

## Part II

# SOME ASPECTS OF THE THEORY OF ANALYTIC FUNCTIONS OF A COMPLEX VARIABLE

### I. DETECTION OF SINGULARITIES OF AN ANALYTIC FUNCTION GIVEN BY A TAYLOR SERIES

In his lectures delivered at the Rice Institute in 1926 (15) the author gave some aspects of the general theory of singularities of an analytic function defined by its Taylor expansion.

Since the appearance of these lectures, many mathematicians have worked on this subject. We shall give here only those of the new results which are related to certain general principles discovered very recently.

Let

$$(1) \quad \sum a_n z^n \quad (z = x + iy),$$

be a Taylor series with a finite radius of convergence:  $0 < R < \infty$ . There exists a well known expression for  $R$ , called the Cauchy-Hadamard formula (15),

$$(2) \quad \frac{1}{R} = \overline{\lim}_{n \rightarrow \infty} |a_n|^{1/n}.$$

It is also well known that there exists at least one singularity of the function  $f(z)$ , obtained by analytic continuation of the series (1), which lies on the circle of convergence  $|z| = R$ . But we shall first revise the definition of a singularity, and especially that of a “*singularity of a Taylor series.*”

Let  $D$  be a domain bounded by a Jordan curve  $C$ , and let  $f(z)$  be an analytic function, holomorphic in  $D$ . A point  $z'$  of

## 120 Infinitely Differentiable Functions

$C$  will be said to be singular (or a singularity) for  $f(z)$ , if  $z_0$  being any point of  $D$ , and  $L_{z_0}^{z'}$  a curve of Jordan joining  $z_0$  to  $z'$ , and having all its points in  $D$  (except of course the point  $z'$ ), the radius of convergence of the Taylor series at a point  $\zeta$  of  $L_{z_0}^{z'}$  tends to zero, when  $\zeta$  tends (on  $L_{z_0}^{z'}$ ) to  $z'$ .

If  $z_0$  is a point of  $D$ , bounded by a simple curve of Jordan,  $C$ , if  $f(z)$  is holomorphic in  $D$ , if

$$(3) \quad \sum d_n(z-z_0)^n$$

is the Taylor series of  $f(z)$  in  $z_0$ , and if  $D$  contains the circle of convergence of (3), a singular point of  $f(z)$  on  $C$  is said to be a singular point of the series (3). A singular point of (3), on the circle of convergence, is therefore a point  $z'$ , situated on this circle, and such that the radius of convergence of the Taylor series of the function given by the sum of (3), in a point  $\zeta$ , inside this circle, tends to zero when  $\zeta$  tends to  $z'$  on a curve of Jordan lying inside the circle of convergence of (3). This definition is permissible since, if  $D_1$  and  $D_2$  are two domains, both limited by Jordan curves, respectively,  $C_1$  and  $C_2$ , containing each the circle of convergence of (3), and if  $f_1(z)$  and  $f_2(z)$  are respectively holomorphic in  $D_1$  and  $D_2$  and have, both, in  $z_0$  the Taylor expansion (3), then the singularities of the two functions on the circle of convergence of (2) are the same.  $f_1(z)$  is the analytic continuation of (3) in  $D_1$ ,  $f_2(z)$  that of (3) in  $D_2$ . If  $D = D_1 \cup D_2$  ( $D$  is composed of the points of  $D_1$  and those of  $D_2$ ), the function  $f(z)$  equal to  $f_1(z)$  in  $D_1$ , and equal to  $f_2(z)$  in  $D_2$ , is holomorphic in  $D$ . It is, in  $D$ , the analytic continuation of  $f_1(z)$ , and that of  $f_2(z)$ . It is also the analytic continuation of the sum (3) in  $D$ . The statement we have given above has to be formulated in the following manner:

*The series (1) has at least one singular point on its circle of convergence.*

## Functions of a Complex Variable 121

In other words, the smallest distance to the origin of the set of all singularities of the series (1) is equal to  $R$  given by the equality (2).

We shall now give a theorem which allows the actual determination of a singularity of a Taylor series on its circle of convergence. This theorem will furnish the smallest argument of the singularities of (3) situated on the circle of convergence. This theorem was proved by the author (13).

THEOREM I. *Let us suppose that*

$$(1) \quad f(z) = \sum_{n=0}^{\infty} a_n z^n$$

*has a finite radius of convergence,  $0 < R_0 < \infty$ , and let us put for  $h \geq 0$ ,*

$$D(h) = \overline{\lim}_{n \rightarrow \infty} |a_n h^n + C_n^1 a_1 h^{n-1} + \dots + a_n|^{1/n}.$$

*$D(h)$  is a continuous function at  $h=0$  and has a right-hand derivative at  $h=0$ ,*

$$D'_+(0) = \lim_{h \rightarrow +0} \frac{D(h) - D(0)}{h},$$

*which satisfies the inequality*

$$|D'_+(0)| \leq 1.$$

*If we write*

$$D'_+(0) = \cos \varphi,$$

*one of the two points,  $R_0 e^{i\varphi}$ ,  $R_0 e^{-i\varphi}$ , is a singular point for the series  $\sum a_n z^n$ , and precisely that of its singularities which is, on the circle of convergence  $|z| = R_0$ , the nearest to the point  $z = R_0$ .*

Let us consider the series

$$(4) \quad g_0(z) = \sum \frac{a_n}{z^n}.$$

## 122 Infinitely Differentiable Functions

If  $z_0$  is a singularity of the series  $f(z)$ ,  $z_0^{-1}$  is a singularity of  $g_0(z)$  and conversely.<sup>1</sup> The radius of convergence of (4) is equal to

$$\frac{1}{R_0} = D(0).$$

Since the function  $g_0(z) \frac{z+h}{z}$  is holomorphic, in  $(z+h)$ , for large  $|z+h|$ , and regular at infinity, we may write (for large  $|z+h|$ ),

$$(5) \quad g_0(z) \frac{z+h}{z} = \sum_{n=0}^{\infty} \frac{d_n(h)}{(z+h)^n},$$

with

$$d_n(h) = \frac{-1}{2\pi i} \oint \frac{g_0(z)}{z} (z+h)^n dz = \frac{-1}{2\pi i} \oint \frac{g_0(z)}{z} (z^n + C_1^n z^{n-1} h + \dots + h^n) dz = a_n + C_1^n a_{n-1} h + \dots + a_0 h^n.$$

The radius of convergence of the series

$$(6) \quad g_h(z) = \sum_{n=0}^{\infty} \frac{d_n(h)}{(z+h)^n}$$

is equal to  $D(h)$ .  $g_0(z)$  and  $g_h(z)$  have the same singularities, except, possibly, the point  $z_0 = -h$ , which may be a singularity for  $g_0(z)$  without being a singularity for  $g_h(z)$ .

Let us denote by  $S^+(h)$  the region  $|z+h| > D(h)$ , by  $S^-(h)$  the region  $|z+h| < D(h)$ , and by  $C(h)$  their common boundary. Since  $g_h(z)$  has all its singularities in  $\bar{S}^-(h) = S^-(h) + C(h)$ , and there is at least one singularity of  $g_h(z)$  on  $C(h)$ , we see that  $D(h)$  is positive for small positive values of  $h$ , and that the intersection

$$I(h) = S^-(h) \cap \bar{S}^-(0)$$

is not empty, and that all the singularities of  $g_0(z)$  are in  $I(h)$  closed:  $\bar{I}(h)$ .

