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THE PRINCIPLE OF HARMONIC MEASURE AND ITS APPLICATIONS

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An important concept useful in the investigation of the properties of analytic functions is the fact that analytic transformations of analytic functions constitute a group, of which those functions which remain invariant under the transformations play an important role. Of particular interest is the question of whether distinct non-Euclidean measure functions may be determined which remain invariant under certain groups of transformations. Among the first to recognize the significance of such measure concepts were Poincare and Klein, and since the time of their investigations into the subject the use of these concepts has become more and more helpful in the development of function theory.

One such measure determination is called harmonic measure, which Rolf Nevanlinna has discussed in several articles and of which he has presented a detailed development in his book, Eindendige Analytische Functionen, which has served as one of the principal references for this paper.

We have the following definition of this measure: Let $G$ be a connected region and $\Gamma$ its boundary; on $\Gamma$ consider a point - set $\alpha$ and its complementary boundary set $\beta = \Gamma - \alpha$. Let $\omega(\tilde{z}, \alpha, G)$ be a function satisfying the following conditions:

1. $\omega$ is harmonic and bounded in the interior of $G$.
2. $\omega$ takes the value 1 in the boundary points $\alpha$ and the value 0 in the boundary points $\beta$. 

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The function \( \omega(z, \alpha, \sigma) \) is then called the harmonic measure of the point set \( \alpha \) in the points \( z \), measured with respect to the region \( G \).

The problem of constructing the function \( \omega \) suggests immediately the solution of the Dirichlet problem of finding a function harmonic in a closed region and taking on the boundary the values of a preassigned function.

What restrictions must be made on the region \( G \) and the boundary set \( \alpha \) in order that the harmonic measure exist is a question that requires much detailed investigation before a thorough answer can be given. We shall now consider some simple cases in which the existence of the function \( \omega \) follows from well known results in potential theory.

In case \( G \) is the unit circle and \( \alpha \) is composed of a finite number of arcs, then if we set \( z = \rho e^{i\phi} \) and \( \Sigma = e^{i\phi} \), we see that \( \omega(\Sigma) \) is bounded and, except for a finite number of points, continuous on the boundary, so that, as was first shown by Schwarz\(^1\), the Poisson integral

\[
\omega(z) = \frac{1}{2\pi} \int_{0}^{2\pi} \omega(e^{i\phi}) \frac{1 - \rho^2}{1 + \rho^2 - 2\rho \cos(\phi - \theta)} \, d\theta
\]

is a function harmonic in the interior of the unit circle for which \( \omega(z) \to \omega(S_0) \) as \( z \to S_0 \) for each boundary point of continuity \( S_0 \) of \( \omega(S) \). Then since
\[
\omega(S) = 1 \quad \text{for } S \in \alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_n
\]
and \( \omega(S) = 0 \) for \( S \in \beta \),
\[
\omega(z) = \frac{1}{2\pi} \sum_{i=1}^{n} \int \frac{1 - \lambda^2}{1 + \lambda^2 - 2\lambda \cos(\theta - \varphi)} \, d\theta.
\]

If the region \( G \) is bounded by one Jordan curve \( \Gamma \), then according to the Riemann mapping theorem for a simply connected region whose boundary consists of at least two points, \( G \) can be mapped one-to-one and conformally onto the unit circle \( K \). The theorem states further that the map is one-to-one and continuous on the circumference of the circle for a simply connected region bounded by a Jordan curve.

Hence if we take \( \alpha \) to be a finite number of arcs of \( \Gamma \), then since \( \omega(S) \) is, for \( S \in \Gamma \), continuous except at a finite number of points, then if \( S' \) is the image point of under the mapping, \( \omega(S') \) will be a function continuous except at a finite number of points on the circumference of \( K \).

\( \omega(z') \) is then determined by means of the Poisson integral for \( z' \in K \). Under the inverse map \( z' \to z \), the harmonic function \( \omega(z') \), because of the invariance of the harmonic property under conformal transformations, is carried into the function \( \omega(z) \) harmonic in the interior of \( G \) and taking the prescribed values on the boundary \( \Gamma \).
Hence $\omega(z)$ is a function whose properties are those of the desired harmonic measure function in $G$.

If $G$ is a connected region whose boundary consists of a finite number of Jordan curves $\Gamma_1, \Gamma_2, \ldots, \Gamma_p$ and if $\alpha$ is in turn composed of a finite number of arcs of the boundary, then we employ the mapping of the universal covering surface into the unit circle $K$. This mapping is accomplished by means of an automorphic function, and under the mapping each curve $\Gamma_j$ (a) becomes in the case $p = 2$ a single arc $\gamma_j$ of the circumference $|x| = 1$, (b) in the case $p > 2$ is ordered into an infinite sequence of non-intersecting open arcs $\gamma_j^i$ on the circumference. The transformation of the arc $\Gamma_j$ into each boundary arcs is single-valued and continuous.

For $p = 2$, consider two fixed points $a_1$ and $a_2$ exterior to $G$ and such that $a_1$ is interior to the simply connected region bounded by the innermost Jordan curve and $a_2$ is exterior to the outermost curve.

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2 A comprehensive treatment of this subject is found in W. Threlfall's *La notion de recouvrement, L'Enseignement Mathematique* (1934-36)
The function \( \omega(z) = \log \frac{z-a_1}{z-a_2} = \log \left| \frac{z-a_1}{z-a_2} \right| + i \arg \frac{z-a_1}{z-a_2} \) is multiple-valued and analytic in \( G \), and \( M = \log \left| \frac{z-a_1}{z-a_2} \right| \leq M \). so that \( G \) is mapped into a simply-connected region \( G_1 \) in the infinite strip \( m = u \leq M \), \( \omega = u + i\phi \).

By the Riemann mapping theorem, \( G_1 \) can be mapped conformally into the unit circle. Under the two mappings \( \Gamma_1 \) and \( \Gamma_2 \) go into two respective arcs \( \gamma_1 \) and \( \gamma_2 \) of the circumference.
For \( p = 3 \), we choose three fixed points \( a_1, a_2, a_3 \).

If they do not lie on a straight line, we may map the circle through them conformally into the upper half plane. By the Riemann mapping theorem we may choose three points of the boundary of the upper half plane which may be mapped respectively into three prescribed points on the circumference of the unit circle. We choose three points \( b_1^1, b_2^1, b_3^1 \) on the circumference such that the angles of the triangle \( T \) of which the three points are vertices are all zero in magnitude.
Since both the upper half-plane and $T$ may be mapped one-to-one and conformally onto the unit circle, then the half-plane may be so mapped onto $T$, with $a_1 \rightarrow b_1$, $a_2 \rightarrow b_2$, and $a_3 \rightarrow b_3$.

Now we reflect the upper half-plane about each of the segments $a_1a_2$, $a_2a_3$, and $a_3a_1$, so that $T$ is reflected about each of the circular arcs $b_1b_2$, $b_2b_3$, and $b_3b_1$. By continuing this process of reflections we map the Riemann surface covering the plane into the unit circle, such that a half-plane of each sheet goes into a triangle in the unit circle with its three vertices $b_\nu$ on the circumference. If $H_1$, $H_2$, and $H_3$ are the three simply connected regions exterior to $G$, then each $H_\nu$, bounded by the Jordan curve $\Gamma_\nu$, goes into an infinite sequence of simply connected regions $L_\nu^i$ in the unit circle, where each $L_\nu^i$ is bounded by a Jordan curve $\gamma_\nu^i$ which intersects the circumference in a vertex $b_\nu$.
If we extract from the interior of the unit circle the regions $L^i_j$, then the remainder of the unit circle is a simply connected region which can be mapped one-to-one and conformally into the unit circle. Then each $\gamma^i_j$ goes into an arc $\gamma^i_j$ on the circumference. Hence we see that each Jordan curve $\Gamma^i_j$ bounding $G$ is mapped into an infinite sequence of open arcs $\gamma^i_j$ of the unit circle.

For the case $p > 3$, the nature of the mapping is obviously much more complicated, since more than three fixed points are required, and since it may be impossible to choose them so that they lie on a circle. The statement that the mapping is possible is justified by the fact that in a domain bounded by more than two points, and hence surely in a domain bounded by at least three Jordan curves, there exists a function $\omega(\xi)$ regular and schlicht and assuming every value $|\omega(\xi)| < 1$.

Now if $\omega(\xi)$ is the function defined in the points $\xi$ of the boundary of $G$ and taking the values 1 in the arcs $\alpha$ and 0 in the arcs $\beta$, we define a boundary value set $\overline{\omega}(\xi)$.

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^3 Bieberbach, L.: *Lehrbuch der Funktionentheorie*, p. 16
on the arcs $\gamma^i_J$ in which we set $\bar{\omega}(\zeta) = \omega^i_J(\zeta)$ where $\zeta$ is the point of uniqueness determined by $S$.

In order to complete the definition of $\bar{\omega}(\zeta)$, we set $\bar{\omega} = 0$ for each point of the circle $|\zeta| = 1$ not belonging to any of the open arcs $\gamma^i_J$.

