Rice University

Numerical Solution of a Parabolic Equation
With Unusual Boundary Conditions

By

Judy S. Dupont

A Thesis Submitted
In Partial Fulfillment of the Requirements for the Degree of

Master of Arts

Thesis Director's signature:

Houston, Texas
October, 1967
NUMERICAL SOLUTION OF A PARABOLIC EQUATION
WITH UNUSUAL BOUNDARY CONDITIONS

Judy S. Dupont

The problem considered is that of a parabolic equation with space variables in a rectangle D. It is given that the normal derivative of the solution is zero on three sides of D and that the solution is a function of time alone on the fourth side. The solution and its normal derivative are otherwise unspecified on this fourth side but an integral condition on the solution is given there. The solution is given initially. Let

\begin{align*}
D &= \{(x,y)/0<x<c_1, \ 0<y<c_2\}, \\
\Gamma &= \{(x,y)/(x,y) = (0,y), \ 0<y<c_2\}, \\
abla \cdot (a(x,y)v_x) - u_t &= a(u_{xx} + u_{yy}) + a_x u_x + a_y u_y - u_t = 0 \text{ on } D, \\
a(x,y) &\geq a_o > 0 \text{ on } D, \\
\frac{\partial u}{\partial n}(x,y,t) &= 0 \text{ on } \partial D \setminus \Gamma, \\
u(0,y,t) &= f(t) \text{ on } \Gamma, \ f(t) \text{ unknown}, \\
- \int_\Gamma b(y)u_x(0,y,t)dy &= g(t), \ g(t) \text{ and } b(y) \text{ known}, \\
b(y) &\geq 0 \text{ on } \Gamma, \ b(y) \geq b_o > 0, \ y \in (y_1,y_2), \ 0<y_1<y_2<c_2, \\
u(x,y,0) &= u_o(x,y), \ u_o(x,y) \text{ known}.
\end{align*}
A finite difference analogue of the differential problem is defined and, assuming \( a(x,y) \in C^1(\Omega) \), stability of it, as well as of the differential problem for solution in \( C^2(\Omega \times [0,T]) \), is proved. A proof of convergence as well as a rate of convergence of the solution of the finite problem to that of the continuous problem, assuming the solution of the continuous problem is \( C^2(\Omega \times [0,T]) \) with respect to \( t \) and \( C^4(\Omega \times [0,T]) \) with respect to \( x \) and \( y \), is given.

Finally, two methods of solving the defined finite problem are presented and shown to be convergent.
Numerical Solution of a Parabolic Equation
with Unusual Boundary Conditions

Judy S. Dupont

The problem to be considered is that of a parabolic equation with space variables in a rectangle D. It is given that the normal derivative of the solution is zero on three sides of D and that the solution is a function of time alone on the fourth side. The solution and its normal derivative are otherwise unspecified on this fourth side but an integral condition on the solution is given there. The solution is given initially. Let

\[
D = \{(x,y)/0<x<c_1, \ 0<y<c_2\},
\]
\[
\Gamma = \{(x,y)/(x,y) = (0,y), \ 0<y<c_2\},
\]
\[
\nabla \cdot (a(x,y) \nabla u) - u_t = a(u_{xx} + u_{yy}) + a_x u_x + a_y u_y - u_t = 0 \text{ on } D,
\]
\[
a(x,y) \geq a_0 > 0 \text{ on } D,
\]
\[
\frac{\partial u}{\partial n}(x,y,t) = 0 \text{ on } \partial D \setminus \Gamma,
\]
\[
u(0,y,t) = f(t) \text{ on } \Gamma, \ f(t) \text{ unknown},
\]
\[
- \int_{\Gamma} b(y) u_x (0,y,t) dy = g(t), \ g(t) \text{ and } b(y) \text{ known}
\]
\[
b(y) \geq 0 \text{ on } \Gamma, \ b(y) \geq b_0 > 0, \ y \in (y_1,y_2), \ 0<y_1<y_2<c_2,
\]
\[
u(x,y,0) = u_0(x,y), \ u_0(x,y) \text{ known}.
\]

The results will be presented in six sections. In section I stability, i.e., uniqueness and continuous dependence
on the data, is proved for solutions in $C^2(\overline{D} \times [0,T])$, assuming $a(x,y) \in C^1(\overline{D})$. In section II a finite difference analogue of the differential problem is defined. In section III stability of the solution of the finite difference scheme will be shown. A proof of convergence as well as a rate of convergence of the solution of the finite problem to that of the continuous problem, assuming the solution of the continuous problem is $C^2(\overline{D} \times [0,T])$ with respect to $t$ and $C^4(\overline{D} \times [0,T])$ with respect to $x$ and $y$, is given in section IV.

Finally, in sections V and VI two methods of solving the defined finite problem are presented and shown to be convergent. The first applies only when a specified relationship between the given functions $a(x,y)$ and $b(y)$ is satisfied while the second applies in the general case.

I. Stability for the Differential Problem.

Let

$L(u) = \nabla \cdot (a \nabla u) - u_t$ and

$B(u) = - \int_{\Gamma} b(y)u_x(0,y,t)dy$.

**Theorem 1.** Let $u \in C^2(\overline{D} \times [0,T])$ be a solution of (1):

$L(u) = 0$, $0 \leq t \leq T$, $(x,y) \in D$,

$B(u) = g(t)$,

$u(x,y,0) = u_0(x,y)$, $u_0$ known,

$\frac{\partial u}{\partial n} = 0$ on $\partial D \setminus \Gamma$,

$u(0,y,t) = f(t)$, $f(t)$ unknown.
Then
\[ |u(x,y,t)| \leq \max_D |u_0(x,y)| + (d_1 t + d_2) \max_{0 \leq t_0 \leq t} |g(t_0)| , \]
d_1,d_2 constants independent of u.

Theorem 1 will be proved by proving four lemmas and using an appropriately chosen comparison function. The basic lemma, lemma 1, is a maximum principle for this problem.

**Lemma 1.** If \( w \in C^2(\overline{D} \times [0,T]) \) is such that
\[
Lw \geq 0 , \\
\frac{\partial w}{\partial n} = 0 \quad \text{on} \quad \partial D \setminus \Gamma ,
\]
then
\[
(2) \quad \max_{(\overline{D} \times [0,T])} w(x,y,t) = \max\{\max_{D} w(x,y,0), \max_{(\Gamma \times (0,T))} w(x,y,t)\} .
\]

**Proof:** Case 1: Assume \( Lw > 0 \) on \( \overline{D} \) and
\[
\max_{(\overline{D} \times [0,T])} w(x,y,t) = w(x_0,y_0,t_0), (x_0,y_0) \in D , 0 \leq t_0 \leq T .
\]
Then \( w_x = w_y = 0, w_t \geq 0, w_{xx} \leq 0, w_{yy} \leq 0 \) at \( (x_0,y_0,t_0) \) and thus \( Lw(x_0,y_0,t_0) \leq 0 \), contradicting that \( Lw > 0 \). Thus,
\[
\max_{(\overline{D} \times [0,T])} w(x,y,t) = \max\{\max_{D} w(x,y,0), \max_{\partial D \times (0,T)} w(x,y,t)\} .
\]
Assume \( \max_{(\overline{D} \times [0,T])} w(x,y,t) = w(x_0,y_0,t_0), 0 \leq t_0 \leq T , (x_0,y_0) \in \partial D \setminus \Gamma . \)
Then \( w_t \geq 0, \Delta w \leq 0 \) and \( w_x = w_y = 0 \) at \( (x_0,y_0,t_0) \), since \( \frac{\partial w}{\partial n} = 0 \) on \( \partial D \setminus \Gamma \) and \( (x_0,y_0,t_0) \) is assumed to be a maximum point. We thus again obtain a contradiction to the assumption \( Lw > 0 \). Therefore, equation (2) holds.
Case 2: Assume $Lw \geq 0$.

