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METRIC FUNCTION SPACES AND REFLECTED SPACES

by

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Abstract

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In this paper we first define what is meant by the term metric function space. Basically, a metric function space consists of a set of functions F and a metric ρ on F which satisfies certain axioms. For example, the L_p spaces and the L(p, q) spaces are metric function spaces.

For certain metric function spaces we can form what we will call the reflected space. Theorem 12 states that the reflected space to a metric function space is itself a metric function space. Theorem 13 shows that the reflected space to the reflected space of a metric function space is the original space. Theorem 14 gives a relation between a metric function space and its reflected space, namely, that a metric function space is absolutely continuous if and only if its reflected space has the truncation property.

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METRIC FUNCTION SPACES AND REFLECTED SPACES

Preliminary Definitions

If $f(x)$ is an extended real-valued measurable function defined on a measure space (χ, μ), then the *distribution function of f*, denoted by $m(f, y)$, is defined by $m(f, y) = \mu\{x:|f(x)| > y\}$ for $y \geq 0$. If A is a subset of $[0, \infty)$, then $|A|$ will denote the Lebesgue measure of A. The rearrangement of f, denoted by f^*, is the nonincreasing real-valued function defined on $(0, \infty)$ that is equimeasurable to $|f|$, i.e., for every $a > 0$, $\mu\{x:|f(x)| > a\} = |\{y:f^*(y) > a\}|$. (It can be shown that f^* always exists; in fact, $f^*(t) = \inf \{y > 0:m(f, y) < t\}$). A function is said to be *simple* if it can be expressed in the form $\sum_{i=1}^{n} \alpha_i \cdot \chi_{E_i}(x)$ where $\{E_1,..., E_n\}$ is a finite, disjoint class of measurable sets and $\{\alpha_1,... \alpha_n\}$ is a finite set of real numbers. If E is a set, then $\chi_E(x)$ will denote the characteristic function of E. The symbol ϕ will denote the empty set.

Metric Function Spaces and Reflected Spaces

Let S be the set of all extended real-valued measurable functions defined on a measure space (χ, μ); functions which agree almost everywhere are identified.

Definition: A metric function space (F, ρ) consists of a subset F of S and a function ρ from S to $[0, \infty]$ such that:

(a) f is in F if and only if $\rho(f) < \infty$.

(b) If f and g are in S and $|f(x)| \leq |g(x)|$ p.p., then $\rho(f) \leq \rho(g)$.

(c) If f and g are in S, then $\rho(f + g) \leq \rho(f) + \rho(g)$.

(d) If $\{f_n\}$ is a nondecreasing sequence of nonnegative functions in F such that $\{\rho(f_n)\}$ is bounded and if $f(x) = \lim_{n\to\infty} f_n(x)$ p.p., then f is in (F, ρ) and $\rho(f) = \lim_{n\to\infty} \rho(f_n)$.

(e) $\rho(f) = 0$ if and only if $f(x) = 0$ p.p.

(f) $\rho(f)$ depends only on f^*, i.e., if $f^*(t) = g^*(t)$, then $\rho(f) = \rho(g)$.

The function ρ, when considered as a function from F to $[0, \infty)$, is said to be a metric on F.

The following spaces are examples of metric function spaces:

1) The L_p spaces, $0 \leq p \leq \infty$

2) The L(p, q) spaces, $0 \leq p, q \leq \infty$ (See O'Neil, p. 85 f.f.)

3) The Orlicz spaces (See Krasnosel'skii and Rutickii, p. 67 f.f.).

Note: The idea of a metric function space is somewhat similar to that of a Banach function space. (See Zaanen, page 205). There are two main differences between these concepts. The first is that in metric function spaces we have a metric whereas in Banach function spaces we are dealing with a norm. The second difference is that metric function spaces have the property that the metric depends only on the rearrangement of a function. Another type of space similar to both of the above spaces is a normed Köthe space. (See Zaanen, page 441.)

The following are true for any metric function space (F, ρ):

1° If f and g are in F, then $(f + g)$ is in F.

2° If f is in F and n is a natural number, then $n \cdot f$ is in F.

3° If f is in F, then $\rho(f) = \rho(|f|)$. Hence, $|f|$ is in F.

Proof: Since f and $|f|$ have the same rearrangement, it follows from axiom (f) that $\rho(f) = \rho(|f|)$.

4° If f is in F and k is an integer, then $k \cdot f$ is in F.

Proof: If k is negative, then $(-k)|f|$ is in F. But $\rho(k \cdot f) = \rho(|k \cdot f|) = \rho((-k)|f|)$.

5° If f is in F and c is a real number, then $c \cdot f$ is in F.

Proof: There exists a natural number b such that $|c| \leq b$. Hence, $\rho(c \cdot f) \leq \rho(b \cdot f) < \infty$.

6° If f is in F and k is an integer, then $\rho(k \cdot f) \leq |k| \rho(f)$.

Proof: $\rho(k \cdot f) = \rho(|k| \cdot f) = \rho(f + \ldots + f)$ ($|k|$ times) $\leq \rho(f) + \ldots + \rho(f)$ ($|k|$ times) $= |k| \rho(f)$.

7° If f is in F and $\frac{1}{k}$ is an integer, then $\rho(k \cdot f) \geq |k| \cdot \rho(f)$.

Proof: $\rho(\frac{1}{k} \cdot kf) \leq |\frac{1}{k}| \rho(k \cdot f)$

Therefore, $|k| \cdot \rho(f) \leq \rho(k \cdot f)$.

8° If f and g are in F, then $\rho(f - g) \geq |\rho(f) - \rho(g)|$.

Proof: Replacing f in axiom (c) by $(f - g)$ we get that $\rho(f) \leq \rho(f - g) + \rho(g)$.

Hence $\rho(f) - \rho(g) \leq \rho(f - g)$. Similarly, replacing g in axiom (c) by $(g - f)$ gives us that $\rho(g) - \rho(f)$

$\leq \rho(f) - \rho(g)$.