<sup>1</sup>If  $\overline{\lim} |b_n|^{1/n} = \rho$ , the series  $F(z) = \sum \frac{b_n}{(z-t)^n}$  converges for  $|z-t| > \rho$ , and diverges for  $|z-t| < \rho$ . A point  $z'$  is a singularity for the series  $F(z)$ , if in a domain,  $D$ , bounded by a curve of Jordan,  $C$ , and containing the region  $|z-t| > \rho$ , there exists a holomorphic function  $F_1(z)$ , equal to  $F(z)$ , in  $|z-t| > \rho$ , and admitting  $z'$  as singularity ( $z'$  is then on  $C$ ).

## Functions of a Complex Variable 123

If  $z=R_0$  is a singular point of  $f(z)$ ,  $z=D(0)$  is a singular point of  $g_0(z)$ , and obviously (since  $h > 0$ ),

$$D(h) = D(0) + h.$$

Our theorem is in this case obvious. Suppose now that  $z=h_0$  is a regular point of  $f(z)$ , then  $z=D(0)$  is a regular point of  $g_0(z)$ . Since  $I(h)$  is not empty, we have

$$D(0) - h \leq D(h) \leq D(0) + h.$$

Therefore  $D(h)$  is continuous at  $h=0$ .

Let us denote by  $p=R_0e^{i\varphi}$  the singular point on  $|z|=R_0$  of  $f(z)$ , having the smallest absolute value of the argument (or one of them, if both points,  $R_0e^{i\varphi}$  and  $R_0e^{-i\varphi}$  are singular points of  $f(z)$ ). The point  $q=D(0)e^{-i\varphi}$  is the singular point of  $g_0(z)$  on  $|z|=D(0)$  with the smallest absolute value of the argument. Let  $A(h)$  denote the arc of  $C(h)$  which is inside  $C(0)$ . Since there is no singularity of  $g_h(z)$  on  $C(h) - \bar{A}(h)$ , there exists at least one singularity of  $g_h(z)$  on  $\bar{A}(h)$ . There is no singularity of  $g_0(z)$  on the part of  $C(0)$  which is outside  $C(h)$ . Since the points  $-h+D(h)e^{i\theta}$  of  $C(h)$  tend uniformly, with respect to  $\theta$ , to the points of  $C(0)$ , the points of  $A(h)$  tending to points of the arc  $D(0)e^{i\psi}$ , with  $|\psi| \leq \varphi$ , and, since the set of singularities of a series is closed, we see immediately that the points of intersection of  $C(0)$  and  $C(h)$  tend respectively to  $D(0)e^{i\varphi}$  and  $D(0)e^{-i\varphi}$ .

The circles  $C(h)$  have in common with  $C(0)$  the points  $z=x_h \pm iy_h$  where

$$x_h = \frac{D^2(h) - D^2(0)}{2h} - \frac{h}{2},$$

and, by what precedes,

$$\lim_{h \rightarrow +0} x_h = \lim_{h \rightarrow +0} \frac{D^2(h) - D^2(0)}{2h} = D(0) \lim_{h \rightarrow +0} \frac{D(h) - D(0)}{h} = D(0) \cos \varphi,$$

which proves the theorem.

## 124 Infinitely Differentiable Functions

This theorem, proved by the author in 1937 (13), has two different aspects: a geometrical one, which is independent of the form by which is expressed the greatest (or the smallest) distance of a closed set to a point lying on a given straight line, and an analytical one expressed in the simplicity of the formula for  $\cos \varphi$ .

The geometrical principle has lately been generalized by many authors: von Mises, Paul Lévy, Bouligand, Perron, Dvoretzky and others. But the most remarkable generalization of the two aspects of the theorem was given by Denjoy (6). We shall also remark that the geometrical part could have been proved by using certain results of Goutcharoff. We shall give now some parts of Denjoy's work.

### 2. GENERALIZATION OF THE GEOMETRICAL PRINCIPLE

Let  $(a, b)$  be a point varying in a closed region  $D$ , and let  $C(a, b)$  be a simple closed Jordan curve, depending on the two parameters  $a$  and  $b$ . We shall say that the curve  $C(a, b)$  varies continuously with  $a$  and  $b$ , if  $\Delta a, \Delta b$  being such that  $(a+\Delta a, b+\Delta b) \in D$ , the least upper bound of quantities  $\delta$ , such that every circle of radius  $\delta$  about a point of  $C(a, b)$  contains a point of  $C(a+\Delta a, b+\Delta b)$ , and every circle of radius  $\delta$  about a point of  $C(a+\Delta a, b+\Delta b)$ , contains a point of  $C(a, b)$ , tends to zero with  $|\Delta a| + |\Delta b|$ .

Suppose now that, if  $(a, b)$  varies in  $D$ , the set of all of the points  $(x, y)$ , of all the curves  $C(a, b)$ , is included in a closed region  $D_1$ ; let  $\Delta$  be a closed region in the four-dimensional space containing all the points which have coordinates  $(a, b, x, y)$ ,  $(a, b)$  being any point of  $D$ ,  $(x, y)$  being any point of  $D_1$ . Let now  $F(x, y; a, b)$  be a continuous function in  $\Delta$ , such that the equation of  $C(a, b)$  is given, for every  $(a, b) \in D$ , by

$$F(x, y; a, b) = 0.$$

## Generalization of the Principle 125

From the continuity of  $F(x, y; a, b)$  in  $\Delta$ , and from the fact that  $C(a, b)$  is a simple closed Jordan curve when  $(a, b)$  is in  $D$ , the point  $(x, y)$  of  $C(a, b)$  varying in  $D_1$ , it follows that  $C(a, b)$  varies continuously with  $a$  and  $b$ .

Indeed,  $(a, b)$  being a point of  $D$ , if we trace about each point,  $(x, y)$ , of  $C(a, b)$  a circle with a given radius  $\rho > 0$ , and consider, by the Borel-Lebesgue theorem, a finite number of these circles, covering  $C(a, b)$ , these circles form around  $C(a, b)$  a channel, the function  $F(x, y; a, b)$  being, for all points of  $D_1$ , which are outside this open channel, different from zero. Therefore, if  $|\Delta a| + |\Delta b|$  is sufficiently small, this is also true for  $F(x, y; a + \Delta a, b + \Delta b)$ , and the curve  $C(a + \Delta a, b + \Delta b)$  given by  $F(x, y; a + \Delta a, b + \Delta b) = 0$  is inside the channel, which proves our statement.

We shall suppose that the function  $F(x, y; a, b)$  satisfies the following conditions

- (1)  $\frac{\partial F}{\partial a}, \frac{\partial F}{\partial b}$  exist and are continuous in  $\Delta$ .
- (2)  $\frac{\partial F}{\partial b}$  is different from zero, in  $\Delta$ . We shall, for instance,

suppose that in  $\Delta$

$$\frac{\partial F}{\partial b} < 0.$$

Since, on a curve  $C(a, b)$ ,  $F(x, y; a, b) = 0$ , we see, by (2), that on the same curve we have for  $\Delta b > 0$ ,

$$F(x, y; a, b + \Delta b) < 0.$$

- (3) We shall suppose that on no arc of  $C(a, b)$  the quantity

$$\frac{\partial F}{\partial a} : \frac{\partial F}{\partial b}$$

is constant.

