



RICE UNIVERSITY

A NON-LINEAR ELLIPTIC PROBLEM ARISING IN
PETROLEUM ENGINEERING

by

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ABSTRACT

The problem of determining the temperature distribution of a body heated by radiation is formulated as a variational nonlinear partial differential equation. An existence and uniqueness theorem is proved for this problem. It is shown that Newton's method applied to this nonlinear problem gives a sequence of linear Dirichlet problems. Convergence results for Newton's method are also derived.

CONTENTS

	page
Introduction 1: Physical Origin of the Problem	1
2. Notation and Basic Theorems	5
3. Variational Formulation	7
4. Reduction to an Unconstrained Variational Problem	16
5. Existence and Uniqueness of the Solution of the Variational Problem	20
6. A Related Linear Problem	25
7. Newton's Method and Monotone Convergence	28
8. A Necessary Condition for the Existence of a Function Satisfying Property (p)	38

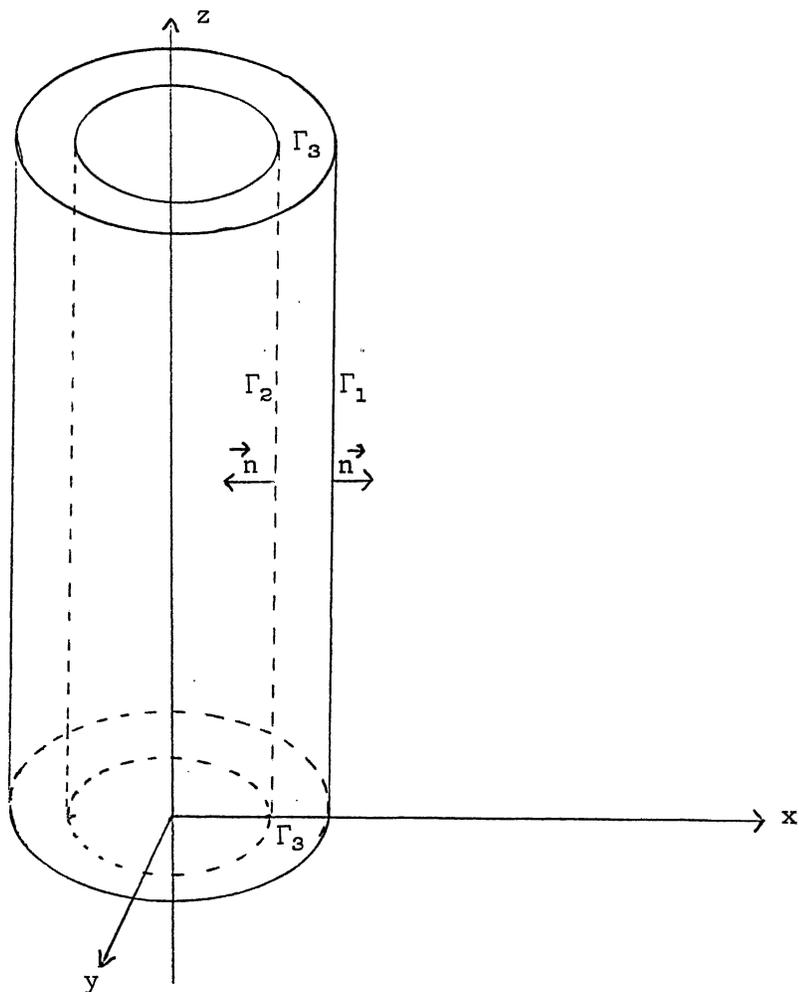
INTRODUCTION

1. PHYSICAL ORIGIN OF THE PROBLEM.

The heat equation with linear boundary conditions corresponds to the physical situation when conduction is the leading phenomenon at the surface of the studied body. However, in petroleum engineering and in other industrial applications, it is not uncommon to heat an object by means of a flame. In this case, the leading phenomenon is no longer conduction, but radiation. Consequently, since radiant energy is proportional to the fourth power of the temperature, the equations governing the distribution of the temperature are no longer linear on the boundary.

