RICE UNIVERSITY

STELLAR WIND INTERACTION
WITH PLANETARY NEBULAE

by

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It has been proposed by other authors that a stellar wind might provide a pressure support at the inner edge of a planetary nebula to prevent the inward motion of material and to insure the ring-like appearance common to most planetary nebulae. The interaction of a supersonic stellar wind with a planetary nebula is examined with three models.

In the shock-relation model the shock relations are generalized to allow for arbitrary changes in the mass, momentum and energy flux of the stellar wind. It is found that there exists a restrictive lower limit on the amount of energy that can be removed from a supersonic flow and still satisfy the shock relations. It is concluded that a supersonic stellar wind cannot flow into a planetary and cool to observed planetary temperatures ($\sim 10^4$°K) and that a shock front must exist between the central star and planetary to reduce the flow to subsonic velocities before reaching the planetary.

In the radiation model the subsonic stellar wind is allowed to lose energy by radiation only. It is concluded that the stellar wind does not significantly cool in a
distance the order of a planetary radius ($\sim 10^4$ A.U.), and that in order for the flow to be cooled and slowed to observed planetary expansion velocities (10-30 km/sec), the flow must first transfer energy and momentum to the planetary.

In the third model the flow is allowed to transfer energy in coulomb collisions and momentum through a viscous force term to a stationary planetary that is heated by ionization-recombination processes and cooled by radiating in the $N_1$ and $N_2$ forbidden lines of OIII. The distances involved in the solutions are less than the mean free path of the flow particles. It is concluded that processes other than those dominant in the main body of the planetary may need to be taken into account in the region of interaction and that the assumption that fluid dynamics may be applied to this flow problem may not be valid.
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CHAPTER 1 - GENERAL SURVEY

1.1 INTRODUCTION

There has been much interest in planetary nebulae because of the role they may play in the evolution of low and intermediate mass stars. Osterbrock (1964) has suggested that a planetary nebula is a stage late in the life of a low mass (1.2 $M_\odot$) star and that it is plausible, but not proved, that all such stars may pass through this stage. Much work has been done on the spectra, physical processes, central stars, shapes and spatial distribution of planetary nebulae (Abell and Goldreich 1966, Aller 1956, Aller and Liller 1968, Gurzadyan 1969, Menzel 1962, Osterbrock 1964, Seaton 1960). Only recently have dynamic models been constructed by Mathews (1966) and Sofia and Hunter (1968). Since the central stars are very hot and at least once in their lives have ejected mass, it is not unreasonable to assume that they are still ejecting mass in the form of a stellar wind. Indeed, Mathews found a stellar wind was necessary to account for the observed shapes of planetary nebulae. It is the purpose of this thesis to present some simple models of the region of interaction between the stellar wind and planetary nebula.
By neglecting many interesting details, it is possible to make a first approximation in describing a planetary nebula, its principal processes and its evolution. This is done in Section 1.2.

In Chapter 2 we generalize the shock jump relations to allow for the addition or loss of mass, momentum and energy to a subsonic or supersonic flow. We conclude that if a supersonic stellar wind flows into a planetary nebula, a shock wave must form in the region of interaction and move upstream toward the central star to reduce the mach number of the flow to less than one before energy and momentum exchange can occur between the stellar wind and planetary nebula.

In Chapter 3 we examine a model in which a subsonic stellar wind cools by radiation between the shock mentioned above and the planetary. We find that the distance between the shock and the planetary must be much larger than typical planetary radii ($10^4$ A.U.) if the stellar wind is to cool to planetary temperatures ($10^4$°K) by radiation alone.

In Chapter 4 we examine a model in which the stellar wind transfers energy to the planetary material which then, because of its greater density, is able to radiate energy at a greater rate. The model assumes a static planetary and that the region of interaction is in a steady state. For the energy and momentum transfer terms used to model this
interaction, we are not able to find solutions that allow the very hot \(10^7\)°K post-shock stellar wind to cool to planetary temperatures. If lower stellar wind temperatures are assumed \(10^5\)°K, solutions can be found. The reason for this behavior is examined.

In Chapter 5 the conclusions and suggestions for further work are discussed.

1.2 PLANETARY NEBULA

A planetary nebula is a roughly symmetric cloud of gas surrounding a very hot star. Seen through a telescope, its image may resemble the planets Uranus or Neptune. The cloud is usually a spherical shell of almost completely ionized hydrogen with a radius, \(R\), of \(10^4\) A.U. and a thickness of about \(\frac{1}{3} R\). A mean abundance ratio by number is \(H : He : O :: 1000 : 180 : 1\) with an electron density of \(10^4\) cm\(^{-3}\), temperature of \(10^4\)°K, and a total mass of \(0.2 M_\odot\). They expand with velocities of 10-30 km/sec and have lifetimes the order of \(10^4\) years (Aller and Liller 1968, Gurzadyan 1969).

The central stars have very high temperatures, 30,000 to 100,000°K, and usually fit one of five types: 1) Wolf-Rayet type with broad emission lines, 2) Of type with emission features, 3) O type with absorption but no emission
features, 4) Continuous-spectra type with no emission or absorption features, 5) High-excitation type with such features as OVI emission lines (Aller and Liller 1968). They have a mass of about 1 M☉ and absolute magnitudes, Mpg, from 8.5 to -2.5 (O'Dell 1963).

There are about 1000 known planetaries in our galaxy but Seaton (1966) estimates there may be up to 8,500. Most are small and stellar in appearance (Gurzadyan 1969), but those that can be resolved show a wide variety of shapes. Their strong concentration toward the galactic center and moderate concentration toward the galactic plane indicate they are part of the Type II disk population (Minkowski and Abell 1963, Abell 1966). NGC 7078 in the globular cluster M15 shows that they may also be extreme Population II. Planetaries have also been observed in the Magellanic Clouds and M31.

A planetary emits radiation by degrading stellar ultraviolet quanta in ionization-recombination processes and by electron excitation of low-lying metastable states of atoms and ions followed by the emission of forbidden line radiation. Even though the gas is highly ionized, there is sufficient neutral hydrogen to make the planetary optically thick to Lyman radiation. The dilution factor, \( W = R_*/4R \), where \( R_* \) is the radius of the central star and \( R \) is the planetary radius, is small enough that the stellar
radiation density is small and few ionizations take place from excited states. If the nebula is optically thick in the Lyman lines, each Lyman photon (\(\lambda < 912 \text{ Å}\)) absorbed will ultimately be degraded into one Ly\(\alpha\) photon and a Balmer series photon. This provides a means of counting the number of Lyman quanta emitted by the star and thereby determining its temperature. This is the basis of the Zanstra method for determining stellar temperatures.

A free electron shares its energy with other electrons with a mean time of a few seconds, suffers an inelastic collision with an atom or ion with a mean time of a few months and recombines with a mean time of about 10 years. The energy addition and loss rates and the above relative lifetimes maintain a Maxwellian electron velocity distribution with a temperature of about \(10^4\)°K (Bohm and Aller 1947).

The origin of the green nebular lines (Bowen 1927) and the principal cooling mechanism is the excitation of OIII by electron collisions followed by the emission of the nebulium lines, \(\text{N}_1(5007 \text{ Å})\) and \(\text{N}_2(4959 \text{ Å})\), in the forbidden transition \(^1\text{D} \rightarrow ^3\text{P}_{1,2}\) (Seaton 1960). Even if the planetary were pure hydrogen, electron excitation of neutral hydrogen would prevent the electron temperature from rising above 20,000°K.
The planetary evolves from an optically thick, dense gas cloud close to the central star. It expands and becomes optically thinner, forms the familiar ring-shaped structure, and finally becomes very large, tenuous and optically thick due to a drop in luminosity of the central star. During this time, $\sim 10^4$ years, the star may evolve significantly toward the white dwarf stage. According to Seaton (1966), the star evolves in a period of 50,000 years from an initial luminosity of $60 \, L_\odot$ and temperature of $32,000^\circ K \rightarrow [25,000 \, L_\odot, 60,000^\circ K] \rightarrow [25,000 \, L_\odot, 100,000^\circ K] \rightarrow [100 \, L_\odot, 100,000^\circ K]$. The final luminosity drop is a consequence of the onset of degeneracy. O'Dell (1963) suggested that in 25,000 years a $1.2 \, M_\odot$ star rapidly contracts in gravitational collapse from $1 \, R_\odot$ to $0.01 \, R_\odot$ while the effective temperature increases from $40,000^\circ K$ to $150,000^\circ K$. Both these times are short compared to the $10^9$ years required for a $1 \, M_\odot$ star to evolve off the main sequence.

