RICE UNIVERSITY

Linear Dimensions of ℓ_p and L_p spaces

by

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A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF

Master of Arts

Thesis Director's Signature

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Houston, Texas

May, 1971

ACKNOWLEDGMENTS

The author wishes to express his deep thanks to Professor William A. Veech for his guidance in the form of his many helpful suggestions during the course of this work.

ABSTRACT

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Isomorphisms between ℓ_p and L_q spaces exist for some ratios between p and q. For some of other ratios between p and q, no isomorphism exists. Similar results exist for L_p and L_q.

§0. Introduction

The present paper is concerned with linear dimensions of ℓ_p and $L_p(0,1)$ for $p \geq 1$.

Banach defines X to be of smaller linear dimension than that of Y if there is a one-to-one continuous linear map of X onto a closed subspace of Y. He raises the question of when the relation $\dim_\ell \ell_q \leq \dim_\ell L_p$ is true. He shows that it can not be true for $1 < q < p < 2$ or $2 < p < q$. Paley settles the question for the cases $1 \leq q < 2 < p$, $1 \leq p < 2 < q$ and $2 < q < p$. He proves that the relation is impossible in the three cases. For the case $1 \leq p < q < 2$, the answer is affirmative by 3.14.

The purposes of this paper are: (1) To show Theorem (i) Let $1 \leq p \neq q \geq 1$. The relation $\dim_\ell L_q \leq \dim_\ell Lp$ holds only when $1 \leq p < q \leq 2$. (ii) Let $p \geq 1 \leq q$. Then $\dim_\ell \ell_q \leq \dim_\ell L_p$ is true only when $1 \leq p < q < 2$ or $1 \leq p = q$ or $p \geq 1$ and $q = 2$. (iii) The linear dimensions of ℓ_q and ℓ_p are incomparable for $1 \leq p \neq q \geq 1$. (2) To extend Paley's proof to cases when $2 \leq p < q$ or $1 \leq q < p \leq 2$.

Section 1 is concerned with definitions of linear dimensions and a remark about isomorphism and equality of linear dimensions. In section 2, first we will describe Banach's discussion about relations between $\dim_{\ell}\ell_q$ and $\dim_{\ell}L_p$ and then give another proof for the cases $1 \leq q < p \leq 2$ and $2 \leq p < q$. Section 3 is concerned with relations between linear dimensions of ℓ_q and L_p for $1 \leq p < q < 2$. In section 4, we will discuss relations between linear dimensions of ℓ_q and ℓ_p and that of L_p and L_q.

§1.

1.1 Definition Let X and Y be two Banach spaces.

X is said to be of smaller linear dimension than Y if there is a one to one map T in B(X,Y) with T(X) a closed linear subspace of Y, in formula:

(1) $$\dim_\ell X \le \dim_\ell Y.$$

1.2 Definition X and Y are said to be of equal linear dimension, if formulae (1) and

(2) $$\dim_\ell Y \le \dim_\ell X$$

are satisfied.

1.3 Definition Linear dimensions of X and Y are said to be incomparable, if none of the relations (1) and (2) is satisfied.

1.4 Remark By the definition, two spaces that are isomorphic have equal linear dimension. But the converse is not true.

Let C be the space of continuous functions on (0,1) with sup norm, $\|f\| = \sup |f(x)|$. Let ℓ_1 be the space of all sequences $x = \{x_n\}$ of real numbers with the norm $\|x\| = \Sigma|x_n| < \infty$. Denote by $C\times\ell_1$ the direct product of

C and ℓ_1 with norm $\|(x,y)\| = [\|x\|^2 + \|y\|]^{\frac{1}{2}}$ for $x \in C$, $y \in \ell_1$. Then $C \times \ell_1$ is separable Banach space. Clearly, C is isomorphic to a subspace of $C \times \ell_1$. On the other hand, since each separable Banach space is isomorphic to a subspace of C, $C \times \ell_1$ is isomorphic to a subspace of C. Thus $\dim_\ell C = \dim_\ell C \times \ell_1$. Suppose now that C and $C \times \ell_1$ are isomorphic, then so are the spaces C^* and $(C \times \ell_1)^*$. Since $(C \times \ell_1)^*$ is isomorphic to $C^* \times \ell_1^*$, this implies that C^* and $C^* \times \ell_1^*$ are isomorphic. But $\ell_1^* = \ell_\infty$ is not weak complete (IV. 13.5 [1]) and so $C^* \times \ell_1^*$ is not weak complete, while C^* is weak complete (IV. 6.3, 13.22 [1]), a contradiction.

§2

2.1 Proposition For all $p \geq 1$, ℓ_p is isometric to a subspace of L_p and hence $\dim_\ell \ell_p \leq \dim_\ell L_p$.

Proof Define $T: \ell_p \to L_p$ by

$$(Tx)(t) = \begin{cases} 2^{\frac{n}{p}} x_n & \text{if } \frac{1}{2^n} < t \leq \frac{1}{2^{n-1}} \\ 0 & \text{if } t = 0 \end{cases}$$

where $x = (x_1, x_2, \ldots) \in \ell_p$.

Then

$$\|Tx\| = \left[\int_0^1 |Tx(t)|^p \, dt \right]^{\frac{1}{p}} = \left[\sum_{n=1}^{\infty} |x_n|^p \right]^{\frac{1}{p}} = \|x\|,$$

Clearly, T is one-to-one and linear, thus T is an isometric isomorphism.

2.2 Notations Denote by $\varphi_0(t)$, $\varphi_1(t)$, ..., the Rademacher's functions, i.e.,

$$\varphi_0(t) = 1 \text{ if } 0 < t < \tfrac{1}{2};$$
$$-1 \text{ if } \tfrac{1}{2} < t < 1; \quad 0 \text{ if } t = 0, \tfrac{1}{2}, 1,$$
and extend periodically,

$$\varphi_n(t) = \varphi_0(2^n t).$$

2.3 Lemma if $\sum_{n=0}^{\infty} a_n^2 < \infty$, then there are constants $B_1(P)$, $B_2(P)$, depending only on P, such that

$$(3) \qquad B_1 \le \left[\int_0^1 \left| \sum_{n=0}^{\infty} a_n\varphi_n(t)\right|^p dt \Big/ \left[\sum_{n=0}^{\infty} |a_n|^2\right]^{p/2}\right] \le B_2$$

for all $P \ge 1$.

Proof refer lemma 2, p. 304 [2].

2.4 Corollary ℓ_2 is isomorphic to a subspace of L_p for all $p \ge 1$ and thus $\dim_\ell \ell_2 \le \dim_\ell L_p$.

Proof For each a ($a_0, a_1, \ldots$) in ℓ_2, by 2.3 lemma, $f(t) = f_a(t) = \sum_{n=0}^{\infty} a_n\varphi_n(t)$ is in L_p. Define

$$T: \ell_2 \to L_p \quad \text{by } Ta = f.$$

Clearly, T is one-to-one and linear. Thus, by (3), T is an isomorphism between ℓ_2 and a subspace of L_p.