Let $v(x,y,t) = w(x,y,t) + \varepsilon X(x)$, $(x,y) \in \bar{D}$, where $X(x)$ is a solution of

(3):

$$X'' = D_1 X' + D_2 \quad \text{on } [0,c_1],$$

$$X'(c_1) = 0,$$

$$D_2 > 0,$$

$$-D_1 = \max_{\bar{D}} |a_x(x,y)|/\min_{\bar{D}} a(x,y).$$

Note that from (3) $X'(x) < 0$, $x \in [0,c_1)$, and thus

$$aX'' = aD_1 X' + aD_2 \implies aX'' > 0,$$

and

$$aX'' = (a/\min_{\bar{D}} a) \max_{\bar{D}} |a_x|(-X') + aD_2 \geq |a_x X'| + aD_2,$$

since $aX'' > 0$. Thus $aX'' - |a_x X'| \geq aD_2 > 0$. Therefore

$$L(v) = L(w) + \varepsilon L(X)$$

$$= L(w) + \varepsilon(a_x X' + aX'')$$

$$\geq L(w) + \varepsilon(aX'' - |a_x X'|)$$

$$\geq L(w) + \varepsilon aD_2$$

$$> 0$$

and $\frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = 0$ on $\partial D \setminus \Gamma$. Thus, by Case 1, equation (2) holds for $v$. Letting $\varepsilon \to 0$ we obtain equation (2) for $w$.

**Lemma 1'.** If $w \in C^2(\bar{D} \times [0,T])$ is such that

$$Lw \geq 0,$$

$$\frac{\partial w}{\partial n} = 0 \text{ on } \partial D \setminus \Gamma,$$

$$Bw \leq 0,$$

$$w(0,y,t) = f_1(t), f_1(t) \text{ unknown},$$
then \( \max_{(\overline{D} \times [0,T])} w(x,y,t) = \max \overline{D} w(x,y,0) \).

Proof: By lemma 1 either

1) \( \max_{(\overline{D} \times [0,T])} w \) occurs at \( t = 0, (x,y) \in \overline{D} \)
or

2) \( \max_{(\overline{D} \times [0,T])} w \) occurs at \( t > 0, (x,y) \in \Gamma \).

Assume 2) holds; thus, there exists \( t_o, 0 < t_o \leq T \), such that
\( w(0,y,t_o) = \max_{(\overline{D} \times [0,T])} w(x,y,t) \). In particular, we have
\( w(x,y,t_o) \leq w(0,y,t_o) \) and therefore
\[
wx(0,y,t_o) = \lim_{h \to 0} \frac{w(h,y,t_o) - w(0,y,t_o)}{h} \leq 0.
\]

Thus,
\[
(4) \quad Bw(0,y,t_o) = - \int_{\Gamma} b(y)wx(0,y,t_o)dy \geq 0.
\]

Case i: Assume \( Bw < 0 \); thus (4) immediately yields a contradiction and therefore 1) holds, as desired.

Case ii: Assume \( Bw \leq 0 \).

Let \( v(x,y,t) = w(x,y,t) + \epsilon X(x) \) where \( X \) is as in the proof of lemma 1. Then \( X'(x) < 0 \) on \( [0,c_1] \) implies

\[
Bv = Bw + \epsilon BX = - \int_{\Gamma} b(y)wx(0,y,t_o)dy - \epsilon \int_{\Gamma} b(y)X'(0)dy > 0.
\]

Apply Case i to \( v \) and let \( \epsilon \to 0 \) to see that 1) holds for \( w \).

Lemma 2. If \( w \in C^2(\overline{D} \times [0,T]) \) and
\[
Lw \leq 0,
\]
\[
\frac{\partial w}{\partial n} = 0 \quad \text{on} \quad \partial D \setminus \Gamma,
\]
\[
Bw \geq 0,
\]
and \( w(0,y,t) = f_2(t), f_2(t) \) unknown,
then \( \min_{(D \times [0,T])} w(x,y,t) = \min_{(x,y) \in \bar{D}} w(x,y,0). \)

Proof: The function \(-w\) satisfies the hypotheses of lemma 1'.

**Lemma 3.** If \( u \in C^2(\bar{D} \times [0,T]) \) is a solution of (1) and \( w \in C^2(\bar{D} \times [0,T]) \) is such that

\[
\begin{align*}
Lw & \leq 0 , \\
\frac{\partial w}{\partial n} & = 0 \text{ on } \partial D \setminus \Gamma , \\
Bw & \geq |g| , \\
w(0,y,t) = f_3(t) , f_3(t) \text{ unknown} , \\
w(x,y,0) & \geq \max_{\bar{D}} |u_0(x,y)| , \\
\end{align*}
\]

then \( |u(x,y,t)| \leq w(x,y,t) \).

Proof: The functions \( w \pm u \in C^2(\bar{D} \times [0,T]) \) satisfy

\[
\begin{align*}
L(w \pm u) & \leq 0 , \\
\frac{\partial (w \pm u)}{\partial n} & = 0 \text{ on } \partial D \setminus \Gamma , \\
B(w \pm u) & = Bw \pm g \geq 0 , \\
(w \pm u)(0,y,t) & = f_3(t) \pm f(t) , \\
(w \pm u)(x,y,0) & \geq 0 , (x,y) \in \bar{D} .
\end{align*}
\]

Thus by lemma 2 applied to \( w \pm u \),

\[
0 \leq \min_{\bar{D}} (w \pm u)(x,y,0) = \min_{D \times [0,T]} (w \pm u)(x,y,t) ,
\]

and thus \( |u(x,y,t)| \leq w(x,y,t) \).

Thus, to obtain the conclusion of Theorem 1 it remains to choose an appropriate function satisfying the conditions of lemma 3. In doing so, for simplification, it is assumed \( D \) is a unit square. It is also assumed that \( a(x,y) \in C^1(\bar{D}) \).
We have, since \( a(x,y) \in C^1(\bar{D}) \),

\[
L((x-1)^2) = 2a(x,y) + 2(x-1)a_x(x,y) \leq 2\max_D a(x,y) + 2\max_D |a_x| = d.
\]

Let

\[
w_1 = \frac{1}{g_0}(x-1)^2, \quad g_0 = 2 \int \Gamma b(y)dy > 0.
\]

Then

\[
\begin{align*}
L(w_1) &\leq \frac{d}{g_0} = d_1, \\
B(w_1) &= \frac{1}{g_0} B((x-1)^2) = \frac{1}{g_0} \left[ - \int \Gamma b(y)(-2)dy \right] = 1, \\
\frac{\partial w_1}{\partial n} &= \frac{1}{g_0} \frac{\partial}{\partial n}((x-1)^2) = 0 \text{ on } \partial D \setminus \Gamma.
\end{align*}
\]

Let \( \tilde{w} = w_1 + d_1 t \).