It is not known what kind of stars pass through the planetary nebula phase during their evolution toward becoming a white dwarf. Osterbrock (1964) suggested that it is plausible that all low mass stars might. The ejection mechanism and the reasons for the rather well defined inner and outer edges are not well understood.
Planetaries are not related to novae events (Minkowski 1948) because planetary expansion velocities are only 30 km/sec, while novae have velocities the order of 1000 km/sec and observation of novae remnants indicates that the interstellar medium does not significantly slow them. The masses ejected by novae are the order of $10^{-6} - 10^{-4} \, M_\odot$ and more than one ejection usually occurs, while planetaries have no more than two envelopes (Gurzadyan 1969). Various ejection mechanisms have been proposed: pulsational instabilities (Rose 1966), shock waves associated with a helium flash (Hayshi et al. 1962), dynamic instabilities in Red Giants with the release of energy by recombination of H and He (Lucy 1967) and slow mass loss by radiation pressure on dust grains (Swamy and Stecher 1969).

Planetary expansion velocities appear to increase linearly with radius within the shell. Wilson (1958a) and Abell and Goldreich (1966) suggested the present velocities and shell structure may be due to an initial ballistic push followed by the shell's disintegration under the influence of gas and Ly$\alpha$ radiation pressure. Auer (1968) has shown that the doppler shifts caused by thermal and expansion velocities may prevent the Ly$\alpha$ radiation density from building up and having a significant dynamic effect. Zanstra (1958) expressed the belief that the final velocity is not
due to an initial action but is the result of forces that act at a later stage. Weedman (1968), from a study of velocity gradients and the apparent relationship between shell thickness, radius and velocity, suggested that a nebula begins as a sphere of material with low expansion velocity and is then accelerated outward. The velocity and radius increase while the thickness changes little. From this he concluded that there is a pressure acting on the inside of the shell to balance the gas pressure within the shell.

The outer edge may be the boundary of a Strömgren sphere expanding at twice the gas velocity (Schatzman and Kahn 1955) or it may be the physical edge that is prevented from freely expanding into space and becoming diffuse by an interaction with the interstellar medium. Liller et al. (1966) noted that old planetaries close to the galactic plane expand less than the expected rate and attributed this to an interaction with the interstellar medium.

Emission lines shifted toward the violet and broadening of absorption lines have been interpreted to mean that the central stars may be ejecting up to $10^{-4} M_\odot$ per year in the form of a stellar wind with a velocity from 500 to 1500 km/sec (Morton 1967, Wilson 1958b).
The momentum delivered by such a stellar wind in only 600 years would equal the present momentum of a typical planetary. If the central star is contracting at approximately constant temperature, the increasing surface gravity of the star may lead to quenching of the stellar wind to prevent high final expansion velocities.

Mathews (1966) constructed dynamic models of planetary nebulae and found it necessary to have a stellar wind supply pressure support at the inner edge to prevent the inward motion of the planetary material and to insure the ring-like appearance common to most planetaries. He assumed there was a shock between the planetary and central star to slow the supersonic wind and that the high temperatures developed in the post-shock region were decreased by radiative cooling in a distance small compared to the thickness of the nebula, although he noted that this required a rather large cooling rate.

Sofia and Hunter (1968) found that a stellar wind was not necessary in their models and that for numerical reasons only did they add what appeared to be a stellar wind at the inner boundary of the planetary.
2.1 DERIVATION OF GENERALIZED SHOCK RELATIONS

In this chapter we examine generalized shock relations to determine what restrictions, if any, are placed on the addition and loss of energy and momentum to a gas flow.

Consider a one-dimensional flow and let two planes normal to the flow divide it into three regions (Figure 1). Region I is referred to as the front, Region II as the back. Let the flow parameters be steady, independent of time. The shock relations that express the conservation of mass, momentum and energy in this situation are

\[
\rho_2 v_2 = \rho_1 v_1 \quad (1)
\]

\[
\rho_2 v_2^2 + p_2 = \rho_1 v_1^2 + p_1 \quad (2)
\]

\[
\frac{5}{3} \rho_2 v_2^3 + \frac{\gamma}{\gamma-1} p_2 v_2 = \frac{5}{3} \rho_1 v_1^3 + \frac{\gamma}{\gamma-1} p_1 v_1 \quad (3)
\]

where \( \rho \) is the density, \( v \) is the velocity, and \( P \) is the pressure. \( \gamma = 5/3 \) and the subscripts refer to Regions I and II.

Equations (1), (2) and (3) are referred to as the shock relations because they are usually used to find the flow parameters in Region II if the parameters in Region I
Regions to which shock relations are applied.

FIGURE 1
are known and if there is a shock in Region III. But these relations are valid whether or not there is a shock in Region III. All that is required is that the flow be one-dimensional, steady and that no mass, momentum or energy is added or lost by the flow in Region III.

We now generalize the shock relations to allow for addition or loss of mass, momentum and energy in Region III.

\[ p_2 v_2 = p_1 v_1 + S \]  \hspace{1cm} (4)

\[ p_2 v_2^2 + p_2 = p_1 v_1^2 + p_1 + L \]  \hspace{1cm} (5)

\[ \frac{3}{2} p_2 v_2^3 + \frac{\gamma}{\gamma-1} p_2 v_2 = \frac{3}{2} p_1 v_1^3 + \frac{\gamma}{\gamma-1} p_1 v_1 + U \]  \hspace{1cm} (6)

where \( S, L \) and \( U \) are the mass, momentum and energy added to the flow per sec per unit area normal to the flow in a column the length of Region III. \( S \) has dimensions \( \text{gm/cm}^2/\text{sec} \), \( L \) has dimensions \( \text{(gm cm/sec)/cm}^2/\text{sec} \), and \( U \) has dimensions \( \text{erg/cm}^2/\text{sec} \).

Cloutier et al. (1969) examined the situation \( S \neq 0 \) and \( L = U = 0 \) when the flow in Region I is an ionized and magnetized solar wind and the mass added in Region III is due to ions being formed by solar radiation in the upper Martian atmosphere and being carried along with the ion flow. The ions were assumed to be cold and have no net momentum when formed so that the \( L \) and \( U \) terms were zero.
They found that there is an upper limit on $S$ ($\sim 9/16 \rho_1 v_1$ for large incident mach numbers). For values of $S$ above this limiting value the shock relations had complex solutions. They concluded that a shock would form in Region III and propagate upstream into Region I and divert the flow around the Martian atmosphere in much the same way that the earth's magnetic field diverts the solar wind around the earth.

We now examine (4), (5) and (6) to see if such limiting values exist on $L$ and $U$. Rewriting them in terms of the following dimensionless variables

$$\eta = \frac{v_1}{v_2} \quad \phi = \frac{\rho_2}{\rho_1} \quad Y = \frac{p_2}{p_1} \quad M^2 = \frac{\rho_1 v_1^2}{\gamma p_1} \quad \sigma = \frac{S}{\rho_1 v_1} \quad \beta = \frac{L}{\rho_1 v_1^2} \quad \alpha = \frac{U}{\rho_1 v_1^3},$$

we get

$$\frac{\phi}{\eta} = 1 + \sigma \quad (8)$$

$$1 - \frac{\phi}{\eta}^2 + \frac{1}{\gamma M^2} (1 - Y) = -\beta \quad (9)$$

$$\frac{\phi}{\eta} (1 - \frac{\phi}{\eta}^2) + \frac{1}{(\gamma - 1) M^2} (1 - Y/\eta) = -\alpha. \quad (10)$$
Eliminating $Y$ from (9) and (10) with $\gamma = 5/3$ and letting

$$\eta_s = \frac{(\gamma+1)M^2}{2+2/(\gamma-1)M^2}$$  \hspace{1cm} (11)$$

$$A = \alpha \left[ \frac{2(\gamma-1)M^2}{2+2/(\gamma-1)M^2} \right] = \alpha \frac{2(\gamma-1)}{(\gamma+1)} \eta_s = \alpha \frac{\eta_s}{2}$$  \hspace{1cm} (12)$$

$$B = \beta \left[ \frac{2M^2}{2+2/(\gamma-1)M^2} \right] = \beta \frac{2\gamma}{(\gamma+1)} \eta_s = \beta \frac{5\eta_s}{4}$$  \hspace{1cm} (13)$$

and using (8) to eliminate $\xi$ we get

$$\eta^2(1+A) - \eta(1+\eta_s+B) + (1+\sigma)\eta_s = 0$$  \hspace{1cm} (14)$$

which has the solutions

$$\eta = \frac{-(1+\eta_s+B) \pm \sqrt{(1+\eta_s+B)^2 - 4(1+\sigma)(1+A)\eta_s}}{2(1+A)}.$$  \hspace{1cm} (15)$$

For a given mach number, $A$ and $B$ are just constants times the relative energy and momentum addition rates $\alpha$ and $\beta$.  