For p=2 in 2.4, we have

2.5 Proposition $\dim_{\ell}\ell_2 = \dim_{\ell}L_2$. Indeed, ℓ_2 is isometric to L_2.

Proof Let $\{w_i(t)\}_{i=1}^{\infty}$ be a complete orthonormal system in L_2. Define

$$T: L_2 \to \ell_2 \text{ by } Tx = (x_1, x_2, \ldots)$$

where $\{x_i\}_{i=1}^{\infty}$ are the Fourier coefficients of X(t) with respect to the system $\{w_i(t)\}$. Then by theorem 3, VII p. 179 [3], T is an isometry.

From 2.6 to 2.10 are two lemmas by Banach and the method that he used for discussing some relations of linear dimensions of ℓ_q and L_p.

2.6 Lemma Let $\{f_i\}_{i=1}^{\infty}$ be a sequence of elements of $L_p(0,1)$ that converges weakly to 0. Then there is a subsequence $\{f_{i_k}\}_{k=1}^{\infty}$ such that

$$(4) \qquad \left\| \sum_{n=1}^{n} f_{i_k} \right\| = \begin{cases} 0(n^{\frac{1}{p}}) & \text{if } 1 < p \leq 2 \\ 0(n^{\frac{1}{2}}) & \text{if } p \geq 2. \end{cases}$$

Proof For $P > 1$, let

$$\varphi(x) = \frac{|1+x|^P - [1 + PZ + \sum_{i=2}^{E(P)} \binom{P}{i} x^i]}{|x|^P}$$

where $E(P)$ denotes the largest integer less than P. Then $\lim_{x\to 0} \varphi(x) = 0$ by expanding $|1 + x|^P$ into binomial series.

Clearly, $\lim_{x\to\pm\infty} \varphi(x)$ are finite. Thus, there is a constant A such that

$$|\varphi(x)| \leq A \text{ for all } x \neq 0 \text{ real number.}$$

Let $x = \frac{b}{a}$ and using the fact that

$$\binom{P}{i} \leq B(P), \text{ a constant for } 2 \leq i \leq P,$$

we have

$$(\dagger) \quad |a + b|^P \leq |a|^P + P|a|^{P-1}\cdot b\cdot \operatorname{sgn} a + A|b|^P + B\sum_{i=2}^{E(P)} |a|^{P-i}\cdot |b|^i.$$

For $b = 0$, $(\dagger)$ is also true, thus $(\dagger)$ is true for all real numbers a,b.

Define $X_{i_1} = X_1$. For $n > 1$, define X_{i_n} by induction.

Let $S_{n-1}(t) = \sum_{k=1}^{n=1} X_{i_k}(t)$. Clearly, $|S_{n-1}|^{p-1} \in L_{p/p-1} = L^*_p$;

Since $\{x_n\}$ converges weakly to 0, there is i_n such that

$$(*) \qquad P\left|\int_0^1 |S_{n-1}|^{p-1} X_{i_n} \operatorname{sgn} S_{n-1}\, dt\right| \le 1.$$

Choose $\{X_{i_n}\}$ such that i_n's satisfy $(*)$. Let $a = S_{n-1}$,

$b = X_{i_n}$ in $(\dagger)$ and integrate. Then

$$(**) \quad \int_0^1 |S_n|^p dt \le \int_0^1 |S_{n-1}|^p\, dt + p\int_0^1 |S_{n-1}|^{p-1} X_{i_n} \operatorname{sgn} S_{n-1} dt$$

$$+ A\int_0^1 |X_{i_n}|^p\, dt + B \sum_{i=2}^{E(P)} \int_0^1 |S_{n-1}|^{p-i} |X_{i_n}|^i dt.$$

By theorem 1, IX [4], $\{\|x_n\|\}$ is bounded and thus we may suppose that $\|x_n\| \le 1$ for all n.

1. $p > 2$.

For $2 \le i \le p$, we have

$$\int_0^1 |S_{n-1}|^{p-i} |X_{i_n}|^i dt \le [\int_0^1 (|S_{n-1}|^{p-i})^{p/p-i} dt]^{p-i/p}$$

$$[\int_0^1 (|x_{i_n}|^i)^{p/i} dt]^{i/p} \le [\int_0^1 |S_{n-1}|^p dt]^{p-i/p}$$

$$\le 1 + [\int_0^1 |S_{n-1}|^p dt]^{p-2/p}$$

(For $a \geq 1$, $a^{p-i} \leq a^{p-2}$ since $p-2 \geq p-i$; for $0 < a < 1$, $a^{p-i} < 1$.)

Therefore, from (**),

$$\|S_n'\|^p \leq \|S_{n-1}\|^p + 1 + A + BP(1 + \|S_{n-1}\|^{p-2}).$$

Similar results are obtained for n-1,....

Thus

$$\|S_n\|^p \leq (1 + A + Bp)\ n + Bp \sum_{k=1}^{n-1} \|S_k\|^{p-2}$$

$$= Cn + D \sum_{k=1}^{n-1} \|S_k\|^{p-2} .$$

Let $M = C + D + 2$. Then $\|S_1\| \leq 1 \leq M$. Suppose $\|S_m\| \leq M m^{\frac{1}{2}}$ for $m = 1,2,\ldots,n-1$. Then

$$\|S_n\| \leq Cn + D \cdot M^{p-2} \sum_{k=1}^{n-1} K^{p-2/2} \leq Cn + DM^{p-2}\ n^{p/2}$$

since for $p > 2$, $\sum_{k=1}^{n-1} K^{p-2/2} \leq n^{p/2}$.

Thus

$$\|S_n\|^p \leq M^p n^{p/2}(n^{1-\frac{p}{2}} \cdot CM^{-p} + DM^{-2}) \leq M^p n^{p/2}$$

since $n^{1-\frac{p}{2}} \cdot C \cdot M^{-p} + DM^{-2} \leq \frac{1}{2} + \frac{1}{2} = 1.$

By induction, we have

$$\|S_n\|^p = O(n^{p/2}).$$

2. $1 < p \leq 2$.

(**) implies that

$$\int_0^1 |S_n|^p \, dt \leq \int_0^1 |S_{n-1}|^p dt + 1 + A + B.$$

Therefore,

$$\|S_n\|^p \leq \|S_{n-1}\|^p + C, \quad C = 1 + A + B.$$

By induction

$$\|S_n\|^p \leq \|S_1\|^p + C(n-1) \leq cn.$$

This implies

$$\|S_n\| \leq c^{\frac{1}{p}} n^{\frac{1}{p}} = Mn^{\frac{1}{p}} .$$

2.7 Lemma Let $p > 1$ and $\{x_i\}_{k=1}^{\infty}$ converges weakly to 0 in ℓ_p. Then there is a subsequence $\{y_k\}_{k=1}^{\infty}$ such that

(5) $\quad \| \sum_{k=1}^{n} y_k \| = O(n^{1/p}).$

Proof Refer XII, 3 theorem 3 p. 200 [4].