9° If $\{f_n\}$ is a sequence of nonnegative functions in (F, ρ) such that $\{\rho(f_n)\}$ is bounded and if $f(x) = \lim_{n\to\infty} f_n(x)$ exists p.p., then $\rho(f) \leq \underline{\lim}_{n\to\infty} \rho(f_n)$.

Proof: Let $g_n(x) = \inf \{f_n(x), f_{n+1}(x), \ldots\}$. Then $g_1(x) \leq g_2(x) \leq \ldots$ and $\{\rho(g_n)\}$ is bounded since $g_n(x) \leq f_n(x)$ for every x and every n. Furthermore, $\lim_{n\to\infty} g_n(x) = f(x)$ p.p. Therefore, by axiom (a), $\rho(f) = \lim_{n\to\infty} \rho(g_n)$. But for every n $\rho(g_n) \leq \rho(f_n)$. Hence, $\rho(f) \leq \underline{\lim}_{n\to\infty} \rho(f_n)$.

Note: It follows from 1° and 5° that a metric function space is a real vector space.

The major portion of this paper deals with what we will define as the reflected space to a given metric function space. Since distribution functions play a key role in the concept of reflected space, we will first make several observations about distribution functions and metric function spaces containing distribution functions as elements.

Proposition 1: Let $\{f_n\}$ be a nondecreasing sequence of nonnegative measurable functions defined on a measure space (χ, μ). If for every x $f_n(x) \to f(x)$, then $m(f_n, y) \to m(f, y)$ for every y.

Proof: Let y be in $[0, \infty)$ Let $A = \{x: |f(x)| > y\}$ and let $A_n = \{x: |f_n(x)| > y\}$. Since $|f_n(x)| \leq |f(x)|$ for every n, $A_n \subseteq A$ for every n. If x is in A, then $|f(x)| = b > y$. Let $\varepsilon = \frac{b - y}{2}$. Since $f_n(x) \to f(x)$, there exists an n such that $|f_n(x)| > b - \varepsilon > y$.

Hence, x is in A_n for some n. Thus, $A = \bigcup_{n=1}^{\infty} A_n$. Furthermore, since $\{f_n\}$ is a nondecreasing sequence of nonnegative functions, $A_n \subseteq A_{n+1}$ for every n. Hence, $\mu(A) = \lim_{n\to\infty} \mu(A_n)$. Therefore, $m(f_n, y) \to m(f, y)$.

Corollary 2: Let $\{f_n\}$ be a nondecreasing sequence of nonnegative measurable functions defined on a measure space (χ, μ). If $f_n(x) \to f(x)$ p.p., then $m(f_n, y) \to m(f, y)$ for every y.

Proof: Let $g(x) = \lim_{n\to\infty} f_n(x)$. Then $g(x) = f(x)$ p.p.
Hence, for every y $m(g, y) = m(f, y)$. By Proposition 1, $m(f_n, y) \to m(g, y)$. Therefore $m(f_n, y) \to m(f, y)$.

Proposition 3: Let $g(x) = k \cdot f(x)$ where k is a natural number and f is an extended real-valued measurable function defined on a measure space (χ, μ). If $m(f, y)$ is in the metric function space (F, ρ), then $\rho(m(g, y)) \leq k \cdot \rho(m(f, y))$.

Proof: Case 1: f is a nonnegative simple function. Let $f(x) = \sum_{i=1}^{n} \alpha_i \cdot \chi_{E_i}(x)$ where the E_i are pairwise disjoint and $\alpha_1 > \alpha_2 > \ldots > \alpha_n > 0$. Then $k \cdot f(x) = \sum_{i=1}^{n} k \cdot \alpha_i \cdot \chi_{E_i}(x)$. If we let $\alpha_{n+1} = 0$, then

$$m(f, y) = \begin{cases} \mu\left(\bigcup_{i=1}^{j-1} E_i\right) & \text{if } y \text{ is in } [\alpha_j, \alpha_{j-1}),\ j = 2, \ldots, n+1 \\ 0 & \text{if } y \geq \alpha_1 \end{cases}$$

and

$$m(k \cdot f, y) = \begin{cases} \mu\left(\bigcup_{i=1}^{j-1} E_i\right) & \text{if } y \text{ is in } [k \cdot \alpha_j, k \cdot \alpha_{j-1}),\ j = 2, \ldots, n+1 \\ 0 & \text{if } y \geq k \cdot \alpha_1 \end{cases}$$

Let

$$f_1(y) = \begin{cases} m(k \cdot f, y) \text{ if } y \text{ is in } [k \cdot \alpha_j, k \cdot \alpha_j + (\alpha_{j-1} - \alpha_j)), \\ \qquad j = 2, \ldots, n+1 \\ 0 \qquad \text{elsewhere} \end{cases}$$

