RICE UNIVERSITY

A FIRST-QUANTIZED PROOF OF THE SYMMETRIZATION POSTULATE

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ABSTRACT

A First-quantized Proof of the Symmetrization Postulate

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Methods used to prove the existence of superselection rules are studied in detail. It is shown that the known methods of proving superselection rules are applicable only if the superselecting operator is an observable. In particular, place permutations are not observables, so the fact that place permutations commute with all observables does not lead to a superselection rule between vectors of different symmetry types.

It is shown that if states not having a definite symmetry type can exist, then it is possible to have several different states which are eigenstates of the same observables with the same eigenvalues. In this case a maximal set of observables does not exist. Therefore a proof of the Symmetrization Postulate (SP) which assumes a maximal set of observables actually presupposes part of what is to be proved. A rigorous proof of the SP must depend only on the weaker assumption of a non-degenerate set of observables.

The usual formulation of the Transition Probability Postulate (TPP) is used to construct a trivial proof of the assertion that physical states are represented by unique rays in Hilbert space; however the feature of the TPP which is essential to this proof is not essential to the TPP itself. The features of the TPP which are essential for computing
transition probabilities are identified. These essential features of the TPP are then used to prove that states are represented by unique rays.

The difficulties involved in rigorously defining the interchange operator $P_{ij}$ are discussed and a careful definition of $P_{ij}$ is given. The uniqueness of the ray is then used to show that states of systems containing several indistinguishable particles must be eigenstates of $P_{ij}$ and must satisfy the Symmetrization Postulate. Finally, the SP is used to prove the slightly stronger result that a given state must have the same symmetry type for each pair of the same species of particles.
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A fundamental unresolved question is whether every particle is either a boson or a fermion. Experimentally there is conclusive evidence that photons and pions are bosons, whereas electrons and nucleons are fermions. Evidence for the other particles, however, is seriously lacking. The experimental situation is discussed by Messiah and Greenberg. There have been many attempts, all of them unsuccessful, to prove the Symmetrization Postulate (SP) from the other postulates of quantum mechanics. These attempted proofs are also discussed by Messiah and Greenberg, who point out that these proofs always depend on an additional assumption not contained in the quantum mechanical postulates. The most common such assumption is that each physical state is represented by exactly one ray in the Hilbert space.

Our objective in this paper is to use the postulates of quantum mechanics to prove the aforementioned assumption that each physical state is represented by a unique ray. Since the uniqueness of the ray is a commonly assumed property, the postulates tend to be formulated in such a way as to presuppose this uniqueness. Such formulations of the postulates lead to trivial proofs of ray uniqueness which do not use the truly essential features of the postulates, but which instead depend on features which have been inserted in the postulates for mathematical or conceptual convenience. Our first problem, then, is to recast the postulates in a form which preserves their essential elements without presupposing ray uniqueness in any non-essential way. Once this has been accomplished, we shall prove that the really necessary elements
of the postulates are by themselves sufficient to prove that physical states must be represented by unique rays.

The principal postulate which we shall employ is the Transition Probability Postulate (TPP), which is discussed in Section 5. The TPP is a good example of a postulate which in several non-essential ways, implicitly presupposes ray uniqueness. Thus the main objective of Section 5 is to purge the TPP of those extraneous features which if left intact would lead to trivial proofs of ray uniqueness.

One such feature concerns the assumed relationships between physical states and the measurements used to prepare those states. These relationships are explored in Section 2, where it is noted that quantum mechanics usually assumes that a physical state is an eigenstate of a complete set of compatible observables. The word "complete" directly implies that the state is represented by a unique ray, but this assumption is unnecessary. Messiah and Greenberg\textsuperscript{1} replace the notion of a complete set with the milder assumption of a "maximal" set of compatible observables, and at first glance this seems quite acceptable. However, in Section 4, a closer examination reveals that even this is unsatisfactory, and it becomes necessary to use the still weaker assumption of a non-degenerate set of observables (NDSO).

Section 3 is concerned with striking down the proof of a superselection rule between vectors of different symmetry types. Such a superselection rule would actually amount to a weak form of the Symmetrization Postulate, and would thus provide a convenient starting point from which to try to prove the SP. By eliminating this superselection rule, we
therefore make it much more difficult to prove the SP.

Once we have finally stripped the TPP down to its bare essentials in Section 5, we go on in Section 6 to prove the uniqueness of the ray from the TPP. Then in Section 7 we give a careful definition of the interchange operator $P_{ij}$. The reason for doing this is that it is surprisingly difficult to give a satisfactory definition of $P_{ij}$. The two obvious ways of defining it both have major drawbacks. One definition is not clearly self-consistent, and the other does not clearly correspond to particle interchange.

Finally in Section 8 we use ray uniqueness to prove the Symmetrization Postulate. At the end of that section, we go on to prove a slightly stronger result than the SP.

Finally we should mention a basic assumption which pervades our entire discussion. We assume that physical states are prepared by making measurements. Although it is frequently mentioned in Section 2, this supposition is first used in Section 3. However, the results of Section 3 do not depend on this assumption. In fact, without this assumption, it would actually be easier to strike down the proof of the superselection rule, since one of the rule's proofs uses this supposition. On the other hand, the assumption plays a vital role in Section 6. So, strictly speaking, we have proven the SP only for states which can be prepared by making measurements. The significance of this, and the possibility of generalizing our results are discussed in the conclusion.
2. SETS OF COMPATIBLE OBSERVABLES

In quantum mechanics we ordinarily assume that a physical state is prepared by the simultaneous measurement of a complete set of compatible observables. The resulting state is then an eigenstate of those observables. By "complete" one generally means that (for a given combination of eigenvalues) the observables completely determine the physical state, and their operators determine its state vector to within a constant complex factor, i.e., determine its ray. Since our objective is to prove the uniqueness of the ray, we cannot assume that we have complete sets of observables. Instead we might try assuming, as Messiah and Greenberg\(^1\) do, the existence of a "maximal" set of compatible observables, i.e., a set of compatible observables which completely determines the physical state, for a given combination of eigenvalues. However even this is assuming too much. As we shall see later, the existence of a maximal set of compatible observables has important consequences for systems containing several indistinguishable particles.

Instead we shall consider a set of compatible observables \(A_1, \ldots, A_n\) such that if \(B\) is any observable which is compatible with each \(A_i\), and if \(A_1', \ldots, A_n', B'\) and \(A_1'', \ldots, A_n'', B''\) are two possible combinations of eigenvalues of these \(n + 1\) observables, then the statement \(A_1' = A_1'', \ldots, A_n' = A_n''\) implies that \(B' = B''\). In other words the spectrum of simultaneous eigenvalues of the observables \(A_1, \ldots, A_n\) has no eigenvalue degeneracy, and hence there can be no further eigenvalue splitting by the addition of another observable \(B\). The set \(A_1, \ldots, A_n\) is said to be
a non-degenerate set of observables (NDSO). Given a set of compatible observables, we can always enlarge it to an NDSO by including more observables. An NDSO, together with one eigenvalue for each of the observables, is called a non-degenerate set of eigenvalues (NDSE).