The quantity

$$q(x, y; a, b) = - \frac{\partial F}{\partial a} : \frac{\partial F}{\partial b}$$

will be called the rank of the point  $(x, y)$  on  $C(a, b)$ .

## 126 Infinitely Differentiable Functions

The points on a curve  $C(a, b)$  having the same rank  $q$  will be said to be conjugate to each other.

It is seen very easily that, if  $g(X)$  is a function defined for all the values  $X$  taken by  $F(x, y; a, b)$  in  $\Delta$ , continuous and differentiable for these values, and is such that  $g(0) = 0$ , then the function

$$\Phi(x, y; a, b) = g(F(x, y; a, b))$$

has the following properties: (1), (2), (3) are satisfied when  $F$  is replaced by  $\Phi$ , and  $\Phi(x, y; a, b) = 0$  defines the curve  $C(a, b)$ . The rank of every point  $(x, y)$  on  $C(a, b)$  has the same value as when defined by means of the function  $F$ . Thus the rank of a point  $(x, y)$  on  $C(a, b)$  is independent of the choice of the function  $g$ .

If there exists a one to one reciprocal correspondence between the region  $D$  where  $(a, b)$  varies and  $D'$  where  $(c, d)$  varies, the correspondence being such that the quantities  $\frac{\partial a}{\partial c}, \frac{\partial a}{\partial d}, \frac{\partial b}{\partial c}, \frac{\partial b}{\partial d}$  exist, then, with the new variables  $(c, d)$  the rank  $q_1$  is equal to

$$q_1 = \frac{q \frac{\partial a}{\partial c} - \frac{\partial b}{\partial c}}{q \frac{\partial a}{\partial d} - \frac{\partial b}{\partial d}}$$

Thus, if  $(x, y)$  and  $(x_1, y_1)$  are conjugate points on the curve  $C(a, b)$ , they are also conjugate when this curve is given with the parameters  $(c, d)$ :  $C'(c, d) = C(a(c, d), b(c, d))$ .

A necessary and sufficient condition in order that the function

$$F_1(x, y; c, d) = F(x, y; c(a, b), d(a, b))$$

be such that  $\frac{\partial F_1}{\partial d} \neq 0$  in the closed region, where the point  $(x, y, c, d)$  varies, is that in this region



## Generalization of the Principle 127

$$q \frac{\partial a}{\partial d} - \frac{\partial b}{\partial d} \neq 0.$$

Let us suppose this condition satisfied. There exist then two alternatives. From every inequality: (i) *rank of*  $(x, y)$  on  $C(a, b) > \text{rank of } (x_1, y_1)$  on  $C(a, b)$  it follows that: (ii) *rank of*  $(x, y)$  on  $C'(c, d) > \text{rank of } (x_1, y_1)$  on  $C'(c, d)$ ; or from every inequality (i) it follows that: (iii) *rank of*  $(x, y)$  on  $C'(c, d) < \text{rank of } (x_1, y_1)$  on  $C'(c, d)$ . This follows at once from the relation

$$\left| \begin{array}{cc} \frac{\partial a}{\partial c} & \frac{\partial b}{\partial c} \\ \frac{\partial a}{\partial d} & \frac{\partial b}{\partial d} \end{array} \right| \neq 0.$$

By a topological transformation of the  $xy$  plane into a  $x'y'$  plane, the curve  $C'(a, b)$  which is the transform of  $C(a, b)$  has the same equation  $F(x, y; a, b) = 0$ , where  $x$  and  $y$  are expressed as functions of  $x'$  and  $y'$ . The point  $(x', y')$ , on  $C'(a, b)$ , has the same rank as the corresponding point  $(x, y)$ , on the curve  $C(a, b)$ .

The function  $F(x, y; a, b)$  having the described properties, and  $C(a, b)$  being a simple closed Jordan curve, given as above by the equality  $F(x, y; a, b) = 0$ , let us also introduce the following definitions.

The rank of a point  $(x, y)$  on  $C(a, b)$  will be denoted by  $q(x, y; a, b)$ . If  $q(x_1, y_1; a, b) < q(x_2, y_2; a, b)$  we shall say that the point  $(x_1, y_1)$  precedes on  $C(a, b)$ , the point  $(x_2, y_2)$ , and that  $(x_2, y_2)$  follows the point  $(x_1, y_1)$ . If  $E$  is a closed set on  $C(a, b)$ , the subset of  $E$  where  $q$  has the smallest value will be said to be the *initial* set of  $E$ ; if there exists only one point having the smallest rank on  $E$ , this point will be said to be the initial point of  $E$ . The *terminal* set or point of  $E$  is the subset of  $E$  or the point of  $E$  with the greatest rank. We shall denote by  $e(\lambda) = e(\lambda, a, b)$  the set

## 128 Infinitely Differentiable Functions

of points on  $C(a, b)$  with rank  $\lambda$ . All the points of  $e(\lambda)$  are *conjugate* one to another. The subset of  $e(\lambda, a, b)$ , formed of points such that every open arc of  $C(a, b)$  containing such a point contains a point with a rank smaller than  $\lambda$ , and a point with a rank greater than  $\lambda$  will be denoted by  $e'(\lambda) = e'(\lambda, a, b)$ . The points of  $e'(\lambda)$  will be said to be *strongly conjugate*, the points of  $e(\lambda) - e'(\lambda)$  *weakly conjugate*.

$C(a, b)$  being a curve described above, let us denote by  $S(a, b)$  the region where  $F(x, y; a, b) < 0$ . On  $\overline{S(a, b)} = S(a, b) + C(a, b)$ , we have  $F(x, y; a, b) \leq 0$ , and since  $\frac{\partial F}{\partial b} < 0$ , we have in  $S(a, b)$ , if  $\Delta b > 0$ ,  $F(x, y; a, b + \Delta b) < 0$ . Thus  $S(a, b + \Delta b)$  contains  $S(a, b)$ . In other words,  $S(a, b)$ , with  $a$  fixed, increases with  $b$ . This is the reason why  $b$  will be said to be the *dilatation parameter* of  $C(a, b)$ . We shall also denote by  $S'(a, b)$  the region where  $F(x, y; a, b) > 0$ .

Let  $H$  be a closed set of points belonging to  $D_1$ . We shall suppose that there exists values of  $a$ , such that to such a value of  $a$  corresponds at least one value of  $b$ , the point  $(a, b)$  being interior to  $D$ , the curve  $C(a, b)$  containing at least one point of  $H$ ,  $S(a, b)$  containing no point of  $H$ . The set of such values  $a$  will be denoted by  $A(H)$ .

If  $H$  is bounded, and if  $C_R$  is an open circle containing  $H$ , we may equally well replace the  $xy$  plane by  $C_R$ , the closed curve  $C(a, b)$  by a simple Jordan arc  $C(a, b)$  relating two points of the circumference of  $C_R$ , the other points of  $C(a, b)$  being in  $C_R$ .

We shall now prove the following important theorem of Denjoy, which gives a theory of envelopes for a family of curves  $C(a, b)$  (6).

