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Residually Finite Properties of Covering Spaces

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ABSTRACT

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The purpose of this thesis is trying to construct some negative results for Griffith's theorem 1 in (3) by showing theorem 1.1 and a counter-example in Section 2. I also give some general positive results for this work in Section 3, to go with the negative ones.

It is well known that Fuchsian groups are residually finite. Section 4 provides some topological construction of Fuchsian groups which yield a simple proof of their residual finiteness.

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INTRODUCTION

Let P be a non-empty class of groups. A group G is said to be residually P, written $G \varepsilon {}_r P$, provided that for each non-trivial element g of G there is a quotient group of G in P in which the element g is mapped non-trivially.

Residual properties of groups have been studied in [2],[5], [6], etc. In particular residually finite groups (P is the class of all finite groups) have received considerable attention (See [5]), since they are known to have other properties useful in algebra and topology. For example, any finitely grenerated residually finite group has a solvable word problem ([8] + [9]) and is hopfian. ([7]).

Let $P_G = \{ N : N \triangleleft G , G/N \varepsilon P \}$. It is easily seen that $G \varepsilon {}_r P$ if and only if $\bigcap \{ N : N \varepsilon P_G \} = 1$. One need only consider a confinal sequence in P_G, that is if $N_1 > N_2 >$ --- is cofinal in P_G, then $G \varepsilon {}_r P$ if and only if $\bigcap Ni = 1$.

In [3] the following approach was taken in attempt to establish residual finiteness of certain classes of groups. Using the above notation, with P the class of all finite groups, it was noted that the sequence $N_1 > N_2 >$ --- induces an inverse system $N_1/N_1' \leftarrow N_2/N_2' \leftarrow$ ---

of the commutation quotient groups. And that there is a natural map $\Theta : N/N' \rightarrow \varprojlim \{N_i/N_i'\}$ where $N = \bigcap N_i$. It was then shown that if G is the fundamental group of a 2-manifold or is a Fuchsian group then $\lim \{ N_i / N_i' \} = 1$.

Finally, a general theorem (Griffith's theorem) was established asserting that Θ is always a monomorphism. Since the above mentioned groups are known not to contain any perfect subgroups, it was concluded that N = 1 and hence G is residually finite.

This work begins with an indication that Griffith's theorem is incorrect. We do this by showing Theorem 1.1 that $\varprojlim \{ N_i / N_i' \} = 1$ whenever G is any finitely presented group. In Section 2, we provide an example of a finitely presented group for which is not a monomorphism.

In Section 3, we present a general procedure for establishing that a group G is residually P (P an arbitrary class) which involves comparing the intersection N of a cofinal sequence $\{ N_i \}$ in P_G with the inverse limit, $\varprojlim \{ N_i / N_i (W) \}$ of the induced system obtained by factoring out some verbal subgroups.

This procedure involves verification of three conditions. Criteria for establishing some of these conditions is discussed.

In Section 4, we present some related theorems about Fuchsian groups. These theorems are for the most part already known, but the proofs are simplificative over existing ones. In particular we use a simple topological construction to derive information about the subgroup structure of Fuchsian groups which yields a proof of their residual finiteness. We conclude with an application showing that the fundamental groups of a certain class of 3-dimensional manifolds (Seifert-fibered manifolds) are residually finite.

1. Residually Finite Properties

Let P be the class of all finite groups. For a group G, P_G ={all normal subgroups N with finite index in G }. Thus G is residually finite iff $\bigcap \{N : N \in P_G\} = 1$. Specifically, if $N_1 > N_2$ --- is any cofinal sequence in P_G, then G is residually finite iff $\bigcap N_i = 1$.

Theorem 1.1 If G is a finitely generated group, then there exists a cofinal sequence $N_1 > N_2$ --- in P_G such that $\varprojlim N_k/N_k = 1$ with the homomorphism induced by the inclusions $N_{n-1} \supsetneq N_n$.

Proof:

Define $N_i = \{ S \triangleleft G : |G : S| \leq i \}$

Since for any integer r > o, there is only a finite number of subgroups of G with index r, we then have:

(i) $|G : N_i| < \infty$

(ii) N_i is fully invariant, therefore is normal in G.

Thus, we have a sequence:

$$G = N_1 \geq N_2 \geq N_3 --- N_i \geq N_{i+1} ---$$

This sequence has the following properties:

(i) N_i is normal in G and N_{i+1} is normal in N_i for all i.

(ii) $|N_i : N_{i+1}| < \infty$ for all i.

(iii) Since each N_i is of finite index, hence is finitely generated. This implies N_i/N_i' is a finitely generated abelian group for each i.

Now, we have a sequence of mapping induced by the inclusions $N_{n+1} \hookrightarrow N_n$

$$G/G' = N_1/N_1' \xleftarrow{i_1} N_2/N_2' \xleftarrow{i_2} \cdots N_n/N_n' \xleftarrow{i_n} N_{n+1}/N_{n+1}' \cdots$$

$$i_n (g_{N_n'}) = g\, N_{n-1}' \quad \text{if } g \varepsilon\, N_n \quad n = 1,2, \cdots$$

For an arbitrary integer n, there is a sequence

$$N_n/N_n' \geq i_{n+1} \left(N_{n+1}/N_{n+1}'\right) \geq i_{n+1}\, i_{n+2} \left(N_{n+2}/N_{n+2}'\right) \geq \cdots$$

Now, we will prove three facts:

(i) $i_{n+1}\, i_{n+2} \cdots i_{n+k} \left(N_{n+k}/N_{n+k}'\right) \triangleleft N_n/N_n'$

(ii) $\left| N_n/N_n' : i_{n+1}\, i_{n+2} \cdots i_{n+k} \left(N_{n+k}/N_{n+k}'\right) \right| < \infty$

(iii) For any subgroup S of finite index of N_n/N_n', there exists an integer M such that $i_{n+1} i_{n+2} \cdots i_{n+m} \left(N_{n+m}/N_{n+m}'\right) \leq S$.