Then

\[
\begin{align*}
L(\tilde{w}) &= L(w_1) + d_1 L(t) \leq d_1 - d_1 = 0, \\
B(\tilde{w}) &= Bw_1 = 1, \\
\frac{\partial \tilde{w}}{\partial n} &= 0 \text{ on } \partial D \setminus \Gamma.
\end{align*}
\]

Let \( g_T = \max_{0 \leq t \leq T} |g(t)| \) and \( w_o = \max_D |u_o(x,y)| \), \( u_o \) as in (1).

Let \( w = g_T \tilde{w} + w_o \). Then

\[
\begin{align*}
L(w) &= g_T L(\tilde{w}) \leq 0, \\
B(w) &= g_T B(\tilde{w}) = g_T \geq |g|, \\
\frac{\partial w}{\partial n} &= g_T \frac{\partial \tilde{w}}{\partial n} = 0 \text{ on } \partial D \setminus \Gamma, \\
w(x,y,0) &= g_T \tilde{w}(x,y,0) + w_o = g_T \frac{(x-1)^2}{g_0} + w_o \geq w_o = \max_D |u_o(x,y)|.
\end{align*}
\]

Thus, \( w \) satisfies the conditions of lemma 3 and
\begin{align*}
|u(x,y,t)| & \leq w(x,y,t) \leq \max_{\mathcal{D}} |u_o(x,y)| + (d_1 t + d_2) \max_{0 \leq t_o \leq t} |g(t_o)| , \\
d_2 & = \max_{\mathcal{D}} |w_1| = \frac{1}{g_0}.
\end{align*}

This is the desired conclusion of Theorem 1 and obviously implies uniqueness and continuous dependence on the data for solutions in \(C^2(\overline{D} \times [0,T])\).

II. Definition of the Finite Difference Problem.

Place a square grid on \(\mathcal{D}\) of step size \(h\) where \(c_1 = (N+1)h\), \(c_2 = (M+1)h\) and let \(k\) denote the step size in the time variable.

Let

\begin{align*}
D_h & = \{(x,y) = (ih,jh), \ 1 \leq i \leq N, \ 1 \leq j \leq M\} , \\
\Gamma_h & = \{(x,y) = (0,jh), \ 1 \leq j \leq M\} .
\end{align*}

A function \(u_h(x,y,t)\), defined for \((x,y,t) = (ih,jh,nk), 0 \leq i \leq N, 1 \leq j \leq M, 0 \leq n\), is sought which satisfies the following finite difference problem, to be referred to as (5). (5):

1) If \((x,y) \in D_h\)

\begin{align*}
L_h u_h(x,y,(n+1)k) & = h^{-2} \left[ -a(x+h/2,y)-a(x-h/2,y) \\
& \quad -a(x,y+h/2)-a(x,y-h/2) \right] u_h(x,y,(n+1)k) \\
& \quad + a(x+h/2,y) u_h(x+h,y,(n+1)k) + a(x-h/2,y) u_h(x-h,y,(n+1)k) \\
& \quad + a(x,y+h/2) u_h(x,y+h,(n+1)k) + a(x,y-h/2) u_h(x,y-h,(n+1)k) \\
& \quad - k^{-1} [u_h(x,y,(n+1)k)-u_h(x,y,nk)] = 0 ,
\end{align*}

i.e., \(\nabla_h; a(x,y) \nabla_h u_h(x,y,(n+1)k))\)

\begin{align*}
& = k^{-1} [u_h(x,y,(n+1)k)-u_h(x,y,nk)] .
\end{align*}
2) The condition $\frac{\partial u}{\partial n} = 0$ on $\partial D \setminus \Gamma$ in the differential problem is to be interpreted in the finite problem as

a) $u_h(Nh, jh, (n+1)k) = u_h((N+1)h, jh, (n+1)k), \ 1 \leq j \leq M,$

b) $u_h(0, jh, (n+1)k) = u_h(0, (n+1)k), \ 1 \leq i \leq N,$

c) $u_h(ih, Mh, (n+1)k) = u_h(ih, (M+1)h, (n+1)k), \ 1 \leq i \leq N.$

This simply causes certain terms to cancel in $L_h u_h$ if $x = Nh$, $y = h$ or $y = Mh$.

3) If $(x, y) \in \Gamma_h$, then

a) $u_h(0, jh, (n+1)k) = u_h(0, ih, (n+1)k), \ 1 \leq i, j \leq M$,

b) $B_h u_h(0, jh, (n+1)k) = \sum_{i=1}^{M} b(ih)[u_h(h, ih, (n+1)k) - u_h(0, ih, (n+1)k)] = g((n+1)k).$

Note that $b(ih) \geq b_0 > 0, \text{ some i, } 1 \leq i \leq M, \text{ if h is sufficiently small.}$

4) $u_h(x, y, 0) = u_0(x, y), \ u_0 \text{ known for } (x, y) \in \overline{D}_h.$

III. Stability for the Finite Difference Problem.

Theorem 2. Let $u_h$ be a solution of (5), $0 \leq t \leq T$.
Then

$$|u_h(x, y, t)| \leq \max_{\overline{D}_h} |u_0(x, y)| + (d_1 t + d_2) \max_{0 \leq t_0 \leq t} |g(t_0)|,$$

d_1, d_2 \text{ constants independent of } u_h.$$

The proof of Theorem 2 is similar to that of Theorem 1 and follows from three lemmas and the use of an appropriately chosen comparison function. The proofs of lemmas 1 through 3 are almost identical to those of similar lemmas in section I and will not be presented here.
Lemma 1. If $w_h$ is such that

\[
L_h w_h \geq 0 ,
\]
\[
B_h w_h \leq 0 ,
\]
\[
w_h(0,jh,nk) = w_h(0,ih,nk), \ 1 \leq i,j \leq M ,
\]
and if

\[
\max_{0 \leq n \leq T/k} w_h(x,y,nk) = w_h(x_o,y_o,n_o,k), \ n_o > 0, \ (x_o,y_o) \in \overline{D}_h , \ (x,y) \in \overline{D}_h
\]

then

\[
w_h(x,y,nk) = w_h(x,y,0) = \text{constant}, \ 0 \leq n \leq n_o, \ (x,y) \in \overline{D}_h .
\]

Lemma 2. If $w_h$ is such that

\[
L_h w_h \leq 0 ,
\]
\[
B_h w_h \geq 0 ,
\]
\[
w_h(0,jh,nk) = w_h(0,ih,nk), \ 1 \leq i,j \leq M ,
\]
and if

\[
\min_{0 \leq n \leq T/k} w_h(x,y,nk) = w_h(x_o,y_o,n_o,k), \ n_o > 0, \ (x_o,y_o) \in \overline{D}_h
\]

then $w_h(x,y,nk) = w_h(x,y,0) = \text{constant}, \ 0 \leq n \leq n_o, \ (x,y) \in \overline{D}_h .
\]

