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Some Compact Operators on Orlicz Spaces

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ABSTRACT

Some Compact Operators on Orlicz Spaces

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Let f be a kernel in the Orlicz space L_A. What is the necessary and sufficient condition on the Young's functions A, B, C so that the operator

$$h(x) = \int f(x-t)g(t)\, dt$$

be compact from L_B into L_C?

It is shown that the problem is impossible on the real line, or more generally, on a locally compact, commutative, connected but not compact group.

If the group is compact, it is proved that the problem is possible, and the necessary and sufficient condition is that for every $\theta > 0$, there exists a number $\eta > 0$ such that for all $x \geq 1$

$$A^{-1}(x)\ B^{-1}(\eta x) \leq \theta\eta x C^{-1}(\eta x)$$

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Definition 1.1: The function A from $[0,\infty)$ into $[0,\infty]$ is a Young's function if $A(0) = 0$, and A is convex, left-continuous, and non-decreasing. A Young's function is said to be a trivial Young's function if either $A(x) = 0$ for all $x \geq 0$ or $A(0) = 0$ and $A(x) = \infty$ for all $x > 0$.

Definition 1.2: If A is a Young's function then for all $y \in [0,\infty]$

$$A^{-1}(y) = \inf \{x : A(x) > y\}$$

where $\inf \emptyset = \infty$.

It is immediate that A^{-1} is a non-decreasing function from $[0,\infty]$ into $[0,\infty]$ which is continuous to the right with $A^{-1}(\infty) = \infty$. For $x \in [0,\infty)$, $A(x) = \sup \{y : A^{-1}(y) < x\}$, where $\sup \emptyset = 0$.

Property 1.3: If A is a Young's function, and if $x \in [0,\infty)$ and $\varepsilon > 0$

then
$$x \leq A^{-1}(A(x))$$

$$A(A^{-1}(x)) \leq x$$

and
$$A(A^{-1}(x)+\varepsilon) > x$$

Definition 1.4: The Orlicz space $L_A(X,\mu)$ determined by Young's function A is the set of all (real- or complex-valued) measurable functions on the measure space (X,μ) for which there exists a number $\theta > 0$ such that

$$\int_X A(\theta\,|f(x)|)\,d\mu(x) < \infty$$

and the L_A-norm of f is defined as

$$||f||_A = \inf\left\{K > 0 : \int_X A\left(\frac{|f(x)|}{K}\right) d\mu(x) \leq 1\right\}$$

It can be shown that under this norm L_A is a Banach space.

<u>Property 1.5</u>: Let f be a measurable function on a measure space (X,μ) and let A be a Young's function. Then $f \in L_A(X,\mu)$ if and only if $||f||_A < \infty$.

<u>Lemma 1. 6</u>: If A is a Young's function and χ_E is the characteristic function of a set E of measure γ, then

$$||\chi_E||_A = \frac{1}{A^{-1}(\frac{1}{\gamma})}$$

(see 5, p.16)

<u>Proof</u>: Assume $\gamma > 0$

If $$\int A\left(\frac{\chi_E(x)}{K}\right) d\mu(x) \leq 1$$

then $$\int_E A\left(\frac{1}{K}\right) d\mu(x) = \gamma\, A\left(\frac{1}{K}\right) \leq 1$$

$$\frac{1}{K} \leq A^{-1} \left(A \left(\frac{1}{K}\right)\right) \leq A^{-1} \left(\frac{1}{\gamma}\right).$$

Thus $$K \geq \frac{1}{A^{-1}\left(\frac{1}{\gamma}\right)}$$

If $$K \geq \frac{1}{A^{-1}\left(\frac{1}{\gamma}\right)}$$

then $$\gamma\, A\left(\frac{1}{K}\right) \leq \gamma\, A\left(A^{-1}\left(\frac{1}{\gamma}\right)\right)$$

$$\leq \gamma \left(\frac{1}{\gamma}\right) = 1$$

Thus $$\int A \left(\frac{\chi_E(x)}{K}\right) d\mu(x) \leq 1$$

So $$\int A \left(\frac{\chi_E(x)}{K}\right) d\mu(x) \leq 1 \quad \text{iff } K \geq \frac{1}{A^{-1}\left(\frac{1}{\gamma}\right)}$$

Hence by Definition 1.4 it follows that

$$||\chi_E||_A = \frac{1}{A^{-1}\left(\frac{1}{\gamma}\right)}$$

Q.E.D.

<u>Definition 1.7</u>: The Young's complement, $\overline{A}$, of a Young's function A is defined for $x \in [0,\infty)$ by

$$\overline{A}(x) = \sup_{y \in [0,\infty)} (xy - A(y))$$

Then $\bar{A}$ is a Young's function, and the Young's complement of $\bar{A}$ is A.

Property 1.8: Let A be a Young's function, then for all $x \in [0,\infty)$

$$x \leq A^{-1}(x)\ \bar{A}^{-1}(x) \leq 2x$$

and for all $x,\ y \in [0,\infty)$ we have Young's Inequality:

$$xy \leq A(x) + \bar{A}(y)$$

From which we have Hölder inequality in the form

$$\left|\int_X f(x)\ g(x)\ d\mu(x)\right| \leq 2\ ||f||_A\ ||g||_{\bar{A}}$$

Property 1.9: If A is a Young's function and if $0 < \alpha < 1$, then

$$A(\alpha x) \leq \alpha\ A(x) \qquad \text{for all } x$$

if $\alpha > 1$, then

$$A(\alpha x) \geq \alpha\ A(x) \qquad \text{for all } x$$

The following gives a substitute for Jensen's Inequality for Young's function.