(i) is trivial since N_n/N_n' is abelian.

(ii) $i_{n+1}\, i_{n+2} \cdots i_{n+k} \left(N_{n+k}/N_{n+k}\right) = \{ g\, N_n' : g \varepsilon\, N_{n+k} \}$

$$\cong \, {}^{N_{n+k}}\!\Big/_{N_{n+k} \cap N_n'} \;\cong\; {}^{N_{n+k} N_n'}\!\Big/_{N_n'} \;=\; {}^{N_n}\!\Big/_{N_n'}$$

$$0\left({}^{N_n/N_n'}\Big/_{N_{n+k}N_n'/N_n'} \right) \le 0\left({}^{N_n}\Big/_{N_{n+k}N_n'} \right) < \infty \quad .$$

but $0\left({}^{N_n}\Big/_{N_{n+k}N_n'} \right) \le 0\left({}^{N_n}\Big/_{N_{n+k}} \right) < \infty$

So we get $\left| {}^{N_n}\Big/_{N_n'} : i_{n+1} i_{n+2} --- i_{n+k} \left({}^{N_{n+k}}\Big/_{N_{n+k}'} \right) \right| < \infty$

(iii) Let S be a subgroup of ${}^{N_n}\Big/_{N_n'}$ and $\left| {}^{N_n}\Big/_{N_n'} : S \right| = r$

Then $\exists$ T subgroup of N_n with $N_n' \le T$ such that $T/_{N_n'} \equiv S$

Then $\left| {}^{N_n}\Big/_{N_n'} : T/_{N_n'} \right| = r = | N_n : T |$

But $|G : T| = | G : N_n|| N_n : T| < \infty$

Thus $\exists$ r $\ni$ $N_m \le T$ for all m $\le$ r

choose m$'$ $\geq$ max (r, n)

then let $m' = n + k$ we have $N_m' \le T$, i.e., $N_{n+k} \le T$. Since $N_n' \le T$ so

$N_{n+k}N_n' \le T$, ${}^{N_{n+k}N_n'}\Big/_{N_n'} \le T\Big/_{N_n'} = S$,

so $i_{n+1}\, i_{n+2} --- i_{n+k} \left({}^{N_{n+k}}\Big/_{N_{n+k}'} \right) \le S$

Therefore (iii) follows.

It is clear that any finitely generated abelian group is residually finite, thus N_n/N_n' is residually finite; hence the intersection of all normal subgroups of finite index of N_n/N_n' is the identity.

Hence $\bigcap_{k=1}^{\infty} i_{n+1} i_{n+2} --- i_{n+k}\left(N_n/N_n'\right) = 1$

for $n = 1, 2, ---$

Let $< g_n N_n' >$ be an element of $\varprojlim \{ N_n/N_n \}$

then $< g_n N_n' > = < g_1 N_1' , --- , g_n N_n' , g_{n+1} G_{n+1}', --- >$

where $g_n N_n' = i_{n+1}\left(g_{n+1} N_{n+1}'\right)$

$= i_{n+1} i_{n+2} --- i_{n+k}\left(g_{n+k} N_{n+k}'\right)$

for all $k = 1,2, ---$

Therefore, $g_n N_n' \in \bigcap_{k=1}^{\infty} i_{n+1} --- i_{n+k}\left(N_{n+k}/N_{n+k}'\right)$

This implies $g_n N_n' = 1$ for $n = 1, 2, ---$

Hence we get the result $\varprojlim \{ N_n/N_n', i_n \} = 1.$

Now, let G be a group with a cofinal sequence $N_n \leq N_2 \leq ---$ in P_G.

We can naturally identify $N = \bigcap N_i$ with $\varprojlim \{ N_n \}$ where the maps in the sequence $N_1 \xleftarrow{i_1} N_2 \xleftarrow{i_2} N_3 \leftarrow$ --- are inclusions. Then we have a commutative ladder.

$$\begin{tikzcd}
N_1 \arrow[d, "\mu_1"] & N_2 \arrow[l, "i_1"'] \arrow[d, "\mu_2"] & N_3 \arrow[l, "i_2"'] \arrow[d] & \cdots \arrow[l] \\
N_1/N_1' & N_2/N_2' \arrow[l, "\Psi_1"'] & N_3/N_3' \arrow[l, "\Psi_2"'] & \cdots
\end{tikzcd}$$

where $\mu_k : N_k \to N_k/N_k'$ is natural projection

$\psi_k : N_{k+1}/N_{k+1}' \to N_k/N_k'$ is inclusion induced map

Let $\mu : N \to \varprojlim \{ N_k/N_k' \}$ is defined by

$$\mu(g) = \langle g N_k' \rangle \qquad g \in N \quad .$$

Since $\mu(g) = \langle g N_k' \rangle = \langle 1 \rangle$ iff $g \in N_k' \; \forall k$,

we have $\ker \mu = \bigcap N_k'$

But $N' < \bigcap N_k'$, so we have a commutative diagram.

$$\begin{tikzcd}
N \arrow[r] \arrow[dr, "\mu"'] & N/N' \arrow[d, "\Theta"] \\
& \varprojlim \{ N_k/N_k' \}
\end{tikzcd}$$

2. The group $G = | a, b : a^{-1}b^2a = b^3 |$ and its applications.

In order to produce a counter example to the results of [3], we need the following.

<u>Theorem 2.1</u> Let $G = | a,b : a^{-1}b^2a = b^3 |$. Then G is a finitely presented, non-residually finite group which contains no non-trivial perfect subgroup.

<u>Proof:</u> It is shown in [2] that G is not residually finite. Suppose $1 \neq H$ is a perfect subgroup of G, then clearly $H < G'$ and $H = H' = H'' = ---$. By Lemma 2.2 below G'' is a free group. Since $H' < G''$. $H = H'$ is also free. Since a non-trivial free group is not perfect, we have a contradiction.

