^2 = \frac{1}{\pi} \left[ \frac{2M(\Gamma/2)}{(\omega-M^2)^2+(2M\Gamma/2)^2} \right] \]
and
\[ [p'(\omega)]^2 = \frac{1}{4\pi} [\omega-(s^{1/2}+m)^2][\omega-(s^{1/2}-m)^2] \]
where \( \omega \) is the cm energy of the \( \pi N \) system and \( \Gamma(\omega) \) is the width of the \( N^* \) resonance, which is calculated in Appendix A to be
\[ \Gamma(\omega) = \frac{1}{3\pi} \left( \frac{g}{2m} \right)^2 g^3 \left( \frac{m}{M} + \frac{E_1}{M} \right) \]

These changes can also be incorporated simply to equations 6 and 7 for \( D_{ij} \) and \( N_{ij} \).

Thus we have reduced the complexity of the coupling of the NN and NN* channels for the \( T = 1, J = 0 \) amplitudes basically to the solution of one integral equation with kernel given by \( K_{11}(s,s',s_0)\rho_1(s') \), with all the integrals being over intervals in the Re(s) axis.

There is still much that we need to do on this problem, though we will not do it in this thesis. The immediate im-
provements will be the inclusion of the full effects of the complex singularities on the calculations. This can be done by writing the unitarity equations as a function of the $nN$ mass and continuing them analytically until the mass equals that of the $N^*$. To obtain a realistic $NN$ interaction we also need to include the exchanges of bosons than the pion. These exchanges do not introduce additional cuts in the amplitude, they only change the discontinuity across the left hand cut. Another factor which should be taken into account is the $NN^* \rightarrow NN^*$ OPE amplitude. One of the problems that this brings is an additional cut in the physical region (a cut which essentially correspond to an exchange of a real pion). This cut introduces further complications in the $N/D$ equations. The above is always based in considering the $NN\pi$ in the isobar approximation; it also is necessary to try to estimate the importance of this assumption. A more ambitious program would also consider channels with more than one pion, such as the $NN\pi\pi$ channel which could in a first approximation be considered as $N^*N^*$. 
APPENDIX A

In this appendix we calculate the decay rate $\Gamma$ of the $N^*$ in order to fix the value of the $N^*N\pi$ coupling constant. Thus we want to calculate the amplitude for the following diagram.

With the interaction Lagrangian given on page 31, we have that

$$\Gamma = \left(\frac{G^2}{4}\right) \sum_{\text{spins}} \int d^4q_1^\prime \, d^4\bar{q}_2 \left(2\pi\right)^4 \delta^4(p - q_1^\prime - \bar{q}_2) \frac{mM}{pE_1 2\omega} / m^1$$

where

$$\eta = \bar{\psi}(\vec{q}_2) \sigma_\mu q_2^\mu.$$ 

In the CM of the $N^*$, $p = (M,0)$; $q_1 = (E_1, \vec{q})$; $q_2 = (\omega, -\vec{q})$, then

$$\Gamma = \left(\frac{G^2}{4}\right) \sum_{\text{spins}} \int d^4q_1^\prime \, d^4\bar{q}_2 \left(2\pi\right)^4 \delta^4(M - \omega - \vec{q}) \frac{mM}{2\omega} / m^1$$

The projection operator for spin $3/2$ positive energy spinors is

$$\Lambda^{\mu\nu}(\vec{q}) = \frac{P + M}{2m} \left[ \frac{2}{3} q_{\mu
u} - \frac{4}{3} q_{\mu}^\nu - \frac{2}{3m^2} p_{\mu} p_{\nu} + \frac{i}{3m} \left( p_{\mu} q_{\nu} - p_{\nu} q_{\mu} \right) \right]$$