If \( A = B = \sigma = 0 \), then

\[
\eta = 1, \quad \eta_s
\]

(16)

where \( \eta_s \) is the usual shock value (\( \sim 4 \) if \( M >> 1 \)). If \( \eta = 1 \), there is no shock in Region III and there is no change in flow parameters. If \( \eta = \eta_s \), there is a shock in Region III. (See Figure 2 for the bracketed terms in (12) and (13) and \( \eta_s \) as functions of \( M \).)

If \( A = B = 0 \) and \( \sigma \neq 0 \), then

\[
\eta^2 - \eta(1+\eta_s) + (1+\sigma)\eta = 0
\]

(17)

\[
\eta = \frac{1 \pm \sqrt{(1-\eta_s)^2 - 4\sigma\eta_s}}{2}
\]

(18)

and it is easy to see that the upper limit on \( S, \rho_1 v_1 \sigma \), is set by requiring that the term in the square root not be negative

\[
\frac{S}{\rho_1 v_1} = \sigma \leq \sigma_{\text{max}} = \frac{(1-\eta_s)^2}{4\eta_s}
\]

(19)

which for large values of mach number, \( \eta_s \approx 4 \), is \( 9/16 \). For the work below \( \sigma \) is assumed to be zero.
FIGURE 2 - Bracketed terms in (12) and (13) and $\eta_s$ as functions of mach number.
TABLE 1

Conversion Factors

<table>
<thead>
<tr>
<th>M</th>
<th>Col 1</th>
<th>Col 2</th>
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<tbody>
<tr>
<td>0.1</td>
<td>150.0</td>
<td>60.2</td>
</tr>
<tr>
<td>0.2</td>
<td>38.0</td>
<td>15.2</td>
</tr>
<tr>
<td>0.3</td>
<td>17.2</td>
<td>6.9</td>
</tr>
<tr>
<td>0.4</td>
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</tr>
<tr>
<td>0.7</td>
<td>3.5</td>
<td>1.4</td>
</tr>
<tr>
<td>0.8</td>
<td>2.8</td>
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</tr>
<tr>
<td>0.9</td>
<td>2.3</td>
<td>0.94</td>
</tr>
<tr>
<td>1.0</td>
<td>2.0</td>
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</tr>
<tr>
<td>2.0</td>
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<td>0.35</td>
</tr>
<tr>
<td>5.0</td>
<td>0.56</td>
<td>0.22</td>
</tr>
<tr>
<td>10.0</td>
<td>0.51</td>
<td>0.20</td>
</tr>
</tbody>
</table>

To get $U/\rho_1 v_1^3$, multiply $A$ by the value in Col 1.

To get $L/\rho_1 v_1^2$, multiply $B$ by the value in Col 2.
2.2 ADDITION AND LOSS OF ENERGY ONLY

2.2.1 Specializing the Shock Relations

A flow losing energy by radiation or conduction may be characterized as having \( A \neq 0 \) and \( B = \sigma = 0 \), where \( A \) from (12) is a constant times the relative energy addition rate \( \alpha \), \( B \) from (13) is a constant times the relative momentum addition rate \( \beta \), and \( \sigma \) from (7) is the relative mass addition rate. Equations (14) and (15) become

\[
\eta^2(1+A) - \eta(1+\eta_s) + \eta_s = 0
\]

\[
\eta = \frac{1+\eta_s \mp \sqrt{(1-\eta_s)^2 - 4A\eta_s}}{2(1+A)}
\]

(See Figure 3.)

The upper limit on \( A \), \( A_m \), the value at the nose of the curve, is determined by requiring that the term in the square root be zero:

\[
A_m = \frac{(1-\eta_s)^2}{4\eta_s}
\]

For \( A > A_m \), \( \eta \) is complex. It can also be shown that at \( A = A_m \), \( M_2 = 1 \) for any value of \( M_1 \). At the maximum energy addition rate the flow out the back of Region III is sonic.
FIGURE 3 - η from (21) as a function of λ for four different mach numbers. Dashed curve is η_a. As η_a is approached along a given mach curve, M_2 → ∞ and Y → 0.
From Equation (9),

\[ Y = 1 + (1 - \frac{1}{\eta}) \gamma M^2 \]  

(23)

and requiring positive values for pressures, \( P_2/P_1 = Y > 0 \),

\[ \eta > \eta_a = \frac{\gamma M^2}{1+\gamma M^2} \]  

(24)

Since it can be shown that

\[ M_2^2 = \frac{M_1^2}{\eta Y} \]  

(25)

the requirement that \( Y > 0 \) is the same as requiring that \( M_2 \) be finite and positive.

Using (20) to find the value of \( A \) corresponding to \( \eta_a \)

\[ A_a = \frac{\eta_a (1+\eta_s) - \eta_s}{\eta_a^2} - 1 \]  

(26)

(See Figure 4 for \( A_m \) and \( A_a \) as functions of \( M_1 \).)

Below we are interested in three situations:

1) The flow in Region I is subsonic.

2) The flow in Region I is supersonic and there is no shock in Region III.

3) The flow in Region I is supersonic and there is a shock in Region III.
FIGURE 4 - $A_m$ and $A_a$ from (22) and (26) as functions of mach number.

-1 < A < $A_m$  Subsonic Flow

$A_a < A < A_m$  Supersonic Flow - No Shock

-1 < A < $A_m$  Supersonic Flow - With Shock
2.2.2 Subsonic Flow in Region I

If the flow is initially subsonic and with $A = 0$ in Region III, then $\eta = \phi = Y = 1$ and $M_1 = M_2$. The flow is not changed in passing through Region III.

Starting at $(\eta = 1, A = 0)$ and following a subsonic curve as energy is added in Region III, $A$ increasing, we find a maximum energy addition rate, $A_m$, at which $M_2 = 1$ and above which $\eta$ is complex.

If $A$ is decreased from $(\eta = 1, A = 0)$, then as $A \to -1$, $\eta \to \infty$ and $M_2 \to 0$. For a given incident velocity $v_1$, $\eta \to \infty$ means that $v_2 \to 0$, or the flow is stopped and $Y \to 1 + \gamma M^2$ (unsubscripted mach numbers refer to Region I).

From Table 1 we see that for $A = -1$, a small mach number means that $U/\rho_1 v_1^3$ is a large negative value, and $A = -1$ means that

$$U = \frac{1}{2} \rho_1 v_1^3 + \frac{\gamma}{\gamma - 1} \rho_1 v_1$$

or that all of the energy flux must be lost in Region III.

For a subsonic flow: $-1 < A \leq A_m$. 
2.2.3 Supersonic Flow - No Shock in Region III

If the flow is initially supersonic and with no shock in Region III, at $A = 0$ we have $\eta = \frac{1}{2}$, $Y = 1$ and $M_1 = M_2$.

Starting at $(\eta = 1, A = 0)$ and following a supersonic curve as $A$ is increased, we find that there is a maximum addition rate, $A_m$, above which $\eta$ is complex and at which $M_2 = 1$. $\eta > 1$ at $A = A_m$ means that $v_2$ has been decreased by flowing into a region of higher pressure, $Y > 1$ from (23).