2.8 Remarks (a) The relations (4) and (5) no longer hold for all $p>1$ if we replace big "0" by small "o". For reference, see p. 200-201, [4].

(b) Let $p > 1 < q$. Suppose L_p is isomorphic to a subspace of ℓ_q. Let $T: L_p \to \ell_q$ be an isomorphism. Given a sequence $\{y_i\}_{i=1}^{\infty}$ in L_p which converges weakly to 0, then the sequence $\{x_i\}_{i=1}^{\infty}$, where $x_i = Ty_i$, also converges weakly to 0. Therefore, by lemma 2.7, there is a subsequence $\{x_{i_k}\}$ such that

(6) $$\| \sum_{k=1}^{n} x_{i_k} \| = 0(n^{\varphi(q)}) \qquad \text{where } \varphi(q) = \frac{1}{q} .$$

The operator $T^{-1}: T(L_p) \to L_p$ is continuous. Hence $\| \sum_{k=1}^{n} y_{i_k} \| \le M \cdot \| \sum_{k=1}^{n} x_{i_k} \|$ for some constant M, and therefore $\| \sum_{k=1}^{n} y_{i_k} \| = 0(n^{\varphi(q)})$.

As $\{y_i\}$ is an arbitrary sequence that converges weakly to 0, we have

$$(7) \qquad \psi(p) \le \varphi(q), \quad \text{here } \psi(p) = \begin{cases} \frac{1}{p} & \text{if } 1 > p \le 2 \\ \frac{1}{2} & \text{if } p \ge 2 \end{cases}$$

by formula (4).

L_p^* is isometric to $L_{p/(p-1)}$, ℓ_q^* is isometric to $\ell_{q/(q-1)}$. Let $T^*: \ell_{q/(q-1)} \to L_{p/(p-1)}$ be the adjoint operator of T. (Actually T^* is a mapping from ℓ_q^* to L_p^*. But by identifying L_p^* with $L_{p/(p-1)}$ and ℓ_q^* with $\ell_{q/(q-1)}$, T^* can be considered also as a mapping from $\ell_{q(q-1)} \to L_{p/(p-1)}$.) Since $T(L_p)$ is closed, we have

$$\text{range } T^* = \{y^* \mid Ty = o \text{ implies } y^*y = 0\}$$

(refer VI 6.2 [1]). But T is an isomorphism, therefore range $T^* = L_{p/(p-1)}$, a closed set. Hence there is a constant K such that to each y^* in L_p^* corresponds on X^* with $T^*X^* = y^*$ and $\|x^*\| \le K\|y^*\|$. For a sequence $\{y_n^*\}$ in L_p^* which converges weakly to 0, let $\{x_n^*\}$ be the corresponding sequence in ℓ_q^* with $T^*X_n^* = y_n^*$ and $\|x_n^*\| \le K \cdot \|y_n^*\|$ for all n. $\{x_n^*\}$ is a bounded set in the reflexive space ℓ_q^* and therefore there is a subsequence

$\{x^*_{n_i}\}$ which converges weakly to a point x^*_0. Hence $T^* x^*_0 = 0$ because $\{y^*_n\}$ converges weakly to 0. We have $y^*_{n_i} = T^*(X^*_{n_i} - X^*_0)$ and $\{x^*_{n_i} - x^*_o\}$ converges weakly to 0. Write $X^*_i = x^*_{n_i} - x^*_o$ for all i. Then by formula (4), there is a subsequence $\{X^*_{i_k}\}$ such that

$$\left\| \sum_{k=1}^{n} X^*_{i_k} \right\| = 0(n^{\varphi(\frac{q}{q-1})}), \quad \varphi(\tfrac{q}{q-1}) = \frac{q-1}{q} .$$

Recalling that $T^* X^*_{i_k} = y^*_{i_k}$, we have

$$\|y^*_{i_k}\| \le \|T^*\| \cdot \|X^*_{i_k}\| = \|T\| \cdot \|X^*_{i_k}\|$$

and therefore

$$\left\| \sum_{k=1}^{n} y^*_{i_k} \right\| = 0(n^{\varphi(\frac{q}{q-1})}).$$

Since $\{y^*_n\}$ is arbitrary, we have

$$\psi(\tfrac{p}{p-1}) \le \varphi(\tfrac{q}{q-1}), \qquad \psi(\tfrac{p}{p-1}) = \begin{cases} \frac{p-1}{p} & \text{if } 1 < \frac{p}{p-1} \le 2 \\ \frac{1}{2} & \text{if } \frac{p}{p-1} \ge 2. \end{cases}$$

(C) In the discussion above, if we replace L_p by ℓ_p, the argument is still valid except the function ψ shall be replaced by φ. Similarly, we can replace ℓ_q by L_q only by replacing φ by ψ.

2.9 Proposition If $\dim_\ell L_p \le \dim_\ell \ell_q$, $p > 1 < q$, then $p = q = 2$. Proof From formulae (7) and (8), we have

$$\psi(p) \le \frac{1}{q} \quad \text{and} \quad \psi\left(\frac{p}{p-1}\right) \le \frac{q-1}{q} \quad .$$

For $1 < p \le 2$, we have $\frac{p}{p-1} \ge 2$. Thus we have, by the formulae above,

$$\frac{1}{p} \le \frac{1}{q} \quad \text{and} \quad \tfrac{1}{2} \le \frac{q-1}{q} \quad .$$

This implies $2 \le q \le p$. Therefore $p = q = 2$. For $p \ge 2$, $\frac{p}{p-1} \le 2$. We have

$$\tfrac{1}{2} \le \frac{1}{q} \quad \text{and} \quad \frac{p-1}{p} \le \frac{q-1}{q}$$

This implies $p \le q \le 2$. Therefore $p = q = 2$.

2.10 Proposition For $1 < q < p < 2$ or $2 < p < q$, the linear dimensions of the spaces L_p and ℓ_q are incomparable.

Proof By Proposition 2.9, it suffices to consider the possibility that

$$\dim_\ell \ell_q \le \dim_\ell L_p .$$

But by (7) and (8) and 2.8 (C), we have

$$\frac{1}{q} \le \psi(p) \text{ and } \frac{q-1}{q} \le \psi(\frac{p}{p-1}) .$$

For $1 < p \le 2$, this implies

$$\frac{1}{q} \le \frac{1}{p} \quad \text{and } \frac{q-1}{q} \le \tfrac{1}{2}$$

and therefore $p \le q$ and $q \le 1$, a contradiction . For $p \ge 2$, we have

$$\frac{1}{q} \le \tfrac{1}{2} \quad \text{and } \frac{q-1}{q} \le \frac{p-1}{p} .$$

This implies $2 \le q \le p$, also a contradiction.