Lemma 3. If $u_h$ is a solution of (5) and if $w_h$ is such that

\[
L_h w_h \leq 0 ,
\]
\[
B_h w_h \geq |g| ,
\]
\[
w_h(0,ih,t) = w_h(0,jh,t), \ 1 \leq i,j \leq M ,
\]
\[
w_h(x,y,0) \geq \max_{D} |u_o(x,y)| ,
\]

then $|u_h(x,y,t)| \leq w_h(x,y,t)$.
Assuming D is a unit square and \( a(x,y) \in C^1(D) \), an appropriate function \( w_h \) satisfying the hypotheses of lemma 3 is

\[
w_h = g_T \left[ \frac{1}{g_1} (x-1)^2 + d_1 t \right] + w_o
\]

where

\[
g_T = \max_{0 \leq t \leq T} |g(t)|, \quad g(t) \text{ as in (5)},
\]

\[
g_1 = B_h((x-1)^2),
\]

\[
L_h \left( \frac{1}{g_1} (x-1)^2 \right) \leq d_1,
\]

and \( w_o = \max_{D_h} |u_o(x,y)|, \quad u_o \text{ as in (5)}. \) Hence,

\[
|u_h(x,y,t)| \leq w_h(x,y,t) \leq \max_{D_h} |u_o(x,y)| + \max_{0 \leq t \leq T} |g(t_o)| (d_1 t + d_2),
\]

\[
d_2 = \max_{D_h} \left| \frac{1}{g_1} (x-1)^2 \right| = \frac{1}{g_1}.
\]

This is the conclusion of Theorem 2.

The following modification of lemma 3 will be used in section IV.

Let \( \nabla_h, n u_h(x,y,t) \) denote the finite approximation to the inner normal derivative of \( u_h \) on \((\partial D \setminus \Gamma)_h\). Thus,

\[
\nabla_h, n u_h(x,y,t) = h^{-1} [u_h(c_1-h,y,t) - u_h(c_1,y,t)] \text{ if } x = c_1,
\]

\[
= h^{-1} [u_h(x,h,t) - u_h(x,0,t)] \text{ if } y = 0,
\]

\[
= h^{-1} [u_h(x,c_2-h,t) - u_h(x,c_2 t)] \text{ if } y = c_2.
\]

We then have

**Lemma 3'.** If \( w_h \) satisfies
Let $L_h w_h \leq 0$ on $D_h$,

$B_h w_h \geq 0$ on $\Gamma_h$,

$\nabla_h w_h \leq 0$ on $(\partial D \setminus \Gamma)_h$,

$w_h(0, y, t) = f_4(t)$, $f_4(t)$ unknown,

$w_h(x, y, 0) \leq 0$,

then $0 \leq w_h(x, y, t)$.

Proof: The proof is a trivial modification of that of lemma 2, adding a Case 3: $(x_0, y_0) \in (\partial D \setminus \Gamma)_h$ and using that $\nabla_h n_h \leq 0$ on $(\partial D \setminus \Gamma)_h$.

IV. Convergence of the Solution of the Finite Problem to that of the Continuous Problem.

In the following, $D$ is assumed to be a unit square.

Theorem 3. If $u \in C^4(D \times [0, T])$ with respect to $x$ and $y$, $u \in C^2(D \times [0, T])$ with respect to $t$, $a(x, y) \in C^2(D)$ with respect to $x$ and $y$, $b(y) \in C^1(\Gamma)$, and if $u$ is a solution of (1), $u_h$ a solution of (5), then

$$v_h(x, y, t) = (u - u_h)(x, y, t) = O(h^k)$$
on $(D_h \times [0, T])$.

Lemma 1. (1) $L_h v_h(x, y, t) = F_1(x, y, t)$, where

$$|F_1(x, y, t)| \leq \varepsilon_1 = O(h^2 + k)$$in $(D_h \times [0, T])$.

(2) $B_h v_h(x, y, t) = F_2(t)$, where $|F_2(t)| \leq \varepsilon_2 = O(h)$
in $(\Gamma_h \times [0, T])$.

(3) $\nabla_h n_h v_h(x, y, t) = F_3(x, y, t)$, where

$$|F_3(x, y, t)| \leq \varepsilon_3$$on $(\partial D \setminus \Gamma)_h \times [0, T])$.

Proof: (1) is well-known.
(2): \[ B_h v_h(0,y,t) = \int_\Gamma b(y)u_x(0,y,t) \, dy \]
\[ = \left[ \int_\Gamma b(y)u_x(0,y,t) \, dy \right] - \sum_{i=1}^M b(ih)u_x(0,ih,t)h \]
\[ + \left[ \sum_{i=1}^M b(ih)[u_x(0,ih,t) - u(0,ih,t)] - u(0,ih,t) \right] \]
\[ = \left[ \int_\Gamma p(y,t) \, dy \right] - \sum_{i=1}^M p(ih,t)h \right] + O(h) , \]
where \( p(y,t) = b(y)u_x(0,y,t) \in C^1(\overline{\Gamma} \times [0,T]) \) with respect to \( y \).

Thus
\[ p(y,t) = p(ih,t) + (y-ih)p_y(\xi,t) \]
\[ = p(ih,t) + O(h) , \quad \xi, y \in (ih-h/2,ih+h/2) . \]

Therefore,
\[ B_h v_h(0,y,t) = \left[ \int_{-h/2}^{h/2} p(y,t) \, dy \right] + \int_{-h/2}^{1-h/2} p(y,t) \, dy \]
\[ + \sum_{i=1}^M \int_{-h/2}^{ih-h/2} p(y,t) \, dy - \sum_{i=1}^M p(ih,t)h \right] + O(h) \]
\[ = O(h) + \sum_{i=1}^M \int_{ih-h/2}^{ih+h/2} 0(h) \, dy = O(h) . \]

(3):

Case 1: \( x = 1 \).

\[ \nabla_h n v_h(1,y,t) = \nabla_h n u(1,y,t) - \nabla_h n u_h(1,y,t) = \nabla_h n u(1,y,t) \]
\[ = h^{-1}[u(1-h,y,t) - u(1,y,t)] = -u_x(1,y,t) + u_{xx}(\xi,y,t)\frac{h}{2}, \xi \in (1-h,h) \]
\[ \frac{\partial u}{\partial n} = 0 \text{ on } \partial D \setminus \Gamma \]

\[ = 0(h) \]

\[ = \epsilon_3 x \text{, since } x = 1. \]

Case 2: \( y = 0. \)

\[ \nabla_h n \nabla_h (x, 0, t) = \nabla_h n u(x, 0, t) = h^{-1}[u(x, h, t) - u(x, 0, t)] \]

\[ = u_y(x, 0, t) + u_{yy}(x, \xi, t)\frac{h}{2}, \quad \xi \in (0, h) \]

and \( u_{yy}(x, \xi, t) = u_{yy}(0, \varepsilon, t) + u_{yyx}(\varepsilon', \varepsilon, t)x, \quad \varepsilon' \in (0, x) \)

\[ = xu_{yyx}(\varepsilon', \varepsilon, t) \text{ since } u(0, y, t) = f(t). \] Thus

\[ \nabla_h n \nabla_h (x, 0, t) = u_y(x, 0, t) + xu_{yyx}(\varepsilon', \varepsilon, t)\frac{h}{2} \]

\[ = xu_{yyx}(\varepsilon', \varepsilon, t)\frac{h}{2} \]

\[ = xo(h) \]

\[ = \epsilon_3 x. \]

Case 3: \( y = 1. \)

The proof is the same as for Case 2.