If $A$ is decreased from zero, we reach a lower limit, $A_a$, at which $Y = 0$, $M_2 \rightarrow \infty$ and $\eta = \eta_a < 1$. From Figure 4 and the conversion factors in Table 1, we see that a highly supersonic flow can lose only a small fraction of its pre-shock energy and still satisfy the shock relations. The increase in velocity, $\eta < 1$, is due to the acceleration by the decreasing pressure, $Y < 1$. Energy loss accelerates a supersonic flow.

2.2.4 Supersonic Flow - With Shock in Region III

If the flow is initially supersonic and if there is a shock in Region III, with $A = 0$, then $\eta = \eta_s$ and $M_2 < 1$. 
Starting at \((\eta = \eta_s, A = 0)\) and following a supersonic curve as energy is added, \(A\) increases until the maximum rate \(A_m\) is reached at which \(M_2 = 1\).

If \(A\) is decreased, a lower limit, \(A = -1\), is reached at which \(\eta \to \infty\) and \(M_2 \to 0\). This means that the flow is stagnated under the influence of an increasing pressure.

An energy loss in passing through a shock implies a lower velocity, higher pressure and a lower mach number in the post-shock region than if there were no energy loss. For \(M_1 \gg 1\), the lower limit on energy loss, \(A = -1\), means that \(U = -\frac{1}{2} \rho_1 v_1^3\).

2.2.5 Summary

There exists upper and lower limits on energy addition and loss rates in all three situations. For subsonic flow the limits are not very restrictive but for supersonic flow they are.

\[-1 < A \leq A_m \quad \text{Subsonic Flow}\]

\[A_a < A \leq A_m \quad \text{Supersonic Flow - No Shock}\]

\[-1 < A \leq A_m \quad \text{Supersonic Flow - With Shock}\]
2.3 ADDITION AND LOSS OF MOMENTUM ONLY

2.3.1 Specializing the Shock Relations

If \( A = \sigma = 0 \) and \( B \neq 0 \), then equations (14) and (15) are

\[ \eta^2 - \eta(1+\eta_s + B) + \eta_s = 0 \]  \hspace{2cm} (27)

\[ \eta = \frac{1+\eta_s + B \pm \sqrt{(1+\eta_s + B)^2 - 4\eta_s}}{2} \]  \hspace{2cm} (28)

which is plotted as a function of \( B \) in Figure 5.

The lower limit to \( B \) is determined by requiring that the term in the square root be zero. There are two values of \( B \) that satisfy this condition:

\[ B_m = -(1 - \sqrt{\eta_s})^2 \]  \hspace{2cm} (29)

\[ B'_m = -(1 + \sqrt{\eta_s})^2 \]  \hspace{2cm} (30)

If \( B \leq B'_m \), \( \eta \) is negative. If \( B'_m < B < B_m \), \( \eta \) is complex. Only if \( B_m < B \) is \( \eta \) positive and real. If \( B = B_m \), then \( M_2 = 1 \). From (10)

\[ Y = \eta[1 + (1 - 1/\eta^2)(\gamma-1) M^2] \]  \hspace{2cm} (31)
FIGURE 5 - \( \eta \) from (28) as a function \( B \) for four different mach numbers. Dashed curve is \( \eta_b \). As \( \eta_b \) is approached along a given mach curve, \( M_2 \to \infty \) and \( Y \to 0 \).
and \( Y > 0 \) requires

\[
\eta > \eta_b = \frac{(\gamma-1)M^2}{2+(\gamma-1)M^2}.
\]

(32)

The value of \( B \) corresponding to \( \eta_b \) is

\[
B_b = \eta_b + \frac{\eta_s}{\eta_p} - 1 - \eta_s.
\]

(33)

(See Figure 6 for \( B_m \) and \( B_b \) as functions of \( M_1 \)). We now examine the three situations mentioned in Section 2.2.1.

2.3.2 **Subsonic Flow in Region I**

If the flow is initially subsonic, then with \( B = 0, \eta = \xi = Y = 1 \) and \( M_1 = M_2 \).

Starting at \((\eta = 1, B = 0)\) and following a subsonic curve there does not appear to be an upper limit on the amount of momentum that may be added. As \( \eta \to \infty, M_2 \to 0 \) using (25) so that the velocity \( v_2 \) decreases and \( P_2 \) increases as momentum is added to a subsonic flow.

If \( B \) is decreased from \((\eta = 1, B = 0)\) then a lower limit, \( B_m \), is reached at which \( M_2 = 1 \).

For subsonic flow: \( B_m < B \).
FIGURE 6 - $B_m$ and $B_b$ from (29) and (33) as functions of mach number.

- $B_m < B$: Subsonic Flow
- $B_m < B < B_b$: Supersonic Flow - No Shock
- $B_m < B$: Supersonic Flow - With Shock
Figure 6
2.3.3 Supersonic Flow - No Shock in Region III

If the flow is initially supersonic and with no shock in Region III, at $B = 0$ we have $\eta = \frac{\gamma}{\gamma - 1} = Y = 1$ and $M_1 = M_2$.

Starting at $(\eta = 1, B = 0)$ and following a supersonic curve as $B$ is increased an upper limit on $B$, $B^*$, is reached at which $M_2 \to 0$ and $\eta = \eta^*_b < 1$.

If $B$ is decreased a lower limit, $B_m$, is reached at which $M_2 = 1$ and below which $\eta$ is complex. Since $\eta > 1$, momentum loss decelerates a supersonic flow.

2.3.4 Supersonic Flow - With Shock in Region III

If the flow is initially supersonic and if there is a shock in Region III, then with $B = 0$ we have $\eta = \eta_s$ and $M_2 < 1$.

Starting at $(\eta = \eta_s, B = 0)$ and following a supersonic curve as $B$ is increased, there does not appear to be an upper limit on $B$ and $v_2 , M_2 \to 0$.

If $B$ is decreased, a lower limit, $B_m$, is reached at which $M_2 = 1$ and below which $\eta$ is complex.

Momentum loss in passing through a shock implies a higher velocity and mach number and a lower pressure in the post-shock region than if there were no momentum loss.
2.3.5 Summary

There exists the same lower limit on \( B_m \) in the three situations considered. Only for supersonic flow without a shock does there exist an upper limit, \( B^* \).

2.4 Addition and Loss of Energy and Momentum

Since energy losses increase \( v_2 \) while momentum losses decrease \( v_2 \) for a supersonic \( v_1 \), we now examine what happens to the above limits when energy and momentum may both be lost in Region III.

From (15), if the term in the square root is to be positive

\[
A_m = \frac{(1-\eta_s)^2}{4\eta_s} + \frac{B(B+1+\eta_s)}{4\eta_s}
\]  

(34)

The only change from (22) is the addition of the second term. The influence on the lower limit, \( A_a \), and the corresponding \( \eta_a \) is

\[
\eta_a = \frac{\gamma M^2}{1+\gamma M^2 + B[1+\frac{3}{2}(\gamma-1)M^2]}
\]  

(35)

as compared to (24). By making \( B \) negative the denominator of \( \eta_a \) becomes smaller and \( \eta_a \) goes from less than one to greater than one.
Curves similar to those in Figure 3 are shown in Figure 7 for a supersonic flow ($\eta_s \sim 4$) for various values of $B$. $\eta_a$ is also shown.

For a given supersonic flow in Region I and with $A = B = 0$, we are at the point ($\eta = 1$, $A = 0$) in Figure 7. Whether it is possible or not to smoothly alter the conditions in Region III to $A = -0.5$, $B = -2$, for example, and not cause a shock to form when $\eta$ has complex values of $Y$ is zero or negative, depends on the way in which $A$ and $B$ are changed. If $A$ is kept constant at zero and $B$ is decreased from zero toward $-2$ we find that there are no real solutions if $B < -1$. This method of altering the conditions will not work. But if $B$ is reduced to $-0.5$ with $A$ fixed at zero, and then $A$ is reduced to $-0.2$ with $B$ fixed, etc., it is possible to follow the original point from ($\eta = 1$, $A = 0$) to a point where $A = -0.5$, $B = -2$, and at all times staying within the required limits imposed by $A_m$ and $A_a$.