2.11 Lemma Let $(a_0, a_1, \ldots) \in \ell_q$, $f_n(x) \in L_p$. If there are constants $B_1(p,q)$ and $B_2(p,q)$ depending only on p and q such that

$$(9) \qquad B_1 \le \{\int_0^1 |\Sigma a_n f_n(x)|^p dx\}^{\frac{1}{p}} / \{\Sigma |a_n|^q\}^{\frac{1}{q}} \le B_2,$$

then, for some constants $B_3(p,q)$ and $B_4(p,q)$ depending only on p and q, we have

(10) $B_3 \le \{\int_0^1 [\Sigma|a_n|^2|f_n(x)|^2 dx\}^{\frac{1}{p}} / \{\Sigma|a_n|^q\}^{\frac{1}{q}} \le B_4.$

Proof Substitute a_n in (9) by $a_n\varphi_n(t)$, take p-th power and integrate from 0 to 1 with respect to t, we have, by formula (3),

$$B_5 \le [\int_0^1\int_0^1 [\Sigma|a_n|^2|f_n(x)|^2]^{p/2} dx / [\Sigma|a_n|^q]^{p/q} \le B_6 .$$

Take p-th root, we have (10).

2.12 Remark Suppose $\dim_{\ell} \ell_q \le \dim_{\ell} L_p$, $p \ge 1 \le q$. Then there is an isomorphism T: $\ell_q \to Y$, where Y is a closed subspace of L_p. Let (1,0,0,...), (0,1,0,...), (0,0,1,0,..)..be mapped by T to functions $f_1(x)$, $f_2(x)$, $f_3(x)$,... in L_p. Then there are constants M_1 and M_2, depending only on p and q, such that

$$M_1 \le (\int_0^1 |\Sigma a_n f_n|^p \, dx)^{\frac{1}{p}} / (\Sigma|a_n|^q)^{\frac{1}{q}} \le M_2$$

for all $(a_0, a_1,...)$ in ℓ_q. Thus the assumption (9) is satisfied.

Here, if we take $a_m = 1$ and $a_n = 0$ for $n \neq m$, then we have

(11) $$M_1 \leq (\int_0^1 |f_m|^p dx)^{\frac{1}{p}} \leq M_2.$$

2.13 Proposition For $p > 2 > q \geq 1$ or $q > 2 > p \geq 1$ or $2 < q < p$, the relation $\dim_\ell \ell_q \leq \dim_\ell L_p$ is impossible.

Proof

(i) $p > 2 > q$. Then

$$(\Sigma|a_n|^q)^{\frac{1}{q}} \leq B_1[\int_0^1 (\Sigma|a_n|^2 \cdot |f_n|^2)^{p/2} dx]^{1/p}$$

$$= \{[\int_0^1 (\Sigma|a_n|^2|f_n|^2)^{p/2} dx]^{2/p}\}^{\frac{1}{2}} B_1$$

$$\leq B_1\{\Sigma[\int_0^1 (|a_n|^2|t_n|^2)^{p/2} dx]^{2/p}\}^{\frac{1}{2}}$$

$$= B_1[\Sigma|a_n|^2 (\int_0^1 |f_n|^p dx)^{2/p}]^{\frac{1}{2}}$$

$$\leq B_1[\Sigma|a_n|^2 B_2]^{\frac{1}{2}}$$

$$= B_3(\Sigma|a_n|^2)^{\frac{1}{2}}$$

which is impossible.

(ii) $q > 2 > p$. Then

$$(\Sigma|a_n|^q)^{1/q} \geq B_1[\int_0^1(\Sigma|a_n|^2\cdot|f_n|^2)^{p/2}]^{1/p}$$

$$= B_1[(\Sigma|a_m|^2)^{p/2} \int_0^1(\sum_n \frac{|a_n|^2}{\Sigma_m|a_m|^2}|f_n|^2)^{p/2}dx]^{1/p}$$

$$\geq B_1[(\Sigma|a_m|^2)^{p/2} \sum_n \frac{|a_n|^2}{\sum_m|a_m|^2} \int_0^1|f_n|^p\,dx]^{1/p}$$

$$\geq B_1\cdot B_2\ (\Sigma|a_n|^2)^{\frac{1}{2}},$$

which is also impossible.

(iii) $2 < q < p$.

Let $(\sum_{n=2}^{\infty}|a_n|^q)^{1/q} = A$. Then

$$\int_0^1|f_1|^{p-2}(\sum_2^{\infty}|a_n|^2|f_n|^2)\,dx$$

$$= A^{2-p}\int_0^1(A^2|f_1|^2)^{p/2-1}(\sum_2^{\infty}|a_n|^2\cdot|f_n|^2)dx$$

$$\leq A^{2-p}\int_0^1(A^2|f_1|^2 + \sum_2^{\infty}|a_n|^2\cdot|f_n|^2)^{p/2}\,dx$$

$(a^{\alpha-1}b \leq (a+b)^{\alpha}$ for $\alpha > 1$, $a < 0 > b$.)

$$\le A^{2-p}B_1(A^q + \sum_2^\infty |a_n|^q)^{p/q}$$

$$= B_2A^2.$$

Therefore

$$\sum_2^\infty |a_n|^2 \int_0^1 |f_1|^{p-2}|f_n|^2 dx \le B_2(\sum_2^\infty |a_n|^q)^{2/q},$$

and hence

$$\int_0^1 |f_1|^{p-2}|f_n|^2 \, dx \to 0 \quad \text{as } n \to \infty.$$

In a similar way, we have, for any fixed k,

$$\int_0^1 |f_k|^{p-2}|f_n|^2 \, dx \to 0 \quad \text{as } n \to \infty.$$

Let N be a large integer. Thus we may choose $n_1 = 1, n_2, \ldots n_N$ such that

$$\int_0^1 |f_1|^{p-2}|f_{n_k}|^2 \, dx \le e^{-N}, \quad k = 2,\ldots,N,$$

$$\int_0^1 |f_{n_2}|^{p-2}|f_{n_k}|^2 \, dx \le e^{-N}, \quad k = 3,4,\ldots N,$$

$$\ldots$$

and so on. Set $f_{n_k} = g_k$ and consider

$$A = \int_0^1 \left(\sum_{k=1}^{N} |f_{n_k}|^2\right)^{p/2} dx = \int_0^1 \left(\sum_{k=1}^{N} |g_k|^2\right)^{p/2} dx.$$

Let μ be the integer such that $\mu \le p/2 < \mu + 1$.