**THEOREM II.** *To every point  $a$  of  $A(H)$  there corresponds only one value of  $b$ ,  $b = \psi(a)$ , such that there exists at least one point of  $H$  on  $C(a, b)$ , and no point of  $H$  in  $S(a, b)$ .*

## Generalization of the Principle 129

*At every interior point  $a$  of  $A(H)$ ,  $\psi(a)$  is continuous, has a right-hand continuous right-hand derivative,  $\psi'(a+0)$ , and a left-hand continuous left-hand derivative,  $\psi'(a-0)$ . If  $H(a)$  denotes the set of points belonging to  $H$  and situated on  $\Gamma(a) = C(a, \psi(a))$ :  $H(a) = H \cap \Gamma(a)$ ,  $\psi'(a+0)$  is the rank of the initial set  $H_1(a)$  of  $H(a)$ .  $\psi'(a-0)$  is the rank of the terminal set  $H_2(a)$  of  $H(a)$ .*

*Let  $I_1(a)$  be the set of points such that in every circle about a point of the set there exists a point of intersection of  $\Gamma(a+\Delta a)$  with  $\Gamma(a)$ ,  $\Delta a$  being positive and arbitrarily small. Let  $I_2(a)$  be the set of points such that in every circle about a point of the set there exists a point of intersection of  $\Gamma(a+\Delta a)$  with  $\Gamma(a)$ ,  $\Delta a$  being negative and arbitrarily small.*

*If  $\Theta_1(a)$  denotes the subset of  $\Gamma(a)$  on which the rank is  $\psi'(a+0)$ , and  $\Theta_2(a)$  the subset of  $\Gamma(a)$  on which the rank is  $\psi'(a-0)$ , then  $\Theta_1(a)$  contains  $I_1(a)$ , and  $\Theta_2(a)$  contains  $I_2(a)$ .*

*If  $h_1(a)$  is the set of points such that, in every circle about a point of the set there exists a point of  $H(a+\Delta a)$  with  $\Delta a$  positive, arbitrarily small, and if  $h_2(a)$  is the set of points such that in every circle about a point of the set there exists a point of  $H(a+\Delta a)$  with  $\Delta a$  negative, in absolute value arbitrarily small, then  $H_1(a)$  contains  $h_1(a)$ , and  $H_2(a)$  contains  $h_2(a)$ .*

*A point of  $I_1(a)$  will be said to be a characteristic point of  $\Gamma(a)$  corresponding to  $\Delta a > 0$ , and a point of  $I_2(a)$  a characteristic point of  $\Gamma(a)$  corresponding to  $\Delta a < 0$ .*

*A point of  $\Theta_1(a)$  will be said to be a generalized characteristic point of  $\Gamma(a)$  corresponding to  $\Delta a > 0$ , and a point of  $\Theta_2(a)$  a generalized characteristic point of  $\Gamma(a)$  corresponding to  $\Delta a < 0$ .*

It follows from this theorem that

$$h_1(a) \subset H_1(a) \subset \Theta_1(a),$$

$$I_1(a) \subset \Theta_1(a),$$

$$h_2(a) \subset H_2(a) \subset \Theta_2(a),$$

$$I_2(a) \subset \Theta_2(a).$$

## 130 Infinitely Differentiable Functions

Let us now pass to the proof of Theorem II. Suppose that, on  $C(a, b)$ , there exists a point of  $H$ , and that there is no point of  $H$  in  $S(a, b)$ . Since  $S(a, b + \Delta b)$ ,  $\Delta b > 0$ , contains  $S(a, b)$ , there exists at least one point of  $H$  in  $S(a, b + \Delta b)$ . And since  $S(a, b)$  contains  $C(a, b - \Delta b)$  with  $\Delta b > 0$  (for  $S(a, b)$  contains  $\bar{S}(a, b - \Delta b)$ ) there is no point of  $H$  on  $C(a, b - \Delta b)$ . Therefore no point  $(a, b + \Delta b)$  with  $\Delta b \neq 0$  is such that there exists a point of  $H$  on  $C(a, b + \Delta b)$ , and no point of  $H$  in  $S(a, b + \Delta b)$ . Therefore the value  $b$  having the above property is unique and  $b = \psi(a)$ .

Let  $\mu_1(a, b)$  be the rank of the initial set of  $C(a, b)$ , and  $\mu_2(a, b)$  the rank of the terminal set of  $C(a, b)$ . Since  $q(x, y; a, b)$  is continuous in the closed region  $D$  (and therefore uniformly continuous) it follows at once that, if  $|\Delta a|$  and  $|\Delta b|$  are sufficiently small, then the quantities  $\mu_1(a, b)$ ,  $\mu_2(a, b)$  differ as little as we want from the corresponding quantities  $\mu_1(a + \Delta a, b + \Delta b)$ ,  $\mu_2(a + \Delta a, b + \Delta b)$ ; for, as we have seen, every point of  $C(a, b)$  is at a distance less than  $\rho$  from a point of  $C(a + \Delta a, b + \Delta b)$ , and conversely,  $\rho$  depending only on  $\eta$ , when  $|\Delta a| + |\Delta b| < \eta$ .

Let  $(a, b)$  be an interior point of  $D$ , and suppose  $|\Delta a| + |\Delta b| > 0$  is so small that the line segment which joins  $(a, b)$  to  $(a + \Delta a, b + \Delta b)$  is in  $D$ . Let us denote the quantity  $\frac{\Delta b}{\Delta a}$  by  $\tau$ . We have

$$\begin{aligned}
 \Delta F &= F(x, y; a + \Delta a, b + \Delta b) - F(x, y; a, b) \\
 (7) \quad &= \frac{\partial F}{\partial a}(x, y; \alpha, \beta) \Delta a + \frac{\partial F}{\partial b}(x, y; \alpha, \beta) \Delta b \\
 &= -\Delta a \frac{\partial F}{\partial b}(x, y; \alpha, \beta) (q(x, y; \alpha, \beta) - \tau),
 \end{aligned}$$

where  $(\alpha, \beta)$  is a point of the line segment which joins  $(a, b)$  to  $(a + \Delta a, b + \Delta b)$ .

Since  $\frac{\partial F}{\partial b} < 0$ , the sign of  $\Delta F$  is that of  $\Delta a (q(x, y; \alpha, \beta) - \tau)$ .

## Generalization of the Principle 131

A positive quantity  $\epsilon$  being given, if  $\tau < \mu_1(a, b) - \epsilon$ , we have, if  $(x, y)$  is on  $C(a, b)$ ,

$$\tau < q(x, y; a, b) - \epsilon;$$

and since, if  $|\Delta a| + |\Delta b|$  is sufficiently small,

$$|q(x, y; a, b) - q(x, y; \alpha, \beta)| < \frac{\epsilon}{2},$$

we see that, for  $|\Delta a| + |\Delta b|$  sufficiently small,

$$q(x, y; \alpha, \beta) - \tau > \frac{\epsilon}{2},$$

and  $\Delta F$  has the sign of  $\Delta a$ . But, on  $C(a, b)$ ,  $F(x, y; a, b) = 0$ , therefore, if  $\tau < \mu_1(a, b) - \epsilon$ , if  $|\Delta a| + |\Delta b|$  is sufficiently small ( $|\Delta a| > 0$ ), and if  $(x, y)$  is a point of  $C(a, b)$ , then

$$\Delta F = F(x, y; a + \Delta a, b + \Delta b) \neq 0.$$

In other words,  $C(a, b)$ , and  $C(a + \Delta a, b + \Delta b)$  have no common points. The same remark is obviously true for each case

$$(8) \quad \begin{aligned} \tau < \mu_1(a, b) - \epsilon, \quad \tau < \mu_1(a + \Delta a, b + \Delta b) - \epsilon, \\ \tau > \mu_2(a, b) + \epsilon, \quad \tau > \mu_2(a + \Delta a, b + \Delta b) + \epsilon. \end{aligned}$$