We now consider the Poisson integral

$$
\bar{\omega}(x) = \frac{1}{2\pi} \int_0^{2\pi} \bar{\omega}(e^{i\theta}) \frac{1 - \lambda^2}{1 + \lambda^2 - 2\lambda \cos(e - \varphi)} \, d\theta,
$$

where $x = \lambda e^{i\varphi}$, $\zeta = e^{i\theta}$. Although the integrand displays discontinuities of a rather complicated nature, the integral nevertheless exists, since it is a convergent series, each of whose terms is an integral over a single segment $\gamma^i_J$.

The series can be shown to be, in each region interior to the unit circle $K$, absolutely and uniformly convergent, so that by the Weierstrass theorem $\bar{\omega}(x)$ is a function harmonic in $K$.

If in a neighborhood of an arbitrary point of an arc $\gamma^i_J$ there is a point of continuity of $\bar{\omega}(\zeta)$, then $\bar{\omega}(x)$ tends to this boundary value, since the integral over the arc $\gamma^i_J$ can be shown to converge to the given boundary value, whereas the other integrals vanish.

The function $\bar{\omega}(x)$ is a harmonic function which is invariant under the substitutions of the group $(S)$ of transformations with respect to which the automorphic function $x = x(\zeta)$ is defined. For if we consider an interior point $\zeta$ of an arc $\gamma^i_J$ and the corresponding Poisson
integral, then by the definition of \( \bar{\omega} \) we see that \( \bar{\omega}(e^{i\alpha}) \) remains invariant when we perform a substitution \( S \) of the group \((S)\). Since the differential

\[
\frac{1 - \lambda^2}{|1 + \lambda^2 - 2\lambda \cos(\theta - \phi)|} \, d\theta
\]

is an invariant under every transformation preserving the unit circle, then the Poisson integral is preserved by the substitution \( S \).

Now the harmonic function \( \omega(\bar{z}) = \bar{\omega}(\chi(\bar{z})) \) is single-valued in the interior of \( G \). For if a point \( \bar{z} \) be taken, going from an arbitrary point \( \bar{z}_0 \) along a closed path back to its original position, then the mapping function undergoes a substitution of the group \((S)\), and by the invariance of \( \bar{\omega} \), the terminal value of \( \omega(\bar{z}) \) will be the same as its initial value.

From the reciprocal continuity properties of the map \( G \to K \) in the boundary points of \( G \) it follows that \( \omega(\bar{z}) \to \omega(\xi) \) as \( \bar{z} \to \xi \). The harmonic function is thus a solution of our boundary-value problem.

That the function constructed by means of the Poisson integral is the only function which has the properties we require of the harmonic measure of a finite number of boundary arcs of a simply connected region \( G \) follows from an extension of the principle of the maximum and minimum for harmonic functions. It is fundamental in the theory of harmonic
functions that a function harmonic in a closed bounded
region attains its maximum and minimum values only on the
boundary of the region. Then for such a region the function
determined by the Poisson integral is the only function harmonic
in the region and taking on the boundary the values of a pre-
scribed continuous function. For if another such function
exists, the difference of the two would be harmonic in the
closed region, and would vanish in all points of the boundary.
Then zero would be both the maximum and minimum value of the
function throughout the region, so that it vanishes in the
region.

Let us consider now a function \( \mu(\bar{z}) \) harmonic in the
interior of the unit circle, bounded in the closed circle, and,
except for a finite number of points \( S_c \) of the circumference,
continuous in the closed circle. Furthermore let \( \mu(\bar{z}) \)
vanish at all points of the circumference except possibly at
the points \( S_c \).

If we consider the case in which \( \mu(\bar{z}) \) has only one
point of
discontinuity - the point \( S = 1 \) - then the behavior of \( \mu(\bar{z}) \)
for a finite number of points of discontinuity follows immediately.
Let \( S = e^{i\theta} \), \( z = \lambda e^{i\phi} \). In the closed region
\( 0 \leq \lambda \leq 1 \), \( \epsilon \leq \phi \leq 2\pi - \epsilon \), \( \mu(\bar{z}) \) is
continuous and thus uniformly continuous, so that for each
\( \epsilon > 0 \) there exists \( \lambda_\epsilon \) such that for \( \int > \lambda_\epsilon \) and

\(^h\) See, e.g., Kellogg, O.D., Foundations of Potential Theory,
p. 223.
\[ \varepsilon \leq \phi \leq 2\pi - \varepsilon, \quad |\mu(\rho e^{i\phi})| < \varepsilon. \]

Now consider any point \( \mathbb{z}_0 = \rho_0 e^{i\psi_0} \) interior to the unit circle. For each \( \varepsilon > 0 \) we choose \( \rho > \frac{\rho_0 + 1}{2} \) such that \( \rho \varepsilon < \rho < 1 \). In the closed circle \( |z| \leq \rho \), \( \mu(z) \) is harmonic, so that:

\[
\mu(z_0) = \frac{1}{2\pi} \int_0^{2\pi} \mu(\rho e^{i\theta}) \frac{\rho^2 - \rho_0^2}{\rho^2 - 2\rho_0 \rho \cos(\theta - \psi_0) + \rho_0^2} \, d\theta
\]

\[= \frac{1}{2\pi} \int_0^{2\pi-\varepsilon} \mu(\rho e^{i\theta}) P(\rho, \rho_0, \theta - \psi_0) \, d\theta + \frac{1}{2\pi} \int_{2\pi-\varepsilon}^{2\pi} \mu(\rho e^{i\theta}) P(\rho, \rho_0, \theta - \psi_0) \, d\theta.\]

In the closed unit circle \( \mu(z) \) is bounded:

\[|\mu(z)| < M. \quad \text{Since } \rho > \frac{\rho_0 + 1}{2} > \rho_0 \quad \text{and} \quad (\rho - \rho_0)^2 \leq \left[\rho^2 - 2\rho_0 \rho \cos(\theta - \psi_0) + \rho_0^2\right] \leq (\rho + \rho_0)^2\]

then

\[0 < P(\rho, \rho_0, \theta - \psi_0) < \frac{\rho^2 - \rho_0^2}{(\rho - \rho_0)^2} = \frac{\rho + \rho_0}{\rho - \rho_0} < \frac{2}{\rho - \rho_0} < \frac{d}{\rho - \rho_0}.\]

Now \( \frac{1}{2\pi} \int_0^{2\pi} P(\rho, \rho_0, \theta - \psi_0) \, d\theta = 1 \), so that:
Since we may take $\varepsilon$ arbitrarily small, we see that $\omega(\Xi_0) = 0$ for each $|\Xi_0| < 1$. Thus we have proved:

**Theorem:** If $\omega(\Xi)$ is harmonic in the interior of the unit circle, bounded in the closed circle, and, except for a finite number of points $\Xi_i$ of the circumference, continuous in the closed circle, and if $\omega(\Xi)$ vanishes at all points of the circumference except possibly at the points $\Xi_i$, then $\omega(\Xi)$ vanishes in the interior of the unit circle.

The function $\omega(\Xi)$ constructed by means of the Poisson integral, which has the required properties of the harmonic measure function for the unit circle is thus the only such function. For if another function $\omega^1(\Xi)$ exists, then the function $\omega(\Xi) = \omega(\Xi) - \omega^1(\Xi)$ satisfies all of the hypotheses of our theorem, so that $\omega(\Xi) \equiv \omega^1(\Xi)$ in the interior of the circle.

If the region $G$ is simply connected or if it is a $p$-fold connected region, then we use a similar argument for
the function determined by means of the Poisson integral in the unit circle into which the region is mapped. However for $p > 3$ the set of discontinuity points on the boundary is no longer finite. It is shown by Bieberbach (Lehrbuch der Funktionentheorie) that if a function is bounded and integrable Legesgue on the boundary of a circle and is continuous except for a set of measure zero on the boundary, then there exists one and only one function harmonic in the interior of the circle and taking on the boundary, except in the set of measure zero, the values of the given boundary function. Since, as above, we have $\omega(x) \equiv \omega_1(x)$, then

$$\omega(\bar{z}(x)) \equiv \omega_1(\bar{z}(x)).$$

Knowing that the harmonic measure of a boundary point-set with respect to a region exists and is unique for the relatively simple regions and boundary point-sets considered so far, we may now note a fundamental property of the measure function — that it is an additive set function. Let $\alpha_1$ and $\alpha_2$ be disjoint boundary sets of a region $G$. Then for each of the sets, as well as their union $\alpha_1 + \alpha_2$, there is a unique harmonic measure function defined, and it follows directly from the determination by means of the Poisson integral that

$$\omega(\bar{z}, \alpha_1, G) + \omega(\bar{z}, \alpha_2, G) = \omega(\bar{z}, \alpha_1 + \alpha_2, G).$$

If we consider a function $\omega(\bar{z})$ satisfying the hypotheses of the theorem just proved, except that it be identically
a constant $c$ in all but a finite number of points of the boundary, then by the theorem $\mu_1(\bar{z}) = \mu(\bar{z}) - c$ must vanish in the interior of the unit circle, so that $\mu(\bar{z}) \equiv c$. So if, in particular, $\alpha = \beta$, $\alpha = \beta = \pi - \alpha$, we have

$$\omega(\bar{z}, \alpha, G) + \omega(\bar{z}, \beta, G) = \omega(\bar{z}, \pi, G) \equiv 1.$$ 

Since a function which is harmonic in the interior of a region cannot have a maximum or a minimum at an interior point of the region, then for $\omega(\bar{z}, \alpha, G) \neq 1$ and $\omega(\bar{z}, \alpha, G) \neq 0$, we have for each interior to $G$, $0 < \omega < 1$.