<u>Lemma 1.10</u>: If A is a Young's function and if $p(x)$ and $f(x)$ are nonnegative measurable functions with $\int p(x)dx \leq 1$, then

$$A\left(\int p(x)\ f(x)\ dx\right) \leq \int p(x)\ A(f(x))\ dx$$

<u>Lemma 1.11</u> (First Generalized Young's Inequality):

If A, B, C are extended real-valued, nonnegative, nondecreasing, left continuous functions defined on $[0,\infty)$, if for $0 \leq x \leq \infty$, $A^{-1}(x) = \inf\left\{y : A(y) > x\right\}$ ($\inf \emptyset = \infty$), and if for all $0 \leq x < \infty$

$$A^{-1}(x)\ B^{-1}(x) \leq C^{-1}(x)$$

then for all $0 \leq x < \infty$, $0 \leq y < \infty$

$$C(xy) \leq A(x) + B(y)$$

<u>Theorem 1.12</u>: If A, B, C are extended real-valued nonnegative, nondecreasing, left continuous functions defined on $[0,\infty)$, and if

$$A^{-1}(x)\ B^{-1}(x) \leq C^{-1}(x) \ ,$$

then $$\int C(|f(x)\ g(x)|)d\mu \leq \int A(|f(x)|)d\mu + \int B(|g(x)|)d\mu$$

Theorem 1.13 (Generalized Hölder's Inequality):

If A, B, C are Young's functions such that

$$A^{-1}(x)\ B^{-1}(x) \leq C^{-1}(x)$$

and if $f \in L_A$, $g \in L_B$ on a measure space (X,μ), then the product $h(x) = f(x)\ g(x)$ is in L_C and

$$||h||_C \leq 2\ ||f||_A\ ||g||_B \quad .$$

Lemma 1.14) (Second Generalized Young's Inequality):

If A, B, C are Young's functions such that for $x \geq 0$,

$$A^{-1}(x)\ B^{-1}(x) \leq x\ C^{-1}(x)$$

then for $x \geq 0$, $y \geq 0$

$$xy \leq A(x)\ C^{-1}(B(y)) + B(y)\ C^{-1}(A(x))$$

Theorem 1.15: If A, B, C are Young's functions such that for $x \geq 0$

$$A^{-1}(x)\ B^{-1}(x) \leq x\ C^{-1}(x)$$

and if $f \in L_A$, $g \in L_B$ on a locally compact unimodular topological group (G,μ), where μ is the Haar measure, then their convolution (writing dt instead of $d\mu$),

$$h(x) = \int_G f(t)\, g(t^{-1}x)\, dt$$

is in $L_C(G,\mu)$ and

$$\|h\|_C \leq 2\, \|f\|_A\, \|g\|_B\ .$$

(see 4, p. 305-306)

Proof. Let $\varepsilon > 0$

Without loss of generality we may assume

$$\|f\|_A = 1 = \|g\|_B$$

Consider

$$\int C\left(\frac{|h(x)|}{2(1+\varepsilon)^2}\right) dx$$

$$\leq \int C\left(\frac{1}{2}\int \frac{|f(t)|}{1+\varepsilon}\,\frac{|g(t^{-1}x)|}{1+\varepsilon}\, dt\right) dx$$

$$\leq \int C\Big[\frac{1}{2}\int A\left(\frac{|f(t)|}{1+\varepsilon}\right) C^{-1}\left(B\left(\frac{|g(t^{-1}x)|}{1+\varepsilon}\right)\right) dt$$

$$+ \frac{1}{2}\int B\left(\frac{|g(t^{-1}x)|}{1+\varepsilon}\right) C^{-1}\left(A\left(\frac{|f(t)|}{1+\varepsilon}\right)\right) dt\Big]\, dx$$

$$\leq \frac{1}{2}\int C\left[\int A\left(\frac{|f(t)|}{1+\varepsilon}\right) C^{-1}\left(B\left(\frac{|g(t^{-1}x)|}{1+\varepsilon}\right)\right) dt\right] dx$$

$$+ \frac{1}{2}\int C\left[\int B\left(\frac{|g(t^{-1}x)|}{1+\varepsilon}\right) C^{-1}\left(A\left(\frac{|f(t)|}{1+\varepsilon}\right)\right) dt\right] dx$$

$$= P + Q$$

where $$P = \frac{1}{2}\int C\left[\int A\left(\frac{|f(t)|}{1+\varepsilon}\right) C^{-1}\left(B\left(\frac{|g(t^{-1}x)|}{1+\varepsilon}\right)\right)dt\right] dx$$

$$\leq \frac{1}{2}\iint A\left(\frac{|f(t)|}{1+\varepsilon}\right) C\, C^{-1}\left(B\left(\frac{|g(t^{-1}x)|}{1+\varepsilon}\right)\right)dt\, dx$$

for $$\int A\left(\frac{|f(t)|}{1+\varepsilon}\right) dt \leq 1$$ and by Lemma 1.9

$$P \leq \frac{1}{2}\iint A\left(\frac{|f(t)|}{1+\varepsilon}\right) B\left(\frac{|g(t^{-1}x)|}{1+\varepsilon}\right)dt\, dx$$

$$= \frac{1}{2}\int A\left(\frac{|f(t)|}{1+\varepsilon}\right)\left[\int B\left(\frac{|g(t^{-1}x)|}{1+\varepsilon}\right)dx\right] dt$$

$$\leq \frac{1}{2}$$

Similarly, $$Q \leq \frac{1}{2}$$

Thus $$P + Q \leq 1$$

i.e. $$||h||_C \leq 2(1+\varepsilon)^2$$

Let $$\varepsilon \to 0$$

$$||h||_C \leq 2$$

Hence $$||h||_C \leq 2\, ||f||_A\, ||g||_B$$

Q.E.D.

<u>Definition 2.1</u>: A linear operator, T, from one Banach space E into another F which maps every bounded set into a set whose closure is compact is called a <u>compact</u> (or <u>completely continuous</u>) <u>linear operator</u>, that is, if for every infinite sequence $\{f_n\}$ of the elements of E such that $||f_n|| \leq C$, the sequence $\{Tf_n\}$ contains a subsequence which converges in the mean to an element of F.

It is known that every linear operator T which can be approximated in norm arbitrarily closely by a compact linear operator is itself compact.