If  $\mu_1(a, b) < \lambda < \mu_2(a, b)$ , the set  $e(\lambda, a, b)$  is not empty, and the sets  $E_1(\lambda, a, b)$ ,  $E_2(\lambda, a, b)$ , which are respectively composed of points of  $C(a, b)$  where  $q < \lambda$  and  $q > \lambda$ , are not empty, and are composed, each, of a set of open intervals. We have obviously  $C(a, b) = E_1 + e + E_2$ . If, in particular,

$$0 < \epsilon < \frac{\mu_2(a, b) - \mu_1(a, b)}{4},$$

if  $|\Delta a| + |\Delta b|$  is sufficiently small, and if

$$\mu_1 + \epsilon < \tau < \mu_2 - \epsilon \quad \left( \tau = \frac{\Delta b}{\Delta a}, \Delta a \neq 0 \right),$$

then  $E_1(\tau - \epsilon, a, b)$  is not empty and is in  $S(a + \Delta a, b + \Delta b)$ , if  $\Delta a > 0$ , and, in  $S'(a + \Delta a, b + \Delta b)$ , if  $\Delta a < 0$ ;  $E_2(\tau + \epsilon, a, b)$

## 132 Infinitely Differentiable Functions

is not empty and is in  $S'(a+\Delta a, b+\Delta b)$ , if  $\Delta a > 0$ , and in  $S(a+\Delta a, b+\Delta b)$  if  $\Delta a < 0$ . Indeed the fact that, for instance,  $E_1(\tau - \epsilon, a, b)$  is in  $S(a+\Delta a, b+\Delta b)$  is seen by the following considerations: For each point of  $E_1(\tau - \epsilon, a, b)$ , we have  $q < \tau - \epsilon$ ; therefore, if  $|\Delta a| + |\Delta b|$  is sufficiently small in order that

$$|q(x, y; a, b) - q(x, y; \alpha, \beta)| < \frac{\epsilon}{2},$$

where  $\alpha, \beta$  have the same meaning as in (7), we shall have for points  $(x, y)$  of  $E_1(\tau - \epsilon, a, b)$ :

$$q(x, y; \alpha, \beta) - \tau < -\frac{\epsilon}{2},$$

and, by (7),  $\Delta F = F(x, y; a+\Delta a, b+\Delta b) < 0$  for every point  $(x, y)$ , of  $E_1(\tau - \epsilon, a, b)$ ; in other words, these points are in  $S(a+\Delta a, b+\Delta b)$ . The proofs for the other cases are analogous.

Let us denote by  $e(\tau, \epsilon, a, b)$  the closed set of points of  $C(a, b)$  for which we have

$$\tau - \epsilon \leq q(x, y; a, b) \leq \tau + \epsilon.$$

Since, between an open arc of  $E_1(\tau - \epsilon, a, b)$  and an open arc of  $E_2(\tau + \epsilon, a, b)$  there exists necessarily a closed arc belonging to  $e(\tau, \epsilon, a, b)$ , and such that one of its extremities is a frontier point of  $E_1(\tau - \epsilon, a, b)$ , and the other extremity is a frontier point of  $E_2(\tau + \epsilon, a, b)$ , we see that there exist such closed arcs, and that the curves  $C(a, b)$  and  $C(a+\Delta a, b+\Delta b)$  intersect each other, if  $|\Delta a| + |\Delta b|$  is sufficiently small, the points of intersection belonging to the part  $e'(\tau, \epsilon, a, b)$  of  $e(\tau, \epsilon, a, b)$ , composed of closed arcs having each a common frontier with  $E_1(\tau - \epsilon, a, b)$  and with  $E_2(\tau + \epsilon, a, b)$ . Every such closed arc of  $e'(\tau, \epsilon, a, b)$  contains an intersection point of the curves  $C(a, b)$ ,  $C(a+\Delta a, b+\Delta b)$ .

## Generalization of the Principle 133

Suppose now that  $\Delta a$  and  $\Delta b$  tend to zero in such a manner that  $\tau = \frac{\Delta b}{\Delta a}$  tends to  $\lambda$ , with  $\mu_1(a, b) < \lambda < \mu_2(a, b)$ .

The set  $e(\lambda, a, b)$  is obviously the intersection of all the sets  $e(\lambda, \epsilon, a, b)$ , when  $\epsilon$  takes all positive values smaller than an arbitrary given quantity. By definition of the set  $e'(\lambda, a, b)$ , every open arc of  $C(a, b)$  containing a point of the set  $e'(\lambda, a, b)$  contains, for  $\epsilon > 0$  sufficiently small, a point of  $E_1(\lambda - \epsilon, a, b)$ , and a point of  $E_2(\lambda + \epsilon, a, b)$ ; thus, such an open arc of  $C(a, b)$  contains, if  $|\Delta a|$  is sufficiently small, an intersection point of the curves  $C(a, b)$ ,  $C(a + \Delta a, b + \Delta b)$ .

Therefore, the set  $I(\lambda, a, b)$ , composed of all the points  $P$  of  $C(a, b)$  such that in every neighborhood of  $P$  there exists an intersection point of the curves  $C(a, b)$ ,  $C(a + \Delta_i a, b + \Delta_i b)$ , and this for a sequence  $\Delta_i a, \Delta_i b$  with  $\Delta_i a \rightarrow 0, \frac{\Delta_i b}{\Delta_i a} \rightarrow \lambda$ , is contained in  $e(\lambda, a, b)$ , and contains  $e'(\lambda, a, b)$ . We suppose here that  $\mu_1(a, b) < \lambda < \mu_2(a, b)$ .

We see also, by the remarks made above, that if  $\frac{\Delta b}{\Delta a} \rightarrow \lambda$  and if  $|\Delta a|$  is sufficiently small,  $E_1(\lambda - \epsilon, a, b)$  is in  $S(a + \Delta a, b + \Delta b)$  if  $\Delta a > 0$ , and in  $S'(a + \Delta a, b + \Delta b)$  if  $\Delta a < 0$ ;  $E_2(\lambda + \epsilon, a, b)$  is in  $S'(a + \Delta a, b + \Delta b)$ , if  $\Delta a > 0$ , and, in  $S(a + \Delta a, b + \Delta b)$ , if  $\Delta a < 0$ .

We are now in a position to prove systematically all the parts of Theorem II.

We have proved that the function  $\psi(a)$  exists when  $a$  belongs to  $A(H)$ . Let us now prove that this function is continuous at every interior point of  $A(H)$ . If, for a sequence  $\Delta_i a \rightarrow 0$ , we had  $\psi(a + \Delta_i a) \rightarrow b > \psi(a)$ ,  $S(a, b)$  would contain no point of  $H$ , since  $S(a + \Delta_i a, \psi(a + \Delta_i a))$  does not contain such a point, but  $S(a, b)$  contains the curve  $C(a, \psi(a)) = \Gamma(a)$ , which contains, by definition, a point of  $H$ .

## 134 Infinitely Differentiable Functions

There would also be a contradiction in supposing that  $b < \psi(a)$ . Thus  $\psi(a)$  is continuous at every interior point of  $A(H)$ .

This proves also that  $\Gamma(a)$  varies continuously with  $a$ . Every point which is such that in each of its neighborhoods there exists a point of  $H(a+\Delta a)$ , with  $\Delta a$  arbitrarily small, is therefore a point of  $\Gamma(a)$ , and, since  $H$  is a closed set, a point of  $H(a)$ .