The concept of harmonic measure which we have defined and of whose existence we have indicated the method of proof for simple regions has recently found considerable applicability in function-theory. Among the persons who have either directly or indirectly used harmonic measure are Johansson, the Riesz brothers, Carleman, Ostrowski, Julia, Warshawski, Hossjer, Beurling, and Jacqueline Ferrand.

In order to apply this idea we need to know the behavior of harmonic measure under conformal transformations of the given region $G$, i.e., how the harmonic measure with respect to a region $G$ in the $\bar{z}$-plane compares with that of a region $G^1$ in the $w$-plane in which an analytic function $\omega = \omega(\bar{z})$ defined in $G$ takes values.
If one considers a one-to-one mapping \( \omega = \omega(z) \) —
conformal in the interior of \( G \) and continuous on the boundary
\( \Gamma = \alpha + \beta \), where \( \Gamma \) is composed of a finite number
of Jordan arcs — of the region \( G + \Gamma \) into a region \( G' + \Gamma' \)
such that the arcs composing the boundary sets \( \alpha \) and \( \beta \)
are carried into corresponding arcs of the sets \( \alpha' \) and \( \beta' \),
then one sees that the harmonic measure \( \omega(z, \alpha', G') \) is
transformed into a function of the image points \( \omega' \) of \( z \).
Since the harmonic property of a function is preserved under
conformal representations, then the function \( \omega(z, \alpha', G') \)
into which \( \omega(z, \alpha, G) \) is carried is a harmonic function;
furthermore \( \omega \) is bounded in \( G' \) and takes the value 1 in
points of \( \alpha' \) and vanishes on \( \beta' \). The transformed
function is thus the harmonic measure of the set \( \alpha' \) with
respect to \( G' \) and is such that
\[
\omega(z, \alpha, G) = \omega(\omega(z), \alpha', G').
\]
Hence the harmonic measure is invariant under a one-to-one
conformal transformation of the reference region. This prop-
erty holds as well when the regions \( G \) and \( G' \) are multiply-
sheeted, i.e. Riemann surfaces.

However, under a single-valued but not one-to-one con-
formal transformations the harmonic measure in general does not
remain invariant. For example, let the unit circle \( |z| \leq 1 \),
on whose circumference we take certain arbitrary arcs as the
set \( \alpha \), be transformed by the function \( \omega = z^2 \), which
carries \(|z| \leq 1\) into \(|\omega| \leq 1\) and \(\alpha\) into certain arcs \(\alpha'\) of the circle \(|\omega| = 1\). Then in general \(\omega(\tilde{z}, \alpha) \neq \omega(\tilde{\omega}, \alpha')\) where each measure is taken with respect to the respective unit circle. For the circle \(|z| \leq 1\) will be mapped single-valuedly onto the double sheeted circular disc \(|\omega| \leq 1\); and under this mapping to an arc of \(\alpha'\), which with respect to the two-sheeted circle has the harmonic measure \(\omega(\omega, \alpha')\), there corresponds a point-set of \(|z| = 1\) which, in addition to the arc \(\alpha\), contains in general still another arc \(\overline{\alpha}\). According to the invariance of harmonic measure under one-to-one transformations, the harmonic measure of the two point-sets \(\alpha\) and \(\overline{\alpha}\) is the harmonic measure of \(\alpha'\), so that

\[\omega(\omega, \alpha') = \omega(\tilde{z}, \alpha) + \omega(\tilde{\omega}, \overline{\alpha}).\]

Since \(\omega \geq 0\), then

\[\omega(\tilde{z}, \alpha, \overline{\alpha}) \leq \omega(\omega, \alpha', \overline{\alpha'}).\]

This example illustrates the important principle of harmonic measure which states that the harmonic measure does not decrease under a conformal transformation. This principle we now state and prove under hypotheses sufficiently general to make possible many applications in function theory.

In order to prove the principle, we need the following further extension of the principle of the maximum and minimum for harmonic functions:
Lemma. If \( \varphi(z) \) is bounded below in the closed unit circle, is harmonic in the interior, and if in all but a finite number of points of the circumference \(|z| = 1\),
\[
\lim_{z \to z_0} \varphi(z) \geq 0
\]
then \( \varphi(z) \) is non-negative in the interior of the unit circle.

Proof. As in the preceding theorem, we need consider only the point \( z = 1 \) at which \( \varphi(z) \) fails to have a non-negative limit inferior. If we take \( \varepsilon > 0 \) and consider the arc \( \gamma : \varepsilon \leq \theta \leq 2\pi - \varepsilon \), then in each point \( z \) of the arc there exists a circle about \( z \) in which, for each \( z \) interior to the unit circle, \( \varphi(z) > -\varepsilon \). By the Heine-Borel theorem we may cover \( \gamma \) by a finite number of these circles, the union \( U \) of which is such that we may then take \( \lambda_\varepsilon \) sufficiently near to 1 so that the arc \( r = \lambda_\varepsilon, \varepsilon \leq \phi \leq 2\pi - \varepsilon \) is interior to \( U \).
Then for a fixed \( z_0 = \lambda_0 e^{i\theta_0} \) interior to the unit circle and for \( P > \frac{\lambda_0 + 1}{2} \), \( \lambda < P < 1 \), we have

\[
\mu(z_0) = \frac{1}{2\pi} \int_{\epsilon}^{2\pi-\epsilon} \mu(P e^{i\theta}) P(P, \lambda_0, e^{-\theta}) d\theta + \frac{1}{2\pi} \int_{\epsilon}^{2\pi-\epsilon} \mu(P e^{i\theta}) P(P, \lambda_0, e^{-\theta}) d\theta.
\]

Since \( \mu(z) \geq -|m| \) and since \( P(P, \lambda_0, e^{-\theta}) > 0 \),

\[
\mu(z_0) > -\frac{|m|}{2\pi} \int_{0}^{2\pi} P(P, \lambda_0, e^{-\theta}) d\theta - \frac{|m|}{2\pi} \int_{0}^{2\pi} P(P, \lambda_0, e^{-\theta}) d\theta
\]

\[
= -\frac{|m|}{2\pi} \left( \frac{4}{1 - \lambda_0} \right) \epsilon = -\epsilon \left[ \frac{1}{2\pi} + \frac{4|m|}{\pi(1 - \lambda_0)} \right].
\]

Since \( \epsilon \) may be taken arbitrarily small, \( \mu(z_0) > 0 \) for each \( |z_0| < 1 \).

**Principle of Harmonic Measure**: In a region \( G_\omega \) bounded by a finite number of Jordan curves \( \Gamma_\omega \) let there be given a single-valued, regular, analytic function \( \omega = \omega(\zeta) \) satisfying the conditions:

1. In \( G_\omega \), \( \omega(\zeta) \) takes values in a region \( G_\omega \) bounded by a finite number of Jordan curves \( \Gamma_\omega \).

2. In each point \( S \) of a certain subset \( \alpha_\omega \) of \( \Gamma_\omega \), \( \omega(\zeta) \) is continuous and takes its values in a sub-region \( A_\omega \) of \( G_\omega \) either bounded by or composed of a finite number of Jordan arcs \( \alpha_\omega \).
Then in each point \( z \) of \( G_z \) for which \( \omega(z) \) is exterior to \( A_{\omega} \)

\[
\omega(z, \alpha_z, G_z) = \omega(\omega(z), \alpha_{\omega}, G_{\omega}^*)
\]

where \( G_{\omega}^* \) is the sub-region \( G_{\omega} - A_{\omega} \), bounded by \( \Gamma_{\omega} \) and \( \alpha_{\omega} \).

Proof. Let \( G_{z}^* \) be that well-defined sub-region of \( G_z \) in which \( \omega(z) \) takes values exterior to \( A_{\omega} \)
and which, in addition to \( \Gamma_z \) is bounded by \( \alpha_z \), where \( \omega(z) \) takes values in the arcs \( \alpha_{\omega} \). The difference

(1) \( \mu(z) = \omega(\omega(z), \alpha_{\omega}, G_{\omega}^*) - \omega(z, \alpha_z, G_z) \)

is then harmonic and single-valued. There are four cases to consider concerning a boundary point \( z^* \) of \( G_z^* 

1. \( z^* \) is an interior point of \( G_z \) and thus

belongs to the set \( \overline{\alpha_z} \). Here \( \omega(z^*, \alpha_z, G_z) \)

is
harmonic and furthermore \( \leq 1 \). When in the interior of \( G^*_z \), \( z \to z^* \), then

\[
\omega(\omega(z), \alpha_w, G^*_w) \to 1,
\]
and since \( \omega(z^*) \)
takes a value on \( \alpha_w \), then \( \omega(\omega(z^*), \alpha_w, G^*_w) = 1 \).
Thus difference \( \omega(z) \) is continuous and non-negative at \( z = z^* \).

2. \( \z^* \) belongs to an arc \( \alpha_z \). Then as \( z \to z^* \),
both \( \omega(\omega(z), \alpha_z, G^*_z) \to 1 \) and
\( \omega(z, \alpha_z, G_z) \to 1 \), so that \( \omega(z^*) = 0 \)
and hence is non-negative.