Thus \( \nabla_h \) satisfies

\[ L_h \nabla_h = F_1(x, y, t), \quad |F_1| \leq \varepsilon_1 = 0(h^2 + k) \text{ in } (D_h \times [0, T]), \]

\[ B_h \nabla_h = F_2(t), \quad |F_2| \leq \varepsilon_2 = 0(h) \text{ in } (\Gamma_h \times [0, T]), \]

\[ \nabla_h n \nabla_h = F_3(x, y, t), \quad |F_3| \leq \varepsilon_3 x, \quad \varepsilon_3 = 0(h) \text{ on } ((\partial D \setminus \Gamma)_h \times [0, T]), \]

\[ \nabla_h (0, y, t) = F_4(t) \text{ on } \Gamma_h, \]

\[ \nabla_h (x, y, 0) = 0 \text{ on } \bar{D}_h. \]
Let \( v_h = v_1 + v_2 \), where

\[
\begin{align*}
L_h v_1 &= F_1(x, y, t) \quad \text{on } (D_h \times [0, T]) , \\
B_h v_1 &= F_2(t) \quad \text{on } (\Gamma_h \times [0, T]) , \\
\nabla_h n v_1 &= 0 \quad \text{on } ((\partial D \setminus \Gamma)_h \times [0, T]) , \\
v_1(0, y, t) &= F_5(t) \quad \text{on } \Gamma_h , \\
v_1(x, y, 0) &= 0 \quad \text{on } \overline{D_h} .
\end{align*}
\]

\[
\begin{align*}
L_h v_2 &= 0 \quad \text{on } (D_h \times [0, T]) , \\
B_h v_2 &= 0 \quad \text{on } (\Gamma_h \times [0, T]) , \\
\nabla_h n v_2 &= F_3(x, y, t) \quad \text{on } ((\partial D \setminus \Gamma)_h \times [0, T]) , \\
v_2(0, y, t) &= F_6(t) \quad \text{on } \Gamma_h , \\
v_2(x, y, 0) &= 0 \quad \text{on } \overline{D_h} .
\end{align*}
\]

First, let us show that \( v_1(x, y, t) = 0(h+k) \) in \((\overline{D_h} \times [0, T])\).

Let

\[
r(x, y, t) = \begin{cases} 
\frac{1}{g_1} (x-1)^2 , & 0 \leq x < 1 , \quad g_1 = B_h((x-1)^2) \\
\frac{h^2}{g_1} , & x = 1 .
\end{cases}
\]

Then, as in section 3, \( L_h r(x, y, t) \leq d \) for \( x < 1-h \). For \( x = 1-h \) we have

\[
L_h r(1-h, y, t) = h^{-2} \left\{ [ -a(x+h/2, y) - a(x-h/2, y) ] r(1-h, y, t) \\
+ a(x+h/2, y) r(1, y, t) + a(x-h/2, y) r(1-2h, y, t) \right\}
\]

\[
= h^{-2} \left\{ [ -a(x+h/2, y) - a(x-h/2, y) ] \frac{h^2}{g_1} + a(x+h/2, y) \frac{h^2}{g_1} + a(x-h/2) \frac{4h^2}{g_1} \right\}
\]

\[
= \frac{3}{g_1} a(x-h/2, y) ,
\]

which is bounded. Thus, there exists a constant \( d_1 \) such that

\( L_h (r(x, y, t)) \leq d_1 \), \( (x, y, t) \in (D_h \times [0, T]) \).
Let \( \bar{w}_1(x,y,t) = r(x,y,t) + (d_1+1)t \). Thus
\[
L_h \bar{w}_1 = L_h r - (d_1+1) \leq -1 \quad \text{on } (D_h \times [0,T]),
\]
\[
B_h \bar{w}_1 = 1,
\]
\[
\nabla_h \bar{w}_1 = 0, \quad \text{on } (\partial D \setminus \Gamma)_h.
\]

Let \( w_{h,1}(x,y,t) = c\bar{w}_1(x,y,t) \pm v_1(x,y,t) \), where \( c = \max(\epsilon_1, \epsilon_2) \).

Then
\[
L_h w_{h,1} = cL_h(\bar{w}_1) \pm L_h(v_1) \leq -\max(\epsilon_1, \epsilon_2) \pm F_1 \leq 0,
\]
\[
B_h w_{h,1} = cB_h \bar{w}_1 \pm B_h v_1 = \max(\epsilon_1, \epsilon_2) \pm F_2 \geq 0,
\]
\[
\nabla_h n w_{h,1} = 0,
\]
\[
w_{h,1}(0,y,t) = F_7(t),
\]
\[
w_{h,1}(x,y,0) = c\bar{w}_1(x,y,0) = cr(x,y,0) = \frac{c}{g_1} (x-1)^2 \geq 0.
\]

Thus by lemma 2 of section III we have
\[
0 \leq w_{h,1}(x,y,t) \quad \text{on } (\overline{D}_h \times [0,T]),
\]
and thus
\[
|v_1(x,y,t)| \leq c\bar{w}_1(x,y,t) \leq \max(\epsilon_1, \epsilon_2) \{ \max_{\overline{D}} r(x,y,t_0) + (d_1+1)t \}
\]
\[
= \max(\epsilon_1, \epsilon_2) \{ \frac{1}{g_1} + (d_1+1)t \} \quad \text{on } (\overline{D}_h \times [0,T]).
\]

Therefore, \( v_1(x,y,t) = 0(h+k) \) in \( (\overline{D}_h \times [0,T]) \), since \( \epsilon_1 = 0(h^2+k) \) and \( \epsilon_2 = 0(h) \).

Next, let us show that \( v_2(x,y,t) = 0(h) \) in \( (\overline{D}_h \times [0,T]) \).

Let \( q(x,y,t) = x(y-\frac{1}{2})^2 + (x-\frac{1}{2})^2 + dt \), where \( d \) is chosen such that \( L_h q \leq 0 \) on \( (D_h \times [0,T]) \). Then
\[ B_h q = \sum_{i=1}^{M} b(ih)[h(y-h)^2 + (h-h)^2 - 1/4] \]
\[ = -h \sum_{i=1}^{M} b(ih)[y-h)^2 + h - 1] \]
\[ \geq h(3/4-h) \sum_{i=1}^{M} b(ih) > h/2 \sum_{i=1}^{M} b(ih) \geq 0 , \]
if \( h < 1/4 \).

Also, \( v_{h,n}q \leq -\frac{1}{2}x \) on \( \partial D \setminus \Gamma \), since
\[ v_{h,n}q(x,0,t) = h^{-1}[x(h-h)^2 - x(-h/2)^2] = x(h-1) < -\frac{1}{2}x , \]
if \( h < \frac{1}{2} \).

Similarly for \( y = 1 \).
\[ v_{h,n}q(1,y,t) = h^{-1}[(1-h)(y-h)^2 + (-h+h)^2 - ((y-h)^2 + 1/4)] \]
\[ = -(y-h)^2 + h - 1 \leq -\frac{1}{2} = (-\frac{1}{2})x , \]
if \( h < \frac{1}{2} \).

Let \( w_{h,2}(x,y,t) = cq(x,y,t) \pm v_2(x,y,t) \), where \( c = 2\varepsilon_3 \).