and the one for spin $1/2$ spinors is

$$\Lambda_+ (\vec{q}) = \frac{\gamma_5 + m}{2m}$$

and then

$$\sum_{\text{spins}} / m^1 = \sum_{\text{spins}} \left[ \bar{\psi}_\mu (\vec{q}) \psi_\mu (\vec{q}) q_{\mu} q_{\nu} \bar{\psi}_\nu (\vec{q}) \omega_\nu (\vec{q}) \right]$$
\[ T_\alpha \left[ (\gamma_\mu i \gamma_5 \partial_\nu - (\gamma_\nu i \gamma_5 \partial_\mu) ) q_\mu^{\alpha} q_\nu^{\beta} \right] \]

where the \((-1)\) comes from the way in which we have normalized \(\omega_{\mu \nu}\). (See Sec. II, 7) This is easily calculated to be

\[ \sum_{\text{spins}} |\Psi|^2 = \frac{q}{2m} \frac{q^2}{z \left( m + \varepsilon_i \right)} \]

then

\[ \Gamma^I = \frac{\left( \frac{M}{m} \right)^3}{4} \frac{1}{2} \int d^4 p \left( \tau \right) \delta(M - E_1 - \omega) \frac{z}{2\omega E_1} \frac{q^2}{3m} \frac{q^2}{z \left( m + \varepsilon_i \right)} \]

let \( z = E_1 + \omega \), \( dz = q \frac{dq}{E_1} \frac{M}{\omega} \) and thus we get

\[ \Gamma^I = \frac{1}{2} \left( \frac{\alpha}{E_1} \right)^2 \frac{q^2}{24} \frac{q^2}{z \left( m + \varepsilon_i \right)} \]

inserting the experimental values, one obtains

\[ \frac{G}{2} \sim 54 \]
APPENDIX B

ISOSPIN AMPLITUDE FACTORS

a. NN ----- NN  
\[ (N^+_{i\tau} N) \delta_{ij} (N^+_{j\tau} N) \]

b. NN ----- NN*  
\[ (N^+_{i\tau} N) \delta_{ij} (N^+_j N) \]

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<thead>
<tr>
<th></th>
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<th>[pm] ( \langle pm \rangle )</th>
<th>[mp] ( \langle mp \rangle )</th>
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<td>( \lambda_{I=1}^{(a)} )</td>
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\( \lambda_{I=0}^{(a)} = -3 \)
\( \lambda_{I=1}^{(a)} = 1 \)

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\( \lambda_{I=1}^{(b)} = -2 \sqrt{\frac{2}{3}} \)
In this appendix we give the transformation tables between LS amplitudes and the partial wave helicity amplitudes. These were obtained using the following formula to transform the LS states to helicity ones:

\[ |\sqrt{\lambda, \lambda_s}\rangle = \sum_{L,S} \frac{2L+1}{\sqrt{2S+1}} \langle L, S, L; \rho, \lambda, \lambda_s | S, L, S - \lambda, \lambda_s \rangle / \sqrt{L(S)} \], \lambda = \lambda_s - \lambda_s

NN \rightarrow NN \quad (T = 1 \text{ only})

\( J = 0 \)

\[
\begin{array}{c|cc}
\text{NN} & J = 0 & J = 1 \\
\hline
J & \sqrt{\lambda, \lambda_s} & \sqrt{\lambda, \lambda_s} \\
\hline
1 & R_{0^+} & 0 \\
3 & 0 & R_{0^+} \\
\end{array}
\]

\( R_{0^+} = T_{0^+} \)

\( J = 1 \)

\[
\begin{array}{c|cc}
\text{NN} & J = 1 & J = 0 \\
\hline
J & \sqrt{\lambda, \lambda_s} & \sqrt{\lambda, \lambda_s} \\
\hline
3 & R_{1^+} & 0 \\
\end{array}
\]

\( R_{1^+} = T_{1^+} \)

\( J = 2 \)

\[
\begin{array}{c|ccc}
\text{NN} & J = 2 & J = 1 & J = 0 \\
\hline
J & \sqrt{\lambda, \lambda_s} & \sqrt{\lambda, \lambda_s} & \sqrt{\lambda, \lambda_s} \\
\hline
1 & R_{2^+} & 0 & 0 \\
3 & R_{2^+} & R_{2^+} & R_{2^+} \\
\end{array}
\]