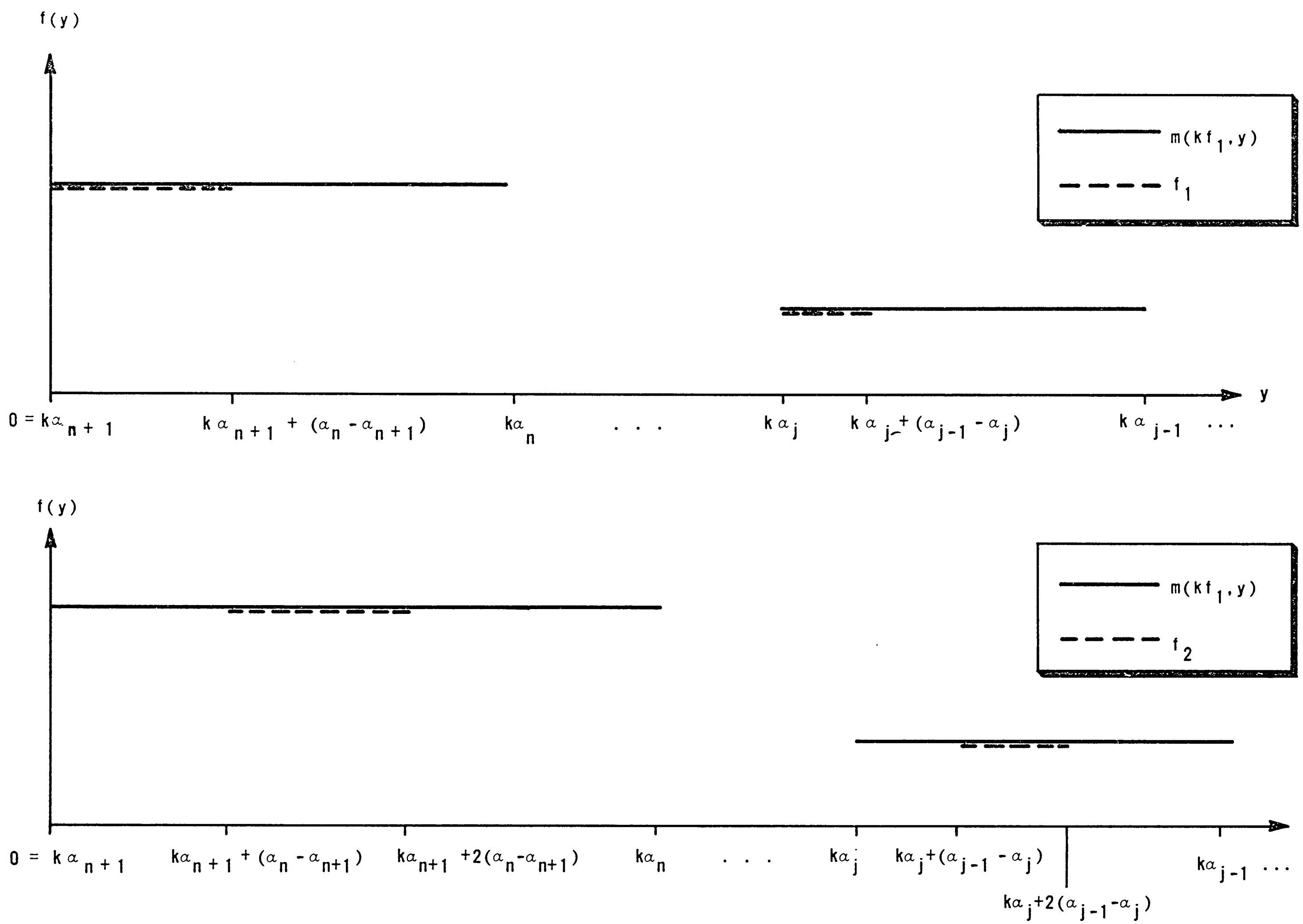
Let

$$f_2(y) = \begin{cases} m(k \cdot f, y) \text{ if } y \text{ is in } [k \cdot \alpha_j + (\alpha_{j-1} - \alpha_j), k \cdot \alpha_j \\ \qquad + 2(\alpha_{j-1} - \alpha_j)),\ j = 2, \ldots, n+1 \\ 0 \qquad \text{elsewhere} \end{cases}$$

Let

$$f_k(y) = \begin{cases} m(k \cdot f, y) \text{ if } y \text{ is in } [k \cdot \alpha_j + (k-1)(\alpha_{j-1} - \alpha_j), \\ \qquad k\ \alpha_{j-1}),\ j = 2, \ldots, n+1 \\ 0 \qquad \text{elsewhere} \end{cases}$$

(See Diagram A)



f(y)
y
m(kf$_1$,y)
f$_1$
0 = kα_{n+1}
kα_{n+1} + (α_n - α_{n+1})
kα_n
. . .
kα_j
kα_{j-}+(α_{j-1} - α_j)
kα_{j-1} . . .
f(y)
y
m(kf$_1$,y)
f$_2$
0 = kα_{n+1}
kα_{n+1}+(α_n - α_{n+1})
kα_{n+1} +2(α_n-α_{n+1})
kα_n
. . .
kα_j
kα_j+(α_{j-1} - α_j)
kα_j+2(α_{j-1}-α_j)
kα_{j-1} . . .

DIAGRAM A

Now for every i, i=1, ..., k, $f_i^*(y) = m(f, y)^*$. In addition $m(k \cdot f, y) = \sum_{i=1}^{k} f_i(y)$.

Therefore, $\rho(m(k \cdot f, y)) = \rho(\sum_{i=1}^{k} f_i) \leq \sum_{i=1}^{k} \rho(f_i) = k\, \rho(m(f, y))$.

Case 2: f is nonnegative p.p.

Since f is measurable, there exists a nondecreasing sequence of nonnegative simple functions $\{g_n\}$ such that $g_n(x) \to f(x)$. (See Halmos, page 85, Theorem B.) Now by Proposition 1, $m(k \cdot g_n, y) \to m(k \cdot f, y)$. By case 1, for every n, $\rho(m(k \cdot g_n, y) \leq k\, \rho(m(g_n, y))$. Since $0 \leq m(g_n, y) \leq m(f, y)$ for every n and every y, $k \cdot \rho(m(g_n, y)) \leq k \cdot \rho(m(f, y)) < \infty$. Hence, $\{\rho(m(k \cdot g_n, y))\}$ is bounded. Therefore, by axiom (d), $\rho(m(k \cdot f, y)) = \lim_{n\to\infty} \rho(m(k \cdot g_n, y))$. Hence we have

$$\rho(m(k \cdot f, y)) \leq \lim_{n\to\infty} k \cdot \rho(m(g_n, y)) \leq k \cdot \rho(m(f, y)).$$

Case 3: f is measurable.

Since $m(k \cdot f, y) = m(k \cdot |f|, y)$, it follows from case 2 that

$$\rho(m(k \cdot f, y)) = \rho(m(k \cdot |f|, y)) \leq k \cdot \rho(m(|f|, y)) = k \cdot \rho(m(f, y)).$$

<u>Corollary 4</u>: Let $g(x) = k \cdot f(x)$ where $k \geq 1$ and f is an extended real-valued measurable function defined on a measure space (χ, μ). If $m(f, y)$ is in the metric function space (F, ρ) and if n is a natural number, $n \geq k$, then $\rho(m(g, y)) \leq n \cdot \rho(m(f, y))$.