Then
\[ L_h w_{h,2} = cL_h q \leq 0 , \]
\[ B_h w_{h,2} = cB_h q \geq 0 , \]
\[ v_{h,n} w_{h,2} = c v_{h,n} q \pm F_3 \]
\[ \leq 2\varepsilon_3(-\frac{1}{2}x) + \varepsilon_3 x \]
\[ = 0 , \]
\[ w_{h,2}(0,y,t) = F_8(t) , \]
\[ w_{h,2}(x,y,0) = c[x(y-h)^2 + (x-h)^2] \geq 0 . \]

Thus \( w_{h,2} \) satisfies the hypotheses of lemma 3' of section III and therefore \( 0 \leq w_{h,2}(x,y,t) \) on \( \overline{D_h} \times [0,T] \). Thus,
\[ |v_2(x,y,t)| \leq 2\varepsilon_3 q(x,y,t) \]
and therefore
\[ v_2(x,y,t) = 0(h) , \]
since \( \varepsilon_3 = 0(h) \) on \( (D_h \times [0,T]) \).

Since \( v_h = v_1 + v_2 \) we have the conclusion of Theorem 3 established.

V. Method I for Solving the Finite Problem.

The first method to be described for solving the finite problem defined in section II is a modification of the well-known Alternating Direction Implicit Method.

Order the points in \( D_h \) from left to right, bottom to top. Thus, \((h,h) = z_1, (2h,h) = z_2, \ldots, (Nh,Mh) = z_{NM}\). Since it is assumed that \( u_h(0,ih,t) = u_h(0,jh,t), 1 \leq i,j \leq M \), we treat the entire line of grid points \( \Gamma_h \) as the \((NM+1)^{st}\) point, \( z_{NM+1} \), in this ordering. Thus, \( D_h \cup \Gamma_h = \{ z_i / i = 1, \ldots, NM+1 \} \).

We set up a matrix \( A \) of coefficients whose \( i^{th} \) row corresponds to coefficients in \( L_h u_h(z_i, (n+1)k) = L_h w_{n+1,i} = 0 \) written in the form

\[
[I - k(\nabla_h \cdot (a \nabla_h))] w_{n+1,i} = w_n,i, \quad 1 \leq i \leq NM.
\]

The \((NM+1)^{st}\) row of \( A \) corresponds to coefficients in the equation \( kh^{-2} B_h u_h(z_{NM+1}, (n+1)k) = kh^{-2} g((n+1)k) \) written in the form

\[
-kh^{-2} \sum_{i=1}^{M} b(ih)(w_{n+1,i-1}N+1 - w_{n+1,i},NM+1) = kh^{-2} g((n+1)k) = kh^{-2} g_{n+1}.
\]

Thus \( A \) is an \((NM+1) \times (NM+1)\) matrix and has the form shown in Figure 1.

Solving the finite problem (5) for \( u_h(x,y, (n+1)k) \), \((x,y) \in D_h \cup \Gamma_h\), is thus equivalent to solving the matrix equation \( A w_{n+1} = d_{n+1} \) where \( w_{n+1} \) and \( d_{n+1} \) are \((NM+1)\) component
vectors such that the \( i \)th component of \( w_{n+1} \), \( 1 \leq i \leq N M + 1 \), is \( w_{n+1, i} \); the \( i \)th component of \( d_{n+1} \) is \( w_{n, i} \), \( 1 \leq i \leq N M \), and the \((N M + 1)\)th component of \( d_{n+1} \) is \( \lambda g_{n+1} \), where \( \lambda = k h^{-2} \).

We write \( A = \tilde{H} + \tilde{V} \) where \( \tilde{H} \) and \( \tilde{V} \) have the forms shown in Figure 1. \( \tilde{H} \) is the matrix consisting of the identity plus the time step \( k \) multiplied by a three point approximation to \( \frac{\partial}{\partial x}(a \frac{\partial u}{\partial x}) \) at points on \( D_h \), together with contribution of the boundary operator \( B_h \) on \( \Gamma \). \( \tilde{V} \) corresponds to the time step multiplied by a three point approximation to \( \frac{\partial}{\partial y}(a \frac{\partial u}{\partial y}) \) at points in \( D_h \).

Define a method of solving \( A w_{n+1} = d_{n+1} \) as follows:

\[
\begin{align*}
(\rho I + \tilde{H}) w_{n+1}^{(i+\frac{1}{2})} & = d_{n+1} - (\tilde{V} - \rho I) w_{n+1}^{(i)} \\
(\rho I + \tilde{V}) w_{n+1}^{(i+1)} & = d_{n+1} - (\tilde{H} - \rho I) w_{n+1}^{(i+\frac{1}{2})} \quad i = 0, 1, \ldots ,
\end{align*}
\]

(10)

where \( \rho > 0 \) is a constant parameter and \( w_{n+1}^{(0)} = w_n \).

**Theorem 4.** If

\[
2\lambda^{-1} + a(h/2, ih) \geq c[b(ih)], \quad 1 \leq i \leq M, \quad \lambda = k n^{-2},
\]

(11)

where \( c \) is defined by

\[
\sum_{i=1}^{M} b(ih) = \sum_{i=1}^{M} a(h/2, ih),
\]

then \( w_{n+1}^{(i)} \) converges to \( w_{n+1} \) as \( i \to \infty \), \( w_{n+1}^{(i)} \) as in (10).

If \( B_h w_{n+1} \) is the heat flow across \( \Gamma \), that is, \( b(y) = a(0, y) \), then conditions (11) are satisfied.

The proof of Theorem 4 follows after a sequence of three lemmas. Note that in the following the last row of \( \tilde{H} \) has been multiplied by the constant \( c \lambda \) for convenience.
If $A$ is an $n \times n$ complex matrix, let $A^\star$ denote the conjugate transpose of $A$.

Definition: If $A = (a_{ij})$ is an $n \times n$ complex matrix then the directed graph of $A$ is a set of $n$ nodes, $P_i$, together with directed paths connecting these nodes such that there is a path from node $P_i$ to node $P_j$ if and only if $a_{ij} \neq 0$.

Definition: The directed graph of $A = (a_{ij})_{i,j=1}^n$ is strongly connected if and only if there exists a sequence of paths from any node $P_i$ to any other node $P_j$ in the directed graph of $A$.

If the $i^{th}$ row of a matrix $A$ consists of coefficients in an equation associated with the $i^{th}$ point in a grid, it is convenient to identify the $i^{th}$ node in the directed graph of $A$ with the $i^{th}$ point in the grid.

Lemma 1. If (11) holds then

1. $\tilde{H} + \tilde{H}^\star$ is positive definite;
2. $\tilde{V}$ is positive semi-definite.

Proof:

1. $\tilde{H} + \tilde{H}^\star$ is clearly Hermetian and therefore has only real eigenvalues. Thus, to prove $\tilde{H} + \tilde{H}^\star$ is positive definite it suffices to show all eigenvalues are strictly positive. This will be done by use of the following well-known theorem of Gerschgorin [2,p.20].

Theorem 5: If $A = (a_{ij})$ is an $n \times n$ irreducible complex matrix then $\lambda$ an eigenvalue of $A$ implies either

$$|\lambda - a_{ii}| = \Lambda_i \quad \text{for all } i = 1, \ldots, n$$

or $|\lambda - a_{ii}| < \Lambda_i$ some $i$, $1 \leq i \leq n$, 


where \( \Lambda_j = \sum_{j=1}^{w} |a_{ij}| \).