Then $p/2 = \mu + q$, $0 \le q < 1$ and

$$\left(\sum_{k=1}^{N} |g_k|^2\right)^{p/2} = \left(\sum_{k=1}^{N} |g_k|^2\right)^{\mu+q} = \left(\sum_{k=1}^{N} |g_k|^2\right)^{\mu}\left(\sum_{k=1}^{N} |g_k|^2\right)^{q}$$

$$\le \left(\sum_{1\le k_i\le N} |g_{k_1}|^2 \cdots |g_{k_\mu}|^2\right) \cdot \left(\sum_{k=1}^{N} |g_k|^{2q}\right)$$

$$\le \sum_{1\le k_i\le N} |g_{k_1}|^2 \cdots |g_{k_\mu}|^2 \cdot |g_{k_{\mu+1}}|^{p-2\mu}.$$

(Minkowski's inequality for $0 < q < 1$ says that $\left\|\sum_{i=1}^{N} f_i\right\|_q \ge \sum_{i=1}^{N} \|f_i\|_q$.)

Therefore

$$A \le \sum_{1\le k_i N} \int_0^1 |g_{k_1}|^2 \cdots |g_{k_\mu}|^2 \cdot |g_{k_{\mu+1}}|^{p-2\mu} dx$$

$$\le \sum_{k=1}^{N} \int_0^1 |g_k|^p dx + N^{\mu+1} \max_{k_i\text{'s not all equal}} \int_0^1 |g_{k_1}|^2 \cdots |g_{k_\mu}|^2 \cdot |g_{k_{\mu+1}}|^{p-2\mu} dx.$$

Now

$$(A_1)\quad \int_0^1 |g_{k_i}|^2 \cdots |g_{k_\mu}|^2 \cdot |g_{k_{\mu+1}}|^{p-2\mu} dx$$

$$\leq \left(\int_0^1 |g_{k_1}|^2 |g_{k_2}|^{p-2} dx\right)^{2/p-2} \left(\int_0^1 |g_{k_1}|^2 |g_{k_3}|^{(2p-4)/p-4} \cdots |g_{k_{\mu+1}}|^{\frac{(p-2\mu)(p-2)}{p-4}} dx\right)^{\frac{p-4}{p-2}} .$$

The second term is less than or equal to

$$\left(\int_0^1 |g_{k_1}|^2 \cdot |g_{k_3}|^{p-2} dx\right)^{2/p-2} \cdot \left(\int_0^1 |g_{k_1}|^2 \cdot |g_{k_4}|^{\frac{2p-4}{p-6}} \cdots |g_{k_{\mu+1}}|^{\frac{(p-2\mu)(p-2)}{p-6}} dx\right)^{\frac{p-6}{p-2}} .$$

Continue the process, we have

$$\int_0^1 |g_{k_1}|^2 \cdots |g_{k_\mu}|^2 \cdot |g_{k_{\mu+1}}|^{p-2\mu} dx$$

$$\leq \left(\int_0^1 |g_{k_1}|^2 \cdot |g_{k_2}|^{p-2} dx\right)^{2/(p-2)} \cdot \left(\int_0^1 |g_{k_1}|^2 \cdot |g_{k_\mu}|^{p-2} dx\right)^{2/(p-2)}$$

$$\cdot \left(\int_0^1 |g_{k_1}|^2 \cdot |g_{k_{\mu+1}}|^{p-2} dx\right)^{(p-2\mu)/(p-2)} .$$

Since $\int_0^1 |g_{k_1}|^2 |g_{k_i}|^{p-2} dx \leq [\int^1 |g_{k_1}|^p dx]^{2/p} [\int^1 |g_{k_i}|^p dx]^{(p-2)/p}$, We have

$$\left(\int_0^1 |g_{k_1}|^2 \cdot |g_{k_i}|^{p-2} dx\right)^{2/p-2} \le B_3, \text{ a constant by (3).}$$

Thus (A_1) does not exceed an expression of the form

$$B_4\left(\int_0^1 |g_k|^2 \cdot |g_{k'}|^{p-2} dx\right)^{(p-2\mu)/(p-2)},$$

where $k \neq k'$. If $k < k'$, then

$$\int_0^1 |g_k|^2 \cdot |g_{k'}|^{p-2} dx \le e^{-N},$$

while if $k > k'$, $p \ge 4$,

$$\int_0^1 |g_k|^2 |g_{k'}|^{p-2} dx$$

$$\le \left(\int_0^1 |g_k|^{p-2} |g_{k'}|^2 dx\right)^{2/(p-2)} \cdot \left(\int_0^1 |g_{k'}|^p dx\right)^{(p-4)/(p-2)}$$

$$\le B_5 e^{-2N/(p-2)}$$

and if $p \le 4$,

$$\int_0^1 |g_k|^2 |g_{k'}|^{p-2} dx$$

$$\le \left(\int_0^1 |g_k|^{p-2} |g_{k'}|^2 dx\right)^{(p-2)/2} \cdot \left(\int_0^1 |g_k|^p dx\right)^{(4-p)/2}$$

$$\leq B_6\ e^{-(p-2)N/2}.$$

Hence

$$A \leq B_7(N + N^{\mu+1}\cdot[e^{-N}+e^{-2N/(p-2)}+e^{-(p-2)N/2}]^{(p-2\mu)/(p-2)})$$

$$\leq B\ N$$

if N is sufficiently large. This contradicts the fact

that $\int_0^1 (\sum_{k=1}^{N} |f_{n_k}|^2)^{p/2}\ dx \geq B'\ N^{p/q}$.

2.14 Another proof for cases (a) $2 \leq p < q$, (b) $1 \leq q < p \leq 2$:

(a) $2 \leq p < q$.

Suppose $\dim_\ell \ell_q \leq \dim_\ell L_p$. Then, by 2.12 remark and 2.11 lemma, we have

$$\{\Sigma|a_n|^q\}^{1/q} \geq B_1[\int_0^1 \{|a_n|^2 |f_n(x)|^2\}^{p/2} dx]^{1/p}.$$

By triangular inequality, we have

$$\{\Sigma|a_n|^p\cdot|f_n(x)|^p\}^{2/p} = \{\Sigma[|a_n|^2\cdot|f_n(x)|^2]^{p/2}\}^{2/p}$$

$$\leq \Sigma\{[|a_n|^2\cdot|f_n(x)|^2]^{p/2}\}^{2/p} = \Sigma|a_n|^2\cdot|f_n(x)|^2.$$

Therefore, recalling the formula (11), we have

$$\{\Sigma|a_n|^q\}^{1/q} \geq B_1\{\int_0^1 \Sigma[|a_n|^p\cdot|f_n(x)|^p dx\}^{1/p}$$

$$\geq B_1\cdot M_1\{\Sigma|a_n|^p\}^{1/p}.$$

Since B_1 and M_1 depend only on p and q, this is impossible.

(b) $1 \leq q < p \leq 2$.

