K-Theory and the Hopf Invariant

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Abstract

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The determination of which spheres, $S^n$, posses the structure of an $H$-space is an important question in Algebraic Topology.

This thesis will discuss a proof of

**Theorem (Adams):** $S^n$ has an $H$-structure if and only if $n = 0, 1, 3$ or $7$.

One defines an invariant called the Hopf Invariant which allows one to prove

**Theorem (Adams):** There is a map $f: S^{2n-1} \to S^n$ of Hopf Invariant $1$ if and only if $n = 1, 2, 4$, or $8$.

The proof of this theorem uses the $K$-theory methods of Adams and Atiyah.
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III. Proof of the Main Theorem
I. Introduction

An important classical question in Algebraic Topology is to determine which spheres, $S^n$, possess the structure of an H-space. This thesis will discuss a proof of

**Theorem 1 (Adams) [1]:** $S^n$ has an H-structure if and only if $n = 0, 1, 3,$ or $7$.

The proof proceeds as follows. First, one associates to each homotopy class of maps $\alpha: S^{n-1} \times S^{n-1} \to S^{n-1}$, a map $h(\alpha): S^{2n-1} \to S^n$ by a construction of H. Hopf [7]. Then for maps $f: S^{2n-1} \to S^n$, one defines an invariant $H(f) \in \mathbb{Z}$, called the Hopf Invariant of $f$. One has that if $\alpha$ is an H-structure that $H(h(\alpha)) = \pm 1$. Then

**Theorem 2 (Adams):** There is a map $f: S^{2n-1} \to S^n$ of Hopf Invariant $1$ if and only if $n = 1, 2, 4$ or $8$.

The original proof by Adams of this theorem [1] uses secondary cohomology operations. The proof in this thesis uses the K-theory methods of Adams and Atiyah in [3].

**The Hopf Invariant**

Let $\alpha: S^{2n-1} \to S^n$, for $n > 1$. Define $C_\alpha = S^n \cup_\alpha D^{2n}$, $C_\alpha$. 
is the mapping cone of $\alpha$ obtained by attaching a $2n$-cell to $S^n$ by the map $\alpha$. This CW complex has the following cohomology:

$$H^i(C_\alpha; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & i = 0 \\ \mathbb{Z}, & i = n \\ \mathbb{Z}, & i = 2n \\ 0, & \text{otherwise} \end{cases}$$

The generators are $x$ and $y$ where $\dim x = n$ and $\dim y = 2n$. Then $x^2 = H(\alpha)y$ for some integer $H(\alpha)$. $H(\alpha)$ is determined up to sign by the homotopy class of $\alpha$ and is called the Hopf Invariant of $\alpha$.

Now given a map $\alpha: S^{n-1} \times S^{n-1} \rightarrow S^n$, we will construct a map $h(\alpha): S^{2n-1} \rightarrow S^n$ and determine its Hopf Invariant. First we need some preliminaries:

**Notation:** The **join** of two spaces $X$ and $Y$ denoted by $X \times Y$ is the identification space of $X \times [0,1] \times Y$ with the elements denoted by $\langle x, t, y \rangle$ and identifications $\langle x, 0, y \rangle = \langle x', 0, y \rangle$ and $\langle x, 1, y \rangle = \langle x, 1, y' \rangle$ for $x, x' \in X$ and $y, y' \in Y$.

The **suspension** $SZ$ of a space $Z$ is the identification space of $Z \times [0,1]$ with the elements denoted by $\langle z, t \rangle$ and identifications $\langle z, 0 \rangle = \langle z', 0 \rangle$ and $\langle z, 1 \rangle = \langle z', 1 \rangle$ for $z, z' \in Z$. 
Lemma [8]: A homeomorphism $\psi: S^{n-1} \times S^{m-1} \to S^{n+m-1}$ is given by $\psi(x,t,y) = \cos(\pi t)y + i \sin(\pi t)x$.

Proof: $\psi$ is a continuous bijection; and since it is defined on compact, Hausdorff spaces, then $\psi$ is a homeomorphism.