We shall determine later a condition on the Young's functions A, B, C in order that every operator, T_f, whose kernel f is in L_A, be compact from L_B into L_C. But this result is not possible on the line, or more generally, on a locally compact, commutative, connected but not compact group.

<u>Example 2.2</u>: For every point x on the real line R',

let $$f(x) = \chi_{[0,1]}(x)$$

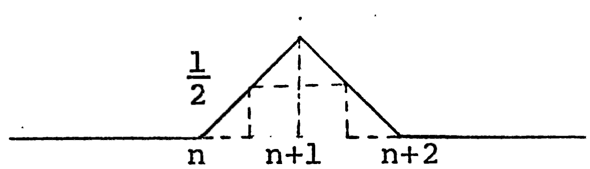
and the integral operator $T = T_f$ defined by $h = Tg$ where

$$h(x) = \int_{-\infty}^{\infty} f(x-t)g(t)dt$$

Let $$g_n(x) = \chi_{[n,n+1]}(x) \quad \text{for all } n = 0,2,4,$$

and $$h_n(x) = \int_{-\infty}^{\infty} f(x-t)\, g_n(t)\, dt$$

then
$$\begin{aligned} h_n(x) &= \int_{n}^{n+1} f(x-t)\, dt \\ &= \int_{x-n-1}^{x-n} \chi_{[0,1]}(u)\, du \\ &= \left| [x-n-1, x-n] \cap [0,1] \right| \\ &= \begin{cases} 0 & \text{if } x < n \\ x-n & \text{if } n \le x \le n+1 \\ n-x+2 & \text{if } n+1 \le x \le n+2 \\ 0 & \text{if } x > n+2 \end{cases} \\ &\ge \frac{1}{2} \chi_{[n+\frac{1}{2}, n+\frac{3}{2}]}(x) \end{aligned}$$



Hence $$h_n(x) > \frac{1}{2} \quad \text{if} \quad x \in \left(n + \frac{1}{2},\ n + \frac{3}{2}\right)$$

Let $$g'(x) = g\ (x-\varepsilon)$$

then
$$\begin{aligned} h'(x) &= \int_{-\infty}^{\infty} f(x-t)\, g'(t)\, dt \\ &= \int_{-\infty}^{\infty} f(x-t)\, g(t-\varepsilon)\, dt \end{aligned}$$

$$= \int_{-\infty}^{\infty} f[(x-\varepsilon)-u]\ g(u)\ du$$

$$= h(x-\varepsilon)$$

Hence we may consider g' instead of h' and so on.

Let A, B, C be any Young's functions

then $$||f||_A = ||\chi_{[0,1]}||_A = \frac{1}{A^{-1}(1)}$$

$$||g_n||_B = ||\chi_{[n,n+1]}||_B = \frac{1}{B^{-1}(1)}$$

and $$||g_n||_B = ||g_m||_B \qquad \text{for all } m,n = 0,2,4,\cdots$$

Consider $$\int C\ (4C^{-1}(1)\ |h_n(x)|)\ dx$$

$$\geq \int_{(n+\frac{1}{2},n+\frac{3}{2})} C\ [4C^{-1}(1)\ |h_n(x)|]\ dx$$

$$> \int_{(n+\frac{1}{2},n+\frac{3}{2})} C\ [2C^{-1}(1)]\ dx$$

$$\geq 1$$

i.e. $$||h_n||_C \geq \frac{1}{4C^{-1}(1)}$$

If $m \neq n$, the supports of h_m and h_n are disjoint.

Hence $$||h_m - h_n||_C \geq ||h_n||_C \geq \frac{1}{4C^{-1}(1)}$$

Thus no convergent subsequence can be extracted from $h_1 = Tg_1$, $h_2 = Tg_2$, $\cdots$. Therefore T is not a compact linear operator for any Orlicz spaces on the real line.

<u>Example 2.3</u>: Let G be a locally compact, commutative, connected, but not compact group. Then there exists in G a discrete subgroup D having a finite system of linearly independent generators, such that the factor group G/D is compact (see 6, p. 153-157) and $\varphi : G \to G/D$ is an evenly covered map. So there exists an open neighborhood U_e of identity $e \in G/D$ such that $\varphi^{-1}(U)$ is a disjoint union of open sets U_α in G, each of which is mapped homeomorphically onto U_e by φ. Then choose closed subset V_e of U_e such that

$$V_e V_e^{-1} \subset U_e$$

and
$$V_e = V_e^{-1}$$

for all $\alpha \in D$,
$$U_\alpha = \alpha \cdot U_e$$
$$V_\alpha = \alpha \cdot V_e$$

Let
$$f(x) = \chi_{V_e}(x)$$

and
$$g_\alpha(x) = \chi_{V_\alpha}(x) \qquad \text{for all } \alpha \in D,\ x \in G$$

Then the integral operator $T = T_f$ defined by $h = T_g$ where

$$h_\alpha(x) = \int_G f(xt^{-1})\, g_\alpha(t)\, dt$$

is not compact

Since $$h_\alpha(x) = \int_G f(xt^{-1})\, g_\alpha(t)\, dt$$

$$= \int_{V_\alpha} f(xt^{-1})\, dt$$

$$= \int_{xV_\alpha^{-1}} f(u)\, du$$

$$= |xV_\alpha^{-1} \cap V_e|$$

But if $$xV_\alpha^{-1} \cap V_e \neq \emptyset$$

i.e. $$x\alpha^{-1} \cdot V_e^{-1} \cap V_e \neq \emptyset$$

then there exist $v, v' \in V_e$ such that

$$x \cdot \alpha^{-1}\, v = v'$$

$$x\alpha^{-1} = v'v^{-1} \in U_e$$

so $$x \in U_\alpha$$

Thus the support of h_α is contained in U_α. Since U_α is disjoint, the supports of h_α and h_β if $\alpha \neq \beta$ are disjoint.