Proof: $|g(x)| \leq |n \cdot f(x)|$. Hence, $m(g, y) \leq m(n \cdot f, y)$. Therefore, $\rho(m(g, y)) \leq \rho(m(n \cdot f, y)) \leq n \cdot \rho(m(f, y))$ with the last inequality following from Proposition 3.

Corollary 5: Let $g(x) = \frac{1}{k} \cdot f(x)$ where k is a natural number and f is an extended real-valued measurable function defined on a measure space (χ, μ). If $m(f, y)$ is in the metric function space (F, ρ), then $\frac{1}{k} \cdot \rho(m(f, y)) \leq \rho(m(g, y))$.

Proof: By Proposition 3, $\rho(m(\frac{k \cdot f}{k}, y)) \leq k \cdot \rho(m(\frac{f}{k}, y))$. Therefore, $\frac{1}{k} \rho(m(f, y)) \leq \rho(m(g, y))$.

Corollary 6: Let $g(x) = \frac{1}{k} \cdot f(x)$ where $k \geq 1$ and f is an extended real-valued measurable function defined on a measure space (χ, μ). If n is a natural number, $n \geq k$, then $\rho(m(g, y)) \geq \frac{1}{n} \rho(m(f, y))$.

Proof: By Corollary 5, $\frac{1}{n} \rho(m(f, y)) \leq \rho(m(\frac{1}{n} \cdot f, y))$. But,

$\rho(m(\frac{1}{n} \cdot f, y)) \geq \rho(m(g, y))$. Hence, $\frac{1}{n} \rho(m(f, y)) \leq \rho(m(g, y))$.

An immediate consequence of the next proposition is the fact that for any measurable function f, $f^*(x) = m(m(f, y), x)$ p.p.

Proposition 7: If $f(x)$ is a nonincreasing, nonnegative extended real-valued measurable function defined on $[0, \infty)$ with Lebesgue measure, then $f(x) = m(m(f, y), x)$ p.p.

Proof: First assume that $f(x)$ is a step function. Then from the assumptions $f(x) = \sum_{i=1}^{n} \alpha_i \cdot \chi_{[a_{i-1}, a_i)}(x)$ p.p. where $0 < a_1 < a_2 < \ldots < a_n$ and $\alpha_1 > \alpha_2 > \ldots > \alpha_n > 0$. Let $a_0 = 0$ and let $\alpha_{n+1} = 0$. Then

$$m(f, y) = \begin{cases} a_i \text{ if } y \text{ is in } [\alpha_{i+1}, \alpha_i),\ i = 1, 2, \ldots n \\ 0 \quad \text{elsewhere} \end{cases}$$

and hence

$$m(m(f, y), x) = \begin{cases} \alpha_i & \text{if } x \text{ is in } [a_{i-1}, a_i),\ i = 1, \ldots, n \\ 0 & \text{elsewhere} \end{cases}$$

So $f(x) = m(m(f, y), x)$ p.p. for f being a step function.

If f is measurable, then there exists a nondecreasing sequence of nonnegative step functions $\{g_n\}$ such that $g_n(x) \to f(x)$. In addition, since $f(x)$ is nonincreasing, each g_n is nonincreasing. (See Halmos, page 85, Theorem B.) Hence, by Proposition 1, $m(g_n, y) \to m(f, y)$. Furthermore, since $g_n(x) \geq 0$ for every x and since $\{g_n\}$ is a nondecreasing sequence, $\{m(g_n, y)\}$ is a nondecreasing sequence. Hence, applying Proposition 1 again, we get that $m(m(g_n, y), x) \to m(m(f, y), x)$. Therefore, $g_n(x) \to m(m(f, y), x)$p.p. since $g_n(x) = m(m(g_n, y), x)$p.p. Hence, $f(x) = m(m(f, y), x)$p.p.

Now we can proceed with the main part of this paper. We begin by defining the reflected space to a given metric function space.

Definition: Let (F, ρ) be a metric function space where the domain of the functions in F is $[0, \infty)$ with Lebesgue measure. Let S be the set of all extended real-valued measurable functions defined on a measure space (χ, μ). Then the reflected space to (F, ρ) is the space $(\tilde{F}, \tau)$ where $\tilde{F} = \{f \in S : m(f, y) \text{ is in } F\}$ and $\tau(f) = \inf \{k > 0 : \rho(m(\frac{f}{k}, y)) \leq k\}$ where $\inf \phi = \infty$.

We will prove that the reflected space to a metric function space is also a metric function space. However, in order to do this we first need to derive some relations between ρ, the metric on the original space F, and the function τ on $\tilde{F}$.

Proposition 8: Let $(\tilde{F}, \tau)$ be the reflected space to (F, ρ) and let $\rho(m(f, y)) = L < \infty$.

i) If $L > 1$, then $\max(1, \frac{\sqrt{L}}{2}) \leq \tau(f) \leq L$.

ii) If $L < 1$, then $L \leq \tau(f) \leq \min(1, 2\sqrt{L})$.

iii) If $L = 1$, then $\tau(f) = 1$.