The Hopf construction $h$ assigns to each map $\alpha: X \times Y \to Z$, a map $h(\alpha): X \times Y \to SZ$ defined by $h(\alpha)(x,t,y) = \langle \alpha(x,y), t \rangle$.

If $\alpha$ preserves base points, then so does $h(\alpha)$.

Proposition: The homotopy class of $h(\alpha)$ depends only on the homotopy class of $\alpha$.

Proof: Let $H: X \times Y \times I \to Z$ be a homotopy with $H(x,y,0) = \alpha_0(x,y)$ and $H(x,y,1) = \alpha(x,y)$ for $x \in X$, $y \in Y$. Then $H$ defines a map $X \times I \times Y \times I \to Z \times I$ by $(x,t,y,s) \mapsto (H(x,y,s), t)$. Since $I$ is compact, then $(X \times Y) \times I$ has the quotient topology, and the function that assigns $h(\alpha)(x,t,y)$ to $(\langle x,t,y \rangle, s)$ is continuous.

The following theorem relates the bidegree of $\alpha: S^{n-1} \times S^{n-1} \to S^{n-1}$ to the Hopf Invariant of $h(\alpha): S^{2n-1} \to S^n$ for even $n$.

Theorem: If $\alpha: S^{n-1} \times S^{n-1} \to S^{n-1}$ is a map of bidegree $(d_1, d_2)$, then the Hopf Invariant of $h(\alpha)$ is $d_1d_2$. 
This theorem follows by a diagram chase. We refer the reader to [8] for a proof.

**Corollary:** If \( q: S^{n-1} \times S^{n-1} \) is an H-structure, then \( H(h(q)) = \pm 1 \).
II. **K-Theory [5]**

For a space $X$, one can define the ring of stable isomorphism classes of virtual bundles, $K(X)$. One then uses $K$ to construct a cohomology theory. This cohomology theory satisfies all the axioms of Eilenberg-Steenrod [6], except the dimension axiom.

If $X$ is a pointed space, one has the notation of reduced groups $\tilde{K}(X)$ which is given as follows: For $x_0 \in X$, there is an imbedding $i: x_0 \to X$ which induces a ring homomorphism $i': K(X) \to K(x_0)$. The kernel of $i'$ is defined to be $\tilde{K}(X)$. For any integer $n \geq 0$ one defines:

1. $\tilde{K}^{-n}(X) = \tilde{K}(S^nX)$
2. $\tilde{K}^0(X) = \tilde{K}(X)$,

where $S^nX$ is the $n$-fold suspension of $X$.

To define $K^n(X)$, for $n > 0$, one uses induction on the Bott Periodicity Theorem: There exists an $\eta \in K^2(S^2)$ so that $\eta \cup (\_): K^n(X) \to K^{n-2}(X)$ is an isomorphism.

For $X$ consisting of a single point one has the following:

$$K^n(X) = \begin{cases} \mathbb{Z}, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

By Bott Periodicity, one can define $K^*(X) = K^0(X) \oplus K^1(X)$ and similarly for $\tilde{K}^*(X)$. Then $K^*(X)$ is a ring under tensor product
and is graded by $\mathbb{Z}_2$ (i.e., $K^0(X) \cdot K^1(X) \subset K^1(X)$ and $K^1(X) \cdot K^1(X) \subset K^0(X)$).

Letting $X^p$ be the $p$-th skeleton of $X$, one filters $K^n(X)$ by defining $K^n_p(X) = \text{Kernel}[K^n(X) \to K^n(X^{p-1})]$. Then \{K^n_p\} is a decreasing filtration in $K(X)$.

The Adams Operations

There exist operations $\psi^k: K(X) \to K(X)$, one for each $k \in \mathbb{Z}$, which preserve the filtration on $K$. They are called the Adams Operations and have the following properties:

1. $\psi^k: K(X) \to K(X)$ is a ring homomorphism
2. $\psi^k(x) = k^n x \mod K_{2n}$ for $x \in K_{2n}$.
3. $\psi^p(x) = x^p \mod p$ for $p$ prime.
4. $\psi^k \psi^\ell = \psi^{k \ell}$ for all $k, \ell$.