\( R_{2^+} = T_{2^+} \)

\[ R_{2^-} = \sqrt{3} T_{2^-} + \sqrt{6} T_{2^-} + \sqrt{6} T_{2^-} \]

\( R_{1^-} = \sqrt{3} T_{1^-} + \sqrt{6} T_{1^-} + \sqrt{6} T_{1^-} \)

\( R_{3^-} = \sqrt{3} T_{3^-} + \sqrt{6} T_{3^-} - \sqrt{6} T_{3^-} \)
NN \to NN^* \quad (T = 1 \text{ only})

\begin{align*}
J = 0 & & \begin{array}{|c|c|c|}
\hline
\text{NN} & S & P \\
\hline
NN^* & A_{\pi}^0 & A_{\pi}^0 \\
\hline
\end{array} \\
& & \begin{array}{|c|c|}
\hline
s_{g^*} & t_{g^*} \\
\hline
A_{\pi}^0 & 1 & 0 \\
A_{\pi}^0 & 0 & 1 \\
\hline
\end{array} \\
J = 1 & & \begin{array}{|c|c|}
\hline
3P_1 & A_{\pi}^+ \\
\hline
3P_2 & A_{\pi}^- \\
3F & A_{\pi}^- \\
\hline
\end{array} \\
& & \begin{array}{|c|c|c|}
\hline
\epsilon_{1a} & \epsilon_{2a} & \epsilon_{3a} \\
\hline
A_{\pi}^+ & \frac{3}{2} & 0 & -\frac{1}{2} \\
A_{\pi}^- & \frac{3}{2} & -\frac{1}{2} & -\frac{3}{2} \\
A_{\pi}^- & \frac{3}{2} & \frac{1}{2} & \frac{3}{2} \\
\hline
\end{array} \\
J = 2 & & \begin{array}{|c|c|c|}
\hline
\text{NN} & 3D_2 & 3P_2 \\
\hline
5S_2 & A_{\pi}^{2+} & - & - \\
3D_2 & A_{\pi}^{2+} & - & - \\
5D_2 & A_{\pi}^{2+} & - & - \\
5G_2 & A_{\pi}^{-+} & - & - \\
3P_1 & - & A_{\pi}^{-+} & A_{\pi}^{-+} \\
5P_1 & - & A_{\pi}^{-+} & A_{\pi}^{-+} \\
3F_2 & - & A_{\pi}^{-+} & A_{\pi}^{-+} \\
5F_2 & - & A_{\pi}^{-+} & A_{\pi}^{-+} \\
\hline
\end{array}
\( J = 2 \)

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<th>( \ell_{5,6}^2 )</th>
<th>( \ell_{7,8}^2 )</th>
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\[ P = \pm 1 \]

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APPENDIX D

CALCULATIONS OF THE BORN AMPLITUDES

NN ----> NN

Here it only rests to calculate $A_D$, $A_E$ for

\[ \lambda_1 = \lambda_3 = \lambda_2 = \lambda_4 = \frac{3}{2} \]

and

\[ \lambda_4 = \lambda_3 = \frac{3}{2} = -\lambda_2 = -\lambda_1 \]

a. $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \frac{3}{2}$

\[ \overline{\mu}(\vec{p}, \gamma_k) Y_2 \mu(-\vec{p}, \gamma_k) = 0 \]

\[ \overline{\mu}(-\vec{p}, \gamma_k) Y_2 \mu(-\vec{p}, \gamma_k) = 0 \]

\[ \therefore \ A_D(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}) = A_E(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}) = 0 \]

b. $\lambda_1 = \lambda_2 = \frac{3}{2}$  \[ \lambda_3 = \lambda_4 = \frac{3}{2} \]