For given loss rates $A$ and $B$ it is only possible to say if a real solution can exist; whether it does or not depends on the history of Region III.
FIGURE 7 - \( \eta \) from (15) with \( \sigma = 0 \) and \( \eta_s = 4 \) as a function of \( A \) for different values of \( B \). Dashed curve is \( \eta_a \). As \( \eta_a \) is approached along a given \( B \) curve, \( M_2 \to \infty \) and \( Y \to 0 \).
2.5 APPLICATION TO STELLAR WIND FLOW

We now examine what loss rates are required to cool and slow a stellar wind to planetary conditions of temperature and observed expansion velocity.

A stellar mass loss rate of $10^{-5} \, M_\odot/yr$ with flow velocity of $10^3$ km/sec corresponds to a mass density of $5 \, m_p/cm^3$ at $10^4$ A.U. if the material is predominantly hydrogen. $m_p$ is the proton mass. For such a stellar wind with a low enough temperature to insure a high mach number, $\sim 10^5\, K$, $\rho_1 v_1^3$ is $\sim 8$ erg/cm$^2$/sec. If $\rho_1 v_1 = \rho_2 v_2$, where Region II is now the planetary nebula that has an expansion velocity of 10 km/sec ($\eta = 100$ if $v_1 = 10^3$ km/sec), we find $\rho_2 = 500 \, m_p/cm^3$. $\rho_2 v_2^3$ is then the order of $8 \times 10^{-4}$, a factor of $\eta^2$ less than the value of $\rho_1 v_1^3$. If

$$P_2 v_2 = \frac{\rho_2 k T_2}{m_p} \cdot v_2$$

(36)

where $T_2$ is the planetary temperature and $k$ is Boltzmann's constant, then for $T_2 \sim 10^4 \, K$, $P_2 v_2$ is $\sim 7 \times 10^{-4}$. Neglecting $P_1 v_1$ compared to $\rho_1 v_1^3$ because the flow is supersonic, we find from (6)
\[
U = \frac{1}{3} \left( \rho_2 v_2^3 - \rho_1 v_1^3 \right) + \frac{\gamma}{\gamma-1} \rho_2 v_2
\]

\[
\approx -\frac{1}{2} \rho_1 v_1^3
\]

(37)

so that \( A \sim -1 \) using Table 1 and a large mach number.

We assume that the above value of \( A \) indicates that it is not possible for a supersonic stellar wind to flow directly into a planetary nebula with the stellar wind parameters being reduced to planetary conditions in a smooth transition. Rather, a shock wave will propagate upstream into Region I and make the flow subsonic before it can exchange energy or momentum with the planetary nebula.
CHAPTER 3 - RADIATION MODEL

We now examine a post-shock, subsonic stellar wind flow to determine the distance required for the stellar wind to cool by radiation alone to a temperature characteristic of a planetary. A steady-state, one-dimensional model with no thermal or momentum transfer to the nebula is assumed.

The time-dependent equations describing the flow are

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \tag{38}
\]

\[
\rho \left[ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right] = -\nabla P \tag{39}
\]

\[
\frac{\partial}{\partial t} \left[ \frac{P}{\gamma - 1} + \frac{1}{2} \rho \vec{v}^2 \right] + \nabla \cdot \left[ \frac{\gamma P}{\gamma - 1} \vec{v} + \frac{\gamma}{2} \rho \vec{v}^2 \vec{v} \right] = -A_o \rho^2 \tag{40}
\]

where we have used \( \frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla \) in (39).

Equation (38) is the conservation of mass equation, (39) is the force equation when only a pressure gradient force is acting and (40) is characteristic of a plasma that radiates at a rate proportional to the product of the ion and electron densities. \( A_o \) has dimensions \( \text{erg-cm}^3/\text{gm}^2/\text{sec} \).
In a steady state the partials with respect to time are zero and in one dimension the divergence and gradient become derivatives with respect to $x$, if the $x$ axis is chosen as the direction of flow. The above equations then become

$$\frac{d}{dx} (p\nu) = 0 \quad (41)$$

$$p\nu \frac{d\nu}{dx} = -\frac{dP}{dx} \quad (42)$$

$$\frac{d}{dx} \left( \frac{\gamma}{\gamma-1} p\nu + \frac{1}{2} p\nu^3 \right) = -A_o \rho^2 \quad (43)$$

where partial derivatives have been replaced by total derivatives. (41) and (42) can be integrated to give

$$p\nu = \text{constant} = S_o \quad (44)$$

$$p\nu^2 + P = \text{constant} = L_o \quad . \quad (45)$$

Letting $\rho_o$ and $v_o$ be the density and velocity at $x = 0$ and rewriting (43), (44) and (45) in terms of the dimensionless variables

$$R = \frac{\rho}{\rho_o} \quad \quad V = \frac{v}{v_o}$$

$$Z = \frac{P}{\rho_o v_o^2} \quad L = \frac{L_o}{\rho_o v_o^3} \quad X = \frac{A_o\rho_o}{v_o} x \quad ,$$

(46)
we get

\[ RV = 1 \] (47)

\[ RV^2 + Z = L \] (48)

\[ \frac{d}{dx} \left( \frac{\gamma}{\gamma-1} ZV + \frac{3}{2} RV^3 \right) = -R^2 \] (49)

which can be solved to give

\[ X = V^4 + \frac{5}{6} L (1 - V^3) - 1 \] (50)

where \( V(X = 0) = 1 \) has been used to evaluate the constant of integration.

It is possible to find other parameters of interest as functions of \( V \) and, using (50), as functions of \( X \).

\[ R = \frac{1}{V} \] (51)

\[ Z = L - V \] (52)

\[ M^2 = \frac{V}{\gamma(L-V)} \] (53)

\[ \frac{T(X)}{T(0)} = \frac{(L-V)V}{L-1} \] (54)

\[ \frac{\Delta E}{p_0 v_0} = \frac{5}{2} \ L (1-V) - 2 (1-V^2) \] (55)

\[ \tau = \frac{4}{3} (v^3-1) - \frac{5}{4} \ L (v^2-1) \] (56)
where $M$ is the local mach number, $T(X)$ is the temperature at $X$, $\tau$ is the time required to travel from $X = 0$ to a point where the velocity is $V$ and $\Delta E/(\rho_0 V_0^3)$ is the total amount of energy radiated by a unit volume in flowing from $X = 0$.

If the pre-shock mach number is large, $P = 3 \rho v^2$ in the post-shock flow. Using this value, $L = 4$. (See Figure 8.)

Using a large value for $A_o$ (Tucker and Gould 1966),

$$A_o = \frac{10^{-22}}{m_p^2}$$

and post-shock parameters for the stellar wind described in Section 2.5, $\rho_o = 20 \text{ m}_p/\text{cm}^3$ and $v_o = 250 \text{ km/sec}$, we find the following relation between the dimensionless distance $X$ and $x$ to be

$$x = 8.7 \times 10^5 X \text{ A.U.} \quad (58)$$

On this scale a typical planetary ($10^4$ A.U.) has a value $X \sim 0.01$.

The post-shock temperature from $P = 3 \rho v^2 = \rho k T/m_p$ at $X = 0$ is about $10^7$°K. If this value is used as $T(0)$ in (54) and if a final temperature the order of $10^4$°K is assumed, then from Figure 8 a distance the order of $X = 2.2$
FIGURE 8 - Equations (50)-(55) as functions of $X$.

Only $\frac{\Delta E}{p_o v_o^3}$ uses the scale to the right; all other curves use the scale to the left.
is needed to cool the flow by radiation. This is much larger than a planetary radius of 0.01.

We conclude that it is not possible for a stellar wind to cool significantly in the time it takes to flow from the shock to planetary. If the stellar wind is to cool to planetary temperatures, it must do so by losing energy, not by radiation, but by transfer to the cooler and denser planetary.
4.1 DESCRIPTION OF MODEL

Since a stellar wind can not cool to planetary

temperatures in a distance less than a planetary radius,
a model is proposed in which the energy is transferred
first to the planetary at a volume rate $A$ and is then
radiated by the planetary at a rate $\Lambda$. In this model
the flowing stellar wind and the planetary gas may also
exchange momentum through a volume force term $F$. Adding
a momentum transfer term, $-F$, to the right side of (39)
and the corresponding energy loss term, $-Fv$, to (40) we
get, after assuming a steady-state, one-dimensional flow,

$$\rho_s v_s = \text{constant}$$  \hspace{1cm} (59)

$$\rho_s v_s \frac{dv_s}{dx} = - \frac{dP_s}{dx} - F$$  \hspace{1cm} (60)

$$\frac{d}{dx} \left( \frac{\gamma}{\gamma-1} \rho_s v_s + \frac{1}{2} \rho_s v_s^3 \right) = -A - Fv_s$$  \hspace{1cm} (61)

where the subscripts denote stellar wind parameters.