Now a maximal set of compatible observables is simply an NDSO, each of whose NDSE's determines only one physical state. Therefore, assuming the existence of maximal sets of compatible observables is equivalent to assuming that only one state corresponds to a given NDSE. We shall investigate the validity of this assumption in Section 4. First, however, we shall need some results about superselection rules. In the meantime, we shall replace the usual assumption that physical states are eigenstates of complete or maximal sets of compatible observables with the much weaker assumption that they are eigenstates of NDSO's.
3. SUPERSELECTION RULES

If no physical measurement can determine the relative phases of the non-colinear Hilbert space vectors \(|\alpha|\) and \(|\beta|\), one generally assumes that there exists a superselection rule between \(|\alpha|\) and \(|\beta|\); that is, that no non-trivial linear combination of \(|\alpha|\) and \(|\beta|\) represents a physical state. Let us analyze the logic on which this conclusion is based.

One way of approaching this is to note that no physical measurement can distinguish between \(|\psi|\equiv a|\alpha| + l|\beta|\) and \(|\phi|\equiv a|\alpha| + e^{i\delta} l|\beta|\); that is, the mathematically computed expectation values for \(|\psi|\) are the same as those for \(|\phi|\). Hence if either \(|\psi|\) or \(|\phi|\) represents a physical state, they must both represent the same state. Clearly \(|\psi|\) and \(|\phi|\) do not belong to the same ray, so they cannot represent the same state; hence the only remaining alternative is that neither of them represents any state. Of course, the falacy of this argument is that while it is commonly assumed that a physical state must be represented by a unique ray in Hilbert space, there is nothing in the postulates which requires this.

Let us try a somewhat different approach. For any observable \(K\), the Hilbert space \(H\) can be decomposed into orthogonal subspaces \(H_K|, H_K|, \ldots, H_K(j)\), \ldots, where \(H_K(j)\) consists of the eigenvectors of \(K\) with the eigenvalue \(K(j)\). For simplicity we consider the case in which \(K\) has only two eigenvalues: \(K'\) and \(K''\). Suppose that \(K\) has the property that if \(|\alpha|\in H_K|\), and \(|\beta|\in H_K|\), then the
state represented by \( |\psi\rangle \equiv \alpha |\alpha\rangle + b |\beta\rangle \) is also represented by \( |\phi\rangle \equiv \alpha |\alpha\rangle + b |\beta\rangle e^{i\delta} \), where the phase \( \delta \) is either partly or completely arbitrary. If we try to compute the expectation value of some observable \( L \) for the state represented by \( |\psi\rangle \) and \( |\phi\rangle \), we obtain\(^*\)

\[
\langle \psi | L | \psi \rangle = |\alpha|^2 \langle \alpha | L | \alpha \rangle + |b|^2 \langle \beta | L | \beta \rangle + 2 \text{Re} (\alpha^* b e^{i\delta} \langle \alpha | L | \beta \rangle)
\]
since \( L \) is hermitian.

On the other hand,

\[
\langle \phi | L | \phi \rangle = |\alpha|^2 \langle \alpha | L | \alpha \rangle + |b|^2 \langle \beta | L | \beta \rangle + 2 \text{Re} (\alpha^* b e^{i\delta} \langle \alpha | L | \beta \rangle)
\]

If \( \langle \alpha | L | \beta \rangle \neq 0 \) and if there is sufficient arbitrariness \( \delta \), then we can generally find a \( |\phi\rangle \) (representing the same state as \( |\psi\rangle \)) such that \( \langle \phi | L | \phi \rangle \neq \langle \psi | L | \psi \rangle \).

Now since \( L \) is an observable, it must have a well-defined expectation value for every physical state. Therefore if every vector with non-vanishing components in both \( H_K \) and \( H_K'' \) actually represents a physical state, then there is no observable \( L \) with a non-zero matrix element between the subspaces \( H_K \) and \( H_K'' \).

To illustrate the consequences of this, we use the particular basis \( \left\{ |\alpha_1\rangle, |\alpha_2\rangle, \ldots, |\beta_1\rangle, |\beta_2\rangle, \ldots \right\} \), where \( \left\{ |\alpha_i\rangle, |\alpha_2\rangle, \ldots \right\} \) is a basis of \( H_K \), and \( \left\{ |\beta_1\rangle, |\beta_2\rangle, \ldots \right\} \) is a basis of \( H_K'' \). For each observable \( L \), \( \langle \alpha_i | L | \beta_j \rangle = 0 \), so the matrix of \( L \) for the above basis is

\[
(L) = \begin{pmatrix}
(I_K) & 0 \\
0 & (I_{K''})
\end{pmatrix}
\]

where \( (I_K) \) and \( (I_{K''}) \) are the matrices of \( L \) in the subspaces \( H_K \) and \( H_K'' \) respectively. In the same basis the matrix of \( K \)

\(^*\) Since \( |\alpha\rangle \) and \( |\beta\rangle \) are orthogonal, \( |\phi\rangle \) is normalized if \( |\psi\rangle \) is.
is given by

\[
(K) = \begin{pmatrix}
(K') & 0 \\
0 & (K'')
\end{pmatrix}
\]

where the matrix \((K')\) has \(K'\)'s along the main diagonal and zeros everywhere else. Hence

\[
(L)(K) = \begin{pmatrix}
(L_K') & 0 \\
0 & (L_K'')
\end{pmatrix}
\begin{pmatrix}
(K') & 0 \\
0 & (K'')
\end{pmatrix}
\]

\[
= \begin{pmatrix}
(L_K')(K') & 0 \\
0 & (L_K'')(K'')
\end{pmatrix}
\]

\[
= \begin{pmatrix}
(K')(L_K') & 0 \\
0 & (K'')(L_K'')
\end{pmatrix}
\]

\[
= (K)(L)
\]

Since their matrices commute, the operators \(K\) and \(L\) commute. Thus \(K\) commutes with every observable; hence no set of compatible observables is an NESO unless \(K\) is a function of these observables. Since every physical state is presumed to be an eigenstate of an NESO, every physical state must be an eigenstate of \(K\); i.e., the vectors representing the state must lie in either \(H_K'\) or \(H_K''\). Therefore, contrary to our assumption, vectors such as \(|\psi\rangle \equiv a|\alpha\rangle + b|\beta\rangle\) do not represent physical states.