<u>Lemma 2.2</u> The second derived subgroup G'' of $G = |a,b :$ $a^{-1}b^2a = b^3 |$ is a free group.

<u>Proof:</u> It is easily established by the Reideweister-Schreier rewriting process (c.f δ 2.3 of [6]) that G' has a presentation

(1) $G' = | b_i : b_i^3 = b_{i+1}^2 \quad -\infty > i > \infty |$

Let $T_n = | b_{-n}, b_{-n+1}, ---, b_n : b_i^3 = b_{i+1}^2 \quad - n \leq i \leq n-1 |$

Then $T_n < T_{n+1}$, $G' = \bigcup_{n=1}^{\infty} T_n$

and $G'' = \bigcup_{n=1}^{\infty} T'_n$

We next show

(2.3) T_n has a presentation of the form $T_n = | \; b_{-n} , b_n \; :$

$b_{-n}^{9n} = b_n^{4n}$, $R_n \; |$ where R_n is a finite set of relations of the form $[\, b_{-n}^r , b_n^s] = 1$ which includes the relations

$$[\, b_{-n}^{\frac{-3(3.9^{n-1}-1)}{2}} , \; b_n^{2.4^{n-1}}] = 1$$

S_n :

$$[b_{-n}^{-3.9^{n-1}\frac{(3.9^{n-1}-1)}{2}} , \; b_n^{2.4^{n-1}(3.9^{n-1}-1)+2}] = 1$$

Proof:

We induct on n. By Tietze transformation:

$$T_1 = | \; b_{-1} \; b_1 \; : \; b_{-1}^9 = b_1^4 , \; [\, b_{-1}^3 \; b_1^2 \,] = 1 \; |$$

Assuming the presentation for T_n, we have

$$T_{n+1} = | \; b_{-(n+1)} , b_{-n} , b_n , b_{n+1} \; : \; b_{-n+1}^3 = b_n^2 ,$$

$$b_{-n}^{9^n} = b^{4^n} , \; b_n^3 = b_{n+1}^2 , \; R_n \; |$$

Now (2) $b_n = b_n^{4^n} \, b_n^{\frac{-3(4^n-1)}{3}} = b_{-n}^{9^n} \, b_{n+1}^{-2(\frac{4^n-1}{3})}$, and we have

$$(3)\quad [\, b_{-n}^{9^n} \;,\; b_{n+1}^{-2\left(\frac{4^{n}-1}{3}\right)} \,] = 1$$

Using (2) and (3), the relation $b_n^{\,3} = b_{n+1}^2$ becomes

$$(4)\quad b_{-n}^{3.9^n} = b_{n+1}^{2.4n}$$

Now $$(5)\quad b_{-n} = b_{-n}^{\,3.9^n \frac{-2(3.9^n-1)}{2}} = b_{n+1}^{2.4^n}\; b_{-(n+1)}^{-3\frac{(3.9^n-1)}{2}}\,,$$

and we have

$$(6)\quad [\; b_{-(n+1)}^{-3\frac{(3.9^n-1)}{2}} \;,\; b_{n+1}^{2.4^n} \;] = 1$$

Putting (5) into (2) we get

$$(7)\quad b_n = b_{-(n+1)}^{-3.9^n\frac{(3.9^n-1)}{2}} \quad b_{n+1}^{2.4^n\,\frac{(3.9^n-1)+2}{3}}$$

Using (5), (7) and the commutative relations, the relation

$b_{-(n+1)}^{\,2} = b_{-n}^2$ becomes

$$(8)\quad b_{-(n+1)}^{9^{n+1}} = b_{n+1}^{4^{n+1}} \quad .$$

The relation (4) becomes:

$$\left(b_{-(n+1)}^{9^{n+1}}\right)^{\frac{3.9^n-1}{2}} = \left(b_{n+1}^{4^{n+1}}\right)^{\frac{3.9^n-1}{2}}$$

which is a consequence of (8).

The relations (3) and (6) are equivalent to the desired relation S_{n+1}.

Using S_{n+1} we compute that for any u, v

$$(9)\quad [b_{-n}^{u} , b_{n}^{v}] = [b_{-(n+1)}^{\frac{-3(3.9^{n}-1)}{2}(u + 9^{n}v)} , b_{n+1}^{2.4^{n}u+(2.4^{n}\frac{(3.9^{n}-1)}{2}+2)v}]$$

Thus using (5), (7), and (9), the relation R_n become relations of the desired form R_{n+1} and the proof of (2.3) is complete.

To complete the Proof of Lemma 2.2, we consider the group $U_n =$

$$| \, b_{-n} , b_{n} : b_{-n}^{9^{n}} = b_{n}^{4^{n}} \, | .$$

It is well known (and easily derived by the Reidemeister-Schreier rewriting process) that U_n is a free group freely generated by the elements $[b_{-n}^{i} , b_{n}^{j}]$ $1 \leq i \leq 9^{n}-1$, $1 \leq j$ $4^{n}-1$.

By 2.3, T_n is a quotient group of U_n. Bu a particularly nice subgroup lying in U_n'. In particular T_n' is a quotient group of the free group U_n' by a free factor of U_n'. Hence T_n' is freely generated by a certain subset of the elements $[b_{-n}^{i} , b_{n}^{j}]$. By (9) a free basis for T_n' extends to a free basis for T_{n+1}'. Hence :

$$G'' = \bigcup_{n=1}^{\infty} T_n' \quad \text{is a free group.}$$

Applications of the group $G = | \, a, b : a^{-1} b^{2} = b^{3} \, |$.

Now we will use Theorem 1.1 and the subgroup G to show that Griffith's theorem is incorrect.

The statement of Griffith's theorem is as follows ([3] Theorem 1).