Let A, B, C be any Young's functions

then $$||f||_A = ||\chi_{V_e}||_A$$

$$= \frac{1}{A^{-1}\left(\frac{1}{|V_e|}\right)}$$

$$\leq \frac{1}{A^{-1}(\frac{1}{|U_e|})} \qquad \text{for } |V_e| \leq |U_e|$$

for all α, $$||g_\alpha||_B = \frac{1}{B^{-1}(\frac{1}{|V_\alpha|})}$$

$$\leq \frac{1}{B^{-1}(\frac{1}{|U_e|})} \qquad \text{for } |V_\alpha| \leq |U_\alpha| = |U_e|$$

But since

$$||h_\alpha||_C \geq \frac{1}{4C^{-1}(\frac{1}{|U_e|})}$$

$$||h_\alpha - h_\beta||_C \geq ||h_\alpha||_C$$

$$\geq \frac{1}{4C^{-1}(\frac{1}{|U_e|})}$$

Hence there exists no convergent subsequence of h_α, h_β, and T is not compact, i.e. the integral operator T is not compact for any Orlicz spaces on a locally compact, commutative, connected and non-compact group.

If the group is compact, we shall establish here a necessary and sufficient condition on the Young's functions in order that the convolution gives a compact operator.

Lemma 3.1: Let A be a Young's function, and (X,μ) be a measure space. Then for every $f \in L_A(X,\mu)$, there exists a sequence of continuous functions, f_n, with compact support such that

$$\lim_{n\to\infty} f_n(x) = f(x) \qquad \text{P.P.}$$

and there exists a function $F \in L_A(X,\mu)$ such that for all n

$$|f_n(x)| \leq F(x) \qquad \text{for all } x \in X .$$

(see 5, p. 53-56)

Lemma 3.2: Let A, B, C be Young's functions which satisfy the condition that for every $\theta > 0$ there exists a number $\eta > 0$ such that for all $x \geq 1$

$$A^{-1}(x)\ B^{-1}(\eta x) \leq \theta\ \eta x C^{-1}(\eta x)$$

If $f \in L_A$, such that $f(x) = f(x+2\pi)$, $g \in L_B$, and E is a measurable subset of $[0,2\pi]$ such that

$$\eta \int_E A(|f(x)|)dx \leq 1$$

$$\eta|E| \leq 1$$

and $$h(x) = \int_0^{2\pi} \chi_E(x-t)\ f(x-t)\ g(t)\ dt$$

where $$||g||_B \leq 1$$

then
$$\int_0^{2\pi} C\left(\frac{|h(x)|}{2\theta}\right) dx \leq (\eta+2\pi\eta^2)\left[\int_E A(|f(x)|)\,dx + |E|\right]$$

<u>Proof</u>: For any $x_1, x_2 \geq 0$

suppose first $$B(x_2) < \eta A(x_1)\ .$$

Then
$$x_2 A^{-1}\left[\frac{1}{\eta} B(x_2) + 1\right]$$
$$\leq B^{-1}(B(x_2))\ A^{-1}\left[\frac{1}{\eta} B(x_2) + 1\right]$$
$$\leq B^{-1}\left[\eta\left(\frac{1}{\eta} B(x_2) + 1\right)\right]\ A^{-1}\left[\frac{1}{\eta} B(x_2) + 1\right]$$
$$\leq \theta\ \eta\left[\frac{1}{\eta} B(x_2) + 1\right]\ C^{-1}(B(x_2) + \eta)$$
$$x_2 \leq \frac{\theta\ \eta\left[\frac{1}{\eta} B(x_2) + 1\right]}{A^{-1}\left[\frac{1}{\eta} B(x_2) + 1\right]}\ C^{-1}\ (B(x_2) + \eta)$$
$$\leq \theta\ \eta\ \frac{A(x_1) + 1}{A^{-1}(A(x_1)+1)}\ C^{-1}\ (B(x_2) + \eta)$$
$$\leq \theta\ \eta\ \frac{A(x_1) + 1}{x_1}\ C^{-1}\ (B(x_2) + \eta)$$

Thus $$x_1 x_2 \leq \theta\ \eta\ [A(x_1) + 1]\ C^{-1}\ [B(x_2) + \eta]$$

Suppose next $$B(x_2) \geq \eta\ A(x_1).$$

Then
$$x_1 B^{-1}[\eta(A(x_1) + 1)]$$
$$\leq A^{-1}[A(x_1) + 1]\ B^{-1}[\eta(A(x_1) + 1)]$$
$$\leq \theta\ \eta\ [A(x_1) + 1]\ C^{-1}[\eta(A(x_1) + 1)]$$
$$x_1 \leq \frac{\theta[\eta A(x_1)+\eta]}{B^{-1}[\eta A(x_1)+\eta]}\ C^{-1}[\eta(A(x_1)+1)]$$
$$\leq \theta\ \frac{B(x_2)+\eta}{B^{-1}[B(x_2)+\eta]}\ C^{-1}[\eta(A(x_1)+1)]$$
$$\leq \theta\ \frac{B(x_2)+\eta}{x_2}\ C^{-1}\ [\eta(A(x_1) + 1)]$$

Thus $$x_1 x_2 \leq \theta\ [B(x_2) + \eta]\ C^{-1}\ [\eta(A(x_1) + 1)]$$

Hence for any $x_1,\ x_2 \geq 0$
$$x_1 x_2 \leq \theta \left\{ \eta[A(x_1) + 1]\ C^{-1}\ [B(x_2) + \eta] \right.$$
$$\left. + [B(x_2) + \eta]\ C^{-1}\ [\eta(A(x_1) + 1)] \right\}$$