\[ \overline{\mu}(\vec{p}, \gamma_k) Y_2 \mu(-\vec{p}, \gamma_k) = i \frac{E + m}{2m} \left( \cos \frac{3}{2}, \sin \frac{3}{2}, -\frac{p \cos \gamma_k}{E + m}, -\frac{p \sin \gamma_k}{E + m} \right) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

\[ = -i \frac{p \cos \gamma_k}{m} \]

\[ \overline{\mu}(-\vec{p}, \gamma_k) Y_2 \mu(\vec{p}, \gamma_k) = i \frac{E + m}{2m} \left( -\cos \frac{3}{2}, \sin \frac{3}{2}, \frac{p \cos \gamma_k}{E + m}, \frac{p \sin \gamma_k}{E + m} \right) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

\[ = -i \frac{p \cos \gamma_k}{m} \]

and analogously,

\[ \overline{\mu}(\vec{p}, \gamma_k) Y_2 \mu(\vec{p}, -\gamma_k) = -i \frac{p \sin \gamma_k}{m} \]

\[ \overline{\mu}(-\vec{p}, \gamma_k) Y_2 \mu(-\vec{p}, -\gamma_k) = i \frac{p \sin \gamma_k}{m} \]
and therefore,

\[ A_d (\% \% \% \% \% \%) = \frac{\mu^2}{m^2} \left( -\cos^2 \theta \right) = \frac{1}{4m^2} \cdot \frac{\mu}{\mu^2} \]

\[ A_e (\% \% \% \% \% \%) = \frac{\mu^2}{m-\mu^2} \left( \cos^2 \theta \right) = \frac{1}{4m^2} \cdot \frac{\mu}{\mu^2} \]

NN → NN*

\[ M_{1,0}^{0+} = \frac{1}{\pi} \int \left\{ \left[ M_{0,0} (\% \% \% \% \% \%) - M_{0,0} (\% \% \% \% \% \%) \right] - \left[ M_{0,0} (\% \% \% \% \% \%) - M_{0,0} (\% \% \% \% \% \%) \right] \right\} \, d\chi \]

where

\[ M_{0,0} (\lambda_1 \lambda_2 \lambda_3 \lambda_4) = \bar{\omega}_{\mu} (\vec{p}, l_1) \mu (\vec{p}, l_2) \bar{\omega}_{\mu} (\vec{p}, l_3) \lambda_5 \mu (\vec{p}, l_4) \]

\[ M_{0,0} (\lambda_1 \lambda_2 \lambda_3 \lambda_4) = \bar{\mu}_{\mu} (\vec{p}, l_1) \mu (\vec{p}, l_2) \bar{\mu}_{\mu} (\vec{p}, l_3) \lambda_5 \mu (\vec{p}, l_4) \]

and

\[ \bar{\mu}_{\mu} (\vec{p}, l_4) = \sqrt{\frac{1}{2}} \epsilon^\mu_{\mu} (\vec{p}, 0) \mu (\vec{p}, l_4) + \sqrt{\frac{1}{2}} \epsilon^\mu_{\mu} (\vec{p}, 1) \mu (\vec{p}, l_4) \]

\[ k_{\mu} = k_{\mu} - p_{\mu} \]