Proof: i) If $L > 1$, then since $\rho(m(\frac{f}{L}, y)) \leq \rho(m(f, y)) = L > 1$, it follows that $1 \leq \tau(f) \leq L$. Since $\sqrt{L} > 1$, there exists a positive integer n such that $\sqrt{L} \leq n \leq 2\sqrt{L}$. Hence, $\frac{1}{2\sqrt{L}} \leq \frac{1}{n} \leq \frac{1}{\sqrt{L}}$. But $\rho(m(\frac{2f}{\sqrt{L}}, y)) \geq \rho(m(\frac{f}{n}, y))$, and by Corollary 5 $\rho(m(\frac{f}{n}, y)) \geq \frac{1}{n}\rho(m(f, y))$. Hence, $\rho(m(\frac{2 \cdot f}{\sqrt{L}}, y)) \geq \frac{L}{n} \geq \frac{\sqrt{L}}{2}$. Therefore, $\tau(f) \geq \frac{\sqrt{L}}{2}$.
So $\max(1, \frac{\sqrt{L}}{2}) \leq \tau(f) \leq L$.

ii) If $L < 1$, then $\rho(m(\frac{f}{L}, y)) \geq \rho(m(f, y)) = L < 1$.
Hence, $L \leq \tau(f) \leq 1$. Since $\frac{1}{\sqrt{L}} > 1$, there exists a positive integer n such that $\frac{1}{2\sqrt{L}} \leq n \leq \frac{1}{\sqrt{L}}$. Thus, $\rho(m(\frac{f}{2\sqrt{L}}, y)) \leq \rho(m(n \cdot f, y))$. But by Proposition 3, $\rho(m(n \cdot f, y)) \leq n \cdot \rho(m(f, y))$. Hence, $\rho(m(\frac{f}{2\sqrt{L}}, y)) \leq n \cdot L \leq \sqrt{L} \leq 2\sqrt{L}$. So $L \leq \tau(f) \leq \min(1, 2\sqrt{L})$.

iii) If $L = 1$, then for every $\varepsilon > 0$, $\rho(m(\frac{f}{1-\varepsilon}, y)) \geq \rho(m(f, y)) = 1 > 1 - \varepsilon$. Hence, $\tau(f) \leq 1$ and for every $\varepsilon > 0$, $\tau(f) > 1 - \varepsilon$.
So $\tau(f) = 1$.

<u>Corollary</u> 9: Let $(\tilde{F}, \tau)$ be the reflected space to (F, ρ) and let $m(f, y)$ be in (F, ρ).

i) If $\tau(f) > 1$, then $\rho(m(f, y)) > 1$.

ii) If $\tau(f) < 1$, then $\rho(m(f, y)) < 1$.

Proof: i) By conclusion ii) of Proposition 8 we get that $\tau(f) > 1$ implies $\rho(m(f, y)) \geq 1$. But by conclusion iii) of Proposition 8, $\tau(f) > 1$ implies $\rho(m(f, y)) \neq 1$. Hence, if $\tau(f) > 1$, then $\rho(m(f, y)) > 1$.

ii) Similarly, by using conclusions i) and iii) of Proposition 8, we get that $\tau(f) < 1$ implies $\rho(m(f, y)) < 1$.

<u>Corollary 10</u>: Let (H, σ) be the reflected space to (G, τ), the reflected space to (F, ρ), where the domain of the functions in F, G, and H is $[0, \infty)$ with Lebesgue measure. Let $\rho(f) = L < \infty$.

i) If $L > 1$, then $\max(1, \frac{\sqrt[4]{L}}{2\sqrt{2}}) \leq \sigma(f) \leq L$.

ii) If $L < 1$, then $L \leq \sigma(f) \leq \min(1, 2\sqrt{2} \cdot \sqrt[4]{L})$.

iii) If $L = 1$, then $\sigma(f) = 1$.

Proof: We know from axiom (f) that $\rho(f) = \rho(f^*)$. Now from Proposition 7 it follows that $f^*(x) = m(m(f, y), x)$ p.p. Hence $\rho(m(m(f, y), x)) = L$. By applying Proposition 8 twice we get the desired inequalities.

<u>Proposition 11</u>: Let $(\tilde{F}, \tau)$ be the reflected space to (F, ρ). If $m(f, y)$ is in (F, ρ), then $\tau(f) = 1$ implies $\rho(m(f, y)) \leq 1$.

Proof: If $\tau(f) = 1$, then for every positive integer n, $\rho(m(\frac{f}{1 + \frac{1}{n}}, y)) \leq 1 + \frac{1}{n}$. But since $\left\{\frac{|f|}{1 + \frac{1}{n}}\right\}$ is a nondecreasing sequence of nonnegative functions such that $\frac{|f(x)|}{1 + \frac{1}{n}} \to |f(x)|$, it follows from Proposition 1 that $m(\frac{|f|}{1 + \frac{1}{n}}, y)$ is a nondecreasing sequence of nonnegative functions such that $m(\frac{|f|}{1 + \frac{1}{n}}, y) \to m(|f|, y)$. Since $\left\{\rho(m(\frac{|f|}{1 + \frac{1}{n}}, y))\right\}$ is bounded by 2, it follows from axiom (d) that $\rho(m(|f|, y)) = \lim_{n\to\infty} \rho(m(\frac{|f|}{1 + \frac{1}{n}}, y)) \leq \lim_{n\to\infty} (1 + \frac{1}{n}) = 1$. So $\rho(m(f, y)) \leq 1$.

We are now finally ready to state and prove the main result of this paper:

<u>Theorem 12</u>: If $(\tilde{F}, \tau)$ is the reflected space to the metric function space (F, ρ), then $(\tilde{F}, \tau)$ is a metric function space.

Proof: We must show that $(\tilde{F}, \tau)$ satisfies all of the axioms for a metric function space.

To show $(\tilde{F}, \tau)$ satisfies axiom (a), first let f be in $\tilde{F}$. Then $m(f, y)$ is in F and so $\rho(m(f, y)) = L < \infty$. But, by Proposition 8, if $L \geq 1$, $\tau(f) \leq L$ and if $L < 1$, $\tau(f) \leq \min(1, 2\sqrt{L})$. So if f is in $\tilde{F}$, then $\tau(f) < \infty$. On the other hand, if $\tau(f) < \infty$, then there exists a real number $k > 0$ such that $\rho(m(\frac{f}{k}, y)) \leq k$. Let n be a natural number such that $n \geq k$. Then $\rho(m(\frac{f}{n}, y)) \leq \rho(m(\frac{f}{k}, y)) \leq k$. Hence, from Corollary 5, it follows that $\rho(m(f, y)) \leq n \cdot k < \infty$. Therefore, $m(f, y)$ is in F and hence f is in $\tilde{F}$. So $(\tilde{F}, \tau)$ satisfies axiom (a).

