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THE DIRICHLET PROBLEM FOR
EQUATIONS OF FINITE DIFFERENCES

by

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Introduction

Along with the recent development of high-speed computing machines has come a renewed interest in the numerical solution of partial differential equations. These machines have made practical various methods of obtaining approximate solutions of the equations, and questions of both practical and theoretical interest which lay behind these methods have arisen.

The problem, of course, is to find the solution of the partial differential equation with certain boundary or initial conditions which will be valid over a certain region. The differential equation, the boundary conditions, or the region itself may be of such a nature as to make an analytical solution impossible. One then turns to approximate methods in an effort to obtain a solution.

One of the most common methods of approach to this problem is to place a lattice over the region, and then to substitute finite difference quotients for the corresponding partial derivatives, making suitable approximations for the boundary conditions. We have then a set of simultaneous algebraic equations, which are then solved numerically. This solution of the set of equations gives us the solution of the partial difference system, and it is defined on the grid points of the lattice.

(1)
This method was advanced by Richardson [1] in 1910. In his paper, Richardson considered such equations as
\[ \nabla^2 u = 0, \quad (\nabla^2 + k^2)v = 0, \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}. \]
He set up the corresponding difference equation, obtained the numerical, and found them to be in very good agreement for particular cases. He further pointed out that this method could be readily extended to other equations, although if he had carried his calculations further, he might have hesitated to make such a recommendation.

Two questions very naturally arise:

(1) Does the solution of the partial difference system converge to the solution of the partial differential system as the distance between the lattice points approaches zero?

(2) Since in practice the distance between the lattice points is always finite, what is the error involved in obtaining the solution of the partial difference system instead of the partial differential system?

Very little general theory has been developed to answer either (1) or (2), and much work still remains to be done in this field. It is the purpose of this thesis to discuss these questions for partial differential equations of the elliptic type, and in particular for Laplace's...
equation.

Liebmann [2] was the first to discuss (1), although Courant, Friedrichs, and Lewy [3] gave the first systematic discussion in 1928. In this paper, a difference equation with auxiliary conditions was defined as convergent provided the solution of the difference equation approached the solution of the differential equation. This proof of the convergence of the Laplace difference equation was followed by papers by Petrowsky [4], Feller [5], and Diaz and Roberts [6], all of which were concerned, at least in part, with equations of elliptic type.

Wasow [7] has published the best consideration to date of question (2) for equations of elliptic type. He defines the truncation error to be the difference between the exact solution of the differential system and the exact solution of the difference system. He considers only the case of Laplace's equation, and obtains a uniform bound on the error. It is to be pointed out here, though, that in general it is impossible to get such an estimate directly, for it involves knowing the analytical solutions, which, of course, is the difficulty in the first place. It is extremely important to have an estimate of this error, since it is believed that it is the principal source of error in the numerical calculations. The other source of error lies in obtaining the solutions of the simultaneous equations, and this in practice has proved to be relative-
ly small. Hence what is needed is some indirect method of
getting an estimate of the truncation error, and preferably
a method involving the numerical solution alone.

We consider first the problem of convergence.
Chapter I.

Convergence Proofs for the Dirichlet Problem

1.1 Partial difference equations of the elliptic type

1.1 Definitions and notation

Let us consider a region \( G \) of the plane and a simple closed curve \( \Gamma \) composed of a finite number of regular arcs. We place a square lattice of mesh width \( h \) over the entire plane, so that each lattice point may be represented by the coordinates \((nh, mh)\), where \( m, n = 0, 1, 2, 3, \ldots \).

By a lattice point of \( G \) we shall mean a point of \( G \) which may be represented in the above form. By a neighboring point of the lattice point \((x_0, y_0)\) we shall mean a point representable as \((x_0 + h, y_0)\) or \((x_0, y_0 + h)\). The lattice region \( G_h \) shall consist of those points which lie in \( G \) and which may be joined by a connected chain of lattice points of \( G \). This is well defined if \( h \) is sufficiently small. By an interior point of \( G_h \) we shall mean a point of \( G_h \) whose four neighboring points belong to \( G_h \). The set of interior points will be denoted by \( G'_h \). All other points of \( G_h \) shall be called boundary points of \( G_h \), and the boundary of \( G_h \) is denoted by \( \Gamma_h \).

We consider the functions \( u, v, w, \ldots \) which will be defined only for the lattice points. These are generally denoted by \( u(x, y) \), \( v(x, y) \), \( w(x, y) \), \( \ldots \). From these functions we form the forward difference quotients
\( \frac{v_x}{v_x} = \frac{v(x+y) - v(x,y)}{h} \) \( (1.1.11) \)
\( \frac{v_y}{v_y} = \frac{v(x, y+h) - v(x, y)}{h} \) \( (1.1.12) \)
and the backward difference quotients
\( \frac{v_x}{v_x} = \frac{v(x,y) - v(x-h,y)}{h} \) \( (1.1.13) \)
\( \frac{v_y}{v_y} = \frac{v(x,y) - v(x,y-h)}{h} \). \( (1.1.14) \)
Difference quotients of higher order are defined in terms of these, as, for example,
\( \left( \frac{v_x}{v_x} \right)^2 = \left( \frac{v_x}{v_x} \right)^2 = \frac{v(x+h,y) - 2v(x,y) + v(x-h,y)}{h^2} \). \( (1.1.15) \)

1.1.2 Green's Formulas

In a manner quite analogous to the study of partial differential equations, let us consider the bilinear expression
\( B(u,v) = au_x v_x + bu_x v_y + cu_y v_x + du_y v_y + gu_x \)
\( + u_y v + vuv_x + guv_y + guv \). \( (1.1.21) \)
formed from the functions \( u(x,y), v(x,y), \) and their forward difference quotients. \( a = a(x,y), \ b = b(x,y), \ldots, \)
\( g = g(x,y) \) are functions defined on the lattice points.

We form the sum
\( h^2 \sum_{G_h} B(u,v) \) \( (1.1.22) \)
over a lattice region \( G_h \), where it is to be understood that the difference quotient between a boundary point and a point not belonging to \( G_h \) is to be set equal to zero.
If we expand this double sum, and regroup the terms with respect to \( v \), we may split it into a sum over the interior points \( G_h \) and the boundary points \( \Gamma_h \). We obtain the fol-
lowing result:

\[
h^2 \sum_{G_h} B(u,v) = -h^2 \sum_{G_h} vL(u) - h \sum_{\Gamma_h} vR(u),
\]

(1.1.23)

where \(L(u)\) is the so-called "linear difference expression" of the second order. Its explicit form is

\[
L(u) = (au_x)_x + (bu_x)_y + (cu_y)_x + (du_y)_y - \alpha u_x
\quad - \rho u_y + (\gamma u)_x + (\delta u)_y - gu.
\]

(1.1.24)

It is defined, of course, for all interior points. \(R(u)\) is a linear difference expression defined for each boundary point, and its particular form is of no immediate importance.

If we regroup \(h^2 \sum_{G_h} B(u,v)\) with respect to \(u\), we obtain

\[
h^2 \sum_{G_h} B(u,v) = -h^2 \sum_{G_h} uM(v) - h \sum_{\Gamma_h} uS(v).
\]

(1.1.25)

\(M(v)\) is called the linear difference expression adjoint to \(L(u)\), and it is given by

\[
M(v) = (sv_x)_x + (sv_y)_y + (cv_x)_x + (dv_y)_y + (\gamma v)_x
\quad + (\delta v)_y - sv_x - \delta v_y - gv.
\]

(1.1.26)

\(S(v)\) is a linear difference expression corresponding to \(R(u)\).

Subtracting (1.1.23) from (1.1.25), we obtain the relationship

\[
h^2 \sum_{G_h} [vL(u) - uM(v)] + h \sum_{\Gamma_h} [vR(u) - uS(v)] = 0.
\]

(1.1.27)

(1.1.23), (1.1.25), and (1.1.27) are called Green's for-
mulas, and obviously are analogous to those obtained in studying partial differential equations.

If the bilinear form \( B(u,v) \) is symmetric, that is, if \( b = c, \alpha = \gamma, \) and \( \beta = \zeta, \) then \( L(u) = M(u). \) In this case, \( L(u) \) is said to be self-adjoint, and may be derived from the quadratic expression

\[
B(u,u) = au_x^2 + 2bu_xu_y + du_y^2 + 2\alpha u_xu + 2\beta u_yu + gu^2.
\]

(1.1.28)

We shall confine our attention to expressions \( L(u) \) which are self-adjoint.

The properties of \( L(u) \) depend primarily on those terms of \( B(u,u) \) which are quadratic in the first difference quotients. We call

\[
P(u,u) = au_x^2 + 2bu_xu_y + du_y^2
\]

(1.1.29)

the characteristic form. If \( P(u,u) \) is positive definite, we say that \( L(u) \) is a difference expression of the elliptic type.