If a group G is of the form $G = \pi_1(X, 0)$ where X denotes a path-connected, LC^1, locally compact metric space with base point 0. Let L be the lattice of all subgroups of G. U is a linearly ordered subset of L such that every $A \in U$ is normal in G. $J = \{A \mid A \in U\}$. Let X_J be the covering space of X such that $\pi_1(X_J) = J$. X_A is a space $X_\infty = \varprojlim \{X_A, P_{AB}\}_{A,B\in U}$ with the usual inverse limit topology on X_∞. (which is induced by the inclusion of X_∞ in the Cartesian product

$\pi\{X_A \mid A \in U\}$). Then there is a monomorphism:

$$H_1(X_J) \to H_i(X_*)$$

where H_1 is the Čech homology.

Since every finitely presented group is of the form $\pi_1(X)$, where X is a finite complex. So there is a finite complex X such that $\pi_1(X) = G = | a,b, : a^{-1}b^2a = b^3|$ Let $G_n = \bigcap\{H : H \subset G, |G : H| \leq n\}$. Then G_n is of finite index and fully invariant for each n. Let $J = \bigcap G_n$. Since G is finitely generated and non-hophian, G is not residually finite. Hence, J cannot be a trivial group.

Then we have

$$1 = \varprojlim \; {G_n}/{G_n'} \cong \varprojlim \; H_1 \; (X_{Gn}) = H_1 \; (X_*)$$

(because H is continuous, see [1] Chapter X)

If Griffith's theorem were true, then we have $H_1 \; (X_J) = 1$. This implies J is a perfect subgroup of G. By Theorem 2.1 G cannot have a non-trivial perfect subgroup, hence forces J = 1. This is a contradiction.

3. General Approach to Residually P

In this section, let P be a non-empty class of groups. For a group G, $P_G = \{ N : N \lhd G \text{ and } G/_N \in P \}$. Thus G is residually P

$(G \in {}_R P)$ iff $\bigcap \{ N : N \in P_G \} = 1$. If $N_1 \geq N_2 \geq$ --- is any co-final sequence in P_G, then $G \in {}_R P$ iff $N_i = 1$. For such a sequence, we can naturally identify $N = \bigcap N_i$ with $\varprojlim \{ N_i \}$ where the maps in the sequence

$$N_1 \xleftarrow{i_1} N_2 \xleftarrow{i_2} N_3 \leftarrow \text{---} \text{ are inclusions.}$$

Now, suppose W is any set of words. For any group K, let K(W) denote the W-verbal subgroup of K, we have a commutative ladder:

$$\begin{tikzcd}
N_1 \arrow[d, "\mu_1"] & N_2 \arrow[l, "i_1"'] \arrow[d, "\mu_2"] & N_3 \arrow[l, "i_2"'] \arrow[d, "\mu_3"] & \text{---} \arrow[l] \\
N_1/N_1(W) & N_2/N_2(W) \arrow[l, "\psi_1"'] & N_3/N_3(W) \arrow[l, "\psi_2"'] & \text{---} \arrow[l]
\end{tikzcd}$$

with $\mu_K : N_K \to N_K/N_{K(W)}$ natural projection

and $\psi_K : N_{K+1}/N_{K+1}(W) \to N_K/N_K(W)$ inclusion induced map (may not one to one)

Thus we have an induced map

$$\mu : N \to \varprojlim \{ N_K/N_K(W) \} \text{ by}$$

$$\mu(g) = \langle g\, N_K(W) \rangle \quad . \quad \text{where } g \varepsilon N.$$

It is easily shown that ker $\mu = \bigcap N_K(W)$.

Since $N(W) < \bigcap N_K(W)$, we have a commutative diagram:

$$\begin{array}{ccc} N & \to & N/N(W) \\ & \searrow \mu & \downarrow \Theta \\ & & \varprojlim \{ N_K/N_{K(W)} \} \end{array}$$

Thus a way to establish that $G \varepsilon {}_R P$ is:

<u>Theorem 3.1</u>. Let P be any non-empty class of groups, G is a group with a cofinal sequence of subgroups in P_G. If G satisfies the following three conditions:

(1) $\varprojlim \{ N_K/N_K(W) \} = 1$

(2) Θ is monic (i.e., $N_i(W) = N(W)$)

(3) G contains no non-trivial subgroup equal to its own W - verbal subgroup. Then G is residually P.

Theorem 3.2 Let P be the class of finite groups , G is any finitely generated gtoup. W is any word in G. Then G has a confian sequence $G = N_1 \geq N_2 \geq --- $ in P_G and if $N_K / N_K(W)$ is residaully finite for each K, then $\varprojlim \{ N_K / N_K(W) \} = 1$.

Proof:

The way to prove this theorem is very similar to theorem 1.1. So we omit it.

Corollary 3.3 If P = { finite groups } , $W = \{\chi^b\}$ $b = 2, 3, 4, 6$, and G is any finitely generated group.

Then $\varprojlim \{ N_K / N_{K(X^b)} \} = 1$.

Proof:

Let G be any finitely generated group and let F be a free group of finite rank and H be a normal subgroup of F such that $G = F/_H$.
Since $G/_{G(X^b)}$ is a quotient group of $F/_{F(X^b)}$, by the Burnside problem, $F/_{F(X^b)}$ is a finite when p = 2,3,4,6. So $G/_{G(X^b)}$ is finite and $N_K / N_{K(X^b)}$ is finite for all K. By Theorem 3.2, we have $\varprojlim N_K / N_{K(X^b)} = 1$.

Lemma 3.4 G is a group. W is any non-trivial set of words. If the k'th W-verbal subgroup $G^{(k)}(W)$ (in the sense of $G^{(k)}(W) = [G^{(k-1)}(W)]$.) is trivial or free for some $k < \infty$. Then G has no non-trivial subgroup equal to its own W-verbal subgroup.