Therefore, if we denote by  $\lambda_1(a)$  and  $\lambda_2(a)$  respectively the minimum and the maximum of the rank  $q$  on  $H(a)$ , the set of the points  $Q$  such that in every neighborhood of  $Q$  there exists a point of the interval  $[\lambda_1(a+\Delta a), \lambda_2(a+\Delta a)]$ , with  $\Delta a$  arbitrarily small, is a point of the interval  $[\lambda_1(a), \lambda_2(a)]$ . In other words, we have

$$(9) \quad \lambda_1(a) \leq \lim_{\Delta a \rightarrow 0} \lambda_1(a+\Delta a) \leq \overline{\lim}_{\Delta a \rightarrow 0} \lambda_2(a+\Delta a) \leq \lambda_2(a).$$

On the other hand, by what we have seen above, if  $\Delta a > 0$  is sufficiently small, the set

$$E_2\left(\frac{\Delta\psi}{\Delta a} + \epsilon, a + \Delta a, \psi(a + \Delta a)\right) = E_2(\tau + \epsilon, a + \Delta a, \psi(a) + \tau\Delta a)$$

$$\left(\tau = \frac{\Delta\psi}{\Delta a}\right), \text{ if it is not empty, is contained in } S(a, \psi(a)), \text{ and}$$

therefore does not contain any point of  $H$ , and, *a fortiori*, any point of  $H(a+\Delta a)$ . Therefore  $H(a+\Delta a)$  is a subset of

$$\Gamma(a) - E_2(\tau + \epsilon, a + \Delta a, \psi(a + \Delta a)).$$

In other words, the greatest rank on  $H(a+\Delta a)$  is not greater than  $\tau + \epsilon$ . Thus, for  $\Delta a > 0$  sufficiently small, we have

$$\lambda_1(a) - \epsilon < \lambda_1(a + \Delta a) \leq \lambda_2(a + \Delta a) \leq \tau + \epsilon = \frac{\Delta\psi}{\Delta a} + \epsilon,$$

$$(10) \quad \frac{\Delta\psi}{\Delta a} > \lambda_1(a) - 2\epsilon.$$



## Generalization of the Principle 135

We see also, since, if  $\Delta a > 0$  is sufficiently small,  $E_1(\tau - \epsilon, a, \psi(a))$  with  $\tau = \frac{\Delta\psi}{\Delta a}$  (if it is not empty) is contained in  $S(a + \Delta a, \psi(a + \Delta a))$ , that the smallest rank on  $H(a)$  is not smaller than  $\tau - \epsilon$ . That is to say,

$$(11) \quad \tau = \frac{\Delta\psi}{\Delta a} \leq \lambda_1(a) + \epsilon.$$

From the second equality (10) and (11) it follows at once that the right-hand derivative  $\psi'_+(a)$  of  $\psi(a)$  exists and is equal to  $\lambda_1(a)$ .

We prove in the same manner that the left-hand derivative  $\psi'_-(a)$  exists and is equal to  $\lambda_2(a)$ . It follows then, from the first inequality (10), that if  $\Delta a$  tends to zero by positive values,  $\lambda_1(a + \Delta a)$  and  $\lambda_2(a + \Delta a)$  tend, both, to  $\lambda_1(a)$ . Thus the right-hand derivative of  $\psi$  at the point  $a + \Delta a$  tends, when  $\Delta a \downarrow 0$ , to the right-hand derivative  $\psi'_+(a)$ . This right-hand derivative is continuous at right, and we have

$$\psi'_+(a) = \psi'_+(a+0) = \lambda_1(a) = \underset{(x,y) \in H(a)}{\text{Min}} q(x, y; a, \psi(a)).$$

We shall write  $\psi'(a+0)$  instead of  $\psi'_+(a+0)$  or instead of  $\psi'_+(a)$ . We can prove in the same manner that  $\psi'(a-0) = \psi'_-(0) = \psi'_-(a-0) = \lambda_2(a) = \underset{(x,y) \in H(a)}{\text{Max}} q(x, y; a, \psi(a))$ .

Now, since, when  $\Delta a \downarrow 0$ ,  $\frac{\Delta\psi}{\Delta a} = \tau$  tends to  $\psi'(a+0)$ , and since the set  $I(\psi'(a+0), a, \psi(a))$  (for the significance of  $I(\lambda, a, b)$  see above) is contained in  $\epsilon(\psi'(a+0), a, \psi(a))$ , we see that

$$\begin{aligned} I(\psi'(a+0), a, \psi(a)) &= I_1(a), \\ \epsilon(\psi'(a+0), a, \psi(a)) &= \Theta_1(a), \\ I_1(a) &\subset \Theta_1(a). \end{aligned}$$

We prove in the same manner that

$$I_2(a) \subset \Theta_2(a).$$

## 136 Infinitely Differentiable Functions

The rank of the points of  $H(a+\Delta a)$  is a quantity belonging to the interval  $[\lambda_1(a+\Delta a), \lambda_2(a+\Delta a)]$ . But we have seen that

$$\lim_{\Delta a \rightarrow +0} \lambda_1(a+\Delta a) = \lim_{\Delta a \rightarrow +0} \lambda_2(a+\Delta a) = \lambda_1(a).$$

This proves that  $h_1(a) \subset H_1(a)$ .

We can prove in the same manner that  $h_2(a) \subset H_2(a)$ . The Theorem II is therefore completely proved.

Let us now consider some applications of Theorem II.

Let  $C(h, D)$  be a circle with center at the point  $z = -h$  ( $z = x + iy$ ), and radius  $D$ . The equation of this circle may be written in the following manner ( $D$  and  $D_1$  are here the half and complete planes, respectively of the points  $(h, D)$  and  $(x, y)$ , except the points at infinity):

$$F(x, y; h, D) = (x+h)^2 + y^2 - D^2 = 0,$$

the parameter of dilatation being  $D$ , the region  $S(h, D)$ , where  $F < 0$ , is the interior of  $C(h, D)$ . The rank,  $q$ , of a point of  $C(h, D)$  is the quantity

$$-\frac{\partial F}{\partial h} : \frac{\partial F}{\partial D} = \frac{x+h}{D},$$

and therefore, if the equation of this circle is given in the form

$$z = -h + De^{i\omega},$$

we see immediately that

$$q = \frac{x+h}{D} = \cos \omega.$$

If  $H$  is a closed set of points in the  $xy$  plane, and if the point  $z = -h$  does not belong to  $H$ , there exists a value  $D$ , such that there is at least one point of  $H$  on  $C(h, D)$ , and no such point in  $S(h, D)$ . But, by Theorem II, there exists then only one such value  $D = D(h)$ . On  $\Gamma(h) = C(h, D(h))$  there exists at least one point of  $H$ , and there is no such point interior to this circle.  $D(h)$  has a right-hand deriva-

## Generalization of the Principle 137

tive  $D'_+(h) = D'(h+0)$ , and a left-hand derivative  $D'_-(h) = D'(h-0)$ , both of these being respectively right-hand and left-hand continuous with respect to  $h$ .  $D'(h+0)$  is therefore the rank of the initial set  $H_1(a)$  of  $H(h) = H \cap \Gamma(h)$ , and  $D'(h-0)$  that of the terminal set  $H_2(a)$  of  $H(h)$ . In other words, since, in writing the equation of  $\Gamma(h)$  in the form

$$z = -h + D(h)e^{i\omega},$$

the rank of a point  $z$  on  $\Gamma(h)$  is  $\cos \omega$ , we may say that the quantity  $D'(h+0)$  is equal to  $\cos \omega_1$ , where  $\omega_1$  is such that one of the two points (or both) of the form

$$z = -h + D(h)e^{\pm i\omega_1}$$

is the point of  $H$ , which is, on  $\Gamma(h)$ , the nearest to the point  $-h - D(h)$ ; and that  $D'(h-0)$  is equal to  $\cos \omega_2$ , where  $\omega_2$  is such that one of the two points (or both) of the form

$$z = -h + D(h)e^{\pm i\omega_2}$$

is the point of  $H$ , which is, on  $\Gamma(h)$ , the nearest to the point  $-h + D(h)$ .