In this section, we shall work with the difference expression

\[
L(u) = u_{xx} + u_{yy} = \Delta u.
\]

(1.1.210)

This may be derived from the expression

\[
B(u,u) = u_x^2 + u_y^2,
\]

(1.1.211)

and is clearly of the elliptic type. The Green's formula corresponding to (1.1.23) is

\[
h^2 \sum_{G_i} (u_x^2 + u_y^2) = -h^2 \sum_{G_i} u \Delta v - h \sum_{G_i} uR(v).
\]

(1.1.212)
and that corresponding to (1.1.27) is
\[ h^2 \sum_{G_h} (v \Delta u - u \Delta v) = -h^2 \sum_{G_h} (vR(u) - uS(v)) = 0. \] (1.1.213)

\( \Delta u \) may be explicitly written in the following manner:
\[ \Delta u = \left[ u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h) \right] / h^2 \] (1.1.214)

If \( \Delta u = 0 \), then \( u(x,y) \) is seen to be the arithmetic mean of the values of the function at the four neighboring points.

1.1.3 The Boundary Value Problem

We formulate the boundary value problem for linear elliptic difference equations of second order in a form analogous to the boundary value problem for partial differential equations.

Suppose we have \( L(u) \), a self-adjoint elliptic difference expression of the second order, and a lattice region \( G_h \). Then the boundary value problem is to find a function \( u \) such that \( L(u) = 0 \) and such that \( u \) takes on pre-assigned values on the boundary points of the lattice region.

That this solution exists and is unique may be seen in the following manner:

Corresponding to each interior point of \( G_h \) we have a homogeneous equation in the functional values at the given point and its four neighboring points. Corresponding to each boundary point we have a non-homogeneous equation
in the functional values at the points in $G$, which neighbor the boundary points. The non-homogeneity arises, of course, from the preassigned boundary values. Hence we have as many equations as we have unknowns.

If we set the boundary values equal to zero, it is seen from (1.1.23), on setting $u = v$, that $B(u, v)$ vanishes. But since $B(u, v)$ is positive definite, then $u_x$ and $u_y$, and consequently $u$, also vanish. The difference equation has therefore the solution $u = 0$ if the boundary values vanish. Hence the solution is uniquely determined by the boundary values, since the difference of two solutions with the same boundary conditions must vanish.

Since we have a linear system of equations with as many unknowns as equations, and since the only solution is $u = 0$ if the boundary values vanish, it follows from Cramer's theorem that if arbitrary boundary values are given, then there must exist exactly one solution. Hence we have the existence of a unique solution of our boundary value problem.

We point out in particular that since $L(u) = \Delta u$ arises from $B(u, u) = u_x^2 + u_y^2$, which is positive definite, the boundary value problem for this expression is always uniquely solvable.

In analogy with Dirichlet's principle, the above boundary value problem may be formulated in the following manner:
We are given a lattice region $G_h$ with boundary $\Gamma_h$, and functions $\psi(x,y)$ defined on $G_h$ and which assume on $\Gamma_h$ the preassigned values $f(x,y)$. We want to find a particular function $\psi(x,y) = u(x,y)$ for which the sum

$$h^2 \sum_{G_h} B(\psi, \psi)$$

assumes the least possible value.

We shall show that such a function exists, and, further, satisfies the equation $L(u) = 0$.

We assume $B(u,u)$ is positive definite. It follows that $h^2 \sum_{G_h} B(\psi, \psi)$ is bounded below. Considered as a function of the values at the lattice points, $h^2 \sum_{G_h} B(\psi, \psi)$ is continuous. From the form of $B(u,u)$, it is seen that it tends to infinity as $u$ tends to infinity. Therefore it follows that (1.1.31) does have a minimum at a finite point, and assumes this minimum. From (1.1.23) it follows that this minimum is assumed for a function such that $L(u) = 0$, for no matter what function is assigned, the sum may be always made less if $L(u) = 0$.

1.2 First Convergence Proof for Dirichlet's Problem

(Proof of Courant, Friedrichs, and Lewy)

The purpose of this section shall be to prove the following theorem:

Suppose that

1. $\Omega$ is a simply-connected region whose boundary
   $\Gamma$ is composed of a finite number of regular arcs;

2. $f(x,y)$ is a function which
a) in the interior of $G$, is continuous and has continuous first and second partial derivatives,

b) is defined and bounded on $G \cup \Gamma$, and is piecewise continuous on $\Gamma$,

c) if $P_0 : (x_0, y_0) \in \Gamma$ and $P : (x, y)$ is an interior point,

\[
\lim_{P \to P_0} f(P) = f(P_0),
\]

except at most a finite number of points,

d) the integral

\[
\iint_G (f_x^2 + f_y^2) \, dx \, dy
\]

exists; and, finally that

(3) $u_h$ is the solution for the boundary value problem for the difference equation $\Delta u = 0$ for the lattice region $G_h$ corresponding to the region and with the boundary values assumed by $f(x, y)$ on the boundary points of $G_h$. Then, as $h \to 0$, and independently of $h$, $u_h$ converges to the solution of the boundary value problem for the differential equation

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,
\]

where $u(x, y)$ assumes the same values as $f(x, y)$ on the boundary of $G$, in the following sense:

if $S_r$ is the boundary strip of the region $G$ all of whose points are at a distance less than $r$ from the boundary, then

\[
\frac{1}{r} \iint_{S_r} (u - f)^2 \, dx \, dy \to 0 \quad \text{as} \quad r \to 0.
\]
The proof of the theorem is based upon the equicontinuity of the sequence of the functions \( u_h \) and their difference quotients.

We make the following definition:

A sequence of functions \( \{\phi_n(P)\} \), \( n = 1, 2, 3, \ldots \) defined upon a lattice region \( G_h \) will be said to be equicontinuous in \( G_h \) if given \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that for all \( n \geq n_0 \), \( |\phi_n(P) - \phi_n(Q)| < \epsilon \) if \( |P - Q| < \delta \) for \( P \) and \( Q \) belonging to \( G_h \).

**Lemma 1:** The sums
\[
h^2 \sum_{G_h} u_{h, x}^2 \quad \text{and} \quad h^2 \sum_{G_h} (u_{h, x}^2 - u_{h, y}^2)
\]
are bounded as \( h \to 0 \).

**Proof:** For the difference expression \( L(u) = \Delta u, \quad B(u, u) = u_x^2 + u_y^2 \). As a consequence of the minimum problem formulation, it follows that
\[
h^2 \sum_{G_h} (u_{h, x}^2 - u_{h, y}^2) \leq h^2 \sum_{G_h} (f_x^2 + f_y^2).
\]
Then, as \( h \to 0 \), the sum on the right tends to
\[
\iint_{G} (f_x^2 - f_y^2) dxdy,
\]
which, by hypothesis, exists.

To show that \( h^2 \sum_{G_h} u_{h, x}^2 \) is bounded, one may proceed in two ways:

1) The greatest and least values of \( u_h \) are assumed on the boundary. This is seen easily, for, supposing \( u_h \) is not a constant, suppose \( u_h \) assumes its maximum (minimum) at an interior point. Since the value of \( u_h \) at an interior point is the arithmetic mean of the values at
the four neighboring points, we are led to the conclusion
that either \( u_h \) is a constant or there is an infinite set
of points in \( G_h \). Either of these, of course, is a con-
tradiction. Hence \( u_h \) is less than some constant \( M^2 \).
Then
\[
\frac{h^2 \sum_{i=1}^{n} u_i^2}{c} \leq h^2 \sum_{i=1}^{n} M \to \int c \, dx dy,
\]
as \( h \to 0 \).

Since this proof uses the fact that the solution
of \( \Delta u = 0 \) assumes its maximum and minimum on the boundary,
it cannot be readily applied to more general equations
of the elliptic type. We may get around this in the
following manner:

2) Let \( \bar{P}_h \notin \Gamma_h \), and let \( d \) be the diameter of the
set \( G_h \). Further, let \( P_h \notin G_h \). Then one may show that the
following inequality holds:
\[
\frac{h^2 \sum_{i=1}^{n} u_i^2}{c} \leq 2dh \frac{u^2}{d} + 2d^2h^2 \sum_{i=1}^{n} (u^2 + u_y^2).
\]
We shall not prove this inequality here since the proof
is exactly the analogue of an inequality we obtain later.
Since by the first part of the lemma, the sum \( h^2 \sum_{i=1}^{n} (u^2 + u_y^2) \)
is bounded as \( h \to 0 \), it is sufficient to show that
\( h \sum_{i=1}^{n} u_i^2(\bar{P}_h) \) is bounded as \( h \to 0 \).