Proof: Suppose there exists a subgroup N such that $N = N^{(k)}(W)$, (k positive integer). Then $N < G^{(k)}(W)$. If for some k, $G^{(k)}(W)$ is free or trivial, then N is free or trivial. But then $N \neq N^{(k)}(W)$ if N is free. Thus, G has no non-trivial subgroup equal to its own W-verbal subgroup.

4. Fuchsian Groups.

Fuchsian groups, or planar discontinuous groups play an important role in complex variable theory, Riemann surface theory, and topology. (See [11]). By representing these groups as fundamental groups of some easily described spaces, we are able to use topological arguments about covering spaces to provide elementary proofs of some of their properties.

It is known that a Fuchsian group G has a presentation in one of the following forms:

$$(1)\quad G = \langle a_1, b_1, \text{---}, a_g, b_g, c_1, \text{---}, c_k \; ;$$

$$c_1^{q_1} = c_2^{q_2} = \text{---} = c_k^{q_k} = c_1 c_2 \text{---} c_k [a_1 b_1] \text{---} [a_g, b_g] = 1\rangle$$

$$(2)\quad G = \langle a_1, \text{---}, a_g, c_1, \text{---}, c_k \; ;$$

$$c_1^{q_1} = c_2^{q2} = \text{---} = c_k^{q_k} = c_1 c_2 \text{---} c_k a_1^2 \text{---} a_g^2 = 1\rangle$$

We proceed to construct a space with fundamental group G.

Definition: Given an integer $n \geq 1$, an n - cap is a space E obtained from the unit disk D (in the complex plane) by identifying each $e^{i\theta} \varepsilon \partial D$ with $e^{i(\theta + \frac{2\pi}{n})}$.

Let $\mu : D \to E$ be the identification map and let $B(E) = \mu(\partial D)$. The following facts are easily established:

(i) $\mu \mid \text{Int } D$ is a homeomorphism of Int D onto $E - B(E)$

(ii) $B(E)$ is a simple closed curve and $\mu \mid \partial D : \partial D \to B(E)$ is an n-sheeted covering.

(iii) $\Pi_1(E)$ is cyclic of order n and the simple closed curve $B(E)$ represents a generator.

Now, let F be a compact, connected 2-manifold with boundary, let J_1, --- J_K be components of ∂F and let q_1, --- q_k be positive integers. Let E_i be a q_i - cap $(1 \leq i \leq k)$ and let $h_i : J_i \to B(E_i)$ be a homeomorphism. The space $X = F_{hi} U E_1 U_{h_2} U$ --- $U_{h_k} E_k$ is called a <u>C-space</u>. The number $C(X) = \max \{ q_i \}$ is called the <u>complexity</u> of X. Note that a C-space of complexity 1 is a compact 2-manifold.

<u>Theorem 4.1</u> Let G be a Fuchsian group (as in (1) or (2) above, then there is a C-space X with $\Pi_1(X) \cong G$.

<u>Proof:</u> Let F be obtained by removing the interiors of k disjoint 2-cells from either a closed 2-manifold of genus g (case (1)) or the

connected sum of g projective planes (case (2)). Then construct X as above. It is an easy application of Van Kampen's theorem to show that $\pi_1(X) \cong G$.

Lemma 4.2 If E is an n-cap and $p : y \to E$ is an (connected) m-sheeted covering, then y is the union of m $(^n/_m)$ - caps E_1, --- E_m such that $E_i \cap E_j = B(E_i) = B(E_j)$, $i \neq j$.

Theorem 4.3 Let G be a Fuchsian group and let H be a subgroup of finite index in G. Put $G \cong \pi_1(X)$ where X is a C-space. Then:

(i) There is a C-space X_1 with $\pi_1(X_1) \cong H$ and with $C(X_1) \leq C(X)$. In particular H is a Fuchsian group.

(ii) If H is normal in G, then $C(X_1) = C(X)$ if and only if for some q_i - cap E_i of X with $q_i = C(X)$, the element of $\pi_1(X)$ represented by $B(E_i)$ lies in H.

Proof: Let $X = F \cup E_1 \cup --- \cup E_k$ as in Theorem 1. Let $p : \tilde{X} \to X$ be the finite sheeted covering such that $p_* (\pi_1(\tilde{X})) = H$. Now $\tilde{F} = p^{-1}(F)$ is a compact, connected, 2-manifold and for each i and each path component y_{ij} of $p^{-1}(E_i)$, Y_{ij} is (by Lemma 2.) a union of $(^{q_i}/_{m_{ij}})$ -caps E_{ij1}, --- , $E_{ijm_{ij}}$ (where $p \mid Y_{ij} : Y_{ij} \to E_i$ is an m_{ij} sheeted covering.

Let $X_1 = \tilde{F} \cup_{ij} \cup E_{ij}$ (i.e., keep just one cap from each Y_{ij}).

It is easily seen that the inclusion $i : X_1 \to \tilde{X}$ induces an isomorphism $i_* : \pi_1(X_1) \to \pi_1(\tilde{X})$. This completes (i).

For (ii), we note that $C(X_1)$ implies that for some i, j with $q_i = C(X)$, $mij = 1$. Since the covering is regular this forces $B(E_i)$ to represent an element of H (since E_i lifts to Y_{ij}).

Theorem 4.4 Let G be a fuchsian group. Then there is a subgroup H of finite index in G which is isomorphic to the fundamental group of a closed 2-manifold S (which may be the 2-sphere if G is finite).

Proof: Let $G = \pi_1(X)$ where X is a C-space. If $C(X) = 1$, then X is already a closed surface and we are done. So assume $C(X) \geq 2$ and that if $G_1 = \pi(X_1)$ is a Fuchsian group with $C(X_1) < C(X)$, then G_1 contains the fundamental group of a closed 2-manifold as a subgroup of finite index.