Note, however, that this does not prove a complete superselection rule. The assumption which we have contradicted was that all vectors with non-vanishing components in both \(H_K'\) and \(H_K''\) represent physical states. Hence we have not proven that no such vector represents a physical state. This deficiency can be remedied by showing directly (i.e., without
assuming that \( \alpha |\alpha\rangle + \beta |\beta\rangle \) represents a state) that 
\[ \langle \alpha | \mathbf{L} | \beta \rangle = 0 \], or equivalently, by showing directly that \( K \) commutes with every observable. It then follows that \( K \) is a function of the observables of any given NDSO; hence every physical state is an eigenstate of \( K \).

To illustrate these ideas, consider the operator \( G \) defined by \( G \equiv -\frac{i}{\hbar} 2\pi \mathbf{J}_z \). This is the operator for a rotation through an angle \( 2\pi \) about the \( z \)-axis. Its adjoint, \( G^\dagger = \frac{i}{\hbar} 2\pi \mathbf{J}_z \), has the same effect as \( G \) upon any vector in the Hilbert space. Hence, within the Hilbert space, 
\[ G^\dagger = G, \]
and so \( G \) is hermitian. \( G \) represents a quantity which can be physically measured — we simply measure \( J_z \) and assign the resulting state a \( G \)-value of +1 when the observed \( J_z \)-value is integral, and a \( G \)-value of -1 when the observed \( J_z \)-value is half-integral. The eigenvectors of \( G \) span the Hilbert space, so \( G \) meets all the requirements of a physical observable.

The Hilbert space decomposes into the orthogonal subspaces \( H_+ \) and \( H_- \) consisting of the eigenvectors of \( G \) with the eigenvalues +1 and -1 respectively. Let \( |\alpha\rangle \in H_+ \) and \( |\beta\rangle \in H_- \). Suppose the vector \( |\psi\rangle \equiv \alpha |\alpha\rangle + \beta |\beta\rangle \), where \( \alpha \neq 0 \neq \beta \), represents a physical state \( \bar{\psi} \). Then \( G|\psi\rangle \) represents the state \( \bar{\psi} \) rotated through \( 2\pi \) about the \( z \)-axis; i.e., represents the state \( \bar{\psi} \). Thus \( \alpha |\alpha\rangle + \beta |\beta\rangle \) and \( G|\psi\rangle = \alpha |\alpha\rangle - \beta |\beta\rangle \) represent the same state. This indeterminacy of \( \pm \pi \) in the relative phase of \( |\alpha\rangle \) and \( |\beta\rangle \) is sufficient to force \( \langle \alpha | \mathbf{L} | \beta \rangle = 0 \) for all observables \( L \). Therefore not every vector with non-vanishing components in both \( H_+ \) and \( H_- \) represents a physical state.
To show that no such vector represents a physical state, we first note that an observable \( L \) can be rotated through an angle \( \Theta \) about the z-axis by rotating the associated measuring apparatus. The result is a new observable whose operator is \( R_z(\Theta) L R_z^{-1}(\Theta) \), where \( R_z(\Theta) = e^{-i\frac{\Theta}{\hbar} J_z} \). If \( \Theta = 2\pi \), then the new observable is the same as \( L \); hence the operator \( R_z(2\pi) L R_z^{-1}(2\pi) \) has the same expectation values as \( L \) for all vectors representing physical states. Therefore, if the Hilbert space has fully coherent physical subspaces,

\[
R_z(2\pi) L R_z^{-1}(2\pi) = L
\]

\[
R_z(2\pi) L = L R_z(2\pi)
\]

\[ G L = L G \]

G thus commutes with every observable \( L \); hence every physical state is an eigenstate of \( G \) and is represented by a vector lying in either \( \mathcal{H}_+ \) or \( \mathcal{H}_- \).

From the foregoing discussion it should be clear that a superselection rule can be proven only when the superselecting operator is an observable. Now Messiah and Greenberg\(^1\) purport to show that a superselection rule exists between vectors of different symmetry types. (Vectors which transform, to within an equivalence, according to a given irreducible representation of the symmetric group are said to have a given symmetry type.) The superselecting operators for this are the place permutations\(^*\). Now place permutations do indeed commute with all physical observables, as Messiah and Greenberg demonstrate. However, for this to lead to a superselection rule, the superselecting operators — the place

\(^*\) For a distinction between place permutations and particle permutations, see Landshoff and Stapp\(^2\).
permutations — must be observables. But this implies that
place permutations commute with place permutations, which
happens not to be true. Thus Messiah and Greenberg have
merely demonstrated that \( \langle \alpha | L | \beta \rangle = 0 \) if \( |\alpha\rangle \) and \( |\beta\rangle \)
are vectors of different symmetry types; they have not proven
that \( a |\alpha\rangle + b |\beta\rangle \) cannot represent a physical state.
4. FAILURE OF MAXIMAL SETS OF OBSERVABLES

We next explore the consequences of the fact that vectors representing physical states need not have a definite symmetry type. Let \( |\lambda\rangle \) be a normalized vector representing a state \( \Lambda \) of a system containing several identical particles. Let \( \Lambda \) be an eigenstate of an NDSO.

We first consider the vector \( P_{ij} |\lambda\rangle \), where \( P_{ij} \) is the operator which interchanges the \( i \)'th and \( j \)'th particles. \( P_{ij} \) is unitary, so \( P_{ij} |\lambda\rangle \) is normalized. If the \( i \)'th and \( j \)'th particles are indistinguishable, then \( P_{ij} \) commutes with every observable of the system. Then if \( |\lambda\rangle \) is an eigenvector of the observable \( A \) with the eigenvalue \( A^* \), so is \( P_{ij} |\lambda\rangle \), and vice versa. Furthermore, for any observable \( U \),

\[
\langle P_{ij} \lambda |U| P_{ij} \lambda \rangle = \langle \lambda | P_{ij}^{-1} U P_{ij} |\lambda\rangle = \langle \lambda | U |\lambda\rangle
\]

Thus, not only are \( |\lambda\rangle \) and \( P_{ij} |\lambda\rangle \) eigenvectors of the same observables with the same eigenvalues, but also their expectation values for any observable are the same. In other words, they represent the same state.