Now if $|f(x)| \leq |g(x)|$ p.p., then for every $k > 0$, $\frac{|f(x)|}{k} \leq \frac{|g(x)|}{k}$ p.p. It follows that $m(\frac{f}{k}, y) \leq m(\frac{g}{k}, y)$ for every $k > 0$ and hence $\rho(m(\frac{f}{k}, y)) \leq \rho(m(\frac{g}{k}, y))$ for every $k > 0$. Therefore, if $\rho(m(\frac{g}{k}, y)) \leq k$, then $\rho(m(\frac{f}{k}, y)) \leq k$. Hence, $\tau(f) \leq \tau(g)$ and so $(\tilde{F}, \tau)$ satisfies axiom (b).

To prove $(\tilde{F}, \tau)$ satisfies axiom (c) we first prove the following:

Claim: If $a + b = 1$, $a \geq 0$, $b \geq 0$, then $m(f + g, y) \leq m(f, a \cdot y) + m(g, b \cdot y)$.

Proof of claim: $\{x: |f(x) + g(x)| > y\} \subseteq \{x: |f(x)| + |g(x)| > y\} \subseteq \{x: |f(x)| > a \cdot y\} \cup \{x: |g(x)| > b \cdot y\}$ since $a + b = 1$. Hence, $m(f + g, y) \leq m(f, a \cdot y) + m(g, b \cdot y)$ and so we have proven the claim.

Now let $K > 0$ and $L > 0$. Then $m(\frac{f + g}{K + L}, y) = m(\frac{K \cdot \frac{f}{K} + L \cdot \frac{g}{L}}{K + L}, y) = m(\frac{K}{K + L} \cdot \frac{f}{K} + \frac{L}{K + L} \cdot \frac{g}{L}, y)$. Applying the claim (with $a = \frac{K}{K + L}$, $b = \frac{L}{K + L}$) it follows that $m(\frac{K}{K + L} \cdot \frac{f}{K} + \frac{L}{K + L} \cdot \frac{g}{L}, y) \leq m(\frac{K}{K + L} \cdot \frac{f}{K}, \frac{K}{K + L} \cdot y) + m(\frac{L}{K + L} \cdot \frac{g}{L}, \frac{L}{K + L} \cdot y)$. Now for every $a > 0$ $m(f, y) = m(a \cdot f, a \cdot y)$ since $\{x: |f(x)| > y\} = \{x: |a \cdot f(x)| > a \cdot y\}$. Hence $m(\frac{f + g}{K + L}, y) \leq m(\frac{f}{K}, y) + m(\frac{g}{L}, y)$. Therefore $\rho(m(\frac{f + g}{K + L}, y)) \leq \rho(m(\frac{f}{K}, y) + m(\frac{g}{L}, y)) \leq \rho(m(\frac{f}{K}, y)) + \rho(m(\frac{g}{L}, y))$. Now let $\tau(f) = K < \infty$ and $\tau(g) = L < \infty$. (The case where either $\tau(f)$ or $\tau(g)$ equals ∞ is trivial.) Then for every $\varepsilon > 0$, $\rho(m(\frac{f + g}{(K + \varepsilon) + (L + \varepsilon)}, y)) \leq \rho(m(\frac{f}{K + \varepsilon}, y)) + \rho(m(\frac{g}{L + \varepsilon}, y)) \leq K + L + 2\varepsilon$. Hence $\tau(f + g) \leq K + L = \tau(f) + \tau(g)$.

To prove axiom (d) let $\{f_n\}$ be a nondecreasing sequence of nonnegative functions in $(\tilde{F}, \tau)$ such that for every n, $\tau(f_n) \leq R < \infty$ and let $f(x) = \lim_{n\to\infty} f_n(x)$ p.p. Now by Corollary 9 $\tau(f_n) < 1$ implies $\rho(m(f_n, y)) \leq 1$, and by Proposition 11 $\tau(f_n) = 1$ implies $\rho(m(f_n, y)) \leq 1$. Also, from Corollary 9 and Proposition 8 it follows that $1 < \tau(f_n) \leq R$ implies $\rho(m(f_n, y)) \leq (2R)^2$. Hence $\{\tau(f_n)\}$ bounded by R implies $\{\rho(m(f_n, y))\}$ is bounded by $\max(1, 4R^2)$. Hence, it follows from Corollary 4 and Corollary 6 that $\{\tau(f_n)\}$ bounded implies $\{\rho(m(\frac{f_n}{K}, y))\}$ is bounded for any $K > 0$.
From Proposition 1, $m(\frac{f_n}{K}, y) \to m(\frac{f}{K}, y)$. Furthermore, since $\{\frac{f_n}{K}\}$ is a nondecreasing sequence of nonnegative functions, $\{m(\frac{f_n}{K}, y)\}$ is a nondecreasing sequence. Hence, by axiom (d), $\rho(m(\frac{f}{K}, y) = \lim_{n\to\infty} \rho(m(\frac{f_n}{K}, y))$ for every $K > 0$. Let $L = \lim_{n\to\infty} \tau(f_n)$. Then $\tau(f_n) \leq L$ for every n since $\{f_n\}$ is a nondecreasing sequence of nonnegative functions. Therefore, for every n and every $\varepsilon > 0$, $\rho(m(\frac{f_n}{L + \varepsilon}, y)) \leq L + \varepsilon$. Hence, $\rho(m(\frac{f}{L + \varepsilon}, y)) \leq L + \varepsilon$ since $\rho(m(\frac{f}{L + \varepsilon}, y)) = \lim_{n\to\infty} \rho(m(\frac{f_n}{L + \varepsilon}, y))$. So $\tau(f) \leq L$. But for every n $|f_n(x)| \leq |f(x)|$ p.p. So $\tau(f) \geq \tau(f_n)$ for every n and thus $\tau(f) \geq L$. Therefore, $\tau(f) = \lim_{n\to\infty} \tau(f_n)$.