To see that \( \tilde{H} + \tilde{H}^* \) is irreducible is equivalent to showing that the directed graph of \( \tilde{H} + \tilde{H}^* \) is strongly connected [2,p.20]. It is clear that if \( P_i \) and \( P_j \) are nodes in the directed graph corresponding to grid points \( z_i \) and \( z_j \), respectively, lying on the same horizontal line in \( \bar{D}_h \) then there exists a sequence of paths from \( P_i \) to \( P_j \). Since \( z_{NM+1} \), the grid point corresponding to \( \Gamma_h \), lies on every horizontal line, there exists a sequence of paths from any node \( P_i \) to \( P_{NM+1} \). Thus there exists a sequence of paths from any node in the directed graph of \( \tilde{H} + \tilde{H}^* \) to any other node. Thus, \( \tilde{H} + \tilde{H}^* \) is irreducible.

Let \( \tilde{H} + \tilde{H}^* = (a_{ij})_{i,j=1}^{NM+1} \). Then let us show that \( |a_{jj}| \geq \Lambda_j \) for all \( j \), \( 1 \leq j \leq NM+1 \), with inequality for some \( j \). Note that \( |a_{jj}| = \alpha_{jj} \).

If \( j \) is such that the \( j \)th row in \( \tilde{H} + \tilde{H}^* \) corresponds to the grid point associated with \( \Gamma_h \), i.e., \( j = NM+1 \), then

\[
\begin{align*}
\alpha_{jj} &= 2c_\lambda \sum_{i=1}^{M} b(ih) \\
\Lambda_j &= c_\lambda \sum_{i=1}^{M} b(ih) + \lambda \sum_{i=1}^{M} a(h/2,ih). \text{ By (11)} \\
c_\lambda \sum_{i=1}^{M} b(ih) &= \lambda \sum_{i=1}^{M} a(h/2,ih) \text{ and thus} \\
\alpha_{jj} &= \Lambda_j.
\end{align*}
\]

If the \( j \)th row corresponds to a point in \( D_h \) immediately to the right of \( \Gamma_h \), i.e., \( j=tN+1 \), \( 0 \leq t \leq M-1 \), then
\[ \alpha_{jj} = 2 + 2\lambda [a(h/2, ih) + a(h+h/2, ih)], \quad i = t+1, \text{ and} \]
\[ \Lambda_j = \lambda [2a(h+h/2, ih) + c[b(ih)] + a(h/2, ih)]. \]
By (11)
\[ 2\lambda^{-1} + a(h/2, ih) \geq c[b(ih)] \text{ and thus} \]
\[ 2 + 2\lambda a(h/2, ih) \geq \lambda b(ih) + \lambda a(h/2, ih) \text{ so that} \]
\[ \alpha_{jj} \geq \Lambda_j. \]
If the \( j \)th row corresponds to a point on the right hand side of \( D_h \), i.e., \( j = pN, 1 \leq p \leq M \), then
\[ \alpha_{jj} = 2 + 2\lambda a(x-h/2, y) \text{ and} \]
\[ \Lambda_j = 2\lambda a(x-h/2, y), \text{ where } z_j = (x, y), \text{ so that} \]
\[ \alpha_{jj} > \Lambda_j. \]
Finally, if the \( j \)th row corresponds to a point in \( D_h \) which is not one of the above then
\[ \alpha_{jj} = 2 + 2\lambda [a(x-h/2, y) + a(x+h/2, y)] \text{ and} \]
\[ \Lambda_j = 2\lambda [a(x-h/2, y) + a(x+h/2, y) \text{ and thus} \]
\[ \alpha_{jj} > \Lambda_j. \]

Thus, if \( \mu \) is an eigenvalue of \( \tilde{H} + \tilde{H}^\ast \) then since \( \alpha_{jj} \geq \Lambda_j \)
for all \( j, 1 \leq j \leq NM+1 \), by Theorem 5 we have \( \mu \geq 0 \). Furthermore, \( \mu = 0 \) is not an eigenvalue since there exists a \( j, 1 \leq j \leq NM+1 \)
such that \( \alpha_{jj} > \Lambda_j \).
Therefore, \( \tilde{H} + \tilde{H}^\ast \) is positive definite.

(2): \( \tilde{V} \) is clearly Hermetian and thus has only real eigenvalues.
It thus suffices to show all eigenvalues of \( \tilde{V} \) are non-negative.
Letting \( \tilde{V} = (v_{ij})_{i,j=1}^{NM+1} \) and \( V_j = \sum_{i=1}^{NM+1} |v_{ij}| \), simple calculation
shows \( v_{jj} = V_j \) for all \( j, 1 \leq j \leq NM+1 \), and therefore, by
Theorem 5, \( \tilde{V} \) is positive semi-definite.
The following two lemmas are trivial modifications of lemmas appearing in [1].

**Definition:** A matrix $Q$ is norm decreasing [non-increasing] with respect to some norm, $\|x\|$, if and only if $\|Qx\| < \|x\|$ for all $x \neq 0$.

**Lemma 2.** Let $P$ be an $n \times n$ complex matrix such that $((P+P^*)x,x) > 0$ for $x \neq 0$ and $((P+P^*)x,x) \geq 0$ for $x \neq 0$. Then $Q = (P-\rho I)(P+\rho I)^{-1}$ is norm decreasing with respect to $\|x\| = (x,x)^{1/2}$ [norm non-increasing with respect to $\|x\| = (x,x)^{1/2}$] if $\rho > 0$.

**Proof:** By definition, $Q$ is norm decreasing if and only if $\|(P-\rho I)y\| < \|(P+\rho I)y\|$ for $y = (P+\rho I)^{-1}x$ if and only if $\|(P-\rho I)y\|^2 < \|(P+\rho I)y\|^2$.

Using the specific norm $\|x\| = (x,x)^{1/2}$ we have $Q$ is norm decreasing with respect to $\|x\| = (x,x)^{1/2}$ if and only if

$$\|(P-\rho I)y\|^2 = ((P-\rho I)y,(P-\rho I)y) < ((P+\rho I)y,(P+\rho I)y) = \|(P+\rho I)y\|^2$$

for all $y \neq 0$ if and only if

$$(Py,Py) - \rho(Py,y) - \rho(y,Py) + \rho^2(y,y) < (Py,Py) + \rho(Py,y) + \rho(y,Py) + \rho^2(y,y)$$

if and only if

$$(Py,y) + (y,Py) = ((P+P^*)y,y) > 0.$$ 

Since this is part of our hypotheses the lemma is proved.