\[ k_{\mu} = p_{\mu} - p_{\mu} \]
Multiplying out the spinors, we get

\[
\{\widetilde{\mu}(p', \gamma) \mu(-p, \gamma)\} \{\widetilde{\nu}(-p', \gamma) \nu(p, \gamma)\} = -\frac{\gamma}{2} (1 - \cos \theta) \text{ABC}_- \\
\{\widetilde{\mu}(p', \gamma) \mu(-p, \gamma)\} \{\widetilde{\nu}(-p', \gamma) \nu(p, \gamma)\} = -\frac{\gamma}{2} \sin \theta \text{ABC}_- \\
\{\widetilde{\mu}(p', \gamma) \mu(p, \gamma)\} \{\widetilde{\nu}(-p', \gamma) \nu(-p, \gamma)\} = -\frac{\gamma}{2} (1 + \cos \theta) \text{ABC}_- \\
\{\widetilde{\mu}(p', \gamma) \mu(p, \gamma)\} \{\widetilde{\nu}(-p', \gamma) \nu(-p, \gamma)\} = -\frac{\gamma}{2} \sin \theta \text{ABC}_- \\
\{\widetilde{\mu}(p', \gamma) \mu(-p, \gamma)\} \{\widetilde{\nu}(-p', \gamma) \nu(p, \gamma)\} = -\frac{\gamma}{2} (1 - \cos \theta) \text{ABC}_- \\
\{\widetilde{\mu}(p', \gamma) \mu(-p, \gamma)\} \{\widetilde{\nu}(-p', \gamma) \nu(p, \gamma)\} = -\frac{\gamma}{2} \sin \theta \text{ABC}_-
\]

where

\[
A = \frac{i E + m}{2m} \sqrt{\frac{(E^2 + m)(E^2 + m)}{2m}} \\
B_{\pm} = 1 \pm \gamma' \left(\frac{E + m}{E^2 - m^2}\right)^{1/2} \\
C_{\pm} = \rho (E + m)^{1/2} \pm \rho' (E - m)^{1/2}
\]

Using eqs. (6) and (8) on pages 20, we also get

\[
\epsilon_{\mu}(p', \gamma) \gamma_{\mu} = \frac{1}{\sqrt{\gamma}} \rho \sin \theta \\
\epsilon_{\mu}^*(p, \gamma) \gamma_{\mu} = \frac{1}{M} (\rho' E - \rho E \cos \theta) \\
\epsilon_{\mu}(p', \gamma) \gamma_{\mu} = \frac{1}{\sqrt{\gamma}} \rho \sin \theta \\
\epsilon_{\mu}^*(p, \gamma) \gamma_{\mu} = \frac{1}{M} (\rho' E + \rho E \cos \theta)
\]
Now, let $x = \cos \theta$, then
\[ k = -\beta + \mu^2 - \omega r \]
\[ \omega = -\beta + \mu^2 + \alpha r \]  
\[ z = -\beta \alpha \]

and then one has that
\[ \frac{1}{2} \int_{-1}^{1} \frac{dx}{y^2} = \frac{1}{2} \int_{-1}^{1} \frac{dx}{\mu^2} = \frac{x}{\alpha} Q_0(z) \]
\[ \frac{1}{2} \int_{-1}^{1} \frac{x \, dx}{y^2} = \frac{1}{2} \int_{-1}^{1} \frac{x \, dx}{\mu^2} = \frac{x^2}{\alpha} Q_0(z) - \frac{x}{\alpha} \]
\[ \frac{1}{2} \int_{-1}^{1} \frac{x' dx}{y^2} = \frac{1}{2} \int_{-1}^{1} \frac{x' dx}{\mu^2} = \frac{x'^2}{\alpha} Q_0(z) - \frac{x'}{\alpha} \]

We also need to define
\[ \Delta_{1} = B_1 C_1 + B_2 C_2 \]
\[ \Delta_{2} = B_1 C_1 - B_2 C_2 \]

Then, putting all of these together one gets
\[ M_{10} = \sqrt{\frac{\gamma}{\alpha}} \left\{ \frac{\rho E}{M} \Delta_i \int_{-1}^{1} \frac{dx}{y^2} + \frac{\rho E}{M} \Delta_i \int_{-1}^{1} \frac{x \, dx}{y^2} + \frac{\rho E}{M} \Delta_i \int_{-1}^{1} \frac{x' \, dx}{y^2} \right\} \]
\[ + \frac{\rho E}{M} \Delta_i \int_{-1}^{1} \frac{x \, dx}{y^2} - \frac{\rho E}{M} \Delta_i \int_{-1}^{1} \frac{x' \, dx}{y^2} \]
\[ = \sqrt{\frac{\gamma}{\alpha}} \left\{ \frac{Q_0(z)}{\alpha} \left[ \frac{x^2}{\alpha} \left( \frac{\rho E}{M} \Delta_i + \frac{\rho}{\alpha} \Delta_i^+ \right) + z \left( \frac{\rho E}{M} \Delta_i - \frac{\rho E}{M} \Delta_i^+ \right) \right] \right\} \]
\[ + \left[ -\frac{\rho E}{M} \Delta_i^+ + \frac{\rho E}{M} \Delta_i \right] + \left[ -\frac{\rho E}{M} \Delta_i^+ - \frac{\rho E}{M} \Delta_i \right] \]