3. \( \z^* \) belongs to an arc of \( \beta_z \), the complementary boundary set of \( \alpha_z \). Then since \( \omega(z^*, \alpha_z, G_z) = 0 \)
and \( \omega(\omega(z), \alpha_w, G^*_w) \geq 0 \),
\[ \lim_{z \to z^*} \omega(z) \geq 0. \]

4. \( \z^* \) is an abutting point of \( \alpha_z \) and \( \beta_z \). The number of these discontinuity points is finite.

Thus we see that the bounded harmonic function \( \omega(z) \)
has in all the boundary points of \( G^*_z \), except for at most a finite number, a non-negative lower bound. Then since \( \omega(z) \)
cannot take its minimum value in the interior of \( G^*_z \),
\( \omega(z) \geq 0 \) in \( G^*_z \), where \( \omega(z) \) takes values in \( G^*_w \),
and our theorem is proved:

\[
\omega(z, \alpha_z, G_z) \leq \omega(\omega(z), \alpha_w, G^*_w).
\]
The applicability of the principle of the increasing of harmonic measure is suggested by the freedom with which the configurations \( C_z, \alpha_z \) and \( C_\omega, \alpha_\omega \) can be chosen. If \( \omega(\bar{z}) \) is an arbitrary single-valued analytic function, one may exclude from its region of definition in the \( \mathbb{C} \)-plane an arbitrary system of regions or arcs and from the image-region in the \( \mathbb{W} \)-plane the images of the excluded \( \mathbb{C} \)-regions or \( \mathbb{C} \)-arcs. Then according to the principle, the harmonic measure of an arbitrary subset of the boundary arcs of the remaining regions in the \( \mathbb{C} \)-plane is at most equal to the harmonic measure of the image arcs, measured with respect to the remaining region in the \( \mathbb{W} \)-plane.

The significance of the principle becomes especially clear when one considers the level curves of the harmonic measure function. If a value \( \lambda \) of the interval \( 0 < \lambda < 1 \) is taken, we can consider the curve \( \omega(\bar{z}, \alpha_z, C_z) = \lambda \) which divides \( G_z \) into two not necessarily connected parts. That part which contains the arcs \( \alpha_z \) in its boundary is characterized by \( 0 < \omega < \lambda \), and the other part by \( 0 < \omega < 1 \). We may proceed in an analogous manner in the \( \mathbb{W} \)-plane. Then the principle of harmonic measure it is immediately seen that if the point \( \Xi \) belongs to the region whose characteristic function is \( \lambda \leq \omega(\bar{z}, \alpha_z, C_z) < 1 \) then the image point \( \omega = \omega(\Xi) \) belongs to the region described by \( \lambda \leq \omega(\omega, \alpha_\omega, C_\omega) < 1 \).
The question arises as to whether in the relationship
\[
\omega(z, \alpha_{\omega}, G_{\omega}) \leq \omega(\omega(z), \alpha_{\omega}, G_{\omega}^*)
\]
equality may occur. We have already seen that such is the case when \( w(z) \) represents a one-to-one conformal mapping. Furthermore if equality occurs at any interior point \( z \) of \( G \), then by the principle of the minimum for harmonic functions the difference
\[
\omega(\omega(z), \alpha_{\omega}, G_{\omega}^*) - \omega(z, \alpha_{\omega}, G_{\omega})
\]
must vanish throughout \( G \). In order to investigate the type of function \( w = w(z) \) for which equality holds, we consider the analytic function \( \varphi = \omega + i\overline{\omega} \), where \( \overline{\omega} \) is the conjugate harmonic function of \( \omega \), uniquely determined except for an arbitrary real constant. Then it can be shown that the equality
\[
\omega(z, \alpha_{\omega}, G_{\omega}) = \omega(\omega(z), \alpha_{\omega}, G_{\omega}^*)
\]
is equivalent to
\[
(2) \quad \varphi'(\omega, \alpha_{\omega}, G_{\omega}^*) = \varphi(z, \alpha_{\omega}, G_{\omega}) + i\mu
\]
where \( \mu \) is a real parameter. The zeros of the derivative \( \varphi'(\omega, \alpha_{\omega}, G_{\omega}^*) \) determine the behavior of \( w(z) \) in \( G_{\omega} \), and it can be shown from this that (2) holds for multiple-valued functions which are not too strongly branched.
Then the two level curves
\[ \omega(\mathbb{R}, \alpha, \mathbb{G}) = \lambda \]
and
\[ \omega(\alpha, \beta^{(0)}, \mathbb{G}^{*}) = \lambda \]
bounding the regions
\[ \lambda \leq \omega(\mathbb{R}, \alpha, \mathbb{G}) < 1 \]
\[ \lambda \leq \omega(\alpha, \beta^{(0)}, \mathbb{G}^{*}) < 1 \]
respectively correspond to each other by the one-parameter family of extremal-functions \[ \omega = \omega(\mathbb{Z}, \mu) \]
defined by (2).

We are now able to apply the principle of harmonic measure in demonstrating the validity of the so-called two-constant theorem. In the region \( G \) let a single-valued bounded analytic function \( w(z) \) be given; \( |w| \leq M \); on a given boundary set \( \alpha \) let its modulus not exceed a preassigned constant \( m < M \). Geometrically this means that for \( z \in G \) the point \( w = w(z) \) is in a circle with radius \( M \) and center at the origin, i.e., the circle \( C(O, M) \) is the region \( G \).

On \( \alpha \), furthermore, the function takes values in a concentric circle with radius \( m \), which thus becomes the region \( A \), so that \( \alpha \) is the circumference \( |w| = M \), and \( G^{*} \), the remainder of the region, in the annulus \( m < |w| < M \). The function
\[ \log \frac{M}{|w|} \]
is obviously harmonic and bounded in \( m < |w| < M \)
and takes the value 1 on \( \alpha \) and 0 on \( \beta^{(0)} : |w| = M \).

Hence it is the function \[ \omega(\alpha, \beta, \mathbb{G}^{*}) \]. Then for
\[ \lambda \leq \omega(\alpha, \beta, \mathbb{G}^{*}) < 1 \]
we have the equivalent relationship.

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\[ \lambda < \frac{\log M - \log |w|}{\log M - \log m} < 1 \]

so that, since \( \log M - \log m > 0 \),
\[ \lambda \log M - \lambda \log m < \log M - \log |w| < \log M - \log m. \]

The left-hand side of the inequality yields
\[ \log |w(z)| < (1 - \lambda) \log M + \lambda \log m. \]

On the level curve \( \omega(z, \alpha, G_\omega) = \lambda \), \( 0 < \lambda < 1 \), by the principle of harmonic measure it follows that
\[ \lambda \leq \omega(\omega, \alpha, G_\omega^*) \]
so that here
\[ \log |w(z)| \leq (1 - \lambda) \log M + \lambda \log m. \]

By the argument given above for the equality of \( \omega(z, \alpha, G_\omega) \) and \( \omega(\omega, \alpha, G_\omega^*) \), then if equality in (3) occurs at a point \( z \) it must occur for all \( z \) of \( G_\omega \), and then it can be shown that if \( \varphi(z) \) is the analytic function whose real part is \( \omega(\omega, \alpha, G) \) then \( \omega(z) \) must have the form
\[ \omega(z) = C \frac{e^{-\lambda} |w(z)|^2}{\varphi(z) (1 - \varphi(z))}. \]

Thus we have used the principle of harmonic measure to show:

The Two-Constant Theorem: If \( \omega(z) \) is a regular analytic function in a region \( G \) where \( |\omega(z)| < M \), and if on certain given boundary arcs \( \alpha \) of \( G \)
then in each point of the region \(0 < \lambda < \omega(z, \alpha, \sigma) < 1\)
\[
\log |\omega(z)| < \lambda \log m + (1-\lambda) \log M.
\]
On the level curve \(\omega(z, \alpha, \sigma) = \lambda\), \(0 < \lambda < 1\),
\[
\log |\omega(z)| \leq \lambda \log m + (1-\lambda) \log M.
\]
If equality holds for a point \(z\), then it holds for all \(z \in \Omega\)
and all \(\lambda\) \((0 \leq \lambda \leq 1)\) and \(\omega(z)\) has the form
\[
\omega(z) = e^{i\mu} m \phi(z)^{1-\phi(z)},
\]
where \(\mu\) is an arbitrary real number and \(\phi(z)\) is the
analytic function whose real part is \(\omega(z, \alpha, \sigma)\).

This theorem enables the demonstration of a property of
the maximum modulus \(M_{\lambda}\) of a function \(\omega(z)\) on the level
curves \(\omega(z, \alpha, \sigma) = \lambda\) of the region in which the
function is defined. Consider two fixed values \(\lambda_1\) and \(\lambda_2\)
\((0 \leq \lambda_1 < \lambda_2 \leq 1)\) and the corresponding region
\(\lambda_1 < \omega(z, \alpha, \sigma) < \lambda_2\).
The harmonic measure of the curve \(\lambda_2\) with respect to this
region \(\Omega'\) is obviously
\[
\omega(z, \lambda_2, \Omega') = \frac{\omega - \lambda_1}{\lambda_2 - \lambda_1},
\]
where $\omega$ is the harmonic measure of the original boundary set $\alpha$ with respect to the initial region $G$. Furthermore, in $G^1, M_{\alpha_2}, M_{\alpha_1},$ and $M_{\alpha}$ play the roles of $m, M,$ and $|\omega(z)|$ respectively in $G$, with $\omega = \lambda_1$ corresponding to $\alpha_1$, so that by the second inequality of the two-constant theorem we have for $\lambda_1 \leq \lambda \leq \lambda_2$:

$$\log M_{\lambda} = \frac{(\lambda - \lambda_1) \log M_{\lambda_2} + (\lambda_2 - \lambda) \log M_{\lambda_1}}{\lambda_2 - \lambda}.$$  

If we consider the function

$$y = \gamma(\lambda) = \log M_{\lambda}$$  

over the interval $\lambda_1 \leq \lambda \leq \lambda_2$, we see that:

The logarithm of the maximum modulus $\log M_{\lambda}$ is a convex function of the parameter $\lambda$.