Suppose a pipe with fluid passing through it is heated. We make the following basic assumptions. On the inner surface of the pipe radiation is negligible while on the outer surface conduction is negligible. The distribution of the radiant energy on the outer surface is known and is given by a positive function. The pipe radiates energy outwardly at a rate proportional to the fourth power of the surface temperature (Stefan's Law). The temperature distribution of the fluid is known. Finally, the heat-flow through the ends of the pipe is nil, while the heat-flow on the inner surface is proportional to the difference of the temperature at the inner surface and the temperature of the fluid. The problem we now consider is that of determining the temperature distribution in the pipe and the amount of heat that effectively passes through the pipe.

The previous assumptions lead us to the following model.



Let Ω be the volume bounded by the external cylinder Γ_1 , the internal cylinder Γ_2 , and the two sections Γ_3 . Let $u(x,y,z)$ be the (unknown) temperature distribution defined in Ω . Finally, let Q be the flow of heat radiated on the pipe by the flame, Q_1 the flow of heat that passes effectively through the pipe, and Q_2 the flow of heat radiated by the pipe. These three functions are defined on the outer surface Γ_1 , and trivially satisfy

$$Q = Q_1 + Q_2$$

Furthermore, if the pipe radiates as a black body, then

$$Q_2 = \mu u^4 ,$$

where μ is a strictly positive constant. Also, the normal derivative of the temperature is proportional to the flow of heat going in, i.e.,

$$\frac{\partial u}{\partial n} = \lambda Q_1 ,$$

where n is the outer normal and λ is a strictly positive constant.

This leads to

$$\frac{\partial u}{\partial n} + \lambda \mu u^4 = \lambda Q \quad \text{on } \Gamma_1 , \quad \text{where } \lambda, \mu > 0.$$

Let $t_0(x,y,z)$ be the (known) temperature distribution of the fluid at Γ_2 . On Γ_2 the heat flow is proportional to the difference of the temperature between the pipe and the fluid, i.e.,

$$\frac{\partial u}{\partial n} = k(t_0 - u) ,$$

where k is a strictly positive constant and n the outer normal.

Since the heat flow through the two ends is nil, we have

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_3 ,$$

where again n denotes the outer normal.

As there is no heat source in the pipe itself, the temperature $u(x,y,z)$ must be a harmonic function in the pipe; therefore

$$\Delta u = 0 \quad \text{in } \Omega.$$

This leads to the system of equations

$$\begin{aligned}
 \Delta u &= 0 && \text{in } \Omega, \\
 \frac{\partial u}{\partial n} &= 0 && \text{on } \Gamma_3, \\
 (1.1) \quad \frac{\partial u}{\partial n} + \lambda \mu u^4 &= \lambda Q && \text{on } \Gamma_1, \quad \lambda, \mu > 0, \text{ and} \\
 \frac{\partial u}{\partial n} + k u &= k t_0 && \text{on } \Gamma_2, \quad k > 0,
 \end{aligned}$$

with $u(x,y,z)$ positive. The functions $Q(x,y,z)$ and $t_0(x,y,z)$ are also positive.

Remark, From the solution of this system, i.e., the temperature distribution in the pipe, we can very simply obtain the quantity of heat that passes effectively through the pipe. We merely compute the following integral

$$Q_{eff} = \int_{\Gamma_1} Q_1 \, d\Gamma = \frac{1}{\lambda} \int_{\Gamma_1} \frac{\partial u}{\partial n} \, \partial\Gamma .$$

Also, if the total amount of heat received by the pipe is

$$Q_{tot} = \int_{\Gamma_1} Q \, d\Gamma ,$$

then the heat efficiency is

$$R = \frac{Q_{eff}}{Q_{tot}} .$$

2. NOTATION AND BASIC THEOREMS.

Let Ω denote a bounded convex open subset of R^3 with boundary Γ . We call the topology induced on Γ by R^3 the Γ -topology. We make the basic assumptions on Ω and Γ .