If we put  $D = \frac{1}{L}$ , and if we write the equation of the same circle  $C(h, D)$ , which with the parameters  $h, L$ , will be denoted by  $C_1(h, L)$ , in the form

$$F_1(x, y; h, L) = \frac{1}{L^2} - (x+h)^2 - y^2 = 0,$$

then the region  $S_1(h, L)$  where  $F_1 < 0$  is the exterior region of the circle  $C_1(h, L)$ , and  $L$  is the dilatation parameter, since  $S_1(h, L + \Delta L)$  contains  $S_1(h, L)$  if  $\Delta L > 0$ . Now the rank  $q(x, y; h, L)$  is given by

$$q = -(x+h)L^3 = -L^2 \cos \omega,$$

where  $\cos \omega$  has the same meaning as above. If  $L(h)$  is such that there exists a point of  $H$  on  $C_1(h, L(h))$  and no such point in  $S_1(h, L(h))$ , then, by Theorem II,  $L'(h+0)$  and  $L'(h-0)$  exist, and, for example,  $L'(h+0)$  is equal to the

### 138 Infinitely Differentiable Functions

minimum of  $-L^2 \cos \omega$ , when the point  $-h + D(h)e^{i\omega}$  ( $D(h) = \frac{1}{L(h)}$ ) is any point of  $H$  on  $C_1(h, L(h))$ . But since  $L'(h+0) = -D'(h+0)L^2$ , we see that

$$-D'(h+0)L^2 = \min (-L^2 \cos \omega) = -L^2 \max \cos \omega, \\ (-h + D(h)e^{i\omega} \text{ on } H \cap C_1(h, L)).$$

Thus

$$(12) \quad D'(h+0) = \cos (\min |\omega|), \quad (-h + D(h)e^{i\omega} \text{ on } H \cap C(h, D)).$$

If we recall the geometrical meaning of  $D(h)$  in Theorem I, and if we take, for  $H$ , the set of singularities of the series (4), we see immediately that the last equality is exactly the Theorem I, with the difference that, in Theorem I, we give the analytic expression of  $D(h)$ , and that (12) is given for  $h=0$ .

Consider now, as curves  $C$  of Theorem II, the straight lines  $C(\theta, p)$  given by the equation

$$F(x, y; \theta, p) = p - x \cos \theta - y \sin \theta = 0,$$

$D$  and  $D_1$  being the complete planes of  $(\theta, p)$  and  $(x, y)$  except the point at infinity.<sup>1</sup> The region  $S(\theta, p)$ , where  $F < 0$ , is the open half-plane, bounded by  $C(\theta, p)$ , containing points  $(-p \cos \theta, -p \sin \theta)$  with  $p > 0$ , arbitrarily large. The dilatation parameter is  $p$ .

The rank  $q(x, y; \theta, p)$  is given by

$$(13) \quad q = y \cos \theta + x \sin \theta.$$

The affix,  $z$ , of a point on  $C(\theta, p)$  may be written in the following manner

$$z = e^{i\theta}(p + it),$$

where  $t$  is a real parameter, varying in  $(-\infty, \infty)$ . The

<sup>1</sup>In this case we should replace the  $xy$  plane by a circle  $C_R$ , and consider the corresponding sets in  $C_R$ . But since  $R$  is arbitrarily large, we neglect  $C_R$ . As it is seen below, in this case, we consider a bounded set  $H$ .

## Generalization of the Principle 139

absolute value of  $t$  is the distance from the point  $(x, y)$ ,  $(z = x + iy)$ , on  $C(\theta, p)$  to the point  $pe^{i\theta}$ —the foot of the perpendicular from the origin to the straight line  $C(\theta, p)$ .  $t$  is positive for the points of  $C(\theta, p)$  which lie on the part obtained by rotation of the ray  $z = pre^{i\theta} (r \geq 1)$  about the point  $pe^{i\theta}$  through a positive right angle. It is negative for the other points of  $C(\theta, p)$ . It follows immediately from (13) that  $q = t$ .

If  $H$  is a closed and bounded set of points, the set  $A(H)$  is composed of all the values  $\theta$  ( $-\infty < \theta < \infty$ ) (here  $a$  is replaced by  $\theta$ ). There exists, by Theorem II, a function  $p(\theta)$  such that there is no point of  $H$  in  $S(\theta, p(\theta))$ , but there exists such a point on  $\Delta(\theta) \equiv C(\theta, p(\theta))$ . By this theorem  $p'(\theta+0)$  exists and is equal to the smallest value of  $t$  such that  $z = e^{i\theta}(p(\theta) + it)$  is a point of  $\Delta(\theta)$  belonging to  $H$ ;  $p'(\theta-0)$  exists, and is equal to the greatest value of  $t$  such that  $z = e^{i\theta}(p(\theta) + it)$  is a point of  $\Delta(\theta)$ , belonging to  $H$ . Therefore the set  $H(\theta)$ , composed of the points of  $H$  situated on  $\Delta(\theta)$ , is situated between the two points

$$e^{i\theta}(p(\theta) + ip'(\theta-0)), e^{i\theta}(p(\theta) + ip'(\theta+0)),$$

these two points belonging to  $H$ .

These latter statements were first proved by Pólya (17). A great part of other important theorems of Pólya, relative to the determination of the singularities of a Taylor series (the set  $H$ ) by means of  $p(\theta)$  and its derivatives, have been proved by Denjoy, in the paper cited, starting with Theorem II. Denjoy gives also very interesting relations between the function  $D(h)$ , which appears in Theorem I and Pólya's function  $p(\theta)$ . This allows him to reestablish the relationship between the position of the singularities of the series (4) and the growth of the entire function  $F(z) = \sum \frac{a_n}{n!} z^n$ , the relationship discovered by Pólya (17).

## 140 Infinitely Differentiable Functions

We shall close our lectures with the following theorem, which is essentially based on Theorem I, and which was proved by the present author (14).

**THEOREM III.** *If the radius of convergence of the series  $\sum a_n z^n$  is equal to unity, if  $\lim_{i=\infty} |a_{n_i}|^{1/n_i} = 1$ , and if there exists a constant  $\delta > 0$ , such that in the circle  $|z| < \delta$ , no polynomial of the sequence*

$$d_{n_i}(z) = a_0 z^{n_i} + C_{n_i}^1 a_1 z^{n_i-1} + \dots + a_{n_i} \quad (i \geq 1)$$

*vanishes, then there exists at least one singular point of the Taylor series, of affix  $e^{i\varphi}$  with*

$$\cos \varphi \geq \overline{\lim}_{i=\infty} R \frac{a_{n_i-1}}{a_{n_i}}.$$

We see, by this theorem, that if  $\overline{\lim}_{i=\infty} \frac{a_{n_i-1}}{a_{n_i}} = 1$ , the other conditions of this theorem being satisfied, then the point  $z=1$  is a singular point for the Taylor series.