Now we may take $X = F \cup E_1 \cup \text{---} \cup E_k$ where $B(E_i)$, when properly joined to the base point, represents the element C_i in the presentation (1) or (2) of G. We further assume that:

$$q_1 \leq q_2 \leq \text{---} \leq q_r < q_{r+1} = q_{r+2} = \text{---} q_k \ (=C(X)).$$

By Theorem 4.3 and induction our proof will be finished if we can find a normal subgroup of finite index in G which contains none of the elements $C_{r+1}, \text{---} C_k$. Since the intersection of all subgroups of a given finite index in any finitely generated group is a normal subgroup of finite index, it suffices to find a subgroup of finite index in G which contains none of the elements C_{r+1}, --- C_k. We consider two cases:

Case 1 $r \leq k-2$ (i.e. X contains more than one cap of maximal order). In this case one easily checks that, the quotient group obtained from G by putting the a_i and b_i equal to one and abelianizing is a finite abelian group, A, in which C_{r+1}, ---, C_k are mapped non-trivially. Thus ker $(G \to A)$ is the desired subgroups of finite index.

Case 2 $r = k - 1$. In this case suppose G contains a proper normal subgroup H of finite index. If $C_k \notin H$ we are done. If $C k \in H$, then the C-space X_1 given by Theorem 4.3 with $\pi_1(X_1) \cong H$ has $n^q k$ -caps; where n is the index of H in G. Thus by Case 1 applied to X_1 we get a subgroup of finite index in H, hence of finite index in G which does not contain C_k. Thus, our proof of Theorem 4.4 will be completed by showing that G contains a proper normal subgroup of finite index.

Lemma 4.6 If G is a Fuchsian group, then G contains a proper normal subgroup H of finite index.

Proof: We wish to show that G has a finite, non-trivial quotient group. This is certainly the case if G/G' is non-trivial which holds (in either (1) or (2)) if $g > o$. If $g = o$ and $k \leq 2$, then G is already finite. So we assume $g = o$ and $k \geq 3$, i.e. :

$$G = | \; C_1 , C_2 , --- , C_k \;, C_1^{q_1} = C_2^{q_2} = --- = C_k^{q_k} = 1,$$

$$C_1 --- C_k = 1 \;, \; k \geq 3, \; q_i \geq 2 \; |$$

Now suppose we can find a finite group Q with non-trivial elements T, U such that $T^{q_1} = U^{q_2} = (TU)^{q_3} = 1$. Then the mapping $C_1 \to T$, $C_2 \to U$, $C_3 \to (TU)^{-1}$, $C_i \to 1$ $i \geq 3$. preserves relations and hence defines a non-trivial homomorphism of G onto a finite group.

We obtain Q as a group of 2x2 matrices over a finite field (Galois field) with p^n elements (see [10]). So let $G\,F\,(p^n)$ be the field with p^n elements (p is a prime and n we chose later). Some facts we need are:

(i) The multiplicative group of $G\,F\,(p^n)$ is cyclic of order p^n-1 ; thus for any divisor m of p^n-1, there is an element $a \varepsilon G\,F\,(p^n)$ whose order (o i a) is m.

(ii) Given $a, b, c, d, \varepsilon G\,F\,(p^n)$ such that $a\,d - b\,c = 1$ and $\lambda \varepsilon G\,F\,(p^n)$ $\lambda + \lambda = a+d \neq \pm 2$ then $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has two distinct eigenvalues λ and $\frac{1}{\lambda}$ in $G\,F\,(p^n)$. This is simply because

$$(\chi-\lambda)(\chi-\frac{1}{\lambda}) = \chi^2-(\lambda+\frac{1}{\lambda}) + 1 = \chi^2 - (a+d)2+(ad-bc) = \det$$
$$(\chi-A) \qquad\qquad (\lambda \neq \frac{1}{\lambda} \text{ since } a+d \neq \pm 2)$$

(iii) Given a,b,c,d as in (ii) then

$o\begin{pmatrix} a & b \\ c & d \end{pmatrix} = o\,(\lambda)$. This is because there is a matrix

C such that $C\,A\,C^{-1} = \begin{pmatrix} \lambda & o \\ o & \frac{1}{\lambda} \end{pmatrix}$ and clearly $o(\Lambda) = o(C\,A\,C^{-1} = o(\lambda)$.

Now to get Q and elements T and U we first consider the case where none of q_1, q_2, q_3 is 2.

Let p be a prime which doesn't divide $q_1q_2q_3$. Then p is relatively prime to $q_1q_2q_3$; so $\exists$ n such that $p^n \equiv 1 \bmod q_1q_2q_3$. Thus, each of $q_1q_2q_3$ divides p^n-1. So by (i) there are elements $\lambda, \mu, \nu \in GF(p^n)$ such that $o(\lambda) = q_1$, $o(\mu) = q_2$, and $o(\nu) = q_3$.

Now, let $T = \begin{pmatrix} \lambda + \frac{1}{\lambda} & 1 \\ -1 & o \end{pmatrix}$, by (iii) $o(T) = q_1$.

Let $U = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. We wish to determine x,y,z,w so that $o(U) = q_2$ and $o(TU) = q_3$. By (iii) above, it suffices to get

(a) $\det U = 1$ (c) $\det (TU) = 1$

(b) $\text{Trace } U = \mu + \frac{1}{\mu}$ (d) $\text{trace } (TU) = \nu + \frac{1}{v}$

Substitute x,y,z,w in (a), (b), (c), (d), then we get:

(a) $xw - yz = 1$ (c) follows from (a) and fact that $\det T = 1$

(b) $x+w = \mu + \frac{1}{\mu}$ (d) $(\lambda + \frac{1}{\lambda})x + z = y = v + \frac{1}{v}$.