Let
$$x_1 = |f(x-t)|$$
$$x_2 = |g(t)|$$

Then $C(\frac{|h(x)|}{2\theta})$

$$= C\left(\frac{1}{2\theta}\int_o^{2\pi} \chi_E(x-t)\ |f(x-t)|\ |g(t)|\ dt\right)$$

$$\leq C\left(\frac{1}{2\theta}\int_o^{2\pi} \chi_E(x-t)\theta\left\{\eta[A(|f(x-t)|)+1]C^{-1}[B(|g(t)|)+\eta]\right.\right.$$

$$\left.\left.+\ [B(|g(t)|)\ +\ \eta]C^{-1}[\eta(A(|f(x-t)|)\ +\ 1)]\right\}dt\right)$$

$$= C\left(\frac{1}{2}\int_o^{2\pi} \chi_E(x-t)\left\{\eta[A(|f(x-t)|)\ +\ 1]C^{-1}[B(|g(t)|)+\eta]\right.\right.$$

$$\left.\left.+\ [B(|g(t)|)\ +\ \eta]\ C^{-1}[\eta(A(|f(x-t)|)\ +\ 1)]\right\}\ dt\right)$$

Since $f(x) = f(x + 2\pi)$

$$\frac{1}{2}\int_o^{2\pi} \chi_E\ (x-t)\ \eta\ A(|f(x-t)|)\ dt$$

$$= \frac{\eta}{2}\int_{x-2\pi}^{x} \chi_E\ (u)\ A(|f(u)|)\ du$$

$$\leq \frac{\eta}{2}\int_E A(|f(u)|)\ du$$

$$\leq \frac{1}{2}$$

$$\frac{1}{2}\int_o^{2\pi} \chi_E(x-t)\ \eta\ dt$$

$$= \frac{\eta}{2}\int_{x-2\pi}^{x} \chi_E\ (u)\ du$$

$$\leq \frac{\eta}{2}\ |E|$$

$$\leq \frac{1}{2}$$

$$\therefore \frac{1}{2}\int_0^{2\pi} \chi_E(x-t)\ \eta[A(|f(x-t)|) + 1]\ dt \leq 1$$

Similarly $$\frac{1}{2}\int_0^{2\pi} B(|g(t)|)\ dt \leq \frac{1}{2}$$

$$\frac{1}{2}\int_0^{2\pi} \chi_E(x-t)\ B(|g(t)|)\ dt \leq \frac{1}{2}$$

$$\frac{1}{2}\int_0^{2\pi} \chi_E(x-t)\ \eta\ dt \leq \frac{1}{2}$$

$$\therefore \frac{1}{2}\int_0^{2\pi} \chi_E(x-t)\ [B(|g(t)|) + \eta]\ dt \leq 1$$

By Lemma 1.10 $$C\left(\frac{|h(x)|}{2\theta}\right)$$

$$\leq \frac{1}{2}\int_0^{2\pi} \chi_E(x-t)\Big\{\eta[A(|f(x-t)|)+1]\,CC^{-1}[B(|g(t)|)+\eta]$$

$$+ [B(|g(t)|) + \eta]\ CC^{-1}[\eta(A(|f(x-t)|)+1)]\Big\}\ dt$$

$$\leq \eta\int_0^{2\pi} \chi_E(x-t)\,[A(|f(x-t)|)+1]\,[B(|g(t)|)+\eta]\ dt$$

Hence $$\int_0^{2\pi} C\left(\frac{|h(x)|}{2\theta}\right)\ dx$$

$$\leq \int_0^{2\pi}\Big\{\eta\int_0^{2\pi} \chi_E(x-t)\,[A(|f(x-t)|)+1]\,[B(|g(t)|)+\eta]\,dt\Big\}dx$$

$$= \eta\int_0^{2\pi}\Big\{\int_0^{2\pi} \chi_E(x-t)\,[A(|f(x-t)|)+1]\,[B(|g(t)|)+\eta]\,dx\Big\}dt$$

$$x-t = y \qquad dx = dy$$

$$= \eta\int_0^{2\pi}\Big\{\int_{-t}^{2\pi-t} \chi_E(y)\,[A(|f(y)|)+1]\,[B(|g(t)|)+\eta]\,dy\Big\}dt$$

$$\leq \eta\int_0^{2\pi}\Big\{\int_0^{2\pi} \chi_E(y)\,[A(|f(y)|)+1]\,[B(|g(t)|)+\eta]\,dy\Big\}dt$$

$$= \eta\int_0^{2\pi}\left\{\int_0^{2\pi} \chi_E(y)\,[A(|f(y)|)+1]\,[B(|g(t)|)+\eta]\,dt\right\}dy$$

$$= \eta\int_0^{2\pi}\left\{\int_0^{2\pi} [B(|g(t)|)+\eta]\,dt\right\}\chi_E(y)\,[A(|f(y)|)+1]\,dy$$

$$\leq \eta\int_0^{2\pi} (1+2\pi\eta)\;\chi_E(y)\,[A(|f(y)|)+1]\,dy$$

$$= (\eta+2\pi\eta^2)\left[\int_E A(|f(x)|)\;dx + |E|\right]$$

Q.E.D.

<u>Theorem 3.3</u>: Let A, B, C, be Young's functions which satisfy the condition that for every $\theta > 0$, there exists a number $\eta > 0$ such that for all $x \geq 1$

$$A^{-1}(x)B^{-1}(\eta x) \leq \theta\,\eta\,x\,C^{-1}(\eta x)$$

And let (x,μ) be a measure space, and for fixed periodic function $f \in L_A$ such that

$$f(x) = f(x + 2\pi)$$

define the integral operator, $T = T_f$, by $h = Tg$, where

$$h(x) = \int_0^{2\pi} f(x-t)\;g(t)\;dt \qquad \text{for all } g \in L_B$$

Then T is a compact linear operator from L_B into L_C.

<u>Proof</u>:
$$h(x) = \int_0^{2\pi} f(x-t)\;g(t)\;dt$$
$$= \int_{-\pi}^{\pi} f(x-t)\;g(t)\;dt$$

We select a sequence $\{f_n(x)\}$ of continuous functions and let $T_n = T_{f_n}$.

Then each T_n is compact by Arzela's theorem, as we now show.