Next, since \( |\lambda\rangle \) need not have a definite symmetry type, \( P_{ij} |\lambda\rangle \) can be linearly independent of \( |\lambda\rangle \). Hence we consider the linear combination \( |\gamma\rangle \equiv r |\lambda\rangle + \sigma P_{ij} |\lambda\rangle \), where \( r \) and \( \sigma \) are non-zero complex numbers chosen in such a way that \( |\gamma\rangle \) is normalized to unity. The normalization condition is

\[
1 = \langle \gamma | \gamma \rangle
\]

\[
= |r|^2 + |\sigma|^2 + r^* \sigma \langle \lambda | P_{ij} \lambda \rangle + \sigma^* r \langle P_{ij} \lambda |\lambda\rangle
\]

\[
= |r|^2 + |\sigma|^2 + 2 \text{Re} (r^* \sigma \langle \lambda | P_{ij} \lambda \rangle)
\]

\[
= |r|^2 + |\sigma|^2 + 2 \langle \lambda | P_{ij} \lambda \rangle \text{Re} (r^* \sigma)
\]
The last step depends on the hermiticity of $P_{ij}^\dagger$, which may be seen from $P_{ij}^\dagger = P_{ij}^{-1} = P_{ij}$. (Interchanges are hermitian even though more general permutations are not.)

If $|\lambda\rangle$ is an eigenvector of the observable $\Lambda$ with the eigenvalue $\lambda'$, then clearly so is $|\gamma\rangle$. However, $|\lambda\rangle$ and $|\gamma\rangle$ can have different expectation values for an observable $U$ of which they are not eigenvectors.

$$\langle \gamma | U | \gamma \rangle = |\lambda|^2 \langle \lambda | U | \lambda \rangle + |\beta|^2 \langle \pi_j \lambda | U | \pi_j \lambda \rangle + i \lambda \beta \langle \lambda | U | \pi_j \lambda \rangle + i \lambda \beta \langle \pi_j \lambda | U | \lambda \rangle$$

$$= (|\lambda|^2 + |\beta|^2) \langle \lambda | U | \lambda \rangle +$$

$$i \lambda \beta \langle \lambda | U | \pi_j \lambda \rangle + i \lambda \beta \langle \pi_j \lambda | U | \lambda \rangle$$

$$= (1 - 2 \langle \lambda | \pi_j \lambda \rangle \text{Re}(\lambda \beta) \langle \lambda | U | \lambda \rangle +$$

$$+ \langle \lambda | U \pi_j \lambda \rangle 2 \text{Re}(\lambda \beta)$$

$$= \langle \lambda | U | \lambda \rangle + 2 \left[ \langle \lambda | U \pi_j | \lambda \rangle - \langle \lambda | U | \lambda \rangle \langle \lambda | \pi_j | \lambda \rangle \text{Re}(\lambda \beta) \right]$$

The expression in brackets is not in general equal to zero, and $\text{Re}(r \beta) = 0$ only if $s = r ki$, where $k$ is a real number. Therefore $\langle \gamma | U | \gamma \rangle \neq \langle \lambda | U | \lambda \rangle$. So $|\gamma\rangle$ and $|\lambda\rangle$ have different expectation values.

In connection with this it should be pointed out that Messiah and Greenberg\(^1\) have constructed a "proof" that $\langle \gamma | U | \gamma \rangle = \langle \lambda | U | \lambda \rangle$ if $|\gamma\rangle$ and $|\lambda\rangle$ belong to the same "generalized ray". The "generalized ray" corresponding to a given NDSE is the space of those eigenvectors of the NDSO
which have the eigenvalues associated with that particular
NDSE, so the vectors \(|\gamma>|\) and \(|\lambda>|\) which we have been con-
sidering do indeed belong to the same generalized ray. How-
ever, Messiah and Greenberg's proof is incorrect because it
assumes that the generalized ray is irreducible with respect
to the symmetric group. Messiah and Greenberg appear to ver-
ify this assumption, but they actually verify its converse:
that the eigensubspace corresponding to a degenerate set of
observables is generally reducible.

So \(|\gamma>|\) and \(|\lambda>|\) generally do have different expecta-
tion values*, hence they represent different states. Yet \(|\gamma>|\)
is an eigenvector of the same observables as \(|\lambda>|\), and with
the same eigenvalues. That is, \(|\gamma>|\) and \(|\lambda>|\) are associated
with the same NDSE. So apparently we can have several dif-
ferent states corresponding to the same NDSE, which would
imply that the NDSO is not maximal. I say "apparently" be-
cause we have not proven that symmetry superselection rules
do not exist; we have merely refuted the existing proofs that
they do exist. But unless and until a correct proof of such
superselection rules is given, we cannot confidently assume
that physical states are eigenstates of maximal sets of ob-
servables. At most we can permit ourselves only the weaker
assumption that they are eigenstates of NDSO's. This will
have important consequences when we consider the Transition
Probability Postulate.

* Except in certain special cases. \(s = rki\) is one such case.
Another such case occurs when \(|\lambda>|\) is an eigenvector of \(P_{ij}\)
i.e., when \(|\lambda>|\) obeys the Symmetrization Postulate.
Let $L$ represent the NDSO which consists of the observables $A_1, \ldots, A_n$. Let $A'_1, \ldots, A'_n$ be a possible eigenvalue combination of these observables, and let $L_i$ represent the NDSE associated with that particular combination of eigenvalues of the observables of $L$. If we simultaneously measure the observables $A_1, \ldots, A_n$ on a given state $\Phi$, there will be a definite probability of getting the result $A'_1, \ldots, A'_n$. We refer to this probability as the transition probability from $\Phi$ to $L_i$, and we denote it symbolically by $\Theta(\Phi, L_i)$. Calculating the transition probabilities is the most basic quantitative question answered by the postulates of quantum mechanics. The postulate which is almost invariably used for such computations is the Transition Probability Postulate (TPP). The usual formulation of the TPP is given in Messiah:

If one performs on the quantum system a simultaneous measurement of a complete set of compatible dynamical variables, the probability of finding the system in the state $\ket{\varphi}$ (i.e., of finding the particular values of these variables defining the dynamical state represented by $\ket{\varphi}$) is equal to the square of the modulus of the scalar product of the vector $\ket{\psi}$ (normalized to unity) representing the dynamical state of the system at the instant the measurement is carried out, by $\ket{\varphi}$, namely $|\langle \varphi | \psi \rangle|^2$.

First of all, we note that Messiah's formulation of the postulate implicitly assumes that no more than one state corresponds to a given NDSE, an assumption which we have seen to be rather questionable. Fortunately this assumption is in no way necessary to the postulate. There is an obvious way to restate the postulate in terms of NDSE's rather than states. Secondly, we shall soon see that Messiah's formulation of the
TPP leads to a trivial proof of the uniqueness of the ray. This proof still follows if we restate the postulate in terms of NDSE's. However the part of the TPP used for this proof is not an essential part of the postulate. We consider this second problem first.

Let \( \Psi \) and \( X \) be the states represented by \( |\psi\rangle \) and \( |x\rangle \) respectively, and let \( \mathcal{L}_i \) be the NDSE which defines \( X \). Then Messiah's postulate says that \( \Theta(\Psi, \mathcal{L}_i) = |\langle x|\psi\rangle|^2 \).