From (b) $w = \mu + \frac{1}{\mu} - x$

From (d) $y = (\lambda + \frac{1}{\lambda})\, x + z = (v + \frac{1}{v})$

Then (a) becomes:

$$x(u + \frac{1}{u} - x) - [(\lambda + \frac{1}{\lambda})x + z - (v + \frac{1}{v})]\, z - 1 = o$$

$$x^2 - [(\mu + \frac{1}{\mu}) - (\lambda + \frac{1}{\lambda})z]x + z^2 - (v + \frac{1}{v})\, z + 1 = o$$

So´ x $= \dfrac{[\mu + \frac{1}{\mu} - (\lambda + \frac{1}{\lambda}) \pm [\mu + \frac{1}{\mu} - (\lambda + \frac{1}{\lambda}) z]^2 - 4(z^2 - (v + \frac{1}{v}) 3 + 1)}{2}$

The part under the radical is:

$[\mu + \frac{1}{\mu} - (\lambda + \frac{1}{\lambda}) z]^2 - 4[z^2 - (v + \frac{1}{\nu}) z + 1]$

$$= (m + \frac{1}{\mu})^2 - 2(\mu + \frac{1}{\mu})(\lambda + \frac{1}{\lambda}) z + (\lambda + \frac{1}{\lambda})^2 z^2$$

$$- 4z^2 + 4(v + \frac{1}{\nu}) z - 4$$

$$= (\mu - \frac{1}{\mu})^2 - [2(\mu + \frac{1}{\mu})(\lambda + \frac{1}{\lambda}) - 4(v + \frac{1}{v})] z + (\nu - \frac{1}{\nu})^2 z^2$$

If $2(\mu + \frac{1}{\mu})(\lambda + \frac{1}{\lambda}) - 4(v + \frac{1}{v}) \neq o$

If $2(\mu + \frac{1}{\mu})(\lambda + \frac{1}{\lambda}) - 4(v + \frac{1}{v}) \neq 0$ then z can be chosen as $2(\lambda - \frac{1}{\lambda})^{-1}$

Then z can be chosen as $\dfrac{(\mu - \frac{1}{\mu})^2}{2(\mu + \frac{1}{\mu})(\lambda + \frac{1}{\lambda}) - 4(v + \frac{1}{v})}$

In Case 1. of $q_1 q_2 q_3$ is 2 (say $q_3 = 2$) then we may assume q_1, q_2, are odd. (otherwise $G/G' \neq 1$).

Then choose n such that $2^n = 1 \bmod q_1 q_2$. Thus there are elements λ, μ ε G F (2^n) with $o(\lambda) = q_1$ $o(\mu) = q_2$.

Put

$$T = \begin{pmatrix} \lambda + {}^1/_\lambda & \\ 1 & o \end{pmatrix} \quad v = \begin{pmatrix} o & 1 \\ 1 & \mu + {}^1/_\mu \end{pmatrix}$$

Then $o(T) = q_1$ $o(v) = q_2$, and T U has the form $\begin{pmatrix} 1 & \delta \\ o & 1 \end{pmatrix}$ so $(TU)^2 =$

$\begin{pmatrix} 1 & 2\delta \\ o & 1 \end{pmatrix} = 1$ (since $2\delta = o$)

So Q E D for Lemma 6.

Corollary 4.7 Fuchsian groups are residaully finite.

Proof: The fundamental group of a closed 2-manifold is residually finite, and a finitely generated group which contains a residually finite subgroup of finite index is residually finite.

Theorem 4.8 Let G be a Fuchsian group of the form (1) or (2). Suppose that each $q_i \geq 2$ and let ℓ be the least common multiple $q_1, ---, q_k$. Suppose H is any subgroup of finite index in G which is isomorphic to the fundamental group of a closed 2-manifold S. Then ℓ divides the index n of H in G and $\chi(S)$.

(χ = Euler characteristic) is a positive multiple of

$$\ell\ (2-2g-\sum_{1}^{k}\ (1-\frac{1}{q_i})\ )\text{ in case (1) or of}$$

$$\ell\ (2-g-\sum_{1}^{k}(1-\frac{1}{q_i})\ )\text{ in case (2)}$$

Proof: Let χ, $\tilde{\chi}$, χ_1, etc. be as in Theorem 4.3. Then H is the fundamental group of a closed 2-manifold S if and only if χ_1 is a closed 2-manifold (homeomorphic to S). But if χ_1 is a 2-manifold, then the caps E_{ij}, added to $\tilde{F}$ to form χ_1 must be 2-cells. Thus, each Y_{ij} is the universal cover of E_i (hence a q_i-sheeted cover). Thus q_i divides n and $p^{-1}(E_i)$ has n/q_i components. Since each q_i divides n then surely ℓ divides n. Now χ_1 is obtained by adding $\sum_{1}^{k} \frac{n}{q_i}$ 2-cells to $\tilde{F}$.

So, $\chi(S) = \chi(\chi_1) = \chi(\tilde{F}) + \sum_1^k \frac{n}{q_i}$. But $\tilde{F}$ is an n-sheeted cover of F so $\chi(\tilde{F}) = n\chi(F)$

By construction $\chi(\tilde{F}) = \begin{cases} 2-2g-k & \text{case (1)} \\ 2-g-k & \text{case (2)} \end{cases}$

Putting these together we get:

$$\chi(S) = \begin{cases} n(2-2g-k+\sum_1^k \frac{1}{q_i}) & \text{case (2)} \\ n(2-g-k+\sum_1^k \frac{1}{q_i}) \end{cases}$$

$$= \begin{cases} n \; 2-2g-\sum_1^k (1-\frac{1}{q_i})\} \\ n \; 2-g-\sum_1^k (1-\frac{1}{q_i})\} \end{cases}$$

since ℓ divide n the conclusion follows.

<u>Corollary 4.9</u> A Fuchsian group G is finite if and only

case(1) g = o and either $k \leq 2$ or k = 3 and $\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} > 1$

case(2) either g = 1 and k = 1 and $q_1 = 2$ or

g = o and either $k \leq 2$ or k = 3 and $\frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} > 1$

<u>Proof</u>: By Theorem 4.4, we have a subgroup H of finite index which is isomorphic to the fundamental group of a closed 2-manifold S. Clearly G is finite if and only if H is finite. But H is finite if and only if $\chi(S) > o$. The corollary then follows from Theorem 4.8.