- (i) Γ has the global cone property, see Lions [1, p. 40-48], and A.P. Calderon [1],
- (ii) Ω is locally on one side of its boundary,
- (iii) Γ is the union of three two-dimensional Γ -closed manifolds, $\tilde{\Gamma}_1, \tilde{\Gamma}_2, \tilde{\Gamma}_3$, such that their Γ -interiors, which we denote by $\Gamma_1, \Gamma_2, \Gamma_3$, respectively, have an empty intersection.
- (iv) Each Γ_i ($i = 1, 2, 3$) consists of finitely many connected domains,
- (v) $\partial\Gamma_1, \partial\Gamma_2, \partial\Gamma_3$, the boundary of $\Gamma_1, \Gamma_2, \Gamma_3$ respectively, is a C^∞ one dimensional manifold,
- (vi) the Γ -Lebesgue measure of Γ_2 is strictly positive.

The L^p ($1 < p < \infty$), L^∞ , and Sobolev spaces are defined in Lions [2, ch. 1] and Lions [1, p. 37] for the case when Ω is an open subset of R^n with a smooth boundary Γ . The generalization to the case when Ω is a C^∞ open manifold with C^∞ boundary is straightforward.

Let M denote an open C^∞ manifold (of dimension three or two) embedded in R^3 , with a C^∞ boundary ∂M . We will denote the Sobolev space of order s by $H^s(M)$.

We will need the following three theorems, given in Lions [2, ch. 1] and [1, p. 37].

Trace Theorem. Let u be a function in $H^s(\Omega)$ (with $s > \frac{1}{2}$), and σ a C^∞ Γ -open part of its boundary Γ . The value of u on σ , $u|_\sigma$, can

be defined, and is an element of $H^{s-\frac{1}{2}}(\sigma)$. The function which assigns $u|_{\sigma}$ to u is continuous from $H^s(\Omega)$ to $H^{s-\frac{1}{2}}(\sigma)$.

Sobolev Embedding Theorem. Let n be the dimension of M . Then

$$H^s(M) \subset L^q(M) \quad \text{where} \quad \frac{1}{q} = \frac{1}{2} - \frac{s}{n}$$

and the identity function from $H^s(M)$ into $L^q(M)$ is continuous.

Compactness Property of Sobolev Spaces. If M is bounded, then $H^{s-\epsilon}(M)$ is compact in $H^s(M)$, for any $\epsilon > 0$.

3. VARIATIONAL FORMULATION.

By the trace theorem, every function u of $H^1(\Omega)$ has a trace $u|_{\Gamma_1}$ on Γ_1 . This trace is a function of $H^{\frac{1}{2}}(\Gamma_1)$. Hence, we can define V as the space of functions in $H^1(\Omega)$, with trace on Γ_1 in $L^5(\Gamma_1)$, i.e.,

$$V = \{u \mid u \in H^1(\Omega), \quad u|_{\Gamma_1} \in L^5(\Gamma_1)\}$$

Remark. By the Sobolev embedding theorem, $H^{\frac{1}{2}}(\Gamma_1) \subset L^4(\Gamma_1)$. Since this theorem is "optimal", i.e., $H^{\frac{1}{2}}(\Gamma_1) \not\subset L^5(\Gamma_1)$, the space V is strictly contained in $H^1(\Omega)$.

We define the following norm on V . For $u \in V$, let

$$(3.1) \quad \|u\|_V = \|u\|_{H^1} + \|u|_{\Gamma_1}\|_{L^5(\Gamma_1)}$$

Proposition 3.1. The normed linear space V is a reflexive Banach space.

Proof. Every Cauchy sequence in V is a Cauchy sequence in both $H^1(\Omega)$ and $L^5(\Gamma_1)$; hence, since $H^1(\Omega)$ and $L^5(\Gamma_1)$ are both complete, it converges in both $H^1(\Omega)$ and $L^5(\Gamma_1)$, i.e., in V . Furthermore, the dual of V is

$$V' = (H^1)' + L^{\frac{5}{4}}(\Gamma_1).$$

For the dual of V' we obtain V again, i.e.,

$$V'' = H^1(\Omega) \cap L^5(\Gamma_1).$$

Remark. The norm (3.1) is not strictly convex. However, the equivalent norms

$$\|u\|_q = (\|u\|_{H^1}^q(\Omega) + \|u\|_{L^5}^q(\Gamma_1))^{\frac{1}{q}}$$

for every $q > 1$ are strictly convex.