A steady-state solution may be justified if a
typical time

$$\tau = \frac{d}{V_c}$$  \hspace{1cm} (62)
is much less than the lifetime of a planetary \((10^4 \text{ yr})\)
where \(d\) is the length of the interaction region and \(v_c\) is
the speed of sound. The region is one-dimensional if
\[
d \ll R
\]
where \(R\) is the planetary radius, so that curvature can be
neglected.

For the gas dynamic equations and the concept of
pressure to be valid, the characteristic lengths of the
solution must be larger than a particle mean free path.
For a point particle moving with velocity \(v\) through a
background of stationary particles of density \(n\) and cross-
section \(\sigma\), the mean free path is of the order
\[
\lambda = \frac{1}{n\sigma}.
\]

If the coordinate system is attached to the planetary
and if the gradient of planetary velocity over the region
of interaction is small, the planetary gas is motionless
and the momentum equation for it is
\[
- \frac{dP}{dx} + F = 0.
\]

If \(F > 0\), the force felt by the planetary is parallel
to the flow velocity of the stellar wind, \(\mathbf{v}_s\), and the
force felt by the flow is anti-parallel to \(\mathbf{v}_s\). \(P\) is
the planetary pressure.
Mass addition to the planetary by the stellar wind is neglected and a mass conservation equation for the planetary is not needed.

Steady-state energy conservation required that the net energy gain by a unit volume of planetary material, \( G \), be zero.

\[
    G = 0 
\]  

Equations (59), (60) and (61) can be solved to give

\[
    \frac{dv}{dx} = \frac{F_v - (\gamma - 1)A}{1 - M^2} \cdot \frac{1}{\gamma p_s} 
\]  

\[
    \frac{dT_s}{dx} = \frac{F_v - (\gamma M^2 - 1)A}{1 - M^2} \cdot \frac{-(\gamma - 1)}{\gamma n_s k v} 
\]  

\[
    \frac{dP_s}{dx} = \frac{F_v - (\gamma - 1)M^2 A}{1 - M^2} \cdot \frac{1}{v} 
\]

where the subscript on \( v_s \) has been dropped since there is only one velocity, that of the stellar wind, and \( n_s \) is the number density of stellar wind particles. The stellar wind is assumed to be electrically neutral and not magnetized so that mass addition as treated by Cloutier et al. (1969) can be neglected. The flow particles are assumed to have mass \( m_p \), the proton mass.
Note that a retarding force \((F > 0)\) and an energy loss \((A > 0)\) have opposite effects on the velocity and pressure gradients in subsonic and supersonic flows. This is in agreement with the results of Chapter 2 where, from shock relation considerations, we found that momentum and energy losses had opposite effects on \(v_2\), the velocity in the post-shock region. Whether a flow will be accelerated or decelerated by given values of \(F\) and \(A\) depends on the mach number. If \(A = 0\), a retarding force will accelerate a subsonic flow and decelerate a supersonic flow.

The ratio of \(Fv\) to \(A\) and the mach number determines the sign of the gradient in the above equations. (See Figure 9.)

We now assume specific forms for the terms \(F\), \(A\) and \(G\). The planetary is assumed to have two sources of energy: the central star and the stellar wind. An electron ejected in ionization by the absorption of stellar radiation has a kinetic energy, on the average, of \(kT_*\), where \(T_*\) is the color temperature of the central star (Spitzer 1968, Kahn 1954). The electron quickly shares this energy with other electrons and reaches a Maxwellian temperature of \(T_P\). If the density of electrons equals the density of ions, \(n_P\), then the number of recombinations per cm\(^3\) per sec is \(n_P^2\).
FIGURE 9 - Sign of gradients as determined by \( \frac{F_v}{A} \) and \( M \).
FIGURE 9
TABLE 2

Sign of Gradients in Regions of Figure 9.

<table>
<thead>
<tr>
<th>REGION</th>
<th>( \nabla v )</th>
<th>( \nabla T_s )</th>
<th>( \nabla P_s )</th>
<th>( \nabla M )</th>
</tr>
</thead>
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<tr>
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<td>(+)</td>
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<tr>
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<td>10</td>
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</table>
where $\alpha$ is $3 \times 10^{-10} \frac{T_p^{-3/4}}{T_p}$ (Allen 1955). The net amount of energy deposited in the planetary by this ionization-recombination process is

$$\Gamma = k(T_\ast - T_p) \frac{3 \times 10^{-10}}{T_p^{3/4}} n_p^2$$  \hspace{1cm} (71)$$

and has dimensions erg/cm$^3$/sec.

Conservation of energy requires that all the energy lost by the stellar wind must be deposited in the planetary if the stellar wind does not lose energy by radiation. The energy lost by the stellar wind is $F_v + A$.

The planetary is assumed to lose energy by radiating in the $N_1$ and $N_2$ forbidden lines of OIII.

$$A = \frac{3.88 \times 10^{-21}}{T_p^{1/2}} \exp\left(-\frac{27900}{T_p}\right) n_p^2$$  \hspace{1cm} (72)$$

is an approximation to this loss rate (Mathews 1968).

(See Figure 10.)

The energy transfer rate $A$ is assumed to be due to coulomb collisions between the subsonic stellar wind with a temperature $T_s$ and density $n_s$ and the planetary material with a temperature $T_p$ and density $n_p$ and may be written as.
FIGURE 10 $- \frac{\Gamma - \Lambda}{n_p^2} x 10^{24}$ from (71) and (72) as a function of $T_p$ for two values $T_\ast$. 
\[
A = \frac{4\sqrt{2\pi} e^4 \ln(2/\theta_m^s) (kT_s - kT_p)}{\sqrt{m_p} (kT_s + kT_p)^{3/2}} n_s n_p
\] (73)

if the particles both have mass \(m_p\) (Longmire 1955). (See Figure 11.) \(e\) is the proton charge, \(4.8 \times 10^{-10}\), and \(\ln(2/\theta_m^s)\) is the coulomb logarithm which is about equal to 25.

The energy balance equation for the planetary is

\[
G = \Gamma - \Lambda + Fv + A.
\] (74)

\(F\) is assumed to have the form \(\rho_s v\) times a collision frequency \(\nu\). For a point particle moving with velocity \(v\) through stationary particles of density \(n_p\) and with cross-section \(\sigma\), \(\nu\) is \(n_p \sigma v\). So that

\[
F = \rho_s v^2 \sigma n_p.
\] (75)

Although \(\sigma\) is one of the more accurately known parameters introduced in this model, it may be treated as a variable parameter to adjust the ratio \(Fv/A\) to place the initial conditions of the integration in various regions of Figure 9. \(\sigma\) is the order of the geometric cross-section for an atom, \(10^{-16}\) cm\(^2\).
FIGURE 11  -  \[ \frac{A}{n_s n_p} \] as a function of \[ \frac{T_s}{T_p} \] with the vertical scale in arbitrary units.
\[
\frac{A}{n_s n_p}
\]

FIGURE II

1000 \(T_s / T_p\)

100

10

1
The system of equations to be solved is:

\[ \rho_s v = \text{constant} \quad \text{(76)} \]

\[ \rho_s v \frac{dv}{dx} = - \frac{dP_s}{dx} - F \quad \text{(77)} \]

\[ \frac{d}{dx} \left( \frac{\gamma}{\gamma - 1} p_s v + \frac{1}{2} \rho_s v^3 \right) = - A - Fv \quad \text{(78)} \]

\[ \frac{dP_s}{dx} = F \quad \text{(79)} \]

\[ G = \Gamma - A + Fv + A = 0 \quad \text{(80)} \]

\[ F = \rho_s v^2 \sigma_p \quad \text{(81)} \]

4.2 METHOD OF SOLUTION

The system of equations (76) - (81) can be reduced to three first-order differential equations for \( v \), \( T_s \) and \( T_p \) and two algebraic equations for \( n_s \) and \( n_p \). (See Appendix A.) After the initial conditions at \( x = 0 \) are chosen, these equations are then integrated using the numerical method described in Appendix B.