Let $$h_n = T_n g ;$$

$$h_n(x) = \int_{-\pi}^{\pi} f_n(x-t)\, g(t)\, dt$$

Let $$|\Delta x| \leq 1 ,$$

$$g \in L_B \text{ and } ||g||_B \leq 1$$

Consider $$|h_n(x + \Delta x) - h_n(x)|$$

$$\leq \int_{-\pi}^{\pi} | f_n(x+\Delta x-t) - f_n(x-t) ||g(t)| dt$$

Since f_n is continuous

$$|f_n(x+\Delta x) - f_n(x)| \leq \omega_n(\Delta x)$$

Thus $$|h_n(x + \Delta x) - h_n(x) |$$

$$\leq \int_{-\pi}^{\pi} \omega_n(\Delta x) \; |g(t)| \; dt$$

$$= \omega_n(\Delta x) \int_{-\pi}^{\pi} |g(t) | \; dt$$

$$\leq 2\omega_n(\Delta x) \; ||g||_B \; || \; 1 \; ||_{\bar{B}}$$

$$\leq 2\omega_n(\Delta x) \; || \; 1 \; ||_{\bar{B}}$$

$$= \epsilon$$

where $\epsilon = 2\omega_n(\Delta x) \; || \; 1 \; ||_{\bar{B}}$

Hence $h_n(x)$ is equicontinuous.

Now we claim that $h_n(x)$ is uniformly bounded.

$$|h_n(x)| = \left| \int_0^{2\pi} f_n(x-t) \cdot g(t) \; dt \right|$$

$$\leq \int_0^{2\pi} |f_n(x-t)| \; |g(t)| \; dt$$

$$\leq M_n \int_0^{2\pi} |g(t)| \; dt$$

where $M_n = \max\limits_{x \in [0, 2\pi]} f_n(x)$

$$\leq M_n \; ||g||_B \; || \; 1 \; ||_{\bar{B}}$$

$$\leq M_n \; || \; 1 \; ||_{\bar{B}}$$

Hence each T_n is compact.

We shall show that the sequence T_n tends to T in the operator norm.

Given any $g \in L_B$, such that $||g||_B \leq 1$

Let $h = Tg$, $h_n = T_n g$ and let $L = ||F||_A$ where F is defined as in lemma 3.1.

Given any $\epsilon > 0$, let $\theta = \frac{\epsilon}{24L}$

Then there exists a number $\eta > 0$ such that for all $x \geq 1$

$$A^{-1}(x)\ B^{-1}(\eta x) \leq \theta\ \eta\ x\ C^{-1}(\eta x)$$

Let β be chosen later

and let $$E_n = \left\{x : \frac{1}{\theta} \mid f(x)-f_n(x) \mid \leq \beta\right\}$$

$$E_n' = X - E_n$$

Then $$\lim_{n\to\infty} |E_n'| = 0$$

for $$\lim_{n\to\infty} |f(x)-f_n(x)| = 0 \qquad \text{p.p.} ,$$

and convergence p.p. implies convergence in measure.

Consider

$$\frac{1}{\theta} |h(x) - h_n(x)|$$

$$\leq \frac{1}{\theta} \int_0^{2\pi} |f(x-t)-f_n(x-t)|\ |g(t)|\ dt$$

$$= P(x) + Q(x)$$

where $$P(x) = \frac{1}{\theta} \int_0^{2\pi} |f(x-t)-f_n(x-t)| \chi_{E_n}(x-t)\ |g(t)|\ dt$$

$$\leq \beta \int_0^{2\pi} |g(t)|\ dt$$

$$\leq 2\beta\ ||g||_B\ ||\ 1\ ||_{\bar{B}}$$

$$\leq 2\beta\ ||\ 1\ ||_{\bar{B}}$$

and $$Q(x) = \frac{1}{\theta}\int_0^{2\pi} |f(x-t)-f_n(x-t)|\chi_{E_n'}(x-t)|g(t)|dt$$

$$\leq \frac{2}{\theta}\int_0^{2\pi} F(x-t)\chi_{E_n'}(x-t)|g(t)|dt$$

Thus $$C\left(\frac{|h(x)-h_n(x)|}{\epsilon}\right)$$

$$= C\left(\frac{|h(x)-h_n(x)|}{24L\theta}\right)$$

$$\leq C\left(\frac{P(x)+Q(x)}{24L}\right)$$

$$\leq \frac{1}{2}C\left(\frac{P(x)}{12L}\right) + \frac{1}{2}C\left(\frac{Q(x)}{12L}\right)$$

$$\leq C\left(\frac{\beta||1||_{\bar{B}}}{6L}\right) + C\left(\frac{1}{6L\theta}\int_0^{2\pi} F(x-t)\chi_{E_n'}(x-t)|g(t)|dt\right)$$

Hence $$\int_0^{2\pi} C\left(\frac{|h(x)-h_n(x)|}{\epsilon}\right) dx$$

$$\leq \int_0^{2\pi}\left[C\left(\frac{\beta||1||_{\bar{B}}}{6L}\right)+C\left(\frac{1}{6L\theta}\int_0^{2\pi} F(x-t)\chi_{E_n'}(x-t)|g(t)|dt\right)\right]dx$$

$$\leq 2\pi C\left(\frac{\beta||1||_{\bar{B}}}{6L}\right)+\int_0^{2\pi} C\left(\frac{1}{6\theta}\int_0^{2\pi}\frac{F(x-t)}{L}\chi_{E_n'}(x-t)|g(t)|dt\right)dx$$

choose β so small that

$$2\pi C\left(\frac{\beta||1||_{\bar{B}}}{6L}\right) \leq \frac{1}{2}$$

Since $\lim_{n\to\infty} |E_n'| = 0$, by absolute continuity we may choose N such that if $n \geq N$, then

$$(\eta+2\pi\eta^2)\left[\int_{E_n'} A\left(\left|\frac{F(x-t)}{L}\right|\right)dx + |E_n'|\right] \leq \frac{1}{2}$$

Thus $$\eta \int_{E_n'} A(|\frac{F(x-t)}{L}|)\, dx \leq 1$$

and $$\eta\, |E_n'| \leq 1$$

By lemma 3.2

$$\int_0^{2\pi} C(\frac{|h(x)-h_n(x)|}{\epsilon})\, dx$$

$$\leq 2\pi C(\frac{\beta || 1 ||_{\bar{B}}}{6L}) + (\eta + 2\pi\eta^2)\left[\int_{E_n'} A(|\frac{F(x-t)}{L}|)\, dx + |E_n'|\right]$$

$$\leq \frac{1}{2} + \frac{1}{2}$$

$$= 1 \qquad\qquad \text{if } n \geq N$$

i.e. $$||h-h_n||_C = ||Tg-T_n g||_C \leq \epsilon$$

$$||T-T_n||_C \leq \epsilon$$

Thus T_n tends to T in the operator norm and since each T_n is compact, T is itself a compact linear operator.