Define $a(u,v)$ by

$$(3.2) \quad a(u,v) = a_1(u,v) + a_2(u,v) + a_3(u,v)$$

where

$$a_1(u,v) = \sum_{i=1}^3 \int_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} d\Omega,$$

$$a_2(u,v) = \alpha \int_{\Gamma_1} |u|^3 uv d\Gamma,$$

$$a_3(u,v) = \beta \int_{\Gamma_2} uv d\Gamma,$$

and α and β are strictly positive real constants.

Proposition 3.2. The form $a(u,v)$ is defined everywhere on V . Furthermore it is linear and continuous in the v variable.

Proof. Since V is contained in $H^1(\Omega)$, $a_1(u,v)$ is defined everywhere on V , and is linear in v . Also,

$$|a_1(u,v)| \leq \|u\|_{H^1} \|v\|_{H^1}.$$

Hence, since the norm of V dominates the norm of H^1 ,

$$|a_1(u,v)| \leq \|u\|_{H^1} \|v\|_V.$$

Now, since V is contained in $L^5(\Gamma_1)$, $a_2(u,v)$ is defined everywhere on V and is linear in v . We also have

$$\frac{1}{\alpha} |a_2(u,v)| \leq \left(\int_{\Gamma_1} (|u|^4)^{\frac{5}{4}} d\Gamma \right)^{\frac{4}{5}} \left(\int_{\Gamma_1} |v|^5 d\Gamma \right)^{\frac{1}{5}} \leq \|u\|_{L^5(\Gamma_1)}^4 \|v\|_{L^5(\Gamma_1)} .$$

So,

$$(3.3) \quad |a_2(u,v)| \leq \alpha \|u\|_{L^5(\Gamma_1)}^4 \|v\|_V$$

and $a_2(u,v)$ is continuous in v on V .

From the trace theorem, any function in $H^1(\Omega)$ is in $H^{\frac{1}{2}}(\Gamma_2)$, and by the Sobolev embedding theorem, $H^{\frac{1}{2}}(\Gamma_2)$ is contained in $L^2(\Gamma_2)$. By the continuity of the trace mapping

$$(3.4) \quad \|v\|_{L^2(\Gamma_2)} \leq C \|v\|_{H^1} ,$$

where C is a real constant.

Since u and v in V implies u and v in $L^2(\Gamma_2)$, $a_3(u,v)$ is defined everywhere on V , is linear in v and

$$|a_3(u,v)| \leq K \|u\|_{L^2(\Gamma_2)} \|v\|_{L^2(\Gamma_2)} ,$$

where K is a real constant. From relation (3.4), we have

$$(3.5) \quad |a_3(u,v)| \leq KC \|u\|_{L^2(\Gamma_2)} \|v\|_V ;$$

so $a_3(u,v)$ is continuous in v on V .

Since $a(u,v)$ is the sum of a_1 , a_2 , and a_3 the proposition is proved.

Let $L(v)$ be defined by

$$(3.6) \quad L(v) = L_1(v) + L_2(v) ,$$

where

$$L_1(v) = \int_{\Gamma_1} g_1 v d\Gamma, \quad \text{and}$$

$$L_2(v) = \int_{\Gamma_2} g_2 v d\Gamma$$

for given

$$g_1 \in H^{-\frac{1}{2}}(\Gamma_1) + L^{\frac{5}{4}}(\Gamma_1) \quad g_1 \geq 0, \quad \text{and}$$

$$g_2 \in H^{-\frac{1}{2}}(\Gamma_2) \quad g_2 \geq 0 .$$

Proposition 3.3. The form $L(v)$ is defined everywhere on V . Furthermore, on this space it is linear, continuous and positive.