Since for \( \bar{P}_h \notin \Gamma_h \), \( u = f \), we know that
\[
\int_{\Gamma_h} r^2 ds
\]
exists, where \( ds \) is the differential of arc length along
\( \Gamma_h \). Let \( L_h \) be the length of \( \Gamma_h \). Then
\[
s_1 = L_h \min_{\bar{P}_h} u^2(\bar{P}_h) \leq \sum_{i=1}^{n} u_i^2(\bar{P}_h) h \leq L_h \max_{\bar{P}_h} u^2(\bar{P}_h) \leq S,
\]
But $s_1$ and $S_1$ are respectively lower and upper sums for $\int f^2 ds$. Therefore

$$s_1 \leq \int f^2 ds \leq S_1.$$  

Hence

$$|\sum_{f_h} u^2(f_h) h - \int f^2 ds| \leq S_1 - s_1.$$  

As $h \to 0$,

$$\int f^2 ds \to \int f^2 ds,$$

which exists, since on $f$ is piece-wise continuous, and $\Gamma$ is rectifiable. The conclusion of the lemma then follows.

**Lemma 2:** If $w_h(x,y)$ represents a difference quotient of $u_h(x,y)$, then $w = w_h$ satisfies the difference equation $\Delta w = 0$ in $G_h$. Moreover, if $G_h^*$ represents a lattice region corresponding to $G^*$, $G^* \subset G$, as $h \to 0$, $h^2 \sum_{G_h^*} w^2$ is bounded, and further, for $G_h^{**} \subset G_h^*$ corresponding to $G_h^{**} \subset G^*$,

$$h^2 \sum_{G_h^{**}} (w^2_x + w^2_y)$$

is bounded.

**Proof:** Since $\Delta w = 0$ is a linear equation with constant coefficients, it is clear that $w = w_h$ satisfies the equation, and by Lemma 1, is bounded.

Consider the quadratic sum

$$h^2 \sum_{Q_h} (w^2_x + w^2_y + w^2_y),$$

where the summation is extended over all the interior points of a square $Q_1$. Let $S_1$ be the boundary row, and let $S_0$ be the row of neighboring interior points. Then
the following relationship holds:

\[ h^2 \sum_{Q_i} (w_x^2 + w_y^2 + w_x^2 + w_y^2) \leq \sum_{S_i} w_x^2 - \sum_{S_o} w_x^2 \]

This relationship will be worked out in detail to show how this particular relationship and others similar to this may be derived.

Consider the following square covered by a lattice of mesh width \( h \):

\[
\begin{array}{cccccc}
21 & 22 & 23 & 24 & 25 \\
16 & 17 & 18 & 19 & \ \\
11 & 12 & 13 & 14 & 15 \\
6 & 7 & 8 & 9 & 10 \\
5 & 4 & 3 & 2 & 1 \\
\end{array}
\]

A square with this number of sub-divisions will be seen to be sufficiently general, for the third, fourth, etc., interior rows contribute nothing to the sum. Then

\[
h^2 \sum_{Q_i} (w_x^2 + w_y^2 + w_x^2 + w_y^2) = 2(w_8 - w_7)^2 + 2(w_9 - w_8)^2 + (w_7 - w_6)^2 + (w_10 - w_9)^2 + 2(w_{13} - w_{12})^2 + 2(w_{14} - w_{13})^2 + (w_{12} - w_{11})^2 + (w_{15} - w_{14})^2 + 2(w_{18} - w_{17})^2 + 2(w_{19} - w_{18})^2 + (w_{17} - w_{16})^2 + (w_{20} - w_{19})^2 + 2(w_{12} - w_7)^2 + 2(w_{17} - w_{12})^2 + (w_7 - w_2)^2 + (w_{22} - w_{17})^2 + 2(w_{13} - w_8)^2 + 2(w_{18} - w_{12})^2 + (w_8 - w_3)^2 + (w_{23} - w_{18})^2 + 2(w_{14} - w_9)^2 + 2(w_{19} - w_{14})^2 + (w_9 - w_4)^2 + (w_{24} - w_{19})^2
\]

\[ = 2h^2 \sum_{Q_i} w \Delta w + \sum_{S_i} w^2 - \sum_{S_o} w^2 - A, \]

where \( A \) is a positive constant. Since \( \Delta w = 0 \), we then have the desired result.
We consider now a sequence of concentric squares $Q_0, Q_1, Q_2, \ldots, Q_N$ with the boundaries $S_0, S_1, \ldots, S_N$. Now, evidently,

$$2h^2 \sum_{Q_k} (w^2_x + w^2_y) \leq h^2 \sum_{Q_k} (w^2_x + w^2_y + w^2_x + w^2_y). \quad (k > 1)$$

If we add the $n$ inequalities

$$2h^2 \sum_{Q_k} (w^2_x + w^2_y) \leq \sum_{S_k} w^2 - \sum_{S_k} w^2,$$

we obtain

$$2nh^2 \sum_{Q_k} (w^2_x + w^2_y) \leq \sum_{S_k} w^2 - \sum_{S_k} w^2 \leq \sum_{S_k} w^2.$$

We now sum these inequalities from $n = 1$ to $n = N$. Noting that $\sum_{k=1}^N 2n = N(N + 1) > N^2$, we get that

$$N^2 h^2 \sum_{Q_k} (w^2_x + w^2_y) \leq \sum_{Q_k} w^2.$$

If, as $h \to 0$, we let the two squares $Q_0$ and $Q_N$ tend to two concentric squares a distance $a$ apart, then $Nh \to a$, and independently of the mesh width $h$,

$$h^2 \sum_{Q_k} (w^2_x + w^2_y) \leq h^2 \sum_{Q_k} w^2 / a^2.$$

Then, since this inequality follows, for different $a$, for any two sub-regions of $G$, one of which is included entirely in the interior of the other, the statement of the lemma follows.

**Lemma 3:** From the boundedness of the sums of Lemma 1 and Lemma 2, it follows that all the difference quotients are bounded and equi-continuous.

**Proof:** Let us consider a rectangle $R$ with corner points $P_0Q_0PQ$, with $P_0Q_0$ and $PQ$ parallel to the $x$-axis, and of
length a. This rectangle is covered by a lattice of mesh width h.

By expanding $h^2 \sum \mathbf{x} w_{xy}$, it follows that

$$w(Q_0) - w(P_0) = h \sum_{R_q} w_{x} - h^2 \sum_{R_k} w_{xy}.$$ 

Hence,

$$|w(Q_0) - w(P_0)| \leq h \sum_{R_q} |w_x| + h^2 \sum_{R_k} |w_{xy}|.$$ 

We now let the line PQ run between an initial position $P_1 Q_1$ a distance h from $P_0 Q_0$ and a fixed position $P_2 Q_2$ a distance 2b from $P_0 Q_0$. If we sum the b/h inequalities of the above form which result, we see that

$$|w(Q_0) - w(P_0)| \leq \frac{1}{b + h} \sum_{R_k} |w_{xy}| + h^2 \sum_{R_k} |w_{xy}|,$$

in which we extend the summation over the entire rectangle $R_2 = P_0 Q_0 P_2 Q_2$.

We then apply Schwarz's inequality, and noting that $1/(b + h) < 1/b$, we obtain

$$|w(Q_0) - w(P_0)| \leq \frac{1}{b} \sqrt{2ab} \sqrt{h^2 \sum_{R_k} w_x^2} + \sqrt{2ab} \sqrt{h^2 \sum_{R_k} w_{xy}^2}.$$ 

Since the sums are bounded by Lemma 1 and Lemma 2, it follows that as $a \to 0$, and independently of h,

$$|w(Q_0) - w(P_0)| \to 0,$$

since we may fix b for any $G < G$. 

Hence, we have shown the equi-continuity in the x-direction, and by a similar argument, one may show the equi-continuity in the y-direction for any $G < G$.

Using the lemmas, it follows from the theorem of
Ascoli [9] that a subsequence of functions \( u_n \) exists which, together with its related difference quotients, tends uniformly to a function \( u(x,y) \) in each closed region interior to \( G \). This limiting function is continuous and possesses continuous partial derivatives of any order, and, moreover, satisfies the partial differential equation

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0
\]

It remains, then, to show that this solution satisfies the boundary conditions formulated above. We let \( S_{r,h} \) be the part of the lattice region \( G_h \) which is in the interior of the boundary strip \( S_r \). We shall now show that for each lattice function \( v(x,y) \) the inequality

\[
h^2 \sum_{i,j} v^2 \leq Ar^2 h^2 \sum_{i,j} (v_x^2 + v_y^2) + Brh \sum v^2
\]

(1.22) is satisfied. \( A \) and \( B \) denote constants dependent only on the region and not on the function \( v \) or on the mesh width \( h \).

To prove the inequality, we divide the boundary \( \Gamma \) of \( G \), into a finite number of pieces such that the tangent to \( \Gamma \) at each point makes an angle with either the \( x \)-axis or the \( y \)-axis greater than some positive angle (for example, 30°). Let \( \gamma \) be one such piece of \( \Gamma \). We draw lines parallel to the \( x \)-axis at the end points of \( \gamma \); they cut out of \( \Gamma \) a piece \( \gamma_r \). Then \( \gamma, \gamma_r, \) and
the two parallels bound a piece $s_r$ of the strip $S_r$.