With the same notations as in Theorem I, we have, for  $h \geq 0$ ,

$$(14) \quad \log D(h) = \overline{\lim}_{n=\infty} \frac{\log |d_n(h)|}{n} \geq \overline{\lim}_{i=\infty} \frac{\log |d_{n_i}(h)|}{n_i}.$$

There exist two constants  $k > 0$  and  $L > 0$  such that  $|a_n| < Lk^n (n \geq 0)$ , therefore

$$|d_{n_i}(z)| = |a_0 z^{n_i} + C_{n_i}^1 a_1 z^{n_i-1} + \dots + a_{n_i}| \leq L(|z|^{n_i} + C_{n_i}^1 k |z|^{n_i-1} + \dots + k^{n_i}) = L(|z| + k)^{n_i}.$$

Since  $d_{n_i}(z)$  do not vanish in  $|z| < \delta$ , we can define  $\text{Arg } d_{n_i}(z)$  in this circle in a continuous manner by defining  $\text{Arg } d_{n_i}(0)$  as  $\text{Arg } a_{n_i}$  with the value situated in the interval  $[0, 2\pi)$ .

The functions

$$\log d_{n_i}(z) = \log |d_{n_i}(z)| + i \text{Arg } d_{n_i}(z)$$

are then holomorphic in  $|z| < \delta$ , and their real parts satisfy the inequality

$$R(\log d_{n_i}(z)) = \log |d_{n_i}(z)| \leq \log L + n_i \log(k + \delta).$$

## Generalization of the Principle 141

Thus the family of functions

$$\varphi_{n_i}(z) = \frac{\log d_{n_i}(z)}{n_i}$$

is such that their real parts are bounded, and therefore the family is normal, in  $|z| < \delta$ . It is then possible to extract from the sequence  $\{n_i\}$  a subsequence  $\{m_j\}$  having the two properties

(1) 
$$\lim_{a_{m_j}} R^{a_{m_j}-1} = \overline{\lim} R^{a_{n_i}-1} = a,$$

(2) the sequence  $\varphi_{m_j}(z) = \frac{\log d_{m_j}(z)}{m_j}$  tends in  $|z| \leq \frac{\delta}{2}$  uniformly to a limit, when  $j \rightarrow \infty$ , this limit being holomorphic in this circle, since then

$$\lim \varphi_{m_j}(0) = \lim \frac{\log a_{m_j}}{m_j} = \lim \frac{\log |a_{m_j}| + i \text{Arg } a_{m_j}}{m_j} = 0.$$

We have, on the other hand, the following equality, which is easy to prove

$$d'_n(z) = n d_{n-1}(z),$$

and in  $|z| < \delta$ , we have

$$\log d_n(z) = \log a_n + \int_0^z \frac{d'_n(t)}{d_n(t)} dt = \log a_n + n \int_0^z \frac{d_{n-1}(t)}{d_n(t)} dt.$$

We have therefore by (14), for  $0 \leq h \leq \frac{\delta}{2}$ ,

$$\log D(h) \geq \lim_{j \rightarrow \infty} \frac{\log |d_{m_j}(h)|}{m_j} = \lim \left( \frac{R \log a_{m_j}}{m_j} + R \int_0^h \frac{d_{m_j-1}(t)}{d_{m_j}(t)} dt \right).$$

But

$$\lim \frac{R \log a_{m_j}}{m_j} = \lim \frac{\log |a_{m_j}|}{m_j} = 0.$$

Therefore

$$\log D(h) \geq \lim_{j \rightarrow \infty} R \int_0^h \frac{d_{m_j-1}(t)}{d_{m_j}(t)} dt.$$

## 142 Infinitely Differentiable Functions

We have also, since  $D'_+(0)$  exists (Theorem I), and since  $D(0) = \overline{\lim} |a_n|^{1/n} = 1$ ,

$$\begin{aligned} D'_+(0) &= (\log D(h))'_{+(h=0)} = \lim_{h \rightarrow +0} \frac{\log D(h)}{h} \geq \\ \lim_{h \rightarrow +0} \frac{1}{h} \left( \lim_{j \rightarrow \infty} R \int_0^h \frac{d_{m_j-1}(t)}{d_{m_j}(t)} dt \right) &= \lim_{j \rightarrow \infty} \left( \lim_{h \rightarrow +0} \frac{1}{h} R \int_0^h \frac{d_{m_j-1}(t)}{d_{m_j}(t)} dt \right) = \\ \lim_{j \rightarrow \infty} R \frac{d_{m_j-1}(0)}{d_{m_j}(0)} &= \lim_{j \rightarrow \infty} R \frac{a_{m_j-1}}{a_{m_j}} \\ &= a = \overline{\lim} R \frac{a_{m-1}}{a_{m_i}}. \end{aligned}$$

But, by Theorem I, there exists a singularity of  $\sum a_n z^n$ , of affix  $e^{i\varphi}$ , where  $\cos \varphi = D'_+(0)$ . Our theorem is thus completely proved.

S. MANDELBROJT.

### BIBLIOGRAPHY

1. S. Bernstein, *Leçons sur les propriétés extrémales*, Paris, Gauthier-Villars, 1926.
2. Carleman, *Les fonctions quasi-analytiques*, Paris, Gauthier-Villars, 1926.
3. H. Cartan, *Actualités Scientifiques*, Paris, Hermann, 1940-1941.
4. H. Cartan et Mandelbrojt, *Acta Math.*, LXXII (1940).
5. Denjoy, *Comptes Rendus*, Paris, t. 173 (1921), p. 1329.
6. Denjoy, *Annales de l'Ecole Normale Supérieure*, t. LV, Fasc. 4 (1940).
7. Gorny, *Acta Math.*, LXXI (1939).
8. Hadamard, *Le Problème de Cauchy*, Paris, Hermann, 1932.
9. Mandelbrojt, *Séries de Fourier et classes quasi-analytiques*, Paris, Gauthier-Villars, 1935.
10. Mandelbrojt, *Sur la régularisation des fonctions*, Paris, Hermann, 1938.
11. Mandelbrojt et N. Wiener, two Notes, *Comptes Rendus*, Paris, t. 203 (1936).
12. Mandelbrojt, *Acta Math.*, LXXII (1940).
13. Mandelbrojt, *Comptes Rendus*, Paris, t. 204 (1937), p. 1456.
14. Mandelbrojt, *Comptes Rendus*, Paris, t. 206 (1938).
15. Mandelbrojt, *The Rice Institute Pamphlet*, Vol. XIV (1927).
16. Ostrowski, *Acta Math.*, LXXIII (1930).
17. Pólya, *Math. Zeitschrift*, 29 (1926), p. 550.
18. San Juan, *C. R. Congr. Intern. Math.*, Oslo, v. 2 (1936).
19. de la Vallée-Poussin, *The Rice Institute Pamphlet*, Vol. XVII (1925).
20. Valiron, *Lectures on the General Theory of Integral Functions*, Toulouse, 1923.
21. N. Wiener, *C. R. Congr. Intern. Math.*, Oslo, v. 1 (1936).