Now suppose that \( X \) and \( \Psi \) are the same state. Then \( \Psi \) is defined by \( \mathcal{L}_i \), and so it follows immediately from the definition of transition probability that \( \Theta(\Psi, \mathcal{L}_i) = 1 \). Hence \( |\langle x|\psi\rangle|^2 = 1 \).

Next we let \( |\pi\rangle \) be the vector defined by \( |\pi\rangle \equiv |x\rangle - |\psi\rangle\langle\psi|x\rangle \). Using \( |\langle x|\psi\rangle|^2 = 1 \) together with the fact that \( |x\rangle \) and \( |\psi\rangle \) are normalized to unity, we easily compute

\[
\langle\pi|\pi\rangle = \langle x|x\rangle - \langle x|\psi\rangle\langle\psi|x\rangle - \langle\psi|x\rangle^*\langle\psi|x\rangle + \\
+ \langle\psi|x\rangle^*\langle\psi|\psi\rangle\langle\psi|x\rangle
\]

\[= 1 - |\langle x|\psi\rangle|^2 - |\langle\psi|x\rangle|^2 + |\langle\psi|x\rangle|^2 \]

\[= 0 \]

Hence \( |\pi\rangle = 0 \)

\( |x\rangle = |\psi\rangle\langle\psi|x\rangle \)

We have already seen that \( \langle\psi|x\rangle \) is a complex number of absolute value 1.
Since $X$ and $\Psi$ are the same state, $|X\rangle$ and $|\Psi\rangle$ are just two arbitrarily chosen, normalized vectors representing the state $\Psi$. So we have shown that any two normalized vectors representing a given state differ from each other by at most a constant phase factor. Hence two unnormalized vectors representing $\Psi$ differ by at most a constant complex factor, i.e., belong to the same ray.

Note, however, that the only part of the TPP that we have used in proving the uniqueness of the ray is the part that tells us how to compute the transition probability from a state to itself. It is quite unnecessary to have a postulate which tells us how to compute such a quantity; this particular quantity is by definition equal to 1. Now it is necessary in quantum mechanics to be able to compute transition probabilities, but it is not necessary to have a formula for computing those transition probabilities whose value is a priori obvious. Hence this part of the TPP is not necessary to quantum mechanics and may be dropped. The new TPP can then be succinctly stated by saying that the transition probability from the state $\Psi$ to the different state $X$ is given by $|\langle X|\Psi\rangle|^2$, where $|X\rangle$ and $|\Psi\rangle$ are normalized vectors representing $X$ and $\Psi$ respectively.

At this point it is well to clarify somewhat the nature of what we are seeking to do. Our stated objective is to prove the uniqueness of the ray, and hence the SP, from the postulates of quantum mechanics. However, as we have just seen, the quantum mechanical postulates, including the TPP, are formulated in such a way as to be inherently prejudicial in favor of the uniqueness of the ray, and they readily lend
themselves to simple proofs of the unique ray assumption. Such proofs, however, in no way answer the question of whether the uniqueness of the ray is necessary to quantum mechanics. The only way to answer this question is to determine whether it is possible to salvage quantum mechanics in the absence of unique rays. In other words, we seek to determine whether the postulates can be reformulated so that they do not implicitly assume the uniqueness of the ray, but so that they do give the same results as the usual postulates in the special case of unique rays. Our reformulation of the TPP in terms of different states is a first step in this direction. However, it is not enough.

As we pointed out earlier, this formulation of the postulate presupposes that a given NDSE determines exactly one state. It does this by regarding the outcome of the measurement as a state, whereas the outcome is really the observed values of the observables, i.e., an NDSE. So we should not be talking about the transition probability from a state to a state. After all, the postulate seeks to describe the result of measuring an NDSO on a given state, in terms of the probability of getting a particular NDSE. Therefore we should talk about the transition probability from a state $\Psi$ to an NDSE, $L_i$, remembering that several states may be associated with $L_i$. In our previous formulations of the TPP, in which we assumed there was only one state $\chi$ associated with $L_i$, the transition probability was given by $|\langle \chi | \Psi \rangle|^2$, where $|\Psi\rangle$ and $|\chi\rangle$ were normalized vectors representing $\Psi$ and $\chi$ (i.e., $L_i$) respectively.

We note that $|\chi\rangle$ was an eigenvector of the NDSO, $L$, with the particular eigenvalues associated with $L_i$. It may
turn out, however, that not all such eigenvectors represent $L_i$. In particular, this will be the case if the SP is correct, since the SP asserts that only those eigenvectors which are symmetric or antisymmetric are acceptable. However, the question of whether all the eigenvectors are acceptable, and even the fact that the acceptable vectors are eigenvectors, will be of no concern to us in the next few sections. All we shall need to know is that each NDSE is represented by one or more normalized vectors, and that such vectors satisfy the TPP.

The precise formulation of the TPP can now be given as follows: If $|\Psi\rangle$ and $|\chi\rangle$ are normalized vectors representing the state $\Psi$ and the NDSE $L_i$, respectively, then $\Theta(\Psi, L_i) = |\langle \chi | \Psi \rangle|^2$, provided that $\Psi$ is not associated with $L_i$. This last condition is necessary because of the fact that $\Theta(\Psi, L_i) = 1$ by definition if $\Psi$ is associated with $L_i$. This is, of course, precisely the consideration which led to restricting the previous formulation of the TPP to the transition probability between different states.
Let \( Y \) denote a particular NDSE whose spectrum of simultaneous eigenvalues is everywhere discrete. Let \( Y_1, Y_2, \ldots \) denote the various NDSE's of \( Y \). We choose a particular one of these NDSE's, say \( Y_k \), and focus attention on it. We wish to prove that any two normalized vectors representing \( Y_k \) differ from each other by at most a phase factor.

Let \( \Phi \) be any realizable physical state such that the transition probability \( \Theta(\Phi, Y_k) \) is neither 0 nor 1. Since we are not assuming that \( Y \) is maximal or that physical states are represented by unique rays, it is possible that \( \Phi \) and each of the \( Y_i \) are each represented by several vectors which are not scalar multiples of one another. Let \( |\Phi\rangle \) be any normalized vector representing the state \( \Phi \), and, for each \( i \), let \( |\gamma_i\rangle \) be any normalized vector representing \( Y_i \).