A further result which is useful in computations is the

Integrality Theorem of Atiyah [4]: Suppose $X$ has no torsion, and let $x \in K^i_{2q}(X)$. Then there exist elements $x_i \in K_{2q+2i(p-1)}(X)$ (i = 0, 1, ..., q) such that $\psi^p(x) = \sum_{i=0}^{q} p^{q-i} x_i$ where $p$ is prime. Moreover, one can choose $x_i \equiv x_i p$. 
The Chern Character

The Chern Character, denoted by \( \text{ch} \), is a map from K-theory into cohomology theory with rational coefficients; that is, \( \text{ch}: K^*(X) \to H^*(X; \mathbb{Q}) \). It induces a ring homomorphism

\[
\text{ch}: K(X) \to H^{ev}(X; \mathbb{Q}),
\]

with the property that \( \text{ch} \) takes a generator of \( K(S^{2k}) \) monomorphically onto a generator of the reduced integral cohomology of \( S^{2k} \) for all \( k \geq 1 \) [5]. If \( X \) is a space without torsion, one also gets the following results:

1. (Adams [2]: For \( x \in K(X) \),
   \[
   \text{ch}^k(x) = \sum_q k^q \text{ch}_q(x)
   \]
   where \( \text{ch}_x = \sum \text{ch}_q(x), \text{ch}_q(x) \in H^{2q}(X; \mathbb{Q}). \)

2. (Atiyah Integrality Theorem) [4]: If \( x \in K_{2q}(X) \) and \( p \) is a prime number, then \( p^t \text{ch}^q_{q+n}(x) \) is \( p \)-integral where \( t = \left\lfloor \frac{n}{p-1} \right\rfloor. \)

For \( a \in H^*(X; \mathbb{Q}) \) to be \( p \)-integral, we mean that

\[
a = \frac{b}{d}
\]

for \( h \in H^*(X; \mathbb{Z}) \) and \( d \) some integer prime to \( p \).
III. Proof of the Main Theorem

We now prove that there are maps $S^{2n-1} \to S^n$ of Hopf Invariant 1 only when $n = 1, 2, 4,$ or 8.

Remark: For spheres of odd dimension greater than 1, there is no map of Hopf Invariant 1. In fact, the Hopf Invariant is zero by anti-commutativity of the cup product. Therefore, one needs only to consider spheres of even dimension.

Consider a map $\alpha: S^{4n-1} \to S^{2n}$, and let $C_\alpha = S^{2n} \cup \partial^{4n}$ be the mapping cone of $\alpha$. The cohomology of $C_\alpha$ is

$$
H^i(C_\alpha; \mathbb{Z}) = \begin{cases} 
\mathbb{Z}, & i = 0 \\
\mathbb{Z}, & i = 2n \\
\mathbb{Z}, & i = 4n \\
0, & \text{otherwise.}
\end{cases}
$$

The generators in $\mathbb{H}^*(C_\alpha; \mathbb{Z})$ are $x$ and $y$ where $\dim x = 2n$ and $\dim y = 4n$. One has $\alpha: S^{4n-1} \to S^{2n}$ with Hopf Invariant $\pm 1$, by assumption.

There is a Puppe Sequence

$$S^{4n-1} \to S^{2n} \to C_\alpha \to SS^{4n-1} \to SS^{2n}.$$

This sequence induces the following exact sequence in reduced K-theory:
\[ \ldots \to \tilde{K}(S^{2n+1}) \to \tilde{K}(S^{4n}) \to \tilde{K}(C_\alpha) \to \tilde{K}(S^{2n}) \to \tilde{K}(S^{4n-1}) \to \ldots \]

where \( \tilde{K}(S^{2n+1}) = \tilde{K}(S^{4n-1}) = 0 \) so that

\[ 0 \to \tilde{K}(S^{4n}) \overset{\psi}{\to} \tilde{K}(C_\alpha) \overset{\varphi}{\to} \tilde{K}(S^{2n}) \to 0 \]

is an exact sequence.