We denote the part of the lattice region $G_h$ contained in $s_r$ by $s_{r,h}$.

Consider then a point $P_h \in s_{r,h}$ and the line drawn through it parallel to the x-axis. It will intersect $\Gamma$, the boundary of $G_h$, at a point $P_h^*$. We denote the piece of the parallel which lies in $s_{r,h}$ by $p_{r,h}$. The length of $p_{r,h}$ is less than or since $r$ is the greatest perpendicular distance of a point of $S_r$ from $\Gamma$. Thus $\alpha$ depends only on the angle of inclination of the tangent to $\Gamma$ with the x-axis.

Since

$$v(P_h) = v(P_h^*) + h \sum_{f \in fr} v_x,$$

we see by squaring and the application of Schwarz's inequality that

$$v^2(P_h) \leq 2v^2(P_h^*) + 2crh \sum_{f \in fr} v_x^2.$$

We now sum with respect to $P_h$ in the x-direction.

We see that

$$\sum_{P_h} v^2(P_h) \leq \frac{2cr}{h} v^2(P_h^*) + 2c^2r^2 \sum_{f \in fr} v_x^2(P_h).$$

or

$$h \sum_{f \in fr} v^2 \leq 2cr \cdot v^2(P_h^*) + 2c^2r^2 \sum_{f \in fr} v_x^2.$$
where it is understood that we are in $s_{r,h}$. If we also
sum in the $y$-direction, we obtain
\[ h \sum_{s_r} v^2 \leq 2cr \sum_{s_r} v^2(P_h) + 2c^2r^2h \sum_{s_r} v_x^2. \]

We have then only to extend the summation over the entire boundary strip to obtain
\[ h^2 \sum_{s_{r,h}} v^2 \leq Ar^2h^2 \sum_{s_{r,h}} (v_x^2 + v_y^2) + Brh \sum_{s_{r,h}} v^2. \]

We now let $v_h = u - f_h$, so that $v_h$ vanishes on $\Gamma_h$. Since $h^2 \sum_{s_{r,h}} (v_x^2 + v_y^2)$ is bounded as $h \to 0$, we see that
\[ \frac{h^2}{r} \sum_{s_{r,h}} v^2 \leq \gamma r, \]
where $\gamma$ is a constant independent of $v$ or $h$.

In order to consider the limiting case as $h \to 0$, we extend the sum not over $S_{r,h}$, but over $S_{r,h} - S_{r,h'}$ with $r < r'$. The same inequality holds, and hence as $h \to 0$, we get
\[ \frac{1}{r} \int_{s_r} v^2 \ dx \ dy \leq \gamma r. \quad v = u - f \]

We now let the boundary strip $S_{r'}$ tend to the boundary; then we obtain the inequality,
\[ \frac{1}{r} \int_{s_r} v^2 \ dx \ dy = \frac{1}{r} \int_{s_r} (u - f)^2 \ dx \ dy \leq \gamma r. \]

Hence as $r \to 0$, $u$ converges in the mean to the boundary values.

2.3 Second Convergence Proof for Dirichlet's Problem
(Proof of Petrowsky)

The proof of Petrowsky that the solution of the first boundary value problem for the Laplace difference
equation tends to the solution of the boundary value problem for the Laplace differential equation, involves the proof of the equi-continuity of the sequence of lattice functions, as did the proof of Courant, Friedrichs, and Lewy, but the proof of the equi-continuity depends on the properties of the harmonic functions, and for that reason may not be extended to more general elliptic equations. This proof, however, shows that the solution converges to the exact solution, rather than converging in the mean, as in the previous proof.

We prove first the following theorem:

Let $D$ be a domain bounded by a Jordan curve $\Gamma$ and let $D' \subset D$ be a domain bounded by a Jordan curve $\Gamma' \subset D$. Consider the set of all possible square lattices parallel to a fixed system of coordinates, and also the set of all lattice functions $u(P)$ that satisfy in $D$ the difference equation $\Delta u = 0$ and are uniformly bounded, $|u(P)| < M$, in $D$. Then there exists a constant $M'$ such that in $D'$

$$|u_x| < M', \quad |u_y| < M',$$

provided only that $h$ is sufficiently small ($h < \delta_h$, where $\delta$ is greater than zero). Similarly, provided only that $h$ is sufficiently small, we have

$$|u_{xx}| < M'', \quad |u_{xy}| < M'', \quad |u_{yy}| < M'' ,$$

and so on for all difference quotients.

Proof: Since all difference quotients $u_x, u_y, u_{xx}, \ldots$ are again solutions of $\Delta u = 0$, it is clear that it
suffices to prove the theorem for \( u_x \). The domain \( D' \) can be covered by a finite number of squares contained in \( D \), and each of these squares can be enclosed in a larger square also contained in \( D \). Hence it suffices to prove the result for the case where \( D \) and \( D' \) are concentric squares.

We may suppose without loss of generality that the origin is at the center of \( D' \). Thus we consider the squares

\[ |x| < a, \ |y| < a \hspace{1cm} (D) \]

and

\[ |x| < a - \delta, \ |y| < a - \delta \hspace{1cm} (D') \]

If \( h \) is less than \( \frac{\delta}{3} \), we replace \( D \) by the square \( |x| < b, \ |y| < b \), where \( b \) is a multiple of \( h \) such that \( a - \frac{\delta}{3} < b \). Thus we have reduced the problem to the case of two squares

\[ |x| < b, \ |y| < b \]

and

\[ |x| < b - \frac{\delta}{2}, \ |y| < b - \frac{\delta}{2}, \]

where \( b \) is a multiple of \( h \).

We now show that the function

\[ z(p) = u_x^2 \Phi + c \left[ u_x^2(p) + u_x^2(p_1) + u_x^2(p_2) + u_x^2(p_3) + u_x^2(p_4) \right], \]

where \( p_1, p_2, p_3, \) and \( p_4 \) are the four points adjacent to \( p \), where

\[ \Phi = (x^2 - b^2)(y^2 - b^2), \]

and where \( c \) is a (large) positive constant to be deter-
(24)

mined, satisfies the inequality

\[ \Delta z \geq 0. \]

Since \((fg)_x = f_g + f_xg_x + f_xg + f_gx\),

\[ (u^2)_x = 2uu_x + u_x^2 + u_x^2. \]

Since \(\Delta u = 0\),

\[ \Delta u^2 = u_x^2 + u_y^2 + u_x^2 + u_y^2. \]

Similarly,

\[ (u_x^2)_x = 2uu_x + u_x^2 + u_x^2, \]

and, since \(\Delta u_x = 0\),

\[ \Delta u_x^2 = u_x^2 + u_x^2 + u_x^2 + u_x^2. \]

Then, we get

\[ \Delta(u_x^2) = \Delta u_x^2 + \Delta_x(u_x^2) + \Delta_x(u_x^2), \]

\[ + \Delta y(u_x^2) + \Delta y(u_y^2) + u_x \Delta \phi \]

\[ = \Delta x(u_x^2 + u_x^2 + u_x^2 + u_x^2) + \Delta_y(u_x^2 + u_x^2), \]

\[ + \Delta_x(u_x^2 + u_y^2) + \Delta_x(u_x^2 + u_y^2), \]

\[ + \Delta y(u_x^2 + u_y^2) + u_x \Delta \phi. \]

Now,

\[ \frac{\phi}{x} = 4x(x^2 - b^2)(y^2 - b^2) = 4x(y^2 - b^2)x^2. \]

Hence there is a constant \(\Delta\) such that in \(D\)

\[ \left| \frac{\phi}{x} \right| < \Theta \sqrt{x}, \]

and similarly for \(\phi_x\). Also, \(\Delta \phi\) is bounded. We may suppose \(\Delta\) so chosen that on \(D\)

\[ |\Delta \phi| < \Theta, |\phi_x| < \Theta \sqrt{x}, |\phi| < \Theta \sqrt{x}. \]
Then we have, for \( \varepsilon > 0, \ v \) arbitrary,
\[
|\frac{\partial^2 u}{\partial x^2} \frac{\partial u}{\partial x} - \frac{\partial u}{\partial x} | \leq \frac{u_x^2}{\varepsilon^2} + \varepsilon \frac{2 u_x^2}{\partial^2 u_{xx}}
\]
and similarly for the other terms of the expression for \( \Delta (u_x^2 \Phi) \). This implies, for example,
\[
\frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial x \partial y} \geq \frac{u_x^2}{\varepsilon^2} - \frac{2 u_x^2}{\partial^2 u_{xx}}.
\]

Therefore, we have that
\[
\Delta (u_x^2 \Phi) \geq (1 - 2 \varepsilon^2 \varepsilon^2) u_x^2 + u_x^2 + u_x^2 + u_x^2 + u_x^2 + u_x^2 \]
\[
= - \frac{1}{\varepsilon^2} (4 u_x^2 + u_x^2 + u_x^2 + u_x^2 + u_x^2 + u_x^2) - \frac{\partial u_x}{\partial x}.
\]

If we select \( \varepsilon \) and \( C \) so that \( \varepsilon^2 \varepsilon^2 < \frac{1}{4}, \ C > \frac{1}{\varepsilon^2} + \varepsilon^2 \),
it then follows that
\[
\Delta z = \Delta (u_x^2 \Phi) + C (u_x^2 + u_x^2 + u_x^2 + u_x^2 + u_x^2) \geq 0.
\]

Then \( z(P) \) attains its maximum value on the boundary.