Now, for some application to 3-manifold :

<u>Def.:</u> <u>A closed Seifert fibert 3-manifold</u> is a 3-manifold obtained as follows:

Let M_1 be a fiber bundle over a closed 2-manifold S with fiber a circle. Let J_1, --- , J_k be disjoint fibers and U_1, ---, U_k be disjoint regular neighborhood of J_1, --- J_k respectively. Then $M_1 - \cup \text{ Int } \bar{U}_i$ has k tori in its boundary. Identify the boundary of a solid tories ($S^1 \times B^2$) to each of these tori by an arbitrary hormeomorphism h_i. The resulting 3-manifold

$$M = (M_1 - \cup \text{ Int } \bar{U}_i) \cup_{h_1} S^1 \times B^2 \cup_{h_2} S^1 \times B^2 \cup --- \cup_{h_k} S^1 \times B^2$$

is called a Seifert fibered 3-manifold (with k-singular fibers).

It is well known that the fundamental group of a closed Seifert fiber space has a presentation of one of the following forms:

(3) $\pi_1(M) = \langle a_1, b_1, ---, a_g, b_g, c_1, ---, c_k, z :$

$a_i z a_i^{-1} = z^{E_i}$, $b_i z b_i^{-1} = z^{\partial_i}$ $1 \leq i \leq g$,

$c_1 --- c_k \; z^n = [a_1, b_1] --- [a_g, b_g]$, $z^{P_i} C_i^{q_i} = 1$,

$1 \leq i \leq k$

where each E_i, $\partial_i = \pm 1$, and $(P_i, q_i) = 1$

$$(4) \qquad \pi_1(M) = \langle a_1, \ldots, a_g, c_1, \ldots c_k, z;\ a_i z a_i^{-1} = z^{E_i}$$

$$(1 \leq i \leq g), \quad c_1 \cdots c_k z^n = a_1^2 \cdots a_g^2, \ z^{p_i} c_i^{q_i} = 1$$

$$(1 \leq i \leq k)$$

where each $E_i = \pm 1$ and $(p_i, q_i) = 1$

Case (3) occurs when S is a closed orientable surface of genus g and case (4) occurs when S is the connected sum of g projective planes. When we take k = o (3) and (4) reduce to presentation of the fundamental group of an S^1-fiber bundle over closed orientable or closed non-orientable surface, respectively.

In any case, the cyclic subgroup Z of $\pi_1(M)$ generated by z is normal in $\pi_1(M)$ and the quotient group $\pi_1(M)/Z$ is a Fuchsian group of type (1) in case (3) or of type (2) in case (4).

Theorem 4.10: Let M be a closed Seifert fibered 3-manifold. Then some finite sheeted cover of M is an S^1 bundle over a closed 2-manifold.

Proof: As above, we have an exact sequence:

$$1 \to Z \to \pi_1(M) \xrightarrow{\mu} G \to 1$$

where G is a Fuchsian group. By Theorem 4.4 G contains a subgroup H of finite index with $H \cong \pi_1(S)$ for some closed 2-manifold S. Let $k = \mu^{-1}(H)$. Let $P : \tilde{M} \to M$ be the finite sheeted covering $P_* \quad \pi_1(\tilde{M}) = k$. Then we have an exact sequence:

$$1 \to Z \to \pi_1(M) \to H \to 1$$

if $H = 1$ (i.e. $S = S^2$) then $\pi_1(M) = Z$; so either $\tilde{M} = S^2 \times S^1$, or $\tilde{M}$ is doubly covered by $S^2 \times S^1$. If $H \neq 1$, then by Theorem 1 of [4] ["Hempel & Jaco, 3-manifolds which fiber over a surface"] $\tilde{M}$ is an S^1 bundle over a closed 2-manifold.

<u>Theorem 4.11</u> If M is a closed Seifert fibered 3-manifold, then $\pi_1(M)$ is residually finite.

<u>Proof</u>: By theorem 4.10, it sufficies to prove the theorem in Case M is an S^1-bundle over a closed 2-manifold. Hence $\pi_1(M)$ has a presentation of form (3) or (4) with k = o. Now the cyclic group Z generated by z is normal in $\pi_1(M)$ and hence is central in a subgroup H of index at most two in $\pi_1(M)$. By applying Theorem 1 of [4] again we see that H has a presentation of form (3) or (4) with k = o and with $E_i = \partial_i = +1$. Thus it suffices to show the group.

$$(3^1) \quad H = < a_1, b_1, ---, a_g, b_g, z = a_i z\, a_i^{-1} = z,\ b_i\ z\ b_i^{-1} = z$$
$$[a_1, b_1,] --- [a_n, b_n] = z^n >$$
$$(4^1) \quad H = < a_1, ---, a_g, z : a_i z\, a_i^{-1} = z,\ a_1{}^2 --- a_g{}^2 = z^n >$$

are residually finite.

Hence, we must show that for $1 \neq x \varepsilon H$ there is a subgroup K of finite index in H which doesn't contain x. If $x \varepsilon Z = g_p(z)$, this follows from the fact that $H/_Z$ is residually finite.

If $x \varepsilon Z$, then in case (3') x maps non-trivially to the group

$$L = \langle a_1, b_1, z \; ; \; a_1 z a_1^{-1} = z, \; b_1 z b_1^{-1} = z, \; [a_1, b_1] = z^n \rangle$$

But from the presentation of L, we see that [L,L] lies in the center of L ; hence $\left([L,L],L\right) = 1$. Thus L is nilpotent and hence residually finite, and the existence of k follows. In case (4') $x \varepsilon [H,H]$ so the theorem follows, too.

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