Let \( \mu \) denote \( \psi(\beta) \) in \( K(C_\alpha) \) where \( \beta \) is a generator of \( K(S^{4n}) \cong \mathbb{Z} \) and let \( \xi \) denote any element of \( \tilde{K}(C_\alpha) \) such that \( \varphi(\xi) = \gamma \) where \( \gamma \) is a generator of \( \tilde{K}(S^{2n}) = \mathbb{Z} \). Then \( \tilde{K}(C_\alpha) = \mathbb{Z} \xi + \mathbb{Z} \mu \).

One must show that \( \xi^2 = \pm \mu \). The Chern Character induces a monomorphism.

\[ \text{ch: } \tilde{K}(S^{2n}) \to \tilde{H}^{ev}(S^{2n}; \mathbb{Z}) \subset \tilde{H}^{ev}(S^{2n}; \mathbb{Q}), \]

for \( n \geq 1 \), so that one has the following diagram:

\[ \begin{array}{cccccc}
0 & \to & \tilde{K}(S^{4n}) & \overset{\psi}{\to} & \tilde{K}(C_\alpha) & \overset{\varphi}{\to} & \tilde{K}(S^{2n}) & \to & 0 \\
& & \downarrow \text{ch} & & \downarrow \text{ch} & & \downarrow \text{ch} & & \\
0 & \to & \tilde{H}^{ev}(S^{4n}; \mathbb{Q}) & \to & \tilde{H}^{ev}(C_\alpha; \mathbb{Q}) & \to & \tilde{H}^{ev}(S^{2n}; \mathbb{Q}) & \to & 0.
\end{array} \]

\( \text{ch: } \tilde{K}(C_\alpha) \to \tilde{H}^{ev}(C_\alpha; \mathbb{Q}) \) is a monomorphism by the Five Lemma, since all the other vertical arrows are monomorphisms (by the Chern Character property for even dimensional spheres). Therefore,
\[
\text{ch } \xi = x + \text{higher dimensional terms}
\]
\[
\text{ch } \mu = y
\]
so that \( \xi^2 = \pm \mu \), since ch is a ring homomorphism.

From the nature of the Adams Operations in \( \mathbb{K}(s^{2n}) \), one has that

1. \( \psi^k(\xi) = k^n \xi + \lambda_{k\mu} \)
2. \( \psi^k(\mu) = k^{2n} \mu \)

In particular, for \( k = 2 \), one has that if \( \xi^2 = \pm \mu \) (which is the assumption of Hopf Invariant 1), then \( \lambda_2 \) is odd (since \( \psi^2(\xi) = \xi^2 \mod 2 \)).

Now calculating \( \psi^2 \psi^3(\xi) \) and \( \psi^3 \psi^2(\xi) \) one gets:

\[
\psi^2 \psi^3(\xi) = \psi^2(3^n \xi + \lambda_{3\mu})
= 2^n 3^n \xi + 3^n \lambda_{2\mu} + 2^{2n} \lambda_{3\mu}
\]
\[
\psi^3 \psi^2(\xi) = \psi^3(2^n \xi + \lambda_{2\mu})
= 2^n 3^n \xi + 2^n \lambda_{3\mu} + 2^{2n} \lambda_{2\mu}.
\]

Since \( \psi^k(\xi) = \xi^{\psi^k} \) for all \( k, \xi \), then \( \psi^2 \psi^3(\xi) = \psi^3 \psi^2(\xi) \) so that \( 3^n \lambda_2 + 2^{2n} \lambda_3 = 2^n \lambda_3 + 3^{2n} \lambda_2 \). Simplification yields \( 3^n (3^n - 1) \lambda_2 = 2^n (2^n - 1) \lambda_3 \). Since \( \lambda_2 \) is odd, \( 2^n | 3^n - 1 \). By elementary number theory [2], this can only occur when \( n = 1, 2, 4 \).
Bibliography