Suppose $\dim_\ell \ell_q \leq \dim_\ell L_p$. Then, as in case (a),

$$\{\Sigma|a_n|^q\}^{1/q} \leq B_2[\int_0^1\{\Sigma|a_n|^2\cdot|f_n(x)|^2\}^{p/2}]^{1/p}.$$

By triangular inequality again, we have

$$\{\Sigma|a_n|^2\cdot|f_n(x)|^2\}^{p/2} = \{\Sigma[|a_n|^p\cdot|f_n(x)|^p]^{2/p}\}^{p/2}$$

$$\leq \Sigma\ |a_n|^p|f_n(x)|^p.$$

Therefore, recalling the formula (11),

$$\{\Sigma|a_n|^q\}^{1/q} \leq B_2\cdot[\int_0^1 \Sigma|a_n|^p\cdot|f_n(x)|^p\ dx]^{1/p}$$

$$\leq B_2\ M_2\{\Sigma|a_n|^p\}^{1/p}.$$

This is impossible.

2.15 Proposition For all $p \geq 1$, the relation $\dim_\ell L_p \leq \dim_\ell \ell_1$ is impossible.

Proof. Weakly convergence implies convergence in ℓ_1 while it does not hold in L_p (13.51 [5]), thus L_p can not be isomorphic to a subspace of ℓ_1.

2.16 Proposition For all $q>1$, the relation $\dim_\ell L_1 \leq \dim_\ell \ell_q$ is impossible.

Proof. Any closed subspace of ℓ_q, $q>1$, is reflexive while L_1 is not reflexive. Therefore L_1 can not be isomorphic to a closed subspace of ℓ_q.

§3.

3.1 Definition Let X be a real vector space and $f: X \to R^+$ a map. f is negative definite, or of negative type, if $f(x) = f(-x)$; $f(0) = 0$; $\sum_{i,j\leq n} f(x_i - x_j)\, \alpha_i \alpha_j \leq 0$ for all n, $x_i \in X$, $\alpha_i \in R$ and $\sum_{i\leq n} \alpha_i = 0$.

3.2 Remarks (a) f is negative definite iff $e^{-\lambda f}$ is positive definite for all $\lambda \geq 0$. Here, a map $\Psi: X \to R$ is called positive definite if $\Psi(x) = \Psi(-x)$ and $\sum_{i,j\leq n} \alpha_i \alpha_j \Psi(x_i - x_j) \geq 0$ for all n, $\alpha_i \in R$, $x_i \in X$. For the proof, refer theorem 2 [6].

(b) For $0 < p < 1$, $t \geq 0$, we have

$$\int_0^\infty \frac{1 - e^{-\lambda^2 t^2}}{\lambda^{1+2p}}\, d\lambda = t^{2p} \int_0^\infty \frac{1 - e^{-\lambda^2}}{\lambda^{1+2p}}\, d\lambda \quad \text{with}$$

$$C(P) = \left\{\int_0^\infty \frac{1 - e^{-\lambda^2}}{\lambda^{1+2p}}\, d\lambda\right\}^{-1}$$ a positive constant depending only on P.

If f is of negative type, substitute t^2 by $f(x_i - x_j)$, multiply $\alpha_i \alpha_j$ and take the summation over i and j. Then, for $\sum_{i\leq n} \alpha_i = 0$,

$$\sum_{i,j\le n} \alpha_i\alpha_j \; f^p(x_i - x_j)$$

$$= -C(P) \int_0^\infty \{ \sum_{i,j\le n} \alpha_i\alpha_j e^{-\lambda^2 f(x_i-x_j)} \} \; \lambda^{-1-2p} \, d\lambda$$

$$\le 0$$

since the integrand is positive. Thus, f^p is also of negative type.

(c) Let $f(x)=|y(x)|$, $y\epsilon X'$, $x\epsilon X$. Then,

$$\sum_{i,j\le n} \alpha_i\alpha_j f^2(x_i-x_j)$$

$$= \sum_{i,j\le n} \alpha_i\alpha_j \{y^2(x_i)+y^2(x_j)\} - 2 \sum_{i,j\le n} \alpha_i\alpha_j y(x_i)y(x_j)$$

$$= - 2 \sum_{i,j\le n} \alpha_i\alpha_j y(x_i)y(x_j)$$

$$= - 2 \; \{ \sum_{i=1} \alpha_i y(x_i) \}^2$$

$$\le 0$$

for all n, $\alpha_i \epsilon R$ with $\sum_{i\le n} \alpha_i = 0$.

This shows that f^2 is negative definite. Thus, $f^p=(f^2)^{p/2}$ is negative definite for $1 \le p \le 2$. Since clearly sums of negative definite functions are negative definite, thus $\sum_{i=1}^{n} |y_i(x)|^p$ is negative definite. This shows that $\{\sum_{i=1}^{n} |y_i(x)|^p\}^{1/p}$ is negative definite for $1 \le p \le 2$.

3.3 Definition Let X be a real normed linear space, $f: X \to R$ a non-negative continuous map and $1 \le p \le 2$. f is an L_p-norm if

(i) $f(ax)=|a|f(x)$ for all $a \in R$ and all $x \in X$ and

(ii) f^p is negative definite.

3.4 Theorem Let X be finite dimensional. The function $f: X \to R$ is an L_p-norm if and only if there exists a positive Radon measure μ on the unit sphere S in X' such that

$$f(x) = \left(\int_S |x'(x)|^p d\mu \right)^{1/p}.$$

Proof Refer 40.11 lemma, p. 59, [7].

3.5 Remarks (a) Every separable $L_p(\nu)$-space is isometric to a subspace of $L_p(0,1)$. (Theorem 7.1, p. 636, [8])

(b) Every separable subspace of an $L_p(\mu)$-space is isometric to a subspace of a separable $L_p(\nu)$-space for $1 \leq p < \infty$. (III. 8.5 [1])

3.6 Corollary Let $1 \leq p \leq q \leq 2$. Then for any integer n, ℓ_q^n is isometric to a subspace of $L_p(0,1)$.

Proof Define f: $\ell_q^n \to R$ by

$$f(x) = (\sum_{i=1}^{n} |x_i|^q)^{1/q} \quad \text{if } x = (x_1, \ldots, x_n) \in \ell_q^n.$$

By 3.2 (c), f^q is negative definite. Hence, by 3.2 (b), $f^p = (f^q)^{p/q}$ is negative definite. Therefore, by theorem 3.4,

$$f(x) = [\int_S |x'(x)|^p \, d\mu]^{1/p},$$

where μ is a positive Radon measure on the unit sphere S of X'.

Define T: $\ell_q^n \to L_p(S,\mu)$ by $(Tx)\xi = \xi x$ for all $\xi \in S$, $x \in \ell_q^n$. Then, $\|Tx\| = [\int_S |(Tx)\xi|^p \, d\mu]^{1/p} = f(x) = \|x\|$. And T is therefore an isometry. By remarks 3.5, ℓ_q^n is isometric to a subspace of $L_p(0,1)$.