Conditions at \( x = 0 \) are chosen in the following manner. Stellar wind parameters are chosen and the shock relations
with \( A = B = \sigma = 0 \) (Section 2.1) are used to find the post-shock values of \( n_s \), \( T_s \), and \( v \). Choosing a value of \( T_p \) such that \( \Gamma - A \) is negative and yet less than \( \sim 20,000^\circ \text{K} \) (Section 1.2), \( n_p \) is fixed by using (80). Since \( \Gamma \) and \( A \) are both proportional to \( n_p^2 \) and \( F_v \) and \( A \) are proportional to \( n_p \), \( G = 0 \) can be written as

\[
n_p = \frac{(F_v + A)'}{\left(\Gamma - A\right)'}
\]

(82)

where the prime on the bracketed terms denotes that they are the terms (73) and (75) divided by \( n_p \) and the terms in (71) and (72) divided by \( n_p^2 \).

4.3 RESULTS OF INTEGRATION

When reasonable values of \( n_s \), \( T_s \), \( v \) and \( T_p \) are used in (74), \( n_s = 20 \text{ cm}^{-3}, T_s \sim 10^6^\circ \text{K}, v \sim 200 \text{ km/sec}, T_p \sim 10^4^\circ \text{K} \), the resulting value of \( n_p \) is the order of \( 10^6 - 10^7 \text{ cm}^{-3} \), about 1000 times larger than observed planetary densities. Some increase of \( n_p \) over the average planetary value may be expected in the interaction region due to the compression caused by the impact of the stellar wind. Whether the pressure gradient set up in the planetary to slow the stellar wind is due to a temperature or density gradient depends on the nature of the energy balance equation assumed for the planetary.
If initial conditions as stated above are used to start an integration, the gradient of $T_p$ develops a discontinuity ($\nabla T_p \rightarrow \infty$) while at the same time $\nabla n_p \rightarrow \infty$ and the integration is stopped. The reason for this behavior is that the denominator of $\nabla T_p$ goes to zero while the numerator remains finite. If $T_s$ is reduced to temperatures the order of $10^5 \text{K}$ and $v$ is reduced to $\sim 10^6 \text{cm/sec}$ to approximate the flow after some cooling has occurred, solutions of the type shown in Figure 12 can be found. $\Gamma = \Lambda$, $F$ and $A$ vary as shown in Figure 13. Figure 14 shows the behavior of $T_p$ for various initial values of $T_s$ but with the same initial values of $v$ and $n_s$. For $T_s \geq 1.4 \times 10^5 \text{K}$ the gradient of $T_p$ becomes infinite and the integration is stopped.

The discontinuity may be understood numerically by reference to a figure similar to Figure 15, drawn for given values of $v$ and $n_s$. Knowing $v$ and $n_s$ and for a given $T_s$, it is possible to solve $\nabla T_p = 0$ and $G = 0$ simultaneously for $T_p$ and $n_p$ and draw a curve ($\text{NUM} = 0$) connecting points where $\nabla T_p = 0$. In a similar manner it is possible to draw a curve ($\text{DEN} = 0$) connecting points where the gradient of $T$ is discontinuous. $\nabla T$ may also be discontinuous if the numerator is very large. This happens at the value of $T_s$ that makes the mach number of the flow equal to one, since
FIGURE 12 - \( v, T_s, T_p, \) and \( n_p \) as functions of \( x \).

\( T_s \) and \( T_p \) are drawn on the same scale for \( x > 2.4 \times 10^6 \) cm.
FIGURE 13 - $F$, $|\Gamma - A|$ and $A$ as functions of $x$
for the integration shown in Figure 12.
FIGURE 14 - $T_p$ as a function of $x$ for different initial $T_s$, but with the same $n_s$ and $v$ as shown in Figure 12. For an initial $T_s \geq 1.4 \times 10^5^\circ K$, $T_p$ develops a discontinuity.
FIGURE 15 - $T_p$ vs $T_s$ showing regions of positive and negative $\nabla T_p$ for initial values of $n_s$ and $v$ as shown in Figure 12. $\text{DEN} = 0$ connects points where the denominator of $\nabla T_p$ is zero. $\text{NUM} = 0$ connects points where the numerator is zero. Dashed curves show how solid curves change when the velocity decreases from $3 \times 10^6$ cm/sec for the solid curves to $2.25 \times 10^6$ cm/sec for the dashed curves.
(1 - M^2)^{-1} is part of the numerator. Regions of positive and negative gradient are labeled by (+) and (-). The evolution of the curves as v is decreased is shown by the dashed curves.

For a given initial $T_p$, 20,000°C for example, and $T_s = 1.4 \times 10^5$°K, the initial point in the integration is below the NUM = 0 and DEN = 0 curves and is in a region of positive gradient. As the integration continues $T_p$ increases and $T_s$ decreases and the point moves up and to the left and crosses into the (-) region before crossing the DEN = 0 curve. If the initial $T_s$ had been greater than about $1.4 \times 10^5$°K, the point would have tried to cross the DEN = 0 curve and the integration would have been stopped. Any initial point that lies to the right or above the DEN = 0 curve will eventually have to cross it in order to cool the flow and decrease $T_p$ to observed planetary temperatures.

From Figure 12 it is seen that $T_p$ does approach the planetary temperatures but that the number density $n_p$ does not approach planetary densities.

The mean free path for a stellar wind particle is of the order $\lambda = 10^9$ cm if $n_p = 10^7$ cm$^{-3}$ and $\sigma = 10^{16}$ cm$^2$ in (64). This is much larger than the $3 \times 10^6$ cm in Figure 12. The large values of $n_p$ required by the relative values of $(Fv + A)'$ and $(\Gamma' - \Lambda')'$ and the fact that the energy
transfer rate is \( \frac{n}{p} (Fv + A) \)' makes it possible for the stellar flow to cool by an order of magnitude in a distance less than a mean free path. So the assumption that fluid gas dynamics can be applied to this problem may not be valid. The characteristic distances are not large enough to insure that individual particle behavior is not important.
CHAPTER 5 -
CONCLUSIONS AND SUGGESTIONS FOR FURTHER WORK

There are three basic conclusions that can be drawn from the three models examined:

1) A supersonic stellar wind can not cool and slow to observed planetary expansion velocities unless a shock exists between the central star and the planetary nebula.

2) The stellar wind does not cool significantly by radiation in flowing from the shock to the planetary.

3) The region of interaction where the subsonic stellar wind is cooled and slowed to planetary conditions may be the order of a mean free path and therefore the equations of fluid gas dynamics may not apply.

Physical processes other than those attributed to the main body of the planetary should be considered in constructing a model of the interaction region. The high temperatures in the post-shock stellar wind, \( \sim 10^7 \text{ K} \), correspond to energies the order of 800 ev. Energy losses due to collisional ionization of the small abundance elements such as oxygen, silicon, etc. may be important.

In the models presented, the planetary material is assumed to be in a steady state and static. The latter
condition should be examined to determine the influence of diffusion of neutrals into and ions out of the interaction region where a high degree of ionization is maintained by the high temperatures.

The steady-state assumption may have to be examined if the four fluid nature of the problem (flowing ions and electrons, planetary ions and electrons), and the plasma instabilities and properties in the presence of magnetic fields (Gurzadyan 1969) are considered. The cyclotron radius of a proton with velocity $v$ km/sec in a magnetic field of $B$ gammas is of the order $10 \frac{v}{B}$ km. Even in a moderate magnetic field the cyclotron radius the order of or smaller than the particle mean free path and hydromagnetic transfer of energy between the stellar wind and the planetary may not be excluded as a possible transfer mechanism. In this case it is the cyclotron radius, not the mean free path, that needs to be considered in examining the assumption that fluid dynamics may be applied to the problem.

It has also been assumed that the stellar wind parameters at the inner edge of the planetary are time independent. If the shock between the central star and the planetary does not maintain a fixed position, but propagates toward the star, then the stellar wind conditions at the
planetary may vary significantly as the shock moves toward the star. For a shock moving at 50 km/sec, it takes only about 500 yrs to move half the planetary radius. The problem of a converging spherical shock and post-shock stellar wind parameters as a function of distance from the shock has not been examined.