From this follows immediately the three-circle theorem of Hadamard. For if we take as our region $G$ a circular annulus, then the harmonic function $\omega$ with respect to $G$ will be linear function of $\log |z|$, so that the parameter $\lambda$ may be replaced by a linear expression in $\log r$, and hence the result of the Hadamard three-circle theorem:

$$\log M(\lambda), \text{ where } M(\lambda) = \max_{|z| = \lambda} |\omega(z)|$$  

is a convex function of $\log r$. 

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Now let us consider an analytic function regular in the upper half-plane and such that in each finite point \( z = x \) of the real axis \( w(z) \) is bounded and \( \lim_{z \to x} |w(z)| \leq 1 \).

Let us denote by \( M(r) \) the maximum modulus of \( w(z) \) on the semi-circumference \( |z| = r \) in the upper half-plane. Then if we take the interior of this semi-circle to be the region \( G_z \) and the annular region between \( |w| = 1 \) and \( |w| = M(r) \), as \( G_w \), we may apply the principle of harmonic measure, letting \( \alpha_z \) be the semi-circumference.

If \( z \) is an interior point of the semi-circle, then

\[
\log \frac{z - r}{z + r}
\]

is an analytic function,

so that \( \arg \frac{z - r}{z + r} \) is harmonic. Then

\[
\mu(z) = 2 \left( 1 - \frac{1}{\pi} \arg \frac{z - r}{z + r} \right)
\]
is harmonic and takes the value 1 at interior points of \( A \): 
\[ |z| = R, \quad \mathcal{I} \varphi > 0, \]  
and the value 0 at interior points of \( B : -\pi < \varphi < \pi, \quad \mathcal{I} \varphi = 0, \quad (z = x + iy) \).  
Furthermore for \( z \) interior to \( G \),
\[ 0 < \omega(z) < 1 \]
and hence \( \omega(z) \) is the harmonic measure of \( A \) with respect to \( G \):
\[ \omega(z, A, G) = 2 \left( 1 - \frac{\varphi}{\pi} \right) = 2 \left( 1 - \frac{1}{\pi} \arctan \frac{\pi - 0}{2 + 0} \right), \]
where \( \varphi \) is the angle between the lines joining \( z \) with the endpoints of the interval \((-\pi, \pi)\) of the real axis.

Since in \( A \), \[ |\omega(z)| \leq M(\varphi), \]
we may set \( M = M(\varphi), |M| = 1 \), and apply the two-constant theorem for the level curve
\[ \omega(z, A, G) = 2 \left( 1 - \frac{\varphi}{\pi} \right) = \lambda, \quad \frac{\pi}{2} < \varphi < \pi, \]
to get
\[
(4) \quad \log |\omega(z)| \leq 2 \left( 1 - \frac{\varphi}{\pi} \right) \log M(\varphi).
\]
If the point \( z \) is held fixed, then as \( \varphi \to \infty, \varphi \to \pi \) and \[ \omega \to \mathcal{O}(1 - \frac{\varphi}{\pi}) \to 0. \]
A simple geometric exercise shows that, for \( z = x + iy \),
\[ \omega = \frac{3}{\pi} \left( \pi - \arg \frac{3-\lambda}{2-\lambda} \right) = \frac{3}{\pi} \left( \arctan \frac{4}{\lambda+x} + \arctan \frac{4}{\lambda-x} \right), \]

and since \( x \) and \( y \) are fixed, then as \( \lambda \to \infty \)

\[ \frac{4}{\lambda+x} \to 0, \quad \frac{4}{\lambda-x} \to 0 \]

so that

\[ \frac{\arctan \frac{4}{\lambda+x}}{\lambda+x} \to 1, \quad \frac{\arctan \frac{4}{\lambda-x}}{\lambda-x} \to 1, \quad \text{as} \quad \lambda \to \infty, \]

and hence

\[ \arctan \frac{4}{\lambda+x} \sim \frac{4}{\lambda+x}, \quad \text{and similarly} \]

\[ \arctan \frac{4}{\lambda-x} \sim \frac{4}{\lambda-x}. \]

Thus

\[ \omega \sim \frac{3}{\pi} \left( \frac{4}{\lambda+x} + \frac{4}{\lambda-x} \right) = \frac{4 \, y \, \lambda}{\pi (\lambda^2 - x^2)}. \]

If we set \( \nu = \lim_{\lambda \to \infty} \frac{\log M(\lambda)}{\lambda} \), then by (4)

\[ \log |\omega(z)| \leq 2 \left( 1 - \frac{\nu}{\pi} \right) \log M(\lambda) \sim \frac{4 \, y \, \lambda}{\pi (\lambda^2 - x^2)} \log M(\lambda) \]

\[ \sim \frac{4 \, y}{\pi} \log M(\lambda), \]

so that in the limit

\[ \log |\omega(z)| \leq \frac{4 \, y}{\pi}. \]

Thus we have proved:

**Theorem of Phragmen-Lindelof.** If \( \omega(z) \) is a function regular in the upper half-plane and bounded in each finite point \( z \neq x \) of the real axis, where

\[ \lim_{z \to x} |\omega(z)| \leq 1, \]

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then there are possible only two cases:

Either the modulus $|w(\bar{z})|$ becomes infinite so rapidly as $|z| \to \infty$ that

$$\lim_{r \to \infty} \frac{\log M(n)}{r}$$

is positive, or else the function is bounded, and hence

$$|w(z)| \leq 1$$

in each point $z$ of the half-plane.

An immediate consequence of this theorem is the following:

**Corollary.** If under the hypothesis of the Phragmen-Lindelof theorem, $w(z)$ tends to zero on a sequence of semicircles so that

$$\lim_{r \to \infty} \frac{\log M(n)}{r} = -\infty,$$

then $w(z)$ vanishes identically in the half-plane.

As proof we need only to note that

$$\log |w(\bar{z})| \leq \frac{4\pi}{n} \lim_{r \to \infty} \frac{\log M(n)}{r} = -\infty.$$

Another application of the harmonic measure principle results when we consider a function $w(z)$ which is bounded in the upper half-plane, e.g. $|w(z)| \leq 1$ , and which on the positive real axis is continuous and has a unique
limit, e.g. zero, as $z \to \infty$ on this half-line. Then for a given arbitrary $\varepsilon > 0$ there exists $x_0$ such that for all points $z = x > x_0$, $|\omega(z)| < \varepsilon$. Now let $G_z$ be the upper half-plane and $\alpha_z$ be the half-line $x > x_0$. Then the two-constant theorem applies, with $M = 1$ and $m = \varepsilon$, and yields

$$\log |\omega(z)| \leq \lambda \log \varepsilon$$

for each point of the region $\lambda < \omega(z, \alpha_z, G_z) < 1$.

The function

$$\omega(z) = 1 - \frac{1}{\pi} \arg(z - x_0)$$

is harmonic and bounded, $0 < \omega(z) < 1$, in the upper half-plane, takes the value 1 in $\alpha_z$: $z = x > x_0$, and the value 0 in $\beta_z$: $z = x < x_0$, and thus is the harmonic measure $\omega(z, \alpha_z, G_z)$. Thus the region $\lambda < \omega < 1$ is the region for which $\lambda < 1 - \frac{1}{\pi} \arg(z - x_0) < 1$, i.e. the angular opening in the upper half-plane between the lines $x > x_0$ and $\arg(z - x_0) = \pi(1 - \lambda)$.
If now we consider an arbitrary value \( \gamma \), \( 0 < \gamma < \pi \), then for each \( \lambda \) such that \( 0 < \lambda < \frac{\gamma}{\pi} \) then for each point \( z \) in the angle \( 0 < \arg z < \pi - \lambda \) and above the line \( \arg(z - \kappa_0) = \pi(1 - \lambda) \), the relationship (5) holds, i.e. 
\[ |\omega(z)| \leq e^\lambda, \quad \lambda \text{ arbitrary.} \]
Thus we have proved:

**Theorem:** If a function bounded in the upper half-plane converges to a limit as \( z \to \infty \) along the positive real axis, then \( \omega(z) \) tends uniformly to this limit in each angle
\[ 0 < \arg z < \pi - \gamma \quad (\gamma > 0). \]

We next show Lindelof's principle concerning the level curves of Green's function. Let \( v(z) \) be a regular single-valued analytic function in the interior of a region \( G_z \) taking values in a region \( G_w \) of the \( w \)-plane, and furthermore let \( \bar{z}_0 \) and \( \omega_0 = \omega(\bar{z}_0) \) be two fixed points and \( g(z, \bar{z}_0, G_z) \) and \( g(\omega, \omega_0, G_w) \) the Green's functions of \( G_z \) and \( G_w \) with poles at \( \bar{z}_0 \) and \( \omega_0 \) respectively.