Proof. If we denote by $(\cdot, \cdot)_1$ the duality product between $H^{\frac{1}{2}}(\Gamma_1) \cap L^5(\Gamma_1)$ and $H^{-\frac{1}{2}}(\Gamma_1) + L^{\frac{5}{4}}(\Gamma_1)$, and by $(\cdot, \cdot)_2$ the duality product between $H^{\frac{1}{2}}(\Gamma_2)$ and $H^{-\frac{1}{2}}(\Gamma_2)$, then L_1 and L_2 can be written as

$$L_1(v) = (g_1, v)_1 \quad \text{and}$$

$$L_2(v) = (g_2, v)_2 .$$

Since g_1 and g_2 are positive, the conclusion follows.

Let us now consider the following problem,

$$(3.7) \quad \text{Find a function } u \text{ in } V, \text{ such that } u \geq 0 \text{ and}$$

$$a(u, v) = L(v) \quad \text{for all } v \text{ in } V.$$

Problem (3.7) is the variational formulation of problem (1.1) if

$$g_1 \text{ represents } \lambda Q ,$$

$$g_2 \text{ represents } kt_0 ,$$

$$\alpha \text{ represents } \lambda\mu , \text{ and}$$

$$\beta \text{ represents } k .$$

It is obvious that these two problems are not, strictly speaking, equivalent. Specifically, we have not stated in what space we are trying to solve problem (1.1). If we choose V as this space, then problem (1.1) is not well defined, since the normal derivative on the boundary for a function in H^1 is not defined. However, we will show that any "classical"

solution of problem (1.1) is a solution of problem (3.7), and conversely if the solution of problem (3.7) is "sufficiently smooth", then it must be a solution of problem (1.1). By a "classical" solution of problem (1.1), and a "sufficiently smooth" solution of problem (3.7), we mean the solution is in $C^2(\Omega)$ (the continuous functions on Ω which have continuous first and second order partial derivatives).

It should be noted that, since the discretization will be made on problem (3.7), the regularity of g_1 and g_2 given above is sufficient to obtain a numerical solution.

We restate problem (1.1) in the equivalent form:

$$\begin{aligned}
 \Delta u &= 0 && \text{in } \Omega, \\
 \frac{\partial u}{\partial n} + \alpha |u|^3 u &= g_1 && \text{on } \Gamma_1, \\
 \frac{\partial u}{\partial n} + \beta u &= g_2 && \text{on } \Gamma_2, \\
 \frac{\partial u}{\partial n} &= 0 && \text{on } \Gamma_3, \text{ and} \\
 u &\geq 0 && \text{in } \bar{\Omega}.
 \end{aligned}
 \tag{3.8}$$

Proposition 3.4. Any classical solution of problem (3.8) is a solution of problem (3.7).

Proof. Let C^∞ be the functions defined on Ω with derivatives of all orders. Clearly, C^∞ is dense in V .

Let u be a classical solution of problem (3.7). The Laplacian of u is zero, hence for any v in C^∞

$$\int_{\Omega} \Delta u \cdot v \, d\Omega = 0$$

Since u and v are smooth, we can apply Green's formula to obtain

$$-\int_{\Omega} \Delta u \cdot v \, d\Omega = -\int_{\Gamma} \frac{\partial u}{\partial n} \cdot v \, d\Gamma + \sum_{i=1}^3 \int_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, d\Omega$$

$$= -\int_{\Gamma} \frac{\partial u}{\partial n} \cdot v \, d\Gamma + a_1(u, v)$$

So, for every v in C^{∞} , we have

$$a_1(u, v) - \int_{\Gamma} \frac{\partial u}{\partial n} \cdot v \, d\Gamma = 0 .$$

Furthermore, multiplying the remaining three relations in (3.8) by v and integrating over $\Gamma_1, \Gamma_2, \Gamma_3$ respectively gives

$$(3.9) \quad \begin{aligned} (i) \quad & \int_{\Gamma_1} \frac{\partial u}{\partial n} \cdot v \, d\Gamma + \alpha \int_{\Gamma_1} |u|^3 uv \, d\Gamma = \int_{\Gamma_1} g_1 v \, d\Gamma \\ (ii) \quad & \int_{\Gamma_2} \frac{\partial u}{\partial n} \cdot v \, d\Gamma + \beta \int_{\Gamma_2} uv \, d\Gamma = \int_{\Gamma_2} g_2 v \, d\Gamma \\ (iii) \quad & \int_{\Gamma_3} \frac{\partial u}{\partial n} \cdot v \, d\Gamma = 0 . \end{aligned}$$