3.7 Definition Let X and Y be Banach spaces. Define the distance between X and Y as $d(X,Y) = \inf \|T\| \cdot \|T^{-1}\|$, the infimum is taken over all invertible T in B(X,Y). If X and Y are not isomorphic, define $D(X,Y) = \infty$.

3.8 Remarks (a) Since $\|S\ T\| \leq \|S\|\ \|T\|$, we have $d(X,Y) \geq 1$.

(b) If X and Y are isometric, then $d(X,Y)=1$.

3.9 Proposition Let X be a Banach space, $1 \leq p \leq \infty$ and $\lambda \geq 1$. If for every finite dimensional subspace E of X there is a subspace E' of ℓ_p such that $d(E,E') \leq \lambda$. Then there is a measure μ and a subspace Y of $L_p(\mu)$ such that $d(X,Y) \leq \lambda$.

Proof Refer proposition 7.1, p. 304 [9].

3.10 Remark The map T in the proof of 3.9 satisfies $\lambda^{-1}\|x\| \leq \|Tx\| \leq \|x\|$. Hence, if $\lambda=1$, then T is an isometry and therefore X is isometric to a subspace of $L_p(\mu)$.

3.11 Proposition If X is a separable Banach space that satisfies the assumptions in proposition 3.9, then there is a subspace Y of $L_p(0,1)$ such that $d(X,Y) \leq \lambda$.

Proof By 3.9 proposition, there is a measure μ and a subspace Y_0 of $L_p(\mu)$ such that $d(X,Y_0)\leq\lambda$. Since X is separable, by 3.5 remarks, Y_0 is isometric to a subspace Y of $L_p(0,1)$. Hence $d(X,Y)\leq\lambda$.

3.12 Proposition Let X be a Banach space, $1\leq p<\infty$ and $\lambda\geq 1$. If for every $\varepsilon>0$, there is a measure $\mu(\varepsilon)$ and a subspace $Y=Y(\varepsilon)$ of $L_p(\mu(\varepsilon))$ such that $d(X,Y)\leq\lambda+\varepsilon$, then there is a measure μ and a subspace Y of $L_p(\mu)$ such that $d(X,Y)\leq\lambda$.

Proof Refer cor. 3, p. 306 [9].

3.13 Proposition Every $L_p(\mu)$-space, μ a positive measure, is an $\mathcal{L}_{p,\lambda}$-space for all $\lambda>1$ in the sense that for every finite dimensional subspace B of $L_p(\mu)$ there is a finite dimensional subspace E of $L_p(\mu)$ containing B such that $d(E,\ell_p^n)\leq\lambda$ where $n=\dim E$.

Proof Let T = collection of all disjoint collection $A_1^\pi,\ldots,A_{n_\pi}^\pi$ of measurable sets of finite, non-zero measure.

Let E_π = span ($\chi_{A_1^\pi},\ldots,\chi_{A_{n_\pi}^\pi}$), where $\chi_{A_i^\pi}$ is the characteristic function of A_i^π. Since every finite dimensional subspace in a B-space is closed (IV, 3.2 [10]), E_π is closed. Define T: $E_\pi \to \ell_{n_\pi}^p$ by

$$T(\Sigma\, t_i\chi_{A_i^\pi}) = (\, t_1\mu(A_1^\pi)^{1/p},\ t_2\mu(A_2^\pi)^{1/p},\ldots).$$

Then,

$$\|\Sigma\, t_i\chi_{A_i^\pi}\| = [\int_0^1 |\, \Sigma\, t_i\chi_{A_i^\pi}\,|^p\, d\mu]^{1/p}$$

$$=[\ \Sigma\ |t_i|^p\ \mu(A_i^\pi)\ ]^{1/p}$$

$$= \|\ T(\ \Sigma\ t_i\chi_{A_i^\pi}\ )\ \|.$$

It is clear that T is one-to one and onto. Therefore T is an isometry. And thus $d(E_\pi,\ell_{n_\pi}^p)=1$. Direct T by inclusion. Since the set of simple functions is dense in L_p (III, 3.8 [10]), we have

$$\overline{sp}\ (\ \bigcup_{\pi\in T} E_\pi\ ) = L_p.$$

Let E be a finite dimensional subspace of L_p with basis $e_1,\ldots e_m$. For any $\delta>0$, since the set of simple functions is dense in L_p, there exists $\pi\in T$ and $f_1,\ldots f_m \in E_\pi$ with

$\|e_i - f_i\| < \delta$ for all $1 \leq i \leq m$.

Thus, for arbitrary real numbers $t_1, \ldots t_m$, we have

$$\| \sum_{i=1}^{m} t_i (e_i - f_i) \| \leq \delta \sum_{i=1}^{m} |t_i| .$$

Moreover, by the facts that an m-dimensional B-space is equivalent to E^m and that the Euclidean norm defined by $\|t\| = [\sum_{i=1}^{m} |t_i|^2]^{\frac{1}{2}}$ for $t = (t_1, \ldots t_m)$ is equivalent to the norm defined by $\|t\| = \sum_{i=1}^{m} |t_i|$, there is a constant K, depending only on E, such that

$$\sum_{i=1}^{m} |t_i| \leq K \| \sum_{i=1}^{m} t_i e_i \| .$$

Let τ: $L_p \to E$ be a projection (refer VI, 9.18 [10] for the existence of p). Define σ: $E \to E_\pi$ by

$$\sigma(\sum_{i=1}^{m} t_i e_i) = \sum_{i=1}^{m} t_i f_i .$$

Then, $\|\sigma x - x\| \leq \delta \sum_{i=1}^{m} |t_i| \leq K\delta \|x\|$ for all $x \in E$. This implies that if $x \neq 0$ then $\sigma x \neq 0$. For otherwise we will have $\|x\| \leq K\delta \|x\|$ which is impossible because wecan choose δ small enough such that $K\delta < 1$. Thus σ is an isomorphism between E and $F = \sigma(E)$. (refer IV, 3.4 [10]) Let $W = \{ w \in E_\pi \mid \tau w = 0 \}$. We claim that $E_\pi = F + W$. For this, consider τ: F $\to$ E, restriction of τ on F. If $y=\sigma x\in F$, $x\in E$, then

$$\|\tau y-x\| = \|\tau y-\tau x\| = \| \tau(y-x) \| \leq \|\tau\| \|y-x\|$$

$$= \|\tau\| \|\sigma x-x\| < \|\tau\| \delta \sum_{i=1}^{m} |t_i| \leq \|\tau\| \delta K \|x\|.$$

As δ is arbitrary, we may suppose that $\|\tau\| \delta K < 1$. Thus if $y\neq 0$, then $\tau y\neq 0$. This implies that, as E and F are finite dimensional with same dimension, τ is an isomorphism between F and E. And thus $F\cap W=\{0\}$ and $E_\pi=F+W$. Let $E'=E+W$. Define T: E' $\to$ E by $T(e+w) = \sigma(e)=w$. T is clearly well defined, linear, one-to-one and onto. Since

$$\|(e+w)-T(e+w)\| = \|e-\sigma e\| \leq K\delta \|e\| = K\delta \|\tau e+\tau w\|$$

$$\leq K\delta \|\tau\| \|e+w\|,$$

we have

$$(1-K\delta \|\tau\|) \|e+w\| \leq \|T(e+w)\| \leq (1+K\delta \|\tau\|) \|e+w\|.$$

Therefore $\|T\| \|T^{-1}\| \leq (1+K\delta \|\tau\|)/(1-K\delta \|\tau\|)$. As δ is arbitrary, we have

$$d(E', \ell^p_{n_\pi}) \leq d(E', E_\pi) d(E_\pi, \ell^p_{n_\pi}) \leq 1+\varepsilon \text{ for all } \varepsilon>0.$$

3.14 Proposition $L_r(0,1)$ is isometric to a subspace of

$L_p(0,1)$ for all $1 \le p \le r \le 2$.