Q.E.D.

<u>Theorem 3.4</u>: If A, B, C are Young's functions which fail to satisfy the condition of theorem 3.3, then there exists $f \in L_A(0,2\pi)$ such that the operator $T = T_f$ defined by $h = Tg$ where

$$h(x) = \int_0^{2\pi} f(t) g(x-t)\, dt, \text{for all } g \in L_B$$

is not a compact linear operator from L_B into L_C.

<u>Proof</u>: Suppose the condition of theorem 3.3 is not satisfied, then there exists $\theta > 0$ such that given a sequence $\eta_n = 4^n$ we may find, for each n, $x_n \geq 1$ such that

$$A^{-1}(x_n)\; B^{-1}(\eta_n x_n) > \theta\eta_n x_n\; C^{-1}(\eta_n x_n)$$

Then the sequence x_n converges to ∞.

For, otherwise some sequence x_{n_k} is bounded, say,

$$x_{n_k} \leq \alpha < \infty$$

Then

$$\begin{aligned} \infty &> A^{-1}(\alpha)\; B^{-1}(\eta\alpha) \\ &\geq A^{-1}(x_{n_k})\; B^{-1}(\eta_{n_k} x_{n_k}) \\ &> \theta\; \eta_{n_k} x_{n_k}\; C^{-1}(\eta_{n_k} x_{n_k}) \\ &\geq \theta\; 4^{n_k}\; C^{-1}(1) \\ &\to \infty \quad \text{as} \quad k \to \infty \end{aligned}$$

This is a contradiction.

We may choose a subsequence x_n' such that

$$x_1' > 4\;, \qquad x_n' > 4x_{n-1}'$$

That is, after renaming points we have a sequence $\eta_n \geq 4^n$ and sequence

$$x_1 > 4 \quad \text{and} \quad x_n > 4x_{n-1}$$

such that $\quad A^{-1}(x_n)\ B^{-1}(\eta_n x_n) > \theta\ \eta_n x_n\ C^{-1}(\eta_n x_n)$

where $\quad \eta_n \geq 4^n, \quad x_n > 4^n$

Let E_n be a sequence of disjoint measurable subset of $[0,2\pi]$ such that

$$E_n = [a_n, b_n]$$

$$|E_n| = \frac{1}{2^n x_n}$$

and $$a_{n-1} - b_n = \frac{1}{x_n} + \frac{1}{x_{n-1}}$$

As $$\sum_1^\infty \left(\frac{1}{2^n x_n} + \frac{1}{x_n} + \frac{1}{x_{n-1}}\right) < 2\pi ,$$

there is enough room in $[0,2\pi]$ for the disjoint subsets E_n.

Let $$F_n = [0, \frac{1}{\eta_n x_n}]$$

then $$\sum_1^\infty \frac{1}{\eta_n x_n} < \sum_1^\infty \frac{1}{16^n} < 2\pi$$

Define $$f(x) = \sum_{n=1}^\infty A^{-1}(x_n)\ \chi_{E_n}(x)$$

and $$g_n(x) = B^{-1}(\eta_n x_n)\chi_{F_n}(x) \quad \text{for all } n=1,2,\cdots$$

Then $$\int A\,(|f(x)|)\,dx$$

$$= \int A\,(|\sum_{n=1}^{\infty} A^{-1}(x_n)\,\chi_{E_n}(x)|)\,dx$$

$$= \sum_{n=1}^{\infty} A\,A^{-1}(x_n)\,|E_n|$$

$$\leq \sum_{n=1}^{\infty} x_n \frac{1}{2^n x_n}$$

$$\leq \sum_{n=1}^{\infty} \frac{1}{2^n}$$

$$= 1$$

so $||f||_A \leq 1$

i.e. $f \in L_A$

and $$\int B(|g_n(t)|)\,dt$$

$$= \int B(|B^{-1}(\eta_n x_n)\,\chi_{F_n}(t)|)\,dt$$

$$= B\,B^{-1}(\eta_n x_n)\,|F_n|$$

$$\leq \eta_n x_n \frac{1}{\eta_n x_n}$$

$$= 1$$

Thus $$||g_n||_B \leq 1$$

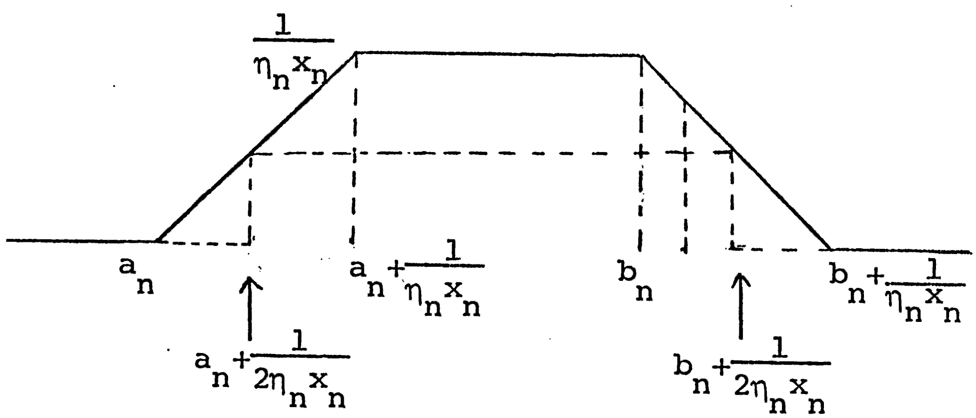
Consider $$h_n(x) = \int f(t)\,g_n\,(x-t)\,dt$$