Note that we are selecting \( |\Phi\rangle \) and each \( |\gamma_i\rangle \) completely independently of one another. Now let \( |\rho\rangle \) be the vector defined by
\[
|\rho\rangle \equiv |\Phi\rangle - \sum_i |\gamma_i\rangle\langle \gamma_i | \Phi \rangle
\]

We next compute
\[
\langle \rho | \rho \rangle = \langle \Phi | \Phi \rangle - \sum_i \langle \Phi | \gamma_i \rangle\langle \gamma_i | \Phi \rangle - \sum_i \langle \Phi | \gamma_i \rangle\langle \gamma_i | \rho \rangle + \\
+ \sum_{i,j} \langle \Phi | \gamma_i \rangle\langle \gamma_i | \gamma_j \rangle\langle \gamma_j | \Phi \rangle
\]

For \( i \neq j \), \( |\gamma_i\rangle \) and \( |\gamma_j\rangle \) represent eigenstates with different eigenvalues of the observables in \( Y \). Hence \( \langle \gamma_i | \gamma_j \rangle = 0 \) if \( i \neq j \). Next, \( \langle \gamma_i | \gamma_i \rangle = 1 \) since \( |\gamma_i\rangle \) is normalized. So we see that \( \langle \gamma_i | \gamma_j \rangle = \delta_{ij} \). Hence it follows that
\[ \langle \rho | \rho \rangle = 1 - \sum_k \langle \phi | \gamma_k \rangle \langle \gamma_k | \rho \rangle - \sum_j \langle \phi | \gamma_j \rangle \langle \gamma_j | \rho \rangle + \sum_i \langle \phi | \gamma_i \rangle \langle \gamma_i | \rho \rangle \]
\[ = 1 - \sum_k |\langle \gamma_k | \rho \rangle|^2 \]

If the state \( \Phi \) were associated with \( \Upsilon_k \), then we would have \( \Theta(\Phi, \Upsilon_k) = 1 \). If \( \Phi \) were associated with one of the other \( \Upsilon_i \), then we would have \( \Theta(\Phi, \Upsilon_i) = 0 \). Since both of these possibilities have previously been ruled out, we conclude that \( \Phi \) is not associated with any of the NDSE's of \( \Upsilon \). Hence, by the TPP, we have \( |\langle \gamma_i | \rho \rangle|^2 = \Theta(\Phi, \Upsilon_i) \) for all \( i \). So,
\[ \langle \rho | \rho \rangle = 1 - \sum_i \Theta(\Phi, \Upsilon_i) \]

Now if we take the state \( \Phi \) and simultaneously measure on it the observables in \( \Upsilon \), the result will be the eigenvalue combination associated with one of the \( \Upsilon_i \). Hence we see that \( \sum_i \Theta(\Phi, \Upsilon_i) = 1 \).

So \( \langle \rho | \rho \rangle = 0 \)

Hence \( |\rho \rangle = 0 \)

\[ |\gamma_i \rangle = \sum_k |\gamma_k \rangle \langle \gamma_k | \rho \rangle \]

Recall that \( |\gamma_i \rangle \) and \( |\gamma_k \rangle \) were any normalized vectors representing \( \Phi \) and \( \Upsilon_i \), respectively, and that \( |\rho \rangle \) and each of the \( |\gamma_i \rangle \) were all chosen completely independently of one another. Hence if we make another choice of a normalized vector \( |\Phi \rangle \) representing the state \( \Phi \), and, for each \( i \), a normalized vector \( |\gamma_i \rangle \) representing \( \Upsilon_i \), we get the relation
\[ |\phi \rangle = \sum_i |\gamma_i \rangle \langle \gamma_i | \rho \rangle \]

If we make the particular choice \( |\phi \rangle = |\rho \rangle \), and \( |\gamma_k \rangle \neq |\gamma_k \rangle \), and \( |\gamma_i \rangle = |\gamma_k \rangle \) for \( i \neq k \), we then get the
relationship

\[ |\rho\rangle = \sum_{k} \gamma_k^* <\gamma_k|\rho\rangle + |\gamma_k\rangle <\gamma_k|\rho\rangle \]

By comparison with \[ |\rho\rangle = \sum_{k} \gamma_k^* <\gamma_k|\rho\rangle \], we obtain

\[ |\gamma_k\rangle <\gamma_k|\rho\rangle = |\gamma_k^*\rangle <\gamma_k|\rho\rangle \]

Now \[ |<\gamma_k|\rho|\rangle|^2 = |<\gamma_k|\rho\rangle|^2 = \mathcal{O}(\Phi, \Psi) \neq 0 \], so \[ <\gamma_k|\rho\rangle \neq 0 \] and \[ <\gamma_k|\rho\rangle \neq 0 \]. Hence

\[ |\gamma_k\rangle = \frac{<\gamma_k|\rho\rangle}{<\gamma_k|\rho\rangle} |\gamma_k\rangle \]

We note that \[ \frac{|<\gamma_k|\rho\rangle|^2}{|<\gamma_k|\rho\rangle|^2} = 1 \]. Thus \[ <\gamma_k|\rho\rangle \]

is merely a phase factor, and the theorem is proved.

The requirement that the spectrum of simultaneous eigenvalues of the operators of \( \mathcal{H} \) be everywhere discrete is not essential. It was inserted for convenience only. If the eigenvalue spectrum of \( \mathcal{H} \) is not everywhere discrete, we can still prove the ray uniqueness result for those NDSE's which do lie in the discrete part of the eigenvalue spectrum. That is, we can prove that

\[ |\gamma_k\rangle = \frac{<\gamma_k|\rho\rangle}{<\gamma_k|\rho\rangle} |\gamma_k\rangle \]

where \[ |\gamma_k\rangle \] and \[ |\gamma_k\rangle \] are any normalized vectors representing an NDSE, \( \mathcal{H}_k \), which lies in the discrete part of the spectrum of \( \mathcal{H} \). To do this, we proceed as before, except that \[ |\rho\rangle \] will now contain integrations over continuous indices as well as summations over discrete ones. Also, when we compute \[ <\rho|\rho\rangle \], the continuous indices will lead to delta functions instead of Kronecker deltas. Most importantly, when we
replace the $|\psi\rangle$'s with $|\gamma\rangle$'s, we must be more careful now than in the purely discrete case, because the continuous eigenvectors cannot all be chosen completely independently of one another. However, the discrete eigenvectors are independent of each other and of the continuous eigenvectors, so we simply choose all the continuous eigenvectors and all the discrete eigenvectors except $|\gamma_k\rangle$ to be the same as before. This leads to the desired result.

So any two normalized vectors representing the same discrete NDSE differ by at most a phase factor. Therefore they have the same expectation values, and consequently they describe the same state. Thus a discrete NDSE characterizes only one physical state, and is therefore maximal. Furthermore, since the normalized vectors representing that NDSE (and hence its associated state) differ by only a phase factor, any two unnormalized vectors representing that state will differ by at most a constant complex factor. So we have proven that states which are characterized by discrete NDSE's are represented by unique rays. The restriction of this result to discrete NDSE's is not a serious limitation, since a state with continuous eigenvalues is not physically realizable. Thus all physically realizable states are represented by unique rays.
In this section we prove that the usual definition of the interchange operator does indeed represent interchange of particles.