Consider a fixed number \( \lambda > 0 \) and the subset of \( G_w \) in which \( v(z) \) takes values for \( g(z, \bar{z}_0, G_z) \geq \lambda \); denote by \( \mu \geq 0 \) the minimum of \( g(\omega, \omega_0, G_w) \) in this subset, so that for \( z \) belonging to \( g(z, \bar{z}_0, G_z) \geq \lambda \) we have corresponding values \( w(z) \) belonging to...
\( g(z, \bar{z}_0, G_{\bar{z}}) \geq \mu \). Then we can apply the principle of harmonic measure to the regions \( G_{\bar{z}} \): \( 0 < g(z, \bar{z}_0, G_{\bar{z}}) < \lambda \) and \( G_w \), in which we take the level curve \( g(z, \bar{z}_0, G_{\bar{z}}) = \lambda \) to be \( \alpha^z \) and \( g(w, \bar{w}_0, G_w) = \mu \) to be \( \beta^w \).

Obviously under these suppositions we have

\[
\omega(z, \alpha^z, G_{\bar{z}}) = \frac{1}{\lambda} g(z, \bar{z}_0, G_{\bar{z}})
\]

\[
\omega(w, \alpha^w, G_w) = \frac{1}{\mu} g(w, \bar{w}_0, G_w),
\]

and hence for \( 0 < g(z, \bar{z}_0, G_{\bar{z}}) < \lambda \)

\[
g(w, \bar{w}_0, G_w) > \frac{\mu}{\lambda} g(z, \bar{z}_0, G_{\bar{z}}).
\]

For the Green's functions we have the representations:

\[
g(z, \bar{z}_0, G_{\bar{z}}) = \log \left| \frac{1}{z - \bar{z}_0} \right| + u(z)
\]

(6)

\[
g(w, \bar{w}_0, G_w) = \log \left| \frac{1}{w - \bar{w}_0} \right| + u_w(w),
\]

where \( u \) and \( u_w \) are continuous at \( \bar{z}_0 \) and \( \bar{w}_0 \) respectively.

Since \( w(z) \) is regular at \( \bar{z}_0 \), then in the neighborhood of \( \bar{z}_0 \)

\[
\left| \frac{w - \bar{w}_0}{z - \bar{z}_0} \right|
\]

is bounded, so that from (6) it follows that as \( z \rightarrow \bar{z}_0 \)

we have:

\[
g(w, \bar{w}_0, G_w) > g(z, \bar{z}_0, G_{\bar{z}}) (1 - \varepsilon)
\]

\( \varepsilon \rightarrow 0 \Rightarrow z \rightarrow \bar{z}_0 \).

Hence:

\[
\lim_{\lambda \rightarrow \infty} \frac{\mu}{\lambda} \geq 1.
\]

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and thus
\[ g(\omega, \omega_0, G_\omega) \geq g(z, z_0, G_z). \]

So we have the:

**Principle of Lindelof.** If \( G_z \) and \( G_\omega \) are two regions and if \( w(z) \) is a single-valued meromorphic function such that for \( z \in G_z \), \( \omega_0 = w(z_0) \in G_\omega \), then for each point of the region \( g(z, z_0, G_z) > \lambda > 0 \) the corresponding point \( w(z) \) belongs to the region \( g(\omega, \omega_0, G_\omega) > \lambda \).

If we set \( z_0 = \omega_0 = 0 \), then the principle of Lindelof becomes Schwarz's lemma, which is proved most simply by the application of the maximum modulus principle to the function \( \frac{w}{z} \) regular for \( |z| < 1 \):

**Schwarz's Lemma:** If \( w(z) \) is regular and bounded for \( |z| < 1 \): 
\[ |\omega(z)| \leq |z| \]
and if \( \omega(0) = 0 \), then
\[ |\omega(z)| \leq |z| \]
for each \( |z| < 1 \). Equality holds for \( z \neq 0 \) only for \( \omega = e^{i\theta}z \) where \( \theta \) is a real constant.

The Lindelof principle and Pick's extension of Schwarz's lemma are useful in establishing the principle of hyperbolic measure, according to which the hyperbolic length of an
arbitrary curvilinear arc \( l_\alpha \) of a smooth region \( G_\alpha \) is at least as great as the hyperbolic length of the arc \( l_\omega \) into which \( l_\alpha \) is carried by an analytic function \( w(z) \) which maps \( G_\alpha \) into a smooth region \( G_\omega \).

An extension of Schwerz's lemma is the following:

**Theorem:** If \( |w(z)| \leq 1 \) for \( |z| < 1 \) and if \( w(z) \) is continuous on an arc \( \alpha_\alpha \) of the circumference and takes there values in an arc \( \alpha_\omega \) of \( |w| = 1 \), then:

An interior point \( \alpha \) of the region whose boundaries are \( \alpha_\alpha \) and the circular arc \( K_\lambda(\alpha_\alpha) \) which cuts the circumference \( |z| = 1 \) in the endpoints of \( \alpha_\alpha \) at angle \( \lambda \pi \) \((0 < \lambda < 1)\) has an image-point \( w = w(\alpha) \) a value lying interior to the region bounded by the arc \( \alpha_\omega \) and \( K_\lambda(\alpha_\omega) \):

A point of \( K_\lambda(\alpha_\alpha) \) and a point of \( K_\lambda(\alpha_\omega) \) correspond to each other if and only if \( w = w(\alpha) \) is a conformal representation of \( |z| \leq 1 \) on \( |w| \leq 1 \) which transforms \( \alpha_\alpha \) into \( \alpha_\omega \).

This theorem is a direct application of the principle of harmonic measure, since \( K_\lambda(\alpha_\alpha) \) is the level curve \( \omega(\alpha, \alpha_\alpha, G_\alpha) = \lambda \) and \( K_\lambda(\alpha_\omega) \) is the level curve \( \omega(\omega, \alpha_\omega, G_\omega) = \lambda \).

If in particular we take \( \omega_0 = 0 \) and \( \omega = 0 \), then under the hypotheses of the above theorem it follows that \( \alpha_\omega \) cannot be shorter than \( \alpha_\alpha \); for otherwise the arcs \( K_\lambda(\alpha_\omega) \) and \( \alpha_\omega \) would bound a region to which \( w = 0 \) could not
belong if \( \lambda \) is so chosen that \( k_\lambda(\alpha_\zeta) \) contains \( \zeta = 0 \). But according to the preceding theorem this is not possible.

This indicates the method of proof for the [Lemma of Lowner](#). If \( |\omega(\zeta)| < 1 \) for \( |\zeta| < 1 \) and \( \omega(0) = 0 \), then when \( w(z) \) is continuous on an arc \( \alpha \) of \( |\zeta| = 1 \) and takes values here which belong to the circumference \( |\omega| = 1 \), then the image arc \( \alpha_\omega \) of \( \alpha_\zeta \) is at least as long as \( \alpha_\zeta \). Equality holds only for

\[
\omega = e^{i\theta_\zeta}.
\]

We have already seen that the harmonic measure of the entire boundary of a region is the constant 1. If we consider the Poisson integral for the unit circle and the solution it affords for the determination of the harmonic measure of an arc \( \alpha \) from \( e^{i\theta_1} \) to \( e^{i\theta_2} \):

\[
(7) \quad \omega(\lambda e^{i\phi}, \alpha, \zeta) = \frac{1}{2\pi} \int_{\theta_1}^{\theta_2} \frac{1 - \lambda^2}{1 + \lambda^2 - 2\lambda \cos(\theta - \phi)} d\theta,
\]

then we see that for \( \zeta = 0 \), then \( r = 0 \), so that we have,

\[
\omega(0, \alpha, \zeta) = \frac{1}{2\pi} \int_{\theta_1}^{\theta_2} d\theta = \frac{\theta_2 - \theta_1}{2\pi},
\]

from which we see that the harmonic measure of an arc of the unit circle measured at the origin is simply the arc-length.
So the lemma of Lowner states that the harmonic measure \( \omega(0, \alpha^2, G_2) \) of an arc \( \alpha^2 \) of the unit circle increases under a transformation \( \omega = \omega(\omega) \) of \( |\omega| < 1 \) into \( |\omega| < 1 \), and hence that the arc-length increases under the transformation. If we consider a fixed point \( z^* = e^{i\phi} \) of \( \alpha^2 \), then according to Schwarz's principle it follows that \( w(z) \) is analytic at \( z^* \), so that if we take \( z^* \) as an end-point of an arc \( \alpha \) of \( \alpha^2 \) and denote by \( \beta \) the image arc of \( \alpha \), then for \( w(z) = e^{i\psi} \) we have

\[
\frac{\omega(\omega(\omega), \beta, G_\omega)}{\omega(\omega, \alpha, G_2)} \geq 1.
\]

Then as the arc-length of \( \alpha \) tends to zero, we see that

\[
\left[ \frac{d \omega(\omega, \beta, G_\omega)}{d \omega(\omega, \alpha, G_2)} \right] \mid_{\alpha = 0} \geq 1.
\]

Since

\[
\frac{1 - \omega^2}{1 + \omega^2 - 2\omega \cos(\theta - \psi)} = \frac{1 - |z|^2}{|e^{i\theta} - z|^2},
\]

then from (7) it follows that

\[
\frac{d \omega(\omega, \beta, G_\omega)}{d \omega(\omega, \alpha, G_2)} = \left( \frac{1 - |w|^2}{|e^{i\psi} - w|^2} \right) \left( \frac{|e^{i\theta} - z|^2}{1 - |z|^2} \right) \frac{d \psi}{d \theta},
\]

so that with (8) we have

\[
\frac{1 - |\omega|^2}{|e^{i\psi} - w|^2} d \psi \geq \frac{1 - |z|^2}{|e^{i\theta} - z|^2}.
\]
Here \( \frac{d\varphi}{d\theta} = |\omega'(z^*)| \). The geometric meaning of the inequality is the following:

When the point \( z \) is interior to the oricircle \( O_{\lambda, \theta} : \)

\[
\frac{1 - |z|^2}{|e^{i\theta} - z|^2} = \lambda \quad (0 < \lambda < \infty)
\]

then \( w = w(z) \) is in the oricircle

\[
O_{\lambda, q, \rho} : \quad \frac{1 - |w|^2}{|e^{i\varphi} - w|^2} = \lambda(q, \rho)
\]

where \( q = \frac{d\theta}{d\varphi} = \left| \frac{d\varphi}{d\omega} \right| \) for \( z = e^{i\theta}, \omega = e^{i\varphi} \).