To prove axiom (e) we must show that $\tau(f) = 0$ if and only if $f(x) = 0$ p.p. But if $f(x) = 0$ p.p., then $m(\frac{f}{K}, y)$ is identically zero for every $K > 0$. Therefore, $\rho(m(\frac{f}{K}, y)) = 0 < K$ for every $K > 0$. Hence $\tau(f) = 0$. If $\tau(f) = 0$, then there exists a sequence $\{K_n\}$ such that $K_n \downarrow 0$, $0 < K_n < 1$, and $\rho(m(\frac{f}{K_n}, y)) \leq K_n$

for every n. But $m(f, y) \leq m(\frac{f}{K_n}, y)$ for every n. Hence $\rho(m(f, y)) \leq \rho(m(\frac{f}{K_n}, y)) \leq K_n$ for every n. Hence, $\rho(m(f, y)) = 0$. Thus $m(f, y) = 0$ p.p. and so $f(x) = 0$ p.p.

To prove axiom (f) let $f^*(x) = g^*(x)$. Then f and g are equimeasurable and hence $\frac{f}{K}$ and $\frac{g}{K}$ are equimeasurable for every $K > 0$. Therefore, $\rho(m(\frac{f}{K}, y)) = \rho(m(\frac{g}{K}, y))$ for every $K > 0$ and so $\tau(f) = \tau(g)$.

Hence, $(\tilde{F}, \tau)$ is a metric function space.

<u>Theorem 13</u>: Let (H, σ) be the reflected space to (G, τ), the reflected space to (F, ρ). Let the domain of the functions in F, G, and H be $[0, \infty)$ with Lebesgue measure. Then (H, σ) and (F, ρ) are the same metric function space, i.e. H = F and σ and ρ generate the same topology.

Proof: Let f be in F. Then f^* is in F since $\rho(f) = \rho(f^*)$. But $f^*(x) = m(m(f, y), x)$ p.p. So $m(m(f, y), x)$ is in F. Hence, $m(f, y)$ is in G and f is in H. So $H \supseteq F$. Now if f is in H then $m(m(f, y), x)$ is in F. Hence, f^* is in F and so f is also in F. So $H \subseteq F$. Then H = F.

To show that σ and ρ generate the same topology it is sufficient to show that for every $\varepsilon > 0$ there exist $\delta_1 > 0$, $\delta_2 > 0$ such that $\sigma(f) < \delta_1$ implies $\rho(f) < \varepsilon$ and $\rho(f) < \delta_2$ implies $\sigma(f) < \varepsilon$. But let $1 > \varepsilon > 0$ and let $\delta_1 = \varepsilon$. If f is in H and $\sigma(f) < \delta$, then $\sigma(f) < \varepsilon < 1$. Thus by applying Corollary 9 twice we get that $\rho(f) < 1$. But by Corollary 10 $\rho(f) < 1$ implies

$\rho(f) \leq \sigma(f)$. Thus $\rho(f) < \varepsilon$. Let $\delta_2 = (\frac{\varepsilon}{2\sqrt{2}})^4$. If f is in F and $\rho(f) < \delta_2$, then $\rho(f) < 1$, and so, by Corollary 10, $\sigma(f) \leq 2\sqrt{2} \cdot \sqrt[4]{\rho(f)} < 2\sqrt{2}\ (\frac{\varepsilon}{2\sqrt{2}}) = \varepsilon$.

The next theorem reveals an interesting relation between a metric function space and its reflected space. First, however, we make the following two definitions.

Definition: Let (F, ρ) be a metric function space. (F, ρ) is said to have the truncation property if for every f in F and every $\varepsilon > 0$ there exists a $\delta > 0$ such that $0 < u < \delta$ implies $\rho(f_u) < \varepsilon$ where $f_u(x) = \left\{\begin{array}{ll} f(x) & \text{if } |f(x)| \leq u \\ u & \text{if } |f(x)| > u \end{array}\right\}$.

Definition: Let (F, ρ) be a metric function space. (F, ρ) is said to be absolutely continuous if for every f in F and every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\mu(E) < \delta$, then $\rho(f \cdot \chi_E) < \varepsilon$.

Theorem 14: A metric function space (F, ρ) is absolutely continuous if and only if its reflected space $(\tilde{F}, \tau)$ has the truncation property.

Proof. Let (F, ρ) be absolutely continuous and let $\varepsilon > 0$. Now if f is in $\tilde{F}$, then $\frac{f}{\varepsilon}$ is also in $\tilde{F}$. Thus, $\rho(m(\frac{f}{\varepsilon}, y)) < \infty$. Since (F, ρ) is absolutely continuous, there exists a $\delta > 0$ such that if $\mu(E) < \delta$, then $\rho(m(\frac{f}{\varepsilon}, y) \cdot \chi_E(y)) < \varepsilon$. In particular, if $0 < u < \delta$ then $|[0, u)| < \delta$ and hence $\rho(m(\frac{f}{\varepsilon}, y) \cdot \chi_{[0, u)}(y)) < \varepsilon$. Let $\delta_1 = \delta \cdot \varepsilon$. If $u < \delta_1$, then $\frac{u}{\varepsilon} < \delta$ and so $\rho(m(\frac{f}{\varepsilon}, y) \cdot \chi_{[0, \frac{u}{\varepsilon})}(y)) < \varepsilon$.

Claim: $m(\frac{f_u}{\varepsilon}, y) = \begin{cases} m(\frac{f}{\varepsilon}, y) & \text{if } y < \frac{u}{\varepsilon} \\ 0 & \text{if } y \geq \frac{u}{\varepsilon} \end{cases} = m(\frac{f}{\varepsilon}, y) \cdot \chi_{[0, \frac{u}{\varepsilon})}(y)$.

Proof of claim: $f_u(x) = \begin{cases} f(x) & \text{if } |f(x)| \leq u \\ u & \text{if } |f(x)| > u \end{cases}$.