But \( \Phi = 0 \) on the boundary; hence in the whole square \( D \)
\[
0 \leq z(P) \leq \frac{5 C u_x^2}{\Phi}.
\]

Hence for \( P \in D^1 \),
\[
u_x^2(P) \leq \frac{5 C u_x^2}{\Phi} \leq \frac{5 C u_x^2}{(\delta)^2}.
\]

Since \( \Phi \geq 0 \), we can take the last expression for \( \Phi \),
and this concludes the proof of the theorem.

We are now in a position to prove the following theorem:

Let \( \Gamma \) be a rectifiable simple closed curve enclosing
a region $D$, and let $s$ be length measured along $\Gamma$. We shall further suppose that at any point $Q$ of $\Gamma$ a circle can be drawn having only the point $Q$ in common with $\Gamma$ and otherwise lying outside. Then given a continuous function $f(s)$ on $\Gamma$ there exists a solution $v(x,y)$ of the differential equation
\[
\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0
\]
in $D$ that tends to $f(s)$ as $(x,y) \to Q$ in $D$.

We shall construct $v(x,y)$ by a passage to the limit from difference equations. In this process we need to reselect the boundary values when the spacing $h$ of the lattice is refined; we shall suppose this done so that the approximate boundary values used tend uniformly to the true boundary values.

From the theorem already proved, we see that any uniformly bounded set of lattice functions $u(P)$ for which $\Delta u = 0$, in particular the sequence of functions $u_h$ with boundary values approaching $f(s)$, is equi-continuous in $D'$. Then there must exist a subsequence $u_{h_1}, u_{h_2}, \ldots$ convergent in $D'$, of the sequence $\{u_h\}$, by Ascoli's theorem. Since $D'$ was any closed region included in $D$, we can pick a sequence of regions $D' \subset D'' \subset D''' \subset \ldots \subset D$ whose

---

We do not need this restriction in two dimensions, but in higher dimensions we do (Lebesgue's example).
sum is $D$, and then find successive subsequences of $u_{h_1}$, $u_{h_1}$, ... converging in each of these regions in turn. The diagonal subsequence will then be convergent in the whole of $D$.

By the same argument, there is a subsequence of the final subsequence above whose first difference quotients also converge in $D$. Proceeding in this fashion, we can find a subsequence any fixed number of whose difference quotients converges in $D$.

These results may be summarized in the lemma:

Suppose we have a sequence of lattices in a domain $D$ such that $h \to 0$, and on each lattice a function $u_h$ such that $\Delta u_h = 0$. If the collection of these $u_h$ is uniformly bounded in $D$, then it contains a subsequence that converges everywhere in $D$ to a function $v(x,y)$ that is harmonic in $D$ and has bounded derivatives of arbitrary order in any closed subregion of $D$.2

This subsequence is simply the final subsequence found above.

It remains to show that the limit $v(x,y)$ of the subsequence $u_{h_1}, u_{h_2}, ...$ that we have chosen satisfies the boundary condition $v = f(s)$ on $\Gamma$.

We fix some point $Q$ of the boundary. By hypothesis

2It is not necessary that the $u_h$'s should approximate the boundary values as long as they are uniformly bounded in $D$. 
we can draw a circle outside $\Gamma$ and meeting $\Gamma$ in $Q$ only. Let $A$ be the center and $\rho$ the radius of such a circle, and let $r$ be the distance from an arbitrary point $P$ to $A$.

We pick any $\varepsilon > 0$ and define the function

$$\tilde{u}(P) = f(Q) + \varepsilon + C \left( \frac{1}{\rho} - \frac{1}{r} \right)$$

where $C$ is a positive constant to be determined later.

On any lattice

$$\Delta \tilde{u}(P) = -C \Delta (1/r) = -C \left[ \frac{\delta^2 (V_r)}{\delta x^2} + \frac{\delta^2 (V_r)}{\delta y^2} + O(h) \right]$$

$$= -C \left[ (V_r)_{rr} + (V_r)_{rr} \cdot \frac{1}{r} + O(h) \right]$$

$$= -C \left[ (V_r)^2 + O(h) \right] < 0$$

in $D$ for $h$ sufficiently small. If $u(P)$ is a solution of the difference equation $\Delta u = 0$ for the lattice, we have

$$\Delta [\tilde{u}(P) - u(P)] < 0 \quad \text{for} \quad P \in D.$$

Hence we may say that $\tilde{u}(P) - u(P)$ assumes its minimum value on the boundary.

Now, since $f$ is continuous, we have for any points $R$ of $D$ sufficiently near $Q$

$$f(Q) + \varepsilon > f(R).$$
Hence we may choose $C$ sufficiently large so that for all $R \epsilon \mathcal{R}$,

$$f(Q) + \epsilon + C \left[1/\rho - 1/r\right] > f(R).$$

If we now suppose that $u(P)$ is an element of the convergent subsequence $\{u_h(P)\}$ we conclude (since the boundary values are approached uniformly) that for $h$ sufficiently small $\tilde{u}(P) - u_h(P) > 0$ on the boundary of the lattice region. Then, since $\Lambda[\tilde{u}(P) - u_h(P)] > 0$, $\tilde{u}(P) > u_h(P)$ in $D$.

On the other hand, for $P$ sufficiently near $Q$ we have $C(1/\rho - 1/r) < \epsilon$, and hence for such $P$

$$u_h(P) < \tilde{u}(P) < f(Q) + 2 \epsilon.$$ 

Hence, by a passage to the limit ($u_h \rightarrow v$)

$$\lim_{P \rightarrow Q} v(P) \leq f(Q) + 2 \epsilon;$$

so that

$$\lim_{P \rightarrow Q} v(P) \leq f(Q).$$

If we now define

$$u(P) = f(Q) - \epsilon - C(1/\rho - 1/r),$$

and carry out the same reasoning as above (with reversed signs) we conclude that for $P$ sufficiently near $Q$

$$u(P) > \tilde{u}(P) > f(Q) - 2 \epsilon.$$ 

Then, as before, we conclude

$$\lim_{P \rightarrow Q} v(P) > f(Q),$$

which concludes the proof.
1.4 Other Convergence Proofs

Diaz and Roberts [6] have given other convergence proofs for the Dirichlet difference boundary value problem. These proofs are the precise analogues of the methods of solution of the Dirichlet differential boundary value problem. We shall outline in brief these proofs. We shall retain the same notation as previously.

1.4.1 Analogue of Poincare's "methode de balayage" [10].

Let \( f(s) \) be the given real-valued function, defined on \( \Gamma \), which is the given boundary value of \( u \) on \( \Gamma \). Consider an "initial function" \( S \) which is a superharmonic function defined on \( G_h \) and which coincides with \( f \) on \( \Gamma \). Then we have

\[
4S(P) \geq S(P_1) + S(P_2) + S(P_3) + S(P_4),
\]

where \( P_1, P_2, P_3, \) and \( P_4 \) are the four neighboring points of \( P \) in \( G_h \), and \( S(P) = f(P) \) for \( P \in \Gamma \). The restriction that \( S \) be superharmonic is not essential and may be removed.

We arrange the points of \( G_h \) in a sequence \( P^{(1)}, P^{(2)}, P^{(3)}, \ldots \) in such a way that each point of \( G_h \) occurs infinitely many times in the sequence, the sequence being arranged in some convenient method.

We now define a sequence of functions on \( G_h \). Let \( w_0 = S \), and define the others in the following manner:

1) \( w_0 = S \) on \( G_h \).
2) $w_1$ is harmonic at $P''$
   
   $w_1 = w_0$ on $G_h - P''$

3) $w_2$ is harmonic at $P^{(2)}$
   
   $w_2 = w_1$ on $G_h - P^{(2)}$

\[ \begin{align*}
\vdots \\
\end{align*} \]

$k+1)$ $w_k$ is harmonic at $P^{(k)}$
   
   $w_k = w_{p-1}$ on $G_h - P^{(k)}$

\[ \begin{align*}
\vdots \\
\end{align*} \]

In other words, we move from point to point in $G_h$ changing the value of the function at each point so that the Laplace difference equation holds at that point.

It is easily shown by induction that each function $w_k$ is superharmonic in $G_h$, and it may also be seen that each function $w_k$ is bounded below by the minimum value of $S$ on $G_h$. Since the sequence was monotonically non-increasing by method of construction, it follows that \{ $w_k$ \} converges at each point of $G_h$. It then only remains to show that

$$\lim_{{k \to \infty}} w_k(P),$$

for $P \in G_h$, is the solution of the boundary value problem.