and also the value of $M_{10}^{0-}$ given on page 33.
APPENDIX E

RIEMANN SURFACE for \( h(z) = \sqrt[2]{(z-a)^2} \log \frac{1+\frac{1}{2}(z-a)}{1-\frac{1}{2}(z-a)} \) \( a \in \mathbb{R} \)

This function \( h \) has the main properties of the function \( f \) studied in Sec. IV but is quite simpler and thus allows one to see better the Riemann surface constructed.

Let \( S \) be the Riemann surface for \( h \), then

\[ h : S \rightarrow \mathbb{C}, \text{ onto and analytic (} \mathbb{C} = \text{ Complex plane} \) \]

Let \( \kappa(z) = \sqrt[2]{(z-a)^2} \) (1)

\[ w(z) = \left[ 1+\kappa(z) \right] / \left[ 1-\kappa(z) \right] \] (2)

The Riemann surface for \( w \) is a two sheeted surface, consisting of two copies of \( \mathbb{C} \) \( \mathbb{C} \cup \{\infty\} \) joined along a cut from \( 0 \) to \( a \). Call these copies \( A \) and \( B \). To have this we define

\[ \theta_1 = \arctan \left( \text{Im} z / \text{Re} z \right) \]

\[ \theta_2 = \arctan \left( (\text{Im} z)/(\text{Re}(z)-a) \right) \]

then \( \theta_1 = \theta_1^\circ + 2m_1 \pi \) \( \theta_2 = \theta_2^\circ + 2m_2 \pi \)

where \( 0 \leq \theta_1^\circ \leq 2\pi \), \( -\pi \leq \theta_2^\circ \leq \pi \) and the \( m_1 \) are integers.

and then \( A = \{ z \mid |m_1| + |m_2| \text{ is even} \} \)

\( B = \{ z \mid |m_1| + |m_2| \text{ is odd} \} \)

and the function \( \alpha \) is then defined by
\[ a(z) = |a(z)| \exp \left[ i(\theta_1 - \theta_2)/2 \right] \]

so for \( z \) on sheet A, \( a(z) \sim 1 \) for large real \( z \) and for \( z \) on sheet B, \( a(z) \sim -1 \) for large real \( z \). Let \( S_a \) be the Riemann surface consisting of A and B as defined above, then

\[ a : S_a \rightarrow \hat{\mathbb{C}} \text{, analytic.} \]

Now let us consider \( \log w(z) \). From (1) and (2) we have that

\[ z = \frac{a}{\bar{a}} + \frac{\bar{a}}{a} \left( w + \frac{1}{w} \right) \quad (3) \]

Thus we see that \( z \) maps the exterior of the unit circle in the \( w \)-plane to \( \hat{\mathbb{C}} \) and the interior onto another copy of the Riemann sphere. These two surfaces are joined along \( z \approx (0, a) \), (which corresponds to \(|w| = 1\)).

Note that the sense of the curves inside the \(|w| = 1\) circle is inverted by the map \( z \). We define the surface for \( \log w \) by having the principal branch of the log for \(-\pi \leq \arg(w) \leq \pi\), i.e., the cut on the \( w \)-plane extends from \(-\infty\) to 0. Now we map this cut by \( z \) into \( S_a \). From (3) we have that

\(-\infty \leq w \leq 0\) is mapped onto \(-\infty < z < 0\) on sheet A

and \(-1 < w < 0\) is mapped onto \(0 < z < -\infty\) on sheet B.