Consider a system of $n$ not necessarily indistinguishable particles. Suppose the set of possible values of each internal variable of the $i$'th particle is the same as that of the corresponding variable of the $j$'th particle. We can then define the interchange operator $P_{ij}$:

Let $r^{(k)}$ represent a particular set of compatible observables which is complete for the $k$'th particle. $r^{(k)}$ is to be regarded as, e.g., the momentum and spin, not as the momentum and spin of the $k$'th particle. The superscripts merely refer to the fact that we may not be using the same observables for every particle. Let $r^{(k)}_{ak}$ be a possible eigenvalue of $r^{(k)}$ for the $k$'th particle. We then define $P_{ij}$ by

$$P_{ij} \sum_{a_1, \ldots, a_m} \langle h^{(c_1)}_{a_1}, \ldots, h^{(c_l)}_{a_l}, \ldots, h^{(m)}_{a_m} | \psi \rangle | h^{(c_1)}_{a_1} \rangle \ldots | h^{(c_l)}_{a_l} \rangle \ldots | h^{(m)}_{a_m} \rangle =$$

$$= \sum_{a_1, \ldots, a_m} \langle h^{(c_1)}_{a_1}, \ldots, h^{(c_l)}_{a_l}, \ldots, h^{(m)}_{a_m} | \psi \rangle | h^{(c_1)}_{a_1} \rangle \ldots | h^{(c_l)}_{a_l} \rangle \ldots | h^{(m)}_{a_m} \rangle$$

Next let $s^{(k)}$ be any set of compatible one-particle observables which is complete for the $k$'th particle, and let $s^{(k)}_{bk}$ be a possible eigenvalue of $s^{(k)}$ for the $k$'th particle. Using the substitution $^* | \delta^{(k)}_{b_k} \rangle = \sum_{a_k} | h^{(k)}_{a_k} \rangle \langle h^{(k)}_{a_k} | \delta^{(k)}_{b_k} \rangle$, (i.e., expanding

* Note that the validity of this expansion formula has been demonstrated in Section 6.
in terms of the particular kets used to define \( P_{ij} \), we see that

\[
P_{ij} \sum_{l_1, \ldots, l_m} \langle \phi^{(1)}_{l_1}, \phi^{(2)}_{l_2}, \ldots, \phi^{(m)}_{l_m} | \psi \rangle | \phi^{(1)}_{l_1} \rangle \cdots | \phi^{(m)}_{l_m} \rangle = \sum_{l_1, \ldots, l_m} \langle \phi^{(1)}_{l_1}, \phi^{(2)}_{l_2}, \ldots, \phi^{(m)}_{l_m} | \psi \rangle | \phi^{(1)}_{l_1} \rangle \cdots | \phi^{(m)}_{l_m} \rangle
\]

Now let \( u \) be a one-particle observable of the \( i \)'th particle. Every observable and eigenvalue which is possible for the \( i \)'th particle is also possible for the \( j \)'th particle, so \( u \) is also an observable of the \( j \)'th particle. Let \( \mathcal{U}_{[ij]} \) be the operator for \( u \) which operates on the state vector of the \( i \)'th particle. Let \( \mathcal{U}_{[ij]} \) be the same thing for the \( j \)'th particle. In choosing the observables \( s^{(1)}, \ldots, s^{(n)} \), let us include \( u \) among the observables in the set \( s^{(j)} \). Then if either the \( i \)'th or \( j \)'th particle is in the state represented by \( | \phi^{(p)}_{l_j} \rangle \), it will be in an eigenstate of \( u \), the eigenvalue for which we write as \( \mathcal{U}_{l_j} \). Hence

\[
P_{ij} \mathcal{U}_{[ij]} P_{ij} | \psi \rangle = \mathcal{U}_{[ij]} | \psi \rangle
\]
Since this is true for any $|\psi\rangle$, we have $P_{i,j} \mu_{[i]} P_{i,j} = \mu_{[i]}$. From this we get $P_{i,j}^{-1} \mu_{[i]} P_{i,j}^{-1} = \mu_{[i]}$. Since $P_{i,j}^{-1} = P_{i,j}$, these relationships become $P_{i,j} \mu_{[i]} P_{i,j}^{-1} = \mu_{[i]}$ and $P_{i,j} \mu_{[i]} P_{i,j}^{-1} = \mu_{[i]}$.

In a similar way we see that if $v$ is a one-particle observable of the $k'$th particle, where $i \neq k \neq j$, and if $\mathcal{N}_{[k]}$ is its operator for the $k'$th particle, then $P_{i,j} \mathcal{N}_{[k]} P_{i,j}^{-1} = \mathcal{N}_{[k]}$.

More generally, any observable $F$ of the $n$-particle system can be written as a function of one-particle observables. Its operator will have the form $F(a_{[1]}, a_{[2]}, \ldots, a_{[n]}), w^{(m)}$ represents a (generally not compatible) set of one-particle observables of the $m'$th particle, and where the subscript $[m]$ means that the operators $\mu_{[m]}$ operate only on the state vector of the $m'$th particle. Then

$$P_{i,j} F P_{i,j}^{-1} =$$
$$= F(P_{i,j} a_{[1]} P_{i,j}^{-1}, \ldots, P_{i,j} a_{[2]} P_{i,j}^{-1}, \ldots, P_{i,j} a_{[n]} P_{i,j}^{-1})$$
$$= F(a_{[1]}, a_{[2]}, \ldots, a_{[n]}).$$

The only change is that the observable $w^{(1)}$ is now measured on the $j'$th particle, and vice versa. That is, the roles of the $i'$th and $j'$th particles in $F$ have been interchanged.

Suppose that $|\psi\rangle$ is an eigenvector of $F$ with the eigenvalue $F'$.

$$F|\psi\rangle = F'|\psi\rangle$$

$$(P_{i,j} F P_{i,j}^{-1}) P_{i,j} |\psi\rangle = P_{i,j} F |\psi\rangle = F' P_{i,j} |\psi\rangle$$
So $P_{ij} |\psi\rangle$ is an eigenvector of $P_{ij} F P_{ij}^{-1}$ with the same eigenvalue $F'$ as before. If we designate the states represented by $|\psi\rangle$ and $P_{ij} |\psi\rangle$ as $\Psi$ and $P_{ij} \Psi$, then $P_{ij} \Psi$ is obtained from $\Psi$ by interchanging the roles of the $i$'th and $j$'th particles in the observables $F$ which define $\Psi$. This is what we mean by interchanging the $i$'th and $j$'th particles in the state $\Psi$, so the operator $P_{ij}$ is indeed the mathematical representation of particle interchange.
8. THE SYMMETRIZATION POSTULATE

Next, suppose that we have two particles which are indistinguishable. That is, no apparatus will be affected by an interchange of these particles. If these two particles are part of a larger state, interchanging them should not observably alter that state, since then the state would be an "apparatus" capable of distinguishing the particles. So states differing only by an interchange of indistinguishable particles are themselves indistinguishable and are hence the same state.