Since there is no question that this limit function assumes the boundary values, since each $w_k$ does, it is sufficient to show that it satisfies the difference equation. Consider a point $P \in G_h$. $P$ occurs infinitely many times in the sequence $P^{(1)}, P^{(2)}, \ldots$, and therefore
there is an infinite sequence of integers \(i_k\) such that 
\[ P = P^{(i_k)} \] for \(k = 1, 2, 3, \ldots\). Consequently, there is a subsequence of functions

\[ w_{i_1}, w_{i_2}, \ldots, w_{i_k}, \ldots \]

of the sequence

\[ w_1, w_2, \ldots, w_k, \ldots \]

such that each function \(w_{i_k}\) of the subsequence is harmonic at \(P\), i.e.,

\[ 4w_{i_k}(P) = w_{i_k}(P_1) + w_{i_k}(P_3) + w_{i_k}(P_3) + w_{i_k}(P_4). \]

Since

\[ \lim_{k \to \infty} w_{i_k}(P) = \lim_{k \to \infty} w_{i_k}(P) \quad \text{for} \ P \in \Omega, \]

it then follows that

\[ 4 \lim_{k \to \infty} w_k(r) = \lim_{k \to \infty} w_k(P_1) + \lim_{k \to \infty} w_k(P_3) + \lim_{k \to \infty} w_k(P_3) + \lim_{k \to \infty} w_k(P_4), \]

i.e., the limit function is a solution of the boundary value problem.

The restriction that \(S\) be superharmonic may be removed by showing that any function defined on \(\Omega\) may be represented as the difference of two superharmonic functions on \(\Omega\).

1.4.2 Analogue of Kellogg's extension of Poincaré's method [11].

This method differs from that of 1.4.1 only in the removal of the restriction that the process of that
method be carried out pointwise at each step. Consider a sequence of subregions $B_1, B_2, B_3, \ldots$ of $G_h$, subject to the following conditions:

a) each point of $G$ is an interior point of an infinite number of the subregions of the sequence.

b) the Dirichlet problem is explicitly solvable for each subregion $B_k$ in terms of arbitrary boundary values on the boundary of $B_k$.

The convergence proof is the same as in 1.4.1. One need only substitute the sequence $B_1, B_2, B_3, \ldots$ of subregions for the sequence $P^0, P^1, P^2, \ldots$ of points.

1.4.3 Method of successive approximations

We start with an initial function $w_0 = g$ which satisfies the prescribed boundary condition, and then define the following sequence of functions $w_k$:

\[
w_0 = g \quad \text{on } G_h;
\]

\[
4 \quad w_1(p) = w_0(p_1) + w_0(p_2) + w_0(p_3) + w_0(p_4) \quad \text{for } p \in G_h;
\]

\[
w_1(p) = g(p) \quad \text{for } p \in \Gamma_h;
\]

\[
4 \quad w_k(p) = w_{k-1}(p_1) + w_{k-1}(p_2) + w_{k-1}(p_3) + w_{k-1}(p_4) \quad \text{for } p \in G_h;
\]

\[
w_k(p) = g(p) \quad \text{for } p \in \Gamma_h;
\]

Again we may suppose, without loss of generality, that $g$ is superharmonic. Hence all the functions $w_k$ are superharmonic, and from the minimum principle of super-
harmonic functions it follows that

\[ w_0 \succ w_1 \succ w_2 \succ \cdots \succ w_k \succ \cdots \succ \min_{r \in A} g \succ \min_{r \in A} f. \]

Thus the sequence converges, and since

\[ \lim_{k \to \infty} w_k = \lim_{k \to \infty} w_k, \]

it follows that the limit function of the sequence is the desired solution of the Dirichlet problem.
Chapter II.
The Truncation Error

We recall that we defined the truncation error to be the difference between the exact solution of the differential equation and the difference equation. In this chapter, we shall obtain the results of Wasow for the Dirichlet problem.

Let $G$ be the rectangle with vertices $(0,0)$, $(1,0)$, $(1,b)$, and $(0,b)$, where $b$ is a rational number. The prescribed boundary function $f(x,y)$ is assumed to be continuous on the boundary $\partial G$, and to possess bounded third derivatives on each closed side of $\partial G$.

By $\nabla^2 u$ we denote $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and by $\Delta u$ we denote as before

$$\frac{\partial^2}{\partial x^2} \left[ \frac{u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h)}{h^2} - 4u(x,y) \right],$$

where $h$ is the mesh width. For any value of $h$ such that $1/h$ and $b/h$ are integers, we denote by $u_h(x,y)$ the function defined at all lattice points $(nh,mh)$ in $G_h + \Gamma_h$ (defined as usual) where $n,m$ are integers, for which $u_h = 0$ in $G_h$ and $u_h = f$, on $\Gamma$. Then if $u$ denotes the solution for the corresponding Dirichlet problem for $\nabla^2 u$, we wish then to consider $(u - u_h)$.

If the boundary values vanish at the vertices, then we may write down readily analytical solutions for
the two problems. Thus we introduce the harmonic polynomial \( q(x,y) \) defined by
\[
bQ(x,y) = A_{00}(x - 1)(y - b) - A_{10}x(y - b) - A_{1b}xy - A_{0b}(x - 1)y,
\]
where \( A_{00}, A_{10}, A_{1b}, A_{0b} \) are the values of \( f \) at the vertices. It is seen therefore that if we replace the boundary values \( f \) by
\[
f^*(x,y) = f(x,y) - Q(x,y)
\]
we obtain a new problem for which the truncation error is the same as for the original problem, since the truncation error corresponding to the boundary problem determined by the values of \( Q(x,y) \) on \( \Gamma \) is zero. We note that \( f^*(x,y) \) vanishes at the vertices, as was desired, and that the second and higher derivatives of \( f^* \) are the same as those of \( f \).

By the superposition principle, the truncation error can be considered the sum of four terms, and corresponding to boundary values that are zero except on one of the four sides of the rectangle. Therefore, we assume temporarily that \( f^* \) is zero except on the side \( y = 0 \). Thus \( f^* = f^*(x) \) is a function of \( x \) alone.

With the problems posed in the above form, then the solution for the problem
\[
\nabla^2 u = 0 \quad \text{in} \ G, \ u = f^* \quad \text{on} \ \Gamma
\]
can be written
\[
u(x,y) = \sum_{n=1}^{\infty} c_n g(y, n \pi) \sin n \pi x \quad (2.12)
\]
and the solution for the problem
\[ \Delta u = 0 \text{ in } G, \ u = f^* \text{ on } \Gamma \]

(2.13)
can be written
\[ u_h(x,y) = \sum_{n=1}^{\infty} \gamma_n g(y, (\pi/n) \sin n\pi x), \]

where
\[ c_n = 2 \int_0^1 f^*(t) \sin n\pi t \, dt \]  
(2.14)
\[ \gamma_n = 2h \left( \frac{\pi}{b \sinh b} \sin n\pi h \right) \]
(2.15)
\[ g(y,t) = \frac{\sinh(b-y)t}{\sinh bt} \]  
(2.16)
and \( \rho_n \) is the solution of the equation
\[ \sinh \hat{a} \rho_n = \sin nh \pi \hat{a}. \]

These formulas may be verified by substitution.

Now
\[ u - u_h = \sum_{n=1}^{\infty} c_n g(y, n\pi) \sin n\pi x - \sum_{n=1}^{\infty} \gamma_n g(y, (\pi/n) \sin n\pi x). \]  
(2.17)

Thus
\[ |u - u_h| \leq \sum_{n=1}^{\infty} |c_n - \gamma_n| g(y, n\pi) + \sum_{n=1}^{\infty} \gamma_n |g(y, n\pi) - g(y, (\pi/n)\sin n\pi x)| \]
\[ + \sum_{n=1}^{\infty} c_n g(y, n\pi) \]
\[ = R_1 + R_2 + R_3 \]  
(2.18)

We now proceed to get estimates on the various quantities.

1) The Fourier coefficients:

Integrating by parts successively, we see that
\[ c_n = 2 \int_0^1 f^*(t) \sin n\pi t \, dt \]
\[ = 2 \left( \frac{2}{(n\pi)^2} \right) f''(1) + \frac{2}{(n\pi)^4} \int_0^1 f''''(t) \cos n\pi t \, dt. \]
Hence
\[ |c_n| < K_n^{-\frac{3}{2}} \quad (2.19) \]

where
\[ K = \frac{2}{\pi^3} \left[ |f''(1)| + |f''(0)| \right] + 4 \max_{-\pi < x \leq \pi} |f'''(x)|. \]

2) The interpolation coefficients
\[ \gamma_n \text{ and } c_n \text{ are related by the following equation:} \]
\[ \gamma_n = c_n + \sum_{k=1}^{\infty} \left[ c_{y_n}^k - c_{y_n}^{k-1} \right], \quad n = 1, 2, 3, \ldots, 1/h \quad (2.101) \]

This relationship may be derived in the following manner:

The Fourier series expansion of \( f''(r) \) converges to the function since the third derivatives of \( f'' \) are bounded. We substitute this series in (2.15), and obtain
\[ \gamma_n = 2h \sum_{\lambda = 1}^{\eta_h} \left[ \frac{1}{3} a_0 + a_1 \cos \pi rh + a_2 \cos 2\pi rh + \ldots \right. \]
\[ + c_1 \sin \pi rh + c_2 \sin 2\pi rh + \ldots \left. \right] \sin n\pi rh, \]
where \( a_k, k = 1, 2, 3, \ldots \) are the Fourier cosine coefficients of \( f'' \), and the \( c_k \) are defined as above.