This is, in part, the result stated in the theorem of Julia-Carathéodory.

Now let us consider a single-valued function \( w(z) \) which is \( G_z \) takes values in \( G_w \), with the following boundary restrictions: On certain boundary arcs \( \alpha_z' \) of the boundary \( w(z) \) takes values in a subset \( A_w' \) of \( G_w \); on certain other boundary arcs \( \alpha_z'' \) it takes values in \( A_w'' \subset G_w \); \( \alpha_z' \) and \( \alpha_z'' \) as well as \( A_w' \) and \( A_w'' \) are disjoint sets. We assume also that \( A_w = A_w' + A_w'' \) is either bounded by or composed of a finite number of Jordan arcs \( \alpha_w' \) or \( \alpha_w'' \).

The principle of the increasingness of harmonic measure applies to \( \alpha_z' \) and \( A_w' \), to \( \alpha_z'' \) and \( A_w'' \), and also to the
unions \( \alpha_2 = \alpha_2' + \alpha_2'' \) and \( A_\nu \). The point \( w(z) \) will therefore always be so situated that the harmonic measure of \( \alpha_w = \alpha_w' + \alpha_w'' \) is at least as large as that of \( \alpha_2' + \alpha_2'' \), measured in the point \( z \). Now let the point \( z \) vary along an arc \( \ell \) interior to \( G_2 \) joining the boundary arcs \( \alpha_2' \) and \( \alpha_2'' \). By the continuity of \( w(z) \), the image point \( w(z) \) will then describe a curve \( \ell_w \) in \( G_w \) joining \( \alpha_w' \) and \( \alpha_w'' \), and for \( z \in \ell_2 \):

\[
\omega(z, \alpha_2, \ell_2) \leq \omega(w, \alpha_w, \ell_w^w).
\]

Let \( m(\ell_2) \) be the minimum of the harmonic measure \( \omega(z, \alpha_2, \ell_2) \) on the arc \( \ell_2 \). Then \( m(\ell_2) \leq 1 \) and \( m(\ell_2) = 1 \) only if \( \alpha_2 \) compose the entire boundary \( \Gamma_2 \) of \( G_2 \), by the maximum principle for harmonic functions. Consider the collection of all the arcs \( \ell_2 \) and let

\[
m_2 = \lim m(\ell_2).
\]
Similarly let \( m(l_w) \) denote the minimum of 
\[ \omega(m, \alpha_l, \sigma_w^*) \] 
for \( m \in l_w \), and 
\[ m_w = \lim_{w \to l_w} m(l_w). \]

According to the principle of harmonic measure 
\[ m_l(z) \leq m_w, \]
so that in the limit we have 
\[ m_l(z) \leq m_w. \]

This relationship describes a significant property of 
contraction of the configuration resulting from the mapping 
of \( \alpha', \alpha'', \) and \( \sigma_z \). If the arcs \( \alpha_l^1 \) and \( \alpha_l^2 \) are relatively "close" to each other, so that in the points \( z \) of 
a path \( l_z \) joining the two the harmonic measure is at least 
\( 1 - \rho \), then the arcs \( \alpha_w^1 \) and \( \alpha_w^2 \) cannot be sufficiently 
"far" apart that in the points \( w \) of \( l_w \) the harmonic measure 
is smaller than \( 1 - \rho \).

In the following figure we have an example of a case which 
obviously cannot occur under the assumptions we have made here.
The level curves $\omega = \lambda$ of the harmonic measures of $\lambda_z' + \lambda_z''$ and of $\lambda_w' + \lambda_w''$ are shown. If the point $z$ runs from $\lambda_z'$ to $\lambda_z''$ in the interior of the region $\omega(z, \lambda_z, \gamma_z) > \lambda$, then the corresponding behavior of the point $w$ is not possible in the region $\omega(w, \lambda_w, \gamma_w^*') > \lambda$, since this region in the $w$-plane is composed of two isolated parts: the arcs $\lambda_w'$ and $\lambda_w''$ are, so to speak, too far apart.

That the ordering of the boundary by the map $w = \omega(z)$ is subject to certain restrictive conditions is well known in the theory of multiply-connected regions; two such regions in general are not carried one-to-one and conformally into each other; certain "modular conditions" must first be satisfied.

We limit ourselves here to a simple application of the concept of contraction. Suppose the arcs $\lambda_z'$ and $\lambda_z''$ have a common point $\rho_z \in \Gamma_z$ which is a possible point of discontinuity of $\omega = \omega(z)$, and consider how the corresponding regions $A_w'$ and $A_w''$ must be situated in the $w$-plane in order that the principle of the increasingness of the harmonic measure be maintained. If we take $l_z$ joining $\lambda_z'$ and $\lambda_z''$ to lie in an arbitrarily small neighborhood of $\rho_z$, then the harmonic measure in all points $z$ of $l_z$ must become arbitrarily close to 1, so that $m_z = 1$. Then since $m_w \geq m_z$, $m_w = 1$. Consequently it follows immediately that only two cases are possible:
1. The distance between the sets $A_w'$ and $A_w''$

is zero, so that $A_w'$ and $A_w''$ have a common point.

2. The arcs $A_w'$ and $A_w''$ constitute the entire

boundary of $G_w$, and thus in $G_w$, $\omega(\omega, A_w', A_w'', \epsilon_w^*) = 1$.

So if $A_w'$ and $A_w''$ are at positive distance from each other,
then if they do not constitute the entire boundary $\Gamma_w$ of
$G_w$, one of the sets, e.g. $A_w'$, must be insulated by the other,
$A_w''$, from $\beta_w = \Gamma_w - (A_w' + A_w'')$.

A simple example of this last case is given by the

exponential function $\omega = e^{z}$, $z = x + iy$. Take as the
region $G_w$ the strip $-1 < x < 1$ in the upper half-plane.
Then the region $G_w$ is composed of the annulus $\frac{1}{e} < \omega < e$.

As arc $A_w'$ take the half-line $x = -1, y \geq 0$ and as
$A_w''$, the positive imaginary axis. The two arcs meet each
other in the point $P$ at $\infty$. The point set $A_w' \equiv A_w'$

is the circumference $|\omega| = \frac{1}{e}$, while $A_w'' \equiv A_w''$

is the unit circle. Then $A_w''$, in accordance with the results
just observed, isolates $A_w'$ from the remaining boundary of
$G_w$, i.e. $|\omega| = e$.

As an application of these results, let us consider a

region $G$ whose boundary contains two Jordan curves $\alpha_1$ and $\alpha_2$

intersecting in the point $P$. Let $w(z)$ be a single-valued analytic function bounded $(|\omega| < M)$ in $G$. Furthermore

let $w(z)$ be continuous in each interior point of $\alpha_1$ and $\alpha_2$

and let $H_1$ be the set of values which $w(z)$ takes on $\alpha_1$. 

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as $\alpha \to P$, and $H_2$ the set of values which $\omega(\alpha)$ takes on $\alpha_2$ as $\alpha \to P$.

Then if $H_1$ and $H_2$ are disjoint sets they must be so situated in the $w$-plane that the boundary of one isolates the boundary of the other from the circumference $|w| = M$. In particular, then, in the limit as $\alpha \to P$, $H_1$ and $H_2$ cannot reduce to two distinct points $w_1 \neq w_2$. Hence the following convergence theorem is proved:

**Theorem of Lindelof.** If in the neighborhood $N$ of a singular point $P$ of an analytic function $w(\alpha)$, the function has two distinct limits as $\alpha \to P$ along two different Jordan arcs, then $w(\alpha)$ is unbounded in $N$.
Extension of the Definition of Harmonic Measure

So far we have considered relatively simple regions \( G \) with simple boundary sets \( \alpha \) for which the harmonic measure is defined. Let us now investigate a more general type of region - an arbitrary connected smooth region \( G \) having at least three boundary points - and let \( \alpha \) be an arbitrary point-set of the boundary \( \Gamma \) of \( G \). In order to construct the harmonic measure \( \omega(2, \alpha, G) \), that is to say a function harmonic in the interior of \( G \) which, in a sense to be explained later, takes the value 1 at points of \( \alpha \) and the value 0 at points of the remaining boundary set \( \beta \), we may use Poincare's method for the solution of the Dirichlet problem.