With these considerations in mind we are ready to derive the Symmetrization Postulate. This time let the operator \( P_{ij} \) interchange two indistinguishable particles. Then the two vectors \( |\psi\rangle \) and \( P_{ij} |\psi\rangle \) represent the same state. Hence, \( P_{ij} |\psi\rangle = C |\psi\rangle \)

where \( C \) is a constant complex factor.

Now \( P_{ij}^2 = 1 \), so \( |\psi\rangle = P_{ij}^2 |\psi\rangle = C^2 |\psi\rangle \)

Hence, \( C = \pm 1 \)

So either \( P_{ij} |\psi\rangle = |\psi\rangle \) or \( P_{ij} |\psi\rangle = -|\psi\rangle \)

This is the Symmetrization Postulate.

Lastly we would like to prove a result which is slightly stronger than the Symmetrization Postulate. The SP says that \( P_{ij} |\psi\rangle = \ell_{ij} |\psi\rangle \) and \( P_{kl} |\psi\rangle = \ell_{kl} |\psi\rangle \), where \( \ell_{ij} = \pm 1 \) and \( \ell_{kl} = \pm 1 \), but it does not say that \( \ell_{ij} = \ell_{kl} \). In other words, a state vector might be symmetric under the exchange of one pair of particles and antisymmetric under the exchange of some other pair. We wish to prove that this never happens.
To do this, we first note that

$$P_{ij} |\psi\rangle = P_{ki} P_{li} P_{kj} P_{lj} P_{ki} |\psi\rangle$$

Hence,

$$e_{ij} |\psi\rangle = e_{ki} e_{lj} e_{jk} e_{li} e_{ki} |\psi\rangle$$

$$= (e_{ki})^2 (e_{lj})^2 e_{kl} |\psi\rangle$$

$$= e_{kl} |\psi\rangle$$

Multiplying by $\langle \psi |$, we obtain $e_{ij} = e_{kl}$.

Q.E.D.
9. CONCLUSION

In conclusion we would like to discuss two questions: First, what is the significance of the fact that we have proven the Symmetrization Postulate? Secondly, have we proven the Symmetrization Postulate?

In answer to the first question, we first point out that proving the SP tells us a great deal about the behavior of particles other than electrons, nucleons, photons, and pions, since the symmetry properties of the less common particles have not been experimentally determined. Furthermore, the various relativistic field theories begin by assuming the SP, either explicitly in a Fock space treatment or implicitly by using the appropriate commutation relations among the creation and destruction operators. Thus proving the SP helps to verify part of the foundation of the relativistic quantum theories. Also, by combining the SP with field theory, it is possible to derive the Exclusion Principle, thereby completing our knowledge of the symmetry properties of the elementary particles. In addition, proving that physical states are represented by unique rays is a valuable result in its own right. For example, as we showed at the beginning of Section 3, this result gives us a simple and convenient criterion for use in investigating superselection rules. Also, the uniqueness of the ray is necessary if we wish to apply Lie Algebras to the study of groups of symmetry transformations. In addition, the uniqueness of the ray is frequently assumed in proving theorems about theoretical physics — see, for example, references.
Now to the second question. In our discussions and theorems we have restricted our consideration to those states which are eigenstates of measurable quantities. Therefore we have strictly verified the SP only for states that can be prepared by making measurements. For that matter, most of the assertions of quantum mechanics have been strictly verified only for such states, because these are the only states for which all predictions can be directly tested. After all, we cannot determine experimentally what percentage of times a given state will behave in a certain way unless we have some means of preparing a large number of exact copies of that state. Nevertheless, the fact remains that most of the physical states that occur in nature are probably not eigenstates of anything for which there exists a measuring apparatus. Can we say anything about the interchange symmetry of such states?

The answer to this question awaits the resolution of the measurement problem. Suppose that this problem had been resolved. The measurement of a complete set of compatible observables on an arbitrary state \( \Psi \) will transform that state into a state \( \Phi \) which is an eigenstate of those observables. \( \Phi \) is the type of state we have been considering, and so it obeys the SP. The interchange operator commutes with the Hamiltonian of the system plus apparatus, so the symmetry properties are conserved. Hence \( \Psi \) has the same symmetry type as \( \Phi \), and therefore obeys the SP. Unfortunately this line of reasoning is not valid because of the existence of the measurement problem, but perhaps some modification of it
can be used once the measurement problem is resolved.

There is also another point we must consider when evaluating the extent to which we have proven the SP. We have apparently used only those features of the Transition Probability Postulate which are essential to that postulate, but is the TPP itself essential to quantum mechanics? Clearly its role — computing the expected distribution of results of a measurement — is essential, but are there no other ways of doing this? In fact, there are. First there is the Expansion Postulate. This postulate asserts that an arbitrary state vector can be expanded in terms of a given set of compatible observables, and that the transition probability from the state to a given eigenvalue combination is equal to the absolute square of the corresponding expansion coefficient. Also there is the Expectation Value Postulate. This postulate is designed to compute just the expectation value rather than the entire distribution, but in principle it can be used for the latter purpose by computing the expectation value of a function of the measured observables which is zero everywhere outside an arbitrarily small region of the space of simultaneous eigenvalues, and equal to unity inside the region. In practice, however, it would be impossible to make any computations in this way.

Given the uniqueness of the ray and the usual assumptions about the Hilbert space, these three postulates are equivalent. However they would give different results if ray uniqueness were violated. Therefore, in the absence of unique rays, quantum mechanics might take any of three forms,
corresponding to these three postulates. We have investigated one of these possible forms, and have shown that it leads us once again to unique rays and the SP. To complete the proof that the SP is essential to quantum mechanics, we would have to exhaust the other two possible forms which quantum mechanics could have in the absence of unique rays.

Thus we can claim to have done two things: First, we have proven Case I of the final result (for eigenstates, at least). Secondly, we have proven that in the absence of the SP, we would lose the most commonly used method of computing the expected distribution of measurement results, and we would apparently not have maximal sets of observables — so quantum mechanics without the Symmetrization Postulate would take on an even more alien character than has been previously suspected.
REFERENCES