Thus we see that the cut goes along the negative real \( z \) on
sheet A and then crosses into sheet B by the (square root) branch point at \( z=0 \) and returns to \( z=-\infty \) on sheet B. Note that the sense of the cut is opposite in both sheets.

Now we have, for \( j=0, \pm 1, \pm 2, \ldots \)

when \( \alpha \) is on sheet A, \( \log w \) on sheet \( A_j \) for

\[ \arg w \in \left[ (z_{j-1})\pi, (z_{j+1})\pi \right] \]

and when \( \alpha \) is on sheet B, \( \log w \) on sheet \( B_j \) for

\[ \arg w \in \left[ (z_{j-1})\pi, (-z_{j+1})\pi \right] \]

and on sheet \( A_j \) \( \log w(z) = \log |w(z)| + 2j\pi i \)

and on \( B_j \) \( \log w(z) = -\log |w(z)| - 2j\pi i \)

Note that we have taken into account the opposite sense of the log cut in A and B.

All of these sheets are cut from \(-\infty\) to 0 (a log cut) and from 0 to \( a \) by a square root cut. The sheets are joined in the following way: \( A_i \) is connected through the right hand cut to \( B_i \) and to \( A_{i+1} \) through the left hand cut, and analogously for \( B_i \).

Now let us look at \( h(z) = \frac{1}{\sqrt{\alpha}} \) and see what Riemann surface it has as a function of \( z \). Note that at \( A_i \) and \( B_i \), \( h(z) \) has the same value. Thus, essentially, the factor \( 1/\alpha(z) \) identifies sheet \( A_i \) with sheet \( B_i \).

Let \( S \) be the Riemann surface of \( h \). \( S \) then consists of an infinite number of copies of \( \mathbb{C} \).

\[ S = \bigcup S^i, \quad i=1,2,\ldots \]

where

\[ S^i = A_{\pm(i-1)} \cup B_{\pm(i-1)} \quad ; \quad S^+ = S^- = S \]
Neighborhoods are then defined in the following way: let 
\( U(x) \) stand for a neighborhood basis about \( x \), then for 
\( x \in S^+_i, \ i \geq 1 \)

\[-\infty < x < 0 \]

\[ U(x+i) = \left\{ z \in S^+_1 \mid |z-x| \lesssim \delta, \ \text{Im} z > 0 \right\} \cup \left\{ z \in S^-_1 \mid |z-x| \lesssim \delta, \ \text{Im} z < 0 \right\} \]

\[ U(x-i) = \left\{ z \in S^+_1 \mid |z-x| \lesssim \delta, \ \text{Im} z < 0 \right\} \cup \left\{ z \in S^-_1 \mid |z-x| \lesssim \delta, \ \text{Im} z > 0 \right\} \]

\[ 0 < x < a \]

\[ U(x+i) = \left\{ z \in S^+_1 \mid |z-x| \lesssim \delta, \ \text{Im} z > 0 \right\} \cup \left\{ z \in S^-_1 \mid |z-x| \lesssim \delta, \ \text{Im} z < 0 \right\} \]

\[ U(x-i) = \left\{ z \in S^+_1 \mid |z-x| \lesssim \delta, \ \text{Im} z < 0 \right\} \cup \left\{ z \in S^-_1 \mid |z-x| \lesssim \delta, \ \text{Im} z > 0 \right\} \]

and for \( z \) not on the branch cuts there are no problems at all. For \( i=1 \), \( S^-_1 \) only has the logarithmic branch cut between \( -\infty \) and 0, and for \( x \) in this region

\[ U(x+i) = \left\{ z \in S^+_1 \mid |z-x| \lesssim \delta, \ \text{Im} z > 0 \right\} \cup \left\{ z \in S^-_2 \mid |z-x| \lesssim \delta, \ \text{Im} z \leq 0 \right\} \]
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