Construct over \( G \) the universal covering surface \( G^\infty \), which coincides with \( G \) if and only if \( G \) is simply connected and which otherwise consists of infinitely many continuous sheets, always simply connected and without interior branch points. Since \( G \) has at least three boundary points, this surface enables a one-to-one conformal representation of \( G \) on the unit circle \( |x| < 1 \). Let the mapping function be denoted by \( x = x(\xi) \). If \( G \) is multiply connected, the order of multiplicity of the function is infinite in the interior of \( G \) and its various spurs are connected by the substitutions \( S \) of a group of linear transformations \( (S) \) which leave the unit circle invariant. The inverse function \( \xi = \xi(x) \)
is, for $|\mathbf{x}| < 1$, single-valued and, with respect to the group $S$ an automorphic function.

The behavior of the boundary under this mapping has been investigated by Caratheodory and Lindelof, and is not at all simple. However, we shall consider here certain fundamental results pertinent to the behavior of the boundary.

Let $z$ be a point of the circumference $|\mathbf{x}| = 1$, and consider the totality of all the points of the circumference which correspond to $z$ under all of the substitutions of the group $(S)$. The set of all such equivalent boundary points is a countable one, which we shall designate as a class. Each point of the circumference belongs to one and only one class, and the set $M$ of all of the classes has the power of the continuum and comprises all the points of the circumference, whereas each class contains only a countable set of points.

Consider now the subset $A$ of the set $M$ of classes, such that $A$ is the collection of classes $S(z)$ for each of which the automorphic transformation function $z(\mathbf{x})$ has a well-defined limit value when the point $\mathbf{x}$ tends radially (or, equivalently, tends in an angular opening) to a point $S(z)$ of the class. This subset $A$ plays an important role in the transformation of the boundary. It can be proved that the classes which constitute the elements of $A$ can be placed in one-to-one correspondence with the accessible boundary points of the region $G$, so that the following facts hold.
1. When the point $x$ tends to a point belonging to a class $S(\xi)$ of the set $A$, then the image point $z = z(x)$ tends to the accessible boundary point $\overline{S}$ of $G$ corresponding to this class.

2. When the point $z$ tends to an accessible boundary point $\overline{S}$ along an arbitrary path in the interior of $G$ with end-point $\overline{S}$, then the point $x(z)$ (selected so as to lie in an arbitrary sheet over $z$) tends to a point of the corresponding class $S(\xi)$.

The following can further be shown about the set $A$ of classes:

1. Either the classes belonging to $A$ comprise all the points of the circumference, except for a set of measure zero, or

2. The set $A$ is itself a point-set of measure zero.

The first case occurs when the set of boundary points of $G$ is "relatively strong," i.e., when it contains an entire continuum. Such a region is referred to as one of bounded type. The corresponding automorphic transformation $z(x)$ is then of bounded characteristic and is also called a function of bounded type, since it can be expressed as the quotient of two bounded functions and thus in many respects displays a behavior analogous to that of a bounded function.

The second case occurs when the boundary point-set of $G$ is "relatively weak," for example when it is composed of only a countable number of points. A region of this type,
for which the set \( A \) of classes is a point-set of measure zero, is called one of unbounded type.

If \( G \) is a region of bounded type, the set of its accessible boundary points is thus ordered one-to-one into the set of almost all of the circumference points of the unit circle, and the accessible points of the given set are ordered into a well-defined point-set \( \alpha_x \) on the circumference. If we assume that the set \( \alpha_x \) is measurable in the Lebesgue sense, then the Poisson integral enables the construction of a function \( \mu(x) \) harmonic in the unit circle which in almost all the points of the set \( \alpha_x \) takes the boundary value 1 and in almost all points of the remaining set \( \beta_x \) of the circumference \(|x| = 1\) takes the value 0. This function is furthermore automorphic with respect to the group \((S)\). Then if we consider the transformation back to the \( z \)-plane, then it is in \( G \) the single-valued and harmonic function \( \omega(z, \alpha, \omega) \) which is defined as the harmonic measure of the set \( \alpha \) in the points \( z \) with respect to the region \( G \).

If the construction exists - i.e., if the image set is measurable Lebesgue - then the initial boundary set is said to be harmonically measurable with respect to \( G \); otherwise it is harmonically immeasurable with respect to \( G \). The harmonic measure in the extended sense then has the property that \( \omega(z, \alpha, \omega) \) tends to the limit 1 as \( z \) tends to an accessible point of \( \alpha \) along each path in the unit circular path.
A boundary point-set $\alpha$ is defined to be of harmonic measure zero if the image set $\alpha^\ast$ is of linear measure zero. In general the nullity of the harmonic measure of a set is dependent upon the region $G$ of which $\alpha$ is a boundary set; i.e. $\alpha$ may be a set of harmonic measure zero with respect to a region $G_1$ and have positive harmonic measure with respect to another region $G_2$.

A special case deserves mention, however, in which $\alpha$ is of harmonic measure zero independently of the region with respect to which the harmonic measure is taken. Let $\alpha$ be closed and of positive distance from its complementary boundary set $\beta$. Then if the harmonic measure of $\alpha$ with respect to a region $G$ vanishes, it will vanish for every region $G'$ for which $\alpha$ is of positive distance from the corresponding set $\beta'$. Hence $\alpha$ is of harmonic measure zero, independent of the region $G$. Such sets are called sets of harmonic measure zero in the absolute sense.

We have considered so far only exceptional regions which are of bounded type. It can be shown actually that regions of unbounded type are identical with those whose boundaries are point-sets of absolute harmonic measure zero. For such a region, however, the Dirichlet problem cannot be solved, so
that all that can be said from the standpoint of the theory of harmonic measure concerning regions of unbounded type is simply that their boundary point-sets are of absolute harmonic measure zero.

With the aid of the above discussed extension of the definition, the principle of harmonic measure can now be stated for quite general regions. Such a general principle leads to certain general theorems concerning the singularities of analytic functions. Among these we mention the following two:

1. When a single-valued analytic function fails, in a region whose boundary is a set of absolute harmonic measure zero, to take values in a set of positive harmonic measure in the absolute sense, then the function reduces to a constant. This result can to a certain extent be considered an extension of the theorems of Liouville and Picard.

2. When, in a region whose boundary is a point-set of positive harmonic measure in the absolute sense, a single-valued analytic function fails to take values in a set of positive harmonic measure, then the function is of bounded type, i.e., it is expressible as the quotient of two bounded functions. Such a function admits a limit in each accessible point of the boundary of the region, except at most for a set of harmonic measure zero. These results are
related to the theorems of Fatou and Riesz on the limit values of bounded functions.

One can use the theory of harmonic measure to obtain Green's function for a region $G$ of bounded type. Taking $z = \infty$ to be interior to $G$, consider a circle $|z| = \rho$ and construct the harmonic measure $\omega_{\rho}(z)$ of the circle with respect to the region of which it and the boundary of $G$ are the boundaries. Then as $\lambda \to \infty$ the expression $\omega_{\lambda} \log \lambda$ tends to Green's function for $G$, has its pole at $z = \infty$.

For a region of unbounded type Green's function does not exist. According to Szego and Myrberg this requires in part that the transfinite diameter, a concept due to Fekete, of the boundary $\Gamma$ vanish. Sets of harmonic measure zero in the absolute sense can thus be shown to be analogous to those whose transfinite diameters vanish, or, to use still another concept, to those whose boundary sets have zero capacity.

It is important to know the relationships which exist between the harmonic measure of a set $\alpha$ with respect to a region $G$ and the Euclidean properties of the geometric configurations formed by $\alpha$ and $G$. Since the harmonic measure remains invariant under a one-to-one conformal transformation, this problem is essential to that of studying the deformations of the interior and boundary of a region under such a transformation, a question which recently has taken an important place in function theory.

Of great applicability along these lines is a principle
of Carleman which states that through appropriate expansion or contraction of the given reference region, harmonic majorants or minorants can be constructed. Important results in the same field follow from the application of a method of Ahlfors, which is based on the application of considerations of arc-length and surface area. In his thesis Beurling has obtained important results by means of a power series development for the mapping function, and for certain extremal problems relating to the circle he has proved that the harmonic measure when expressed, with the aid of the Green's function, as a Stieltjes integral, represents a majorant region.

Certain results due to Lindeberg, Cartan and Ahlfors may be used to show that a point-set is of harmonic measure zero if it has logarithmic measure zero.
Bibliography

1. R. Nevanlinna, Die harmonische Mass von Punktmengen und
   seine Anwendung in der Funktionentheorie (Attonde
   Skandinaviska Matematikerkongressen, 1934).

2. R. Nevanlinna, Eindentige analytische Functionen (Berlin,
   1936).

3. R. Nevanlinna, Sur la mesure harmonique des ensembles de
   points (Comptes Rendus, July - Oct. 1934).

4. R. Nevanlinna, Sur un principe general de l'analyse
   (Comptes Rendus, July - Oct. 1934).

5. W. Threlfall, La notion de recouvrement (L'Enseignement