Clearly (3.9) is equivalent to

$$(3.10) \quad \begin{aligned} (i) \quad & \int_{\Gamma_1} \frac{\partial u}{\partial n} \cdot v \, d\Gamma + a_2(u, v) = L_1(v) \\ (ii) \quad & \int_{\Gamma_2} \frac{\partial u}{\partial n} \cdot v \, d\Gamma + a_3(u, v) = L_2(v) \\ (iii) \quad & \int_{\Gamma_3} \frac{\partial u}{\partial n} \cdot v \, d\Gamma = 0. \end{aligned}$$

By adding the three equations (i), (ii), (iii) in (3.10) we obtain

$$\int_{\Gamma} \frac{\partial u}{\partial n} \cdot v \, d\Gamma + a_2(u, v) + a_3(u, v) = L(v)$$

However, $a_1(u, v) - \int_{\Gamma} \frac{\partial u}{\partial n} \cdot v \, d\Gamma = 0$, so

$$a_1(u, v) + a_2(u, v) + a_3(u, v) = L(v);$$

equivalently

$$(3.11) \quad a(u,v) = L(v).$$

Equation (3.11) must be satisfied by every function in C^∞ . Recalling that C^∞ is dense in V , and that $a(u,v)$ and $L(v)$ are continuous on V , we see that (3.11) is satisfied by all v in V . Since $u \geq 0$, This proves the proposition.

Proposition 3.5. Any smooth solution of problem (3.7) is a solution of problem (3.8).

Proof. Let u be a C^2 solution of problem (3.7). Let C_0^∞ be the space of infinitely differentiable functions which vanish together with their derivatives on the boundary Γ . For any function v in C_0^∞ we have

$$a(u,v) = L(v)$$

However, since v is zero together with its derivatives on Γ , we have

$$a(u,v) = a_1(u,v), \quad \text{and}$$

$$L(v) = 0.$$

So $a_1(u,v) = 0$ for any v in C_0^∞ . Hence, since the two functions are smooth, and v is zero on the boundary Γ , we can apply Green's formula to obtain

$$\int_{\Omega} \Delta u \cdot v \, d\Omega = 0.$$

Since this formula holds for any v in C_0^∞ , Δu is zero. Furthermore

$$a_1(u,v) = \int_{\Gamma} \frac{\partial u}{\partial n} \cdot v \, d\Gamma.$$

Now, let v be a function in C^∞ , which vanishes only on Γ_2 and Γ_3 .

We have

$$a_1(u,v) = \int_{\Gamma_1} \frac{\partial u}{\partial n} \cdot v \, d\Gamma,$$

$$a_2(u,v) = \alpha \int_{\Gamma_1} |u|^3 uv \, d\Gamma,$$

$$a_3(u,v) = 0,$$

$$L_1(v) = \int_{\Gamma_1} g_1 v \, d\Gamma, \quad \text{and}$$

$$L_2(v) = 0 \quad .$$

Therefore

$$\int_{\Gamma_1} \left(\frac{\partial u}{\partial n} + \alpha |u|^3 u - g_1 \right) v \, d\Gamma = 0,$$

for all v in $C_0^\infty(\Gamma_1)$. Also, on Γ_1

$$\frac{\partial u}{\partial n} + \alpha |u|^3 u = g_1 \quad .$$

Analogously, if v is a function in $C_0^\infty(\Gamma_2)$, then

$$a_1(u,v) = \int_{\Gamma_2} \frac{\partial u}{\partial n} \cdot v \, d\Gamma,$$

$$a_2(u,v) = 0,$$

$$a_3(u,v) = \beta \int_{\Gamma_2} uv \, d\Gamma,$$

$$L_1(v) = 0, \quad \text{and}$$

$$L_2(v) = \int_{\Gamma_2} g_2 v \, d\Gamma$$

Therefore

$$\int_{\Gamma_2} \left(\frac{\partial u}{\partial n} + \beta u - g_2 \right) v \, d\Gamma = 0,$$

for all functions v in $C_0^\infty(\Gamma_2)$. Hence, on Γ_2

$$\frac{\partial u}{\partial n} + ku = g_2 \quad .$$

In the same manner, taking a function v in $C_0^\infty(\Gamma_3)$ we obtain

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_3 \quad .$$

This proves the proposition.