$$> \int A^{-1}(x_n)\chi_{[a_n,b_n]}(t)B^{-1}(\eta_n x_n)\chi_{[0,\frac{1}{\eta_n x_n}]}(x-t)\,dt$$

$$= A^{-1}(x_n)B^{-1}(\eta_n x_n)\int \chi_{[a_n,b_n]}(t)\chi_{[0,\frac{1}{\eta_n x_n}]}(x-t)\,dt$$

$$= A^{-1}(x_n)B^{-1}(\eta_n x_n)\int_{a_n}^{b_n} \chi_{[x-\frac{1}{\eta_n x_n},x]}(t)\,dt$$

$$> \theta\,\eta_n x_n\, C^{-1}(\eta_n x_n)\left|[a_n,b_n] \cap [x-\frac{1}{\eta_n x_n},x]\right|$$



$$> \theta\,\frac{1}{2\eta_n x_n}\,\eta_n x_n\, C^{-1}(\eta_n x_n)\chi_{[a_n+\frac{1}{2\eta_n x_n},\,b_n+\frac{1}{2\eta_n x_n}]}(x)$$

$$> \frac{\theta}{2}\, C^{-1}(\eta_n x_n)\ \chi_{[a_n+\frac{1}{2\eta_n x_n},\,b_n+\frac{1}{2\eta_n x_n}]}$$

Hence $\quad h_n(x) > \frac{\theta}{2}\, C^{-1}(\eta_n x_n)$

if $\quad x \in [a_n + \frac{1}{2\eta_n x_n},\ b_n + \frac{1}{2\eta_n x_n}]$

If $m > n$

$$h_m(x) = \int f(t)\, g_m(x-t)\, dt$$

$$= \int f(t) B^{-1}(\eta_m x_m) \chi_{[0,\frac{1}{\eta_m x_m}]}(x-t)\, dt$$

$$= \int f(t) B^{-1}(\eta_m x_m) \chi_{[x-\frac{1}{\eta_m x_m},x]}(t)\, dt$$

Suppose $x \leq a_{n-1}$

and $b_{n+1} < x - \frac{1}{\eta_m x_m}$

Then $h_m(x) = \int A^{-1}(x_n) \chi_{[a_n,b_n]}(t) B^{-1}(\eta_m x_m) \chi_{[x-\frac{1}{\eta_m x_m},x]}(t)\, dt$

$$= A^{-1}(x_n) B^{-1}(\eta_m x_m) \int \chi_{[a_n,b_n]}(t) \chi_{[x-\frac{1}{\eta_m x_m},x]}(t)\, dt.$$

If $x - \frac{1}{\eta_m x_m} = b_n$

i.e. $x = b_n + \frac{1}{\eta_m x_m}$

$$h_m(x) = A^{-1}(x_n) B^{-1}(\eta_m x_m) \int \chi_{[a_n,b_n]}(t) \chi_{[b_n,b_n+\frac{1}{\eta_m x_m}]}(t)\, dt$$

Hence $h_m(x) = 0$ if $x \in (b_n + \frac{1}{\eta_m x_m}, b_n + \frac{1}{\eta_n x_n}]$

But $m > n.$

$$x_m \geq x_{n+1} > 4x_n \,, \qquad \eta_m > \eta_n$$

Thus $\quad h_m(x) = 0 \qquad \text{if } x \in [b_n + \frac{1}{4\eta_n x_n}, b_n + \frac{1}{\eta_n x_n}]$

Consider $\quad \int C\left(\frac{16|h_n(x)-h_m(x)|}{\theta}\right) dx \qquad \text{if } m > n$

$$> \int_{[b_n + \frac{1}{4\eta_n x_n}, b_n + \frac{1}{2\eta_n x_n}]} 4C\left(\frac{4|h_n(x)|}{\theta}\right) dx$$

$$> \int_{[b_n + \frac{1}{4\eta_n x_n}, b_n + \frac{1}{2\eta_n x_n}]} 4C\left(2C^{-1}(\eta_n x_n)\right) dx$$

$$> \frac{4}{4\eta_n x_n} \cdot \eta_n x_n$$

$$= 1$$

Thus $\quad ||h_n - h_m||_C \geq \frac{\theta}{16} \quad \text{if } \; m > n.$

Hence no L_C convergent subsequence can be extracted from $h_1, h_2, \cdots$. Therefore T cannot be compact.

Q.E.D.

BIBLIOGRAPHY

1. M. JODEIT. Some Relations among Orlicz Spaces. Master's thesis, Rice University, Houston, Texas (June 1965).

2. M. A. KRASNOSEL'SKII and Ya. B. RUTICKII. Convex Functions and Orlicz Spaces. (Trans. L. F. Boron). P. Noordhoff, Ltd. (1961).

3. L. A. LIUSTERNIK and V. J. SOBOLEV. Elements of Functional Analysis.

4. R. O'NEIL. Fractional Integration in Orlicz Spaces. I. Trans. Am. Math. Soc. Vol. 115, Issue 3 (1965) p. 300-328.

5. R. O'NEIL. Integral Transforms and Tensor Products on Orlicz Spaces and L(p,q) Spaces. Journal D'Analyse Mathématique Vol. XXI, p. 1-276.

6. L. PONTRJAGIN. Topological Groups (Trans. EMMA LEHMER).

7. F. RIESZ and B. Sz NAGY. Lecons d'analyse Fonctionnelle. Akadémiai Kaido, Budapest (1953).

8. A. ZYGMUND. Trigonometric Series. 2 vols. Cambridge Univ. Press (1959).