Now
\[ \frac{1}{3} a_0 \sum_{\lambda = 1}^{\eta_h} \sin n \pi rh = 0, \]
and further,
\[ a_k \sum_{\lambda = 1}^{\eta_h} \cos \pi rh \sin n \pi rh = 0, \]
because of the orthogonality property of the trigonometric sums. Let us now consider
\[ c_k \sum_{\lambda = 1}^{\eta_h} \sin \pi krh \sin n \pi rh. \]

This is equal to
\[
\sum_{n=1}^{\infty} \left[ \cos \pi rh(k-n) - \cos \pi rh(k+n) \right].
\]

This is zero unless \(k-n=0\) or \(k+n=0\) or \(k+n\) is a multiple of \(2/h\), in these cases, the sum is \(1/h\). Hence (2.101) follows.

Using (2.19), we see from (2.101) that for \(n \leq 1/h\)
\[
\left| (\cos \pi + \sin \pi) \right| \leq K \sum_{k=1}^{\infty} \left\{ \left( \frac{\pi}{h} \right)^3 + \left( \frac{\pi}{h} \right)^{-3} \right\}
\]
\[
= K \frac{h^3}{\pi} \sum_{k=1}^{\infty} \left\{ (k+\frac{\pi}{h})^{-3} + (k-\frac{\pi}{h})^{-3} \right\}
\]
\[
\leq K \frac{h^3}{\pi} \int_{\frac{\pi}{h}}^{\infty} \left[ (x+\frac{\pi}{h})^{-3} + (x-\frac{\pi}{h})^{-3} \right] dx
\]
\[
< K \frac{h^3}{\pi} < 1.40K h^2
\]

(2.102)

3) \(g(y,n) - g(y,n/h)\)

First we express \(\frac{1}{2} q_n\) in terms of \(\sinh \pi\).

If \(f(x) = \sinh^{-1}(\sin x)\), then
\[
f(x) = x + \frac{x^3}{6} \left[ \frac{2 \cos \sqrt{1-2\sin^2(x)}}{(1+\sin^2(x))^{1/2}} \right] \quad 0 < x < x
\]

Since \(\frac{1}{2} q_n = \sinh^{-1}(\sin \frac{1}{2} n \pi)\)
\[
\frac{1}{2} q_n = \frac{1}{2} n \pi + (\frac{1}{2} n \pi)^3 R
\]

where
\[
R = -\frac{\cos \sigma (1-2 \sin^2 \sigma)}{(1+\sin^2 \sigma)^{1/2}} \quad 0 < \sigma < n \pi
\]
\[
\therefore \frac{1}{2} q_n - \frac{1}{2} n \pi = (\frac{1}{2} n \pi)^3 R
\]
\[
\left| q_n - n \pi \right| \leq (n \pi)^3 / 2
\]

(2.103)

Now
\[
g(y,x) = \frac{\sinh(b-y)t}{\sinh bt}
\]

(2.104)

By using the addition formulas for the hyperbolic functions, you may show that
\[
\frac{d}{dt} g(y, t) = (2b-y) \sinh y t - y \sinh (2b-y) t \quad \frac{2}{2 \sinh^2 bt}.
\]

(2.105)

The denominator is positive, and by series expansion, it may be seen that the numerator is not positive. Hence

\[0 \leq \left| \frac{d}{dt} g(y, t) \right| \leq \frac{y \sinh (2b-y) t}{2 \sinh^2 bt} \quad t > 0\]

for

\[\left| \frac{d}{dt} g(y, t) \right| = \frac{y \sinh (2b-y) t - (2b-y) \sinh y t}{2 \sinh^2 bt} \leq \frac{y \sin (2b-y) t}{2 \sinh^2 bt} \]

(2.106)

We use this to express \( g(y, \pi) - g(y, \varphi/n) \) by the mean value theorem. We need first the following inequality:

\[\frac{2}{h} \leq \varphi/n \leq \pi n \quad \text{for} \quad 1 \leq n \leq 1/h. \quad (2.107)\]

Since \( \sin x \leq x \),

\[\sinh \frac{\varphi}{n} = \sin \frac{\varphi}{n} \pi \leq \frac{\varphi}{n} \pi \leq \frac{\pi}{2} \pi.\]

Since \( \sin x/\pi \downarrow \) as \( x \uparrow \),

\[\sinh \frac{\varphi}{n} \leq \frac{\varphi}{n} \pi \sin \frac{\varphi}{n} \pi /\pi \pi \quad \frac{2}{\pi} \leq \frac{\varphi}{n} \pi \sin \frac{\pi}{2} \pi /\pi \pi \]

\[= \frac{\varphi}{n} \pi /\pi \pi = \pi.\]

Since \( \sinh x/\pi \uparrow \) as \( x \uparrow \), \( x > 0 \)

\[\sinh \varphi/\pi \leq \sinh \frac{\varphi}{n} \pi \leq \frac{\varphi}{n} \pi \pi \]

\[\varphi/\pi \leq \varphi/\pi \leq \frac{\varphi}{n} \pi.\]

Hence \( \frac{\varphi}{n} \leq \varphi/\pi \).

Since \( x \leq \sinh x \)

\[\varphi/\pi \leq \sinh \varphi/\pi \leq \frac{\varphi}{n} \pi.\]
\(\frac{\theta_n}{\theta} \leq \eta \pi\)
\[\therefore \quad \frac{\theta_{n+1}}{\theta} \leq \frac{\theta_n}{\theta} \leq \eta \pi\]

Since
\[
\left| \frac{d}{dt} q(y,t) \right| \leq \frac{1}{2} y \frac{\sinh (2b - y) t}{\sinh 2bt} \\
= \frac{1}{2} y \left( e^{(2b - y) t} - e^{-(2b - y) t} \right) \frac{2}{e^{2bt} - e^{-2bt}} \\
= \frac{y e^{-yt} (1 - e^{-2(y - b)t})}{1 - e^{-2bt}} \cdot \frac{1}{1 - e^{-2bt}}
\]

\[\left| \frac{d}{dt} q(y,t) \right| \leq y e^{-yt} (1 - e^{-2b/3})^{-1} \quad \text{for} \quad 1 \leq n \leq \eta, \quad 4t \leq t \]
\[\theta_{n+1} \leq \frac{e^{2bt} - e^{-2bt}}{e^{2bt} - e^{-2bt}} \cdot \frac{1}{1 - e^{-2bt}} \]

\[|g(y, n\pi) - g(y, n\pi)| \leq (1 - e^{-2b/3})^{-1} y e^{-2b/3} \theta^{2/3} \]
\[\text{for} \quad 1 \leq n \leq \eta, \quad \theta \leq 1\]
\[|g(y, n\pi)| \leq 1 \quad 0 \leq y \leq b \quad (2.109)\]

We further note that
\[0 \leq g(y, t) \leq 1 \quad 0 \leq y \leq b \quad (2.109)\]

We are now in a position to get an estimate on \(R_1\),
\[R_2, \text{ and } R_3\]
\[R_1 = \sum_{n=1}^{\eta} \left| c_n - \kappa_n \right| q(y, \pi n) \leq \frac{1.4}{h} k h^3 = 1.4 k h^3\]
\[R_2 = \sum_{n=1}^{\eta} |c_n| \left| q(y, \pi n) - q(y, \pi n) \right| \leq \sum_{n=1}^{\eta} k n^{-3} \leq K \int_{\eta}^{\infty} \frac{dk}{k^3} = \frac{k h^2}{2}\]
\[R_3 = \sum_{n=1}^{\eta} |\chi_n| \left| q(y, \pi n) - q(y, \pi n) \right| \]

Now, here, \(\chi_n |k(n^{-3} + 1.4 h^3) | \leq 2.4 k n^{-3}\). Using the above bound on \(|g(y, n\pi) - g(y, n\pi)|\) and the formula for a geometric progression, we see that
Since for \( y > 0 \), we have
\[
y/(e^{4y/3} - 1) \leq 3/4,
\]
then
\[
R_2 \leq 5.24(1 - e^{-8b/3})^{-1} Kh^2.
\]
\[
|u - u_h| \leq 1.4 Kh^2 + Kh^2 + 5.24(1 - e^{-8b/3})^{-1} Kh^2
\]
\[
= Kh^2(1.9 + 5.24(1 - e^{-8b/3})^{-1}.
\]

If we denote by \( \bar{M}_2 \) the numerical maximum of the second derivative of the given boundary function at the vertices of the rectangle, and by \( \bar{M}_3 \) the numerical maximum, anywhere on \( \Gamma \), of the third derivative of the boundary function, the last inequality can be replaced by
\[
|u - u_h| \leq \left[ 0.297 + 0.676 \left( 1 - e^{-8b/3} \right)^{-1} \right] \left[ \bar{M}_2 + 0.319 \bar{M}_3 \right] h^2.
\]

This formula was derived under the assumption that the function was zero except on the side \( y = 0 \). For the full truncation error, three analogous expressions have to be added to the last inequality. This leads to
\[
|u - u_h| \leq \left[ 0.246 + 0.676 \left( 1 - e^{-8b/3} \right)^{-1} \right] \left[ 2 \bar{M}_2 + \bar{M}_3 \right] h^2
\]
\[
+ \left[ 0.246 + 0.676 \left( 1 - e^{-8b/3} \right)^{-1} \right] 2 \bar{M}_2 + 3 \bar{M}_3 \right] h^2.
\]
Chapter III.