**Lemma 3.** The Alternating Direction Implicit process

$$(H+\rho I)u^{i+1/2} = d - (V-\rho I)u^i,$$

$$(V+\rho I)u^{i+1} = d - (H-\rho I)u^{i+1/2}$$

is convergent provided $H$ is such that $((H+H^*)x,x) > 0$ for $x \neq 0$ and $V$ is such that $((V+V^*)x,x) \geq 0$ for $x \neq 0$. 
Proof: It suffices to show that the spectral radius
\[ \rho(T) = \max \{|\lambda_i| \mid \lambda_i \text{ an eigenvalue of } T\} < 1, \]
where \( T \) is the iterator
\[ T = (V+\rho I)^{-1}(H-\rho I)(H+\rho I)^{-1}(V-\rho I). \]
Equivalently, it suffices to show \( \rho(\tilde{T}) < 1 \)
where \( \tilde{T} = (V+\rho I)(T)(V+\rho I)^{-1} \]
\[ = [(H-\rho I)(H+\rho I)^{-1}][(V-\rho I)(V+\rho I)^{-1}] \]
By lemma 2, \([(H-\rho I)(H+\rho I)^{-1}] \) is norm decreasing and \([(V-\rho I)(V+\rho I)^{-1}] \) is norm non-increasing with respect to \( \|x\| = (x,x)^{\frac{1}{2}} \). Thus their product, \( \tilde{T} \), is norm decreasing with respect to \( \|x\| = (x,x)^{\frac{1}{2}} \).
Thus \( \rho(\tilde{T}) < 1 \).

The value of this method of solution of the problem
\[ A^w_{n+1} = d_{n+1} \] depends on the ease of inverting the matrices \( (\rho I+H) \) and \( (\rho I+V) \). It is well-known that a matrix of form \( (\rho I+V) \) is easily invertible.

That \( (\rho I+H) \) is also easily invertible follows from noting that this matrix consists of \( M \times N \) square blocks \( H_i \), \( 1 \leq i \leq M \), together with an \( NM+1^{st} \) row and an \( NM+1^{st} \) column (Figure 1). Since each block \( H_i \) is positive definite and tridiagonal it is well known that there exist matrices \( L_i \) and \( U_i \) such that
\[ H_i = L_i U_i, \]
where \( L_i \) is lower triangular with non-zero entries only on the main diagonal and first lower off-diagonal; \( U_i \) is upper triangular with non-zero entries only on the main and first upper off-diagonals with ones on the main diagonal. Thus, \( H_i \) is easily invertible since \( L_i \) and \( U_i \) are.
Let

\[
L = \begin{pmatrix}
L_1 & 0 & \cdots \\
0 & L_2 & \cdots \\
0 & 0 & \ddots \\
0 & 0 & 0 & L_M
\end{pmatrix}
\quad \text{and } U = \begin{pmatrix}
U_1 & 0 & \cdots \\
0 & U_2 & \cdots \\
0 & 0 & \ddots \\
0 & 0 & 0 & U_M
\end{pmatrix}
\]

where \( b_i = b(ih) \) and \( v_i \) is an \( N \) component vector whose \( j^{th} \) component is zero, \( j \neq 1 \), and whose first component is \( \lambda a(n/2, ih) \). Then \( L \) and \( U \) are easily invertible and \((\rho I + H) = LU\) implies that \( \rho I + H \) is easily invertible.

VI. Method II for Solving the Finite Problem.

Convergence of this second method for solving the finite problem (5) does not depend on a relationship between the functions \( a(x, y) \) and \( b(y) \). It involves solution by standard procedure, for example the Alternating Direction Implicit Method, of three boundary value problems at each time level.

We solve (5) for \( u_h(x, y, (n+1)k) = w_{n+1}^{(q)}(x, y), (x, y) \in D_h, \) assuming \( w_n(x, y), (x, y) \in D_h, \) is known, by solving the following modification of (5) for \( w_{n+1}^{(q)} = w^{(q)}(x, y, (n+1)k), q = 1, 2, 3, (x, y) \in D_h. \)

\((5'):\)

1) \( \nabla_h \cdot (a(x, y) v_h w_{n+1}^{(q)}) = k^{-1}(w_{n+1}^{(q)} - \nabla_w), (x, y) \in D_h, \)
where the condition \( \frac{\partial u}{\partial n} = 0 \) on \( \partial D \setminus \Gamma \) for the differential problem is interpreted here as in (5).

2) \( w^{(q)}_{n+1} = f^{(q)}_{n+1}, \ f^{(q)}_{n+1} = f^{(q)}((n+1)k) \) is known, \( (x,y) \in \Gamma_n \).

Thus, \( f^{(q)}_{n+1} \) is a \( q \)th approximation to the unknown value of \( w^{(q)}_{n+1} \) on \( \Gamma_h \). Using this approximate boundary value, we solve (5') for \( w^{(q)}_{n+1} \), \( (x,y) \in D_h \), by any standard procedure.

Knowing \( f^{(q-1)}_{n+1} \) and \( f^{(q)}_{n+1} \), we calculate \( f^{(q+1)}_{n+1} \) as follows:

(12):

i) Solve (5') for \( w^{(q-1)}_{n+1} \) and \( w^{(q)}_{n+1} \);

ii) Compute

\[
\gamma(f^{(j)}_{n+1}, w^q) = - \sum_{i=1}^{M} b(ih)[w^{(j)}(h, ih, (n+1)k) - f^{(j)}_{n+1}], \ j=q-1, q;
\]

iii) Set \( f^{(q+1)}_{n+1} = f^{(q)}_{n+1} - C^{-1}[\gamma(f^{(q)}_{n+1}, w^q) - g_{n+1}] \),

where \( C = [\gamma(f^{(q)}_{n+1}, w^q) - \gamma(f^{(q-1)}_{n+1}, w^q)]/[f^{(q)}_{n+1} - f^{(q-1)}_{n+1}] \).

Theorem 6 below yields the termination of this iteration. Given any two different approximations to \( f^{(q)}_{n+1}, f^{(1)}_{n+1} \) and \( f^{(2)}_{n+1} \), having solved (12) for \( f^{(3)}_{n+1} \) and then (5') for \( w^{(3)}_{n+1} \) the result of Theorem 6 is that \( B_h w^{(3)}_{n+1} = g_{n+1} \) and thus \( w^{(3)}_{n+1} \) is the desired solution of (5) at the \( (n+1) \)st time level. Thus, we need only solve three boundary value problems and perform some trivial computations to solve (5) at each time level.

Let \( w^{n+1}(\alpha) \) denote the solution of (5') with \( w^{n+1} = \alpha \) on \( \Gamma_h \).
Theorem 6. The function
\[ \gamma(a, w_n) = \sum_{i=1}^{M} b(ih)[w_{n+1}(a)(h, ih(n+1)k) - a] \]
is an affine function of its first argument.

Proof: We have \( \gamma(a, w_n) = B_h w_{n+1}(a) \). Write \( w_{n+1}(a) = v_0 + v_1 \), where

1) \( v_h(a(x, y)v_h v_0) = k^{-1}(v_0 - w_n) \)
2) \( v_0 = 0 \) on \( \Gamma_h \)
1'') \( v_h(a(x, y)v_h v_1) = k^{-1}v_1 \)
2') \( v_1 = a \) on \( \Gamma_h \).

Clearly \( v_1 \) is a linear function of \( a \), the first argument of \( \gamma(a, w_n) \), and thus \( B_h v_1 \) is a linear function of \( a \): \( B_h v_1 = c a \), \( c \) a constant. Thus,

\[ \gamma(a, w_n) = B_h w_{n+1}(a) = B_h(v_0 + v_1) \]
\[ = B_h(v_0) + c a \]
\[ = \gamma(0, w_n) + c a . \]

Thus, \( \gamma(a, w_n) \) is an affine function of its first argument.
Bibliography