Hence, $\frac{f_u}{\varepsilon}(x) = \begin{cases} \frac{f(x)}{\varepsilon} & \text{if } |f(x)| \leq u \\ \frac{u}{\varepsilon} & \text{if } |f(x)| > u \end{cases}$. Now $m(\frac{f_u}{\varepsilon}, y) =$ $\mu\{x: |\frac{f_u}{\varepsilon}(x)| > y\}$. If $y \geq \frac{u}{\varepsilon}$, then $m(\frac{f_u}{\varepsilon}, y) = 0$. If $y < \frac{u}{\varepsilon}$, then $\{x: |\frac{f_u}{\varepsilon}(x)| > y\} = \{x: |\frac{f(x)}{\varepsilon}| > y\}$. For: $\{x: |\frac{f_u}{\varepsilon}(x)| > y\} \subseteq \{x: |\frac{f(x)}{\varepsilon}| > y\}$ since for every x $|\frac{f_u}{\varepsilon}(x)| \leq |\frac{f(x)}{\varepsilon}|$. But if x is not in $\{x: |\frac{f_u}{\varepsilon}(x)| > y\}$, then $|\frac{f_u}{\varepsilon}(x)| \leq y < \frac{u}{\varepsilon}$. Hence, $|f_u(x)| \leq y \cdot \varepsilon < u$. But if $|f_u(x)| < u$, then $|f_u(x)| = |f(x)|$ and so $|f(x)| \leq y \cdot \varepsilon$. Therefore, x is not in $\{x: |\frac{f(x)}{\varepsilon}| > y\}$. Hence, $\{x: |\frac{f_u}{\varepsilon}(x)| > y$ $=$ $\{x: |\frac{f(x)}{\varepsilon}| > y\}$. Therefore,

$$m(\frac{f_u}{\varepsilon}, y) = \begin{cases} m(\frac{f}{\varepsilon}, y) & \text{if } y < \frac{u}{\varepsilon} \\ 0 & \text{if } y \geq \frac{u}{\varepsilon} \end{cases} = m(\frac{f}{\varepsilon}, y) \cdot \chi_{[0, \frac{u}{\varepsilon})}(y),$$

and so we have proven the claim. It follows from the claim that if $\rho(m(\frac{f}{\varepsilon}, y) \cdot \chi_{[0, \frac{u}{\varepsilon})}(y)) < \varepsilon$, then $\rho(m(\frac{f_u}{\varepsilon}, y)) < \varepsilon$. So for $u < \delta$, $\tau(f_u) < \varepsilon$.

Now let $(\tilde{F}, \tau)$ have the truncation property and let f be in (F, ρ).

Case 1: f is nonincreasing and nonnegative.

By Proposition 7, $f(y) = m(g, y)$ p.p. where $g(x) = m(f, x)$. Hence, g is in $\tilde{F}$. Let $0 < \varepsilon < 1$. Then since $(\tilde{F}, \tau)$ has the truncation property, there exists a $\delta > 0$ such that $0 < u < \delta$

implies $\tau(g_u) < \varepsilon$. Hence, $\rho(m(\frac{g_u}{\varepsilon}, y)) < \varepsilon$ and so $\rho(m(\frac{g}{\varepsilon}, y)\cdot\chi_{[0, \frac{u}{\varepsilon})}(y)) < \varepsilon$. But $\varepsilon < 1$ implies $\rho(m(g, y) \cdot \chi_{[0, \frac{u}{\varepsilon})}(y)) \leq \rho(m(\frac{g}{\varepsilon}, y) \cdot \chi_{[0, \frac{u}{\varepsilon})}(y)) < \varepsilon$. Thus, $\rho(f(y) \cdot \chi_{[0, \frac{u}{\varepsilon})}(y)) < \varepsilon$. So for all sets E of the form $[0, v)$ where $v < \frac{\delta}{\varepsilon}$ we have satisfied the absolute continuity condition. Now if E is measurable and $\mu(E) = v < \frac{\delta}{\varepsilon}$, then, since f is nonincreasing, $m(f \cdot \chi_E, y) \leq m(f \cdot \chi_{[0, v)}, y)$. Hence $(f(y) \cdot \chi_E(y))^* \leq (f(y) \cdot \chi_{[0, v)}(y))^* = f(y) \cdot \chi_{[0, v)}(y)$. So $\rho(f \cdot \chi_E) < \varepsilon$.

Case 2: f is measurable.

If f is measurable then $(f \cdot \chi_E)^*(y) \leq f^*(y) \cdot \chi_{[0, \mu(E))}(y)$ because for every $\alpha > 0$, $\{x: |f(x) \cdot \chi_E(x)| > \alpha\} \subseteq \{x: |f(y)| > \alpha\}$. Hence, $m(f \cdot \chi_E, \alpha) \leq m(f, \alpha)$ for every $\alpha > 0$. Therefore, $(f \cdot \chi_E)^*(y) \leq f^*(y)$ for every $y > 0$ and so $(f \cdot \chi_E)^*(y) \leq f^*(y) \cdot \chi_{[0, \mu(E))}(y)$. Now let $\mu(E) < \frac{\delta}{\varepsilon}$ where $0 < \varepsilon < 1$ is given and δ corresponds to ε in the truncation property for $(\tilde{F}, \tau)$. Then since $\rho(f \cdot \chi_E) = \rho((f \cdot \chi_E)^*) \leq \rho(f^* \cdot \chi_{[0, \mu(E))})$, and for $\mu(E) < \frac{\delta}{\varepsilon}$, $\rho(f^* \cdot \chi_{[0, \mu(E))}) < \varepsilon$, it follows that $\rho(f \cdot \chi_E) < \varepsilon$.

Bibliography

1. P. R. Halmos: Measure Theory, D. Van Nostrand Co., (1950).

2. G. H. Hardy, J. E. Littlewood, and G. Polya: Inequalities, Cambridge, (1952).

3. M. A. Krasnosel'skii and Ya. B. Rutickii: Convex Functions and Orlicz Spaces, P. Noordhoff, (1961).

4. R. O'Neil: Integral Transforms and Tensor Products on Orlicz Spaces and L(p, q) Spaces. (To be published.)

5. A. C. Zaanen: Integration, North Holland Publishing Co., (1967).