A More General Elliptic Equation

We shall show in this chapter how the results of the first chapter may be extended to more general equations of elliptic type.

It is well known that if the coefficients in the expression

\[
L^*(u) = A \frac{\partial^2 u}{\partial x^2} + 2B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} + cu
\]

are analytic, then \( L(u) \) may be written in the following form:

\[
L^*(u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} + cu.
\]

It is also known that, provided that \( c \neq 0 \), and for a sufficiently regular region \( G \), the boundary value problem analogous to the Dirichlet problem has a solution.

We shall establish this result by the use of difference equations in the case where the coefficients are constant. As in Chapter I, we shall find that the solution of the boundary value problem has continuous derivatives of all orders.

We prove the following theorem:

Suppose that

(1) \( G \) is a simply-connected region whose boundary is composed of a finite number of regular arcs, and \( G \) is finite;

(2) \( f(x,y) \) is a function which

a) in the interior of \( G \), is continuous and has
continuous first and second partial derivatives,

b) is defined and bounded on $\Gamma$, and is piecewise continuous on $\Gamma$,

c) if $P_0: (x_0, y_0) \in \Gamma$ and $P: (x, y)$ is an interior point,

$$\lim_{P \to P_0} f(P) = f(P_0),$$

except at most a finite number of points,

d) the integral

$$\int_G (f_x^2 + f_y^2 + 2 \alpha f_x f_y + \beta f_y^2) \, dx \, dy$$

exists; and, finally that

(3) $u_h$ is the solution of $L(u) = 0$ which assumes on the boundary $\Gamma_h$ of the lattice region $G_h$ corresponding to $G$ the values assumed by $f(x, y)$, where

$$L(u) = \Delta u - 2 \alpha u_x - 2 \beta u_y - gu,$$

with $\alpha$, $\beta$, $g(g > 0)$ representing constants.

Then, as $h \to 0$, independently of the value of $h$, $u_h$ converges to the solution of the differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - 2 \alpha \frac{\partial u}{\partial x} - 2 \beta \frac{\partial u}{\partial y} - gu = 0,$$

where $u(x, y)$ assumes the values taken on by $f(x, y)$ on $\Gamma$, in the sense defined in Chapter I.

The proof of this theorem follows the outline of the theorem of Chapter I, and differs only in the details of the proof. We point out in particular that in Lemma 1, we cannot assume that the maximum is assumed on the boundary. We first establish the equi-continuity of the sequence of solutions $u_h$ and their difference quotients.
Lemma 1: The sums 

$$h^2 \sum_{G_h} u_x^2 \text{ and } h^2 \sum_{G_h} (u_x^2 + u_y^2)$$

are bounded as $h \to 0$.

Proof: $L(u)$ is derived from the bilinear form

$$B(u,u) = u_x^2 + u_y^2 + 2\alpha u_x u + 2 \eta u_y u + \gamma u_x u$$

which is assumed to be positive definite. It therefore follows that there exists a constant $\mathcal{K}$ such that

$$u_x^2 + u_y^2 < \mathcal{K} B(u,u).$$

Thus

$$h^2 \sum_{G_h} (u_x^2 + u_y^2) < \mathcal{K} h^2 \sum_{G_h} B(u,u).$$

But by the minimum problem formulation,

$$h^2 \sum_{G_h} B(u,u) \leq h^2 \sum_{G_h} B(f,f) \leq \int \int_G \left( f_x^2 + f_y^2 + 2\alpha f_x f_y \right) \, dx \, dy,$$

which, by hypothesis, exists.

It therefore follows that, as $h \to 0$,

$$h^2 \sum_{G_h} (u_x^2 + u_y^2)$$

is bounded.

Further, since

$$h^2 \sum u_x^2 \leq 2d h \sum u_x^2 \sum F_n + 2d^2 h^2 \sum (u_x^2 + u_y^2),$$

where $d$ is a constant dependent only on the region, it follows, as in the previous theorem, that as $h \to 0$, the sum

$$h^2 \sum u_x^2$$

is bounded.

Lemma 2: If $w_h(x,y)$ represents a difference quotient of
(46)

\[ u_h(x,y), \text{ then } w = w_h \text{ satisfies the difference equation } \]
\[ L(w) = 0 \text{ in } G_h. \text{ Moreover, if } G^h_n \text{ represents a lattice } \]
\[ \text{region corresponding to } G^*, G^* \subset G, \text{ as } h \to 0, \ h^2 \sum_{Q_k} w^2 \]
\[ \text{is bounded, and further, for } G^{**}_n \subset G^*_h \text{ corresponding } \]
\[ \text{to } G^{**} \subset G^*, \]
\[ h^2 \sum_{Q_k} (w_x^2 + w_y^2) \]
\[ \text{is bounded.} \]

**Proof:** Since \( L(u) \) is a linear equation with constant coefficients, it follows that \( w = w_h \) satisfies the equation \( L(u) = 0. \)

We consider the sum
\[ h^2 \sum_{Q_k} (w_x^2 + w_y^2 + 2 \lambda w_x w + 2 \beta w_y w + g w^2 + w_x^2 + w_y^2 + 2 \lambda w_x w + 2 \beta w_y w + g w^2), \]
where the summation is extended over all the interior points of a square \( Q. \) Let \( S_1 \) be the boundary row, and let \( S_0 \) be the row of neighboring interior points. Then, as it may be shown, the above sum is bounded by
\[ \sum_{S_1} w^2 - \sum_{S_0} w^2. \]

The proof of this is carried out in the manner of the lemma of Chapter I, the principal difference being that we use the fact that at each interior point, \( L(w) = 0. \)

We then consider a sequence of concentric squares \( Q_0, Q_1, \ldots, Q_N \) with boundaries \( S_0, S_1, \ldots, S_N. \) As before,
\[ 2h^2 \sum_{Q_k} (w_x^2 + w_y^2) \leq h^2 \sum_{Q_k} (w_x^2 + w_y^2 + w_x^2 + w_y^2) \]
for \( k > 1. \) But
\[ h^2 \sum_{Q_k} (w_x^2 + w_y^2 + w_i^2 + w_j^2) \leq \chi h^2 \sum_{Q_k} \left( w_x^2 + w_y^2 + 2w_x w + 2\rho w_x w + \gamma w^2 \right) + w_x^2 + w_y^2 + 2w_x w + 2\rho w_x w + \gamma w^2 \]

Hence,
\[ 2h^2 \sum_{Q_i} (w_x^2 + w_y^2) \leq \chi \left[ \sum_{s_{k+1}} w_x^2 - \sum_{s_k} w_x^2 \right]. \]

The remainder of the proof of the lemma is exactly the same as in the theorem of Chapter I.

**Lemma 3:** From the boundedness of the sums of Lemma 1 and Lemma 2, it follows that all the difference quotients are bounded and equi-continuous.

**Proof:** Since the proof of the lemma of Chapter I nowhere depends on the particular form of the difference equation, the same proof is valid here.

It then follows from the theorem of Ascoli that a subsequence of functions \( u_n \) exists which tends uniformly to a function \( u(x,y) \) which is continuous in each closed region interior to \( G \). Further \( u(x,y) \) satisfies the differential equation
\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - 2\alpha \frac{\partial u}{\partial x} - 2\alpha \frac{\partial u}{\partial y} - 2u = 0. \]

We note again that the corresponding subsequence of difference quotients of any order tend to the corresponding partial derivatives of \( u(x,y) \), and they are continuous in any closed region interior to \( G \).

We note further that the proof that the solution of
assumes the boundary values does not depend upon the particular form of the differential equation. Hence the same proof is valid. That this is true is, of course, the virtue of the proof of Courant, Friedrichs, and Lewy.
References


[8] See [3]