Since bright inner edges have not yet been observed in planetaries, it is possible that the energy deposited by the stellar wind may not play a significant role in the evolution of a planetary, but the observed distinct inner edges indicate that the pressure support supplied by the stellar wind may be important.
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John Howard — what are you doing? How about some refreshment?
APPENDIX A

EQUATIONS INTEGRATED

The equations to be solved are (76) - (81) in Section 4.1 and they are

\[
\rho_s v = \text{constant} = M_1 \quad (A-1)
\]

\[
\rho_s v \frac{dv}{dx} = - \frac{dp_s}{dx} - F \quad (A-2)
\]

\[
\frac{d}{dx} \left( \frac{\gamma}{\gamma-1} p_s v + \frac{1}{2} \rho_s v^3 \right) = - A - Fv \quad (A-3)
\]

\[
\frac{dp_s}{dx} = F \quad (A-4)
\]

\[
G = \Gamma - \Lambda + A + Fv = 0 \quad (A-5)
\]

\[
F = \rho_s v^2 \sigma_p \quad (A-6)
\]

where \( \Gamma, \Lambda \) and \( A \) are given by (71), (72), and (73).

The first three equations can be combined to give

\[
\frac{dv}{dx} = \frac{Fv - (\gamma-1)A}{1 - M^2} \cdot \frac{1}{\gamma p_s} \quad (A-7)
\]

\[
\frac{dT_s}{dx} = \frac{Fv - (\gamma M^2 - 1)A}{1 - M^2} \cdot \frac{-(\gamma-1)}{\gamma n_s k v} \quad (A-8)
\]
Taking \( \frac{d}{dx} \) of (A-5)

\[
0 = \frac{dT}{dx} \cdot \frac{dT}{dx} + G_2 \frac{dT}{dx} + G_3 \frac{dn}{dx} + G_4 \frac{dn}{dx} \quad (A-9)
\]

where

\[
G_1 = \frac{\partial G}{\partial T_s} = \frac{A}{2} \cdot \left[ \frac{5T_p - T_s}{T_s - T_p} \right] \quad (A-10)
\]

\[
G_2 = \frac{\partial G}{\partial T_p} = -\Gamma \cdot \left[ \frac{1}{T_s - T_p} + \frac{3}{4T_p} \right]
\]

\[
- \Lambda \cdot \left[ \frac{27900}{T_p^2} - \frac{1}{2T_p^2} \right] + \frac{A}{2} \left[ \frac{5T_s - T_p}{T_s^2 - T_p^2} \right] \quad (A-11)
\]

\[
G_3 = \frac{\partial G}{\partial n_p} = \frac{\Gamma - \Lambda}{n_p} \quad (A-12)
\]

\[
G_4 = \frac{\partial G}{\partial n_s} = \frac{A - 2Fv}{n_s} \quad (A-13)
\]

And from (A-4)

\[
n_k \frac{dT_p}{dx} + kT_p \frac{dn_p}{dx} = F \quad (A-14)
\]
Eliminating \( \frac{dn_p}{dx} \) from (A-9) and (A-14) and solving for \( \frac{dT_P}{dx} \)

\[
\frac{dT_P}{dx} = \frac{G_3 F + k T_P \left[ G_1 \frac{dT_s}{dx} - G_4 \frac{n_s}{v} \frac{dv}{dx} \right]}{k (n_p G_3 - T_P G_2)} \quad (A-15)
\]

where

\[
\frac{dv}{dx} + v \frac{dn_s}{dx} = 0 \quad (A-16)
\]

from (A-1) has been used to replace \( \frac{dn_s}{dx} \) in (A-9).

Combining (A-2) and (A-4) and integrating

\[
\rho_s v^2 + p_s + p_p = \text{constant} = M_2 \quad (A-17)
\]

\( M_1 \) and \( M_2 \) can be determined from the initial values of \( T_s, T_P, n_s, n_p \) and \( v \).

The calculations are done in the following manner.

(A-7) and (A-8) are evaluated and are used in calculating (A-16). New values for \( v, T_s, \) and \( T_P \) are determined from

\[
v' = v + \Delta v \quad (A-18)
\]

\[
T_s' = T_s + \Delta T_s \quad (A-19)
\]

\[
T_P' = T_P + \Delta T_P \quad (A-20)
\]
where $\Delta v$, $\Delta T_s$ and $\Delta T_p$ are the derivatives, $k_j$, times the increment $h$ or $h/2$ (Appendix B). $n_s$ and $n_p$ are calculated from

\begin{align*}
n'_s &= \frac{M_1}{m'_{p} v'} \\
n'_p &= \frac{M_2 - p'_s - (\rho_s v^2)'}{kT'_p} \tag{A-22}
\end{align*}

where the $'$ means evaluated using primed quantities.
APPENDIX B

NUMERICAL METHOD

The Runga-Kutta method used to integrate the equations in Appendix A is as follows.

Let the derivatives be written as

\[
\frac{dv}{dx} = f_1(T_s, T_p, v, n_s, n_p) \tag{B-1}
\]

\[
\frac{dT_s}{dx} = f_2(T_s, T_p, v, n_s, n_p) \tag{B-2}
\]

\[
\frac{dT_p}{dx} = f_3(T_s, T_p, v, n_s, n_p) \tag{B-3}
\]

Let \( h \) be the increment in \( x \), then for \( j = 1, 2, 3, \)

\[
k_{j1} = f_j(T_s, i, T_p, i, v, i, n_s, i, n_p, i) \tag{B-4}
\]

\[
k_{j2} = f_j(T_s, i + \frac{h}{2} k_{21}, T_p, i + \frac{h}{2} k_{31}, v, i + h k_{11}, n_s, 2, n_p, 2) \tag{B-5}
\]

\[
k_{j3} = f_j(T_s, i + \frac{h}{2} k_{22}, T_p, i + \frac{h}{2} k_{32}, v, i + h k_{12}, n_s, 3, n_p, 3) \tag{B-6}
\]

\[
k_{j4} = f_j(T_s, i + h k_{23}, T_p, i + h k_{33}, v, i + h k_{13}, n_s, 4, n_p, 4) \tag{B-7}
\]
where \( n_s \) and \( n_p \) are calculated from (A-21) and (A-22) and
the values of \( T_s \), \( T_p \) and \( v \) that precede them in the
argument list for \( f_j \).

These four derivatives are averaged

\[
k_j = \frac{k_{j1} + 2k_{j2} + 2k_{j3} + k_{j4}}{6} \quad \text{(B-8)}
\]

and the values at the next increment are found from

\[
v_{i+1} = v_i + h k_1 \quad \text{(B-9)}
\]

\[
T_{s,i+1} = T_{s,i} + h k_2 \quad \text{(B-10)}
\]

\[
T_{p,i+1} = T_{p,i} + h k_3 \quad \text{(B-11)}
\]

\[
n_{s,i+1} = \frac{M_1}{m v_{i+1}} \quad \text{(B-12)}
\]

\[
n_{p,i+1} = \frac{M_2 - (P_s)_{i+1} - (\rho_s v^2)_{i+1}}{k T_{p,i+1}} \quad \text{(B-13)}
\]

where (B-12) and (B-13) are from Appendix A and the
subscripts \( i+1 \) on \( P_s \) and \( \rho_s v^2 \) mean evaluated using the
values at \( i+1 \).


Then if

\[
\frac{v_{i+1} - v_i}{v_i} < \epsilon \quad (B-14)
\]

\[
\frac{T_{S,i+1} - T_{S,i}}{T_{S,i}} < \epsilon \quad (B-15)
\]

\[
\frac{T_{P,i+1} - T_{P,i}}{T_{P,i}} < \epsilon \quad (B-16)
\]

are satisfied with \( \epsilon \approx 0.05 \), the integration is continued with the \( i+1 \) values being used as the initial values. If any of the three inequalities is not satisfied then the increment \( h \) is divided by two and the integration is restarted using the values at \( i \).

Since the expression for \( G \) is not used in the calculation of the derivatives, even though partial derivatives of \( G \) are used, \( G \) is calculated at each increment and if \( G \) is greater than about 0.1% of \( \Gamma, \Lambda, F_v \) or \( A \), then the integration is restarted with a smaller increment \( h \).
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