Proof Let $\varepsilon>0$ be any number. By 3.13 proposition, for every finite dimensional subspace E of $L_r(0,1)$ there is a subspace B containing E such that

$$d(B, \ell_r^m) \le 1+\varepsilon \text{ where } m=\dim B<\infty.$$

By 3.6 cor., ℓ_r^m is isometric to a subspace E_p of L_p. Apply 3.13, there is a subspace B_p containing E_p such that

$$d(B_p, \ell_p^{m'}) \le 1+\varepsilon \quad \text{where } m' = \dim B_p < \infty.$$

Therefore there is a subspace $\tilde{E}$ of ℓ_p such that

$$d(E, \tilde{E}) \le (1+\varepsilon)^2.$$

By 3.9 proposition, there is a measure $\mu=\mu(\varepsilon)$ and a subspace $Y=Y(\varepsilon)$ of $L_p(\mu)$ such that

$$d(L_r, Y) \le 1+\varepsilon.$$

By 3.12 proposition and 3.10 remark, $L_r(0,1)$ is therefore isometric to a subspace Z of $L_p(\mu)$. Since $L_r(0,1)$ is separable, by 3.5 remarks, $L_r(0,1)$ is isometric to a subspace of $L_p(0,1)$.

3.15 Corollary For $1\le p\le q\le 2$, $\dim_\ell \ell_q \le \dim_\ell L_p$.

Proof Immediately follows from 3.14 proposition.

3.16 Theorem For (1) $1 \leq P < q < 2$ or (2) $1 \leq p = q$ or (3) $p \geq 1$ and $q = 2$, $\dim_{\ell} \ell_q \leq \dim_{\ell} L_p$. For other ratios of $p \geq 1 \leq q$, the relation is impossible.

Proof Follows from 2.1, 2.4, 2.10, 2.13 and 3.15.

§4.

4.1 Proposition The linear dimensions of ℓ_p and ℓ_q, $1 < p \neq q > 1$, are incomparable.

Proof. Suppose $\dim_\ell \ell_p \leq \dim_\ell \ell_q$. Then, by formulae (6) and (7), we must have

$$\frac{1}{p} \leq \frac{1}{q} \quad \text{and} \quad \frac{p-1}{p} \leq \frac{q-1}{q} .$$

But this leads to $p=q$, a contradiction.

4.2 Proposition The relation $\dim_\ell \ell_p \leq \dim_\ell \ell_q$ is impossible for $1 = p < q$ or $1 = q < p$.

Proof.

(a) $1 = p < q$

Since ℓ_q, $q > 1$, is reflexive, any closed subspace of ℓ_q is therefore also reflexive. Since ℓ_1 is not reflexive, it can not be isomorphic to any closed subspace of ℓ_q.

(b) $1 = q < p$

In ℓ_1, every weakly convergent sequence is convergent. But in ℓ_p, $p > 1$, weak convergence does not imply convergence. Thus it is impossible that ℓ_p be isomorphic to a subspace of ℓ_1.

4.3 Proposition If $\dim_\ell L_p \leq \dim_\ell L_q$, $p > 1 < q$, then $q \leq p \leq 2$ or $2 \leq p \leq q$.

Proof. By formulae (7) and (8), we have

$\psi(p) \leq \psi(q)$ and $\psi(p/(p-1)) \leq \psi(q/(q-1))$.

There are four cases:

(i) $1 < p \leq 2$, $1 < q \leq 2$. We have $1/p \leq 1/q$, and therefore $q \leq p \leq 2$.

(ii) $1 < p < 2$, $q \geq 2$. We have $1/p \leq \frac{1}{2}$ and $\frac{1}{2} \leq (q-1)/q$, and therefore $2 \leq p$, a contradiction.

(iii) $p > 2$, $1 < q \leq 2$. We have $\frac{1}{2} \leq 1/q$ and $(p-1)/p \leq \frac{1}{2}$. Therefore $p \leq 2$, a contradiction.

(iv) $p \geq 2$, $q \geq 2$. We have $(p-1)/p \leq (q-1)/q$. And thus $2 \leq p \leq q$.

4.4 Corollary If $\dim_\ell L_p = \dim_\ell L_q$, $p > 1 < q$, then $p=q$.

4.5 Proposition For $q > 1$, the relation $\dim_\ell L_1 \leq \dim_\ell L_q$ is impossible.

Proof. Any closed subspace of L_q, $q > 1$, is reflexive. But L_1 is not reflexive; therefore L_1 can not be isomorphic to a closed subspace of L_q.

4.6 Proposition The relation $\dim_\ell L_q \le \dim_\ell L_p$ holds for $1 \le p \le q \le 2$ but is impossible for $2 < q < p$.

Proof.

(i) $1 \le p \le q \le 2$

It follows immediately from 3.14 proposition that $\dim_\ell L_q \le \dim_\ell L_p$.

(ii) $2 < q < p$.

Suppose L_q is isomorphic to a subspace of L_p. Then ℓ_q will therefore also be isomorphic to a subspace of L_p. This contradicts the results of 2.13 proposition.

4.7 Proposition The relation $\dim_\ell L_q \le \dim_\ell L_1$ holds if and only if $1 \le q \le 2$.

Proof. For $1 \le q \le 2$, this is just 4.6. For $q > 2$, suppose $\dim_\ell L_q \le \dim_\ell L_1$. Then ℓ_q, as a subspace of L_q, will be isomorphic to a subspace of L_1. This contradicts

2.13.

4.8 Theorem For $1 \leq p \neq q \geq 1$, the linear dimensions of ℓ_p and ℓ_q are incomparable.

Proof. This follows directly from 4.1 and 4.2.

4.9 Theorem. Let $1 \leq p \neq q \geq 1$. The relation $\dim_{\ell} L_q \leq \dim_{\ell} L_p$ holds only when $1 \leq p < q \leq 2$.

Proof. This follows immediately from 4.3, 4.5, 4.6, and 4.7.

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