4. REDUCTION TO AN UNCONSTRAINED VARIATIONAL PROBLEM.

We would like to reduce problem (3.6) to the form

Find a function u in V such that

$$(4.1) \quad a(u,v) = L(v) \quad \text{for all } v \text{ in } V.$$

To do this we show that any solution of (4.1) is positive, i.e., the maximum principle holds in V .

Lemma 4.1 (Maximum Principle). Any function u in V can be written in the form

$$u = u^+ - u^-$$

where

$$u^+ \in V, \quad u^+ \geq 0, \quad \text{and}$$

$$u^- \in V, \quad u^- \geq 0.$$

Proof. Define u^+ and u^- by

$$u^+ = \sup(u, 0), \quad \text{and}$$

$$u^- = \sup(-u, 0).$$

Trivially, $u = u^+ - u^-$, and from the maximum principle in H^1 , u^+ and u^- are in H^1 . Furthermore, u^+ and u^- are elements of $L^5(\Gamma_1)$. This proves the lemma.

Lemma 4.2. If σ is a Γ -open part of the boundary of Ω with strictly positive measure, then,

$$(4.2) \quad \sum_{i=1}^3 \int_{\Omega} \left(\frac{\partial u}{\partial x_i} \right)^2 d\Omega + \int_{\sigma} u^2 d\sigma \geq C \|u\|_{H^1}^2$$

where C is a real constant strictly positive which depends only on Ω and σ .

Proof. Let us define, for any distribution u , $\|u\|$ by

$$\|u\|^2 = \sum_{i=1}^3 \int_{\Omega} \left(\frac{\partial u}{\partial x_i} \right)^2 d\Omega + \int_{\sigma} u^2 d\sigma .$$

We will show that $\|u\|$ is a norm. The only property that is not immediate is

$$\|u\| = 0 \quad \text{implies} \quad u = 0 .$$

If $\|u\| = 0$, then $\int_{\Omega} \left(\frac{\partial u}{\partial x_i} \right)^2 d\Omega = 0$ for $i = 1, 2, 3$; and u is constant in Ω .

Also, since $\int_{\sigma} u^2 d\sigma = 0$, this constant must be zero. Hence $\|u\|$ is a norm on $H^1(\Omega)$. To show that the two norms $\|u\|$ and $\|u\|$ on $H^1(\Omega)$ are equivalent, it is sufficient to show that they induce the same algebraic space, i.e., the distributions bounded in the sense of $\|u\|$ are bounded in the sense of $\|u\|$.

Clearly $\|u\| \leq b$ implies

$$\sum_{i=1}^3 \int_{\Omega} \left(\frac{\partial u}{\partial x_i} \right)^2 d\Omega \leq B, \quad \text{and}$$

$$\int_{\sigma} u^2 d\sigma \leq B .$$

Also these two inequalities imply that $\int_{\Omega} u^2 dx$ is bounded. This proves the lemma.

Proposition 4.1. Any solution of problem (4.1) is positive, i.e., problems (3.7) and (4.1) are equivalent.

Proof. Suppose u is a solution of problem (4.1). By lemma 4.1, if we let $v = -u^-$ then

$$a(u, -u^-) = -L(u^-) .$$

However,

$$a_1(u^+ - u^-, -u^-) = -a_1(u^+, u^-) + a_1(u^-, u^-)$$

$$\text{and, since } a_1(u^+, u^-) = \sum_{i=1}^n \int_{\Omega} \frac{\partial u^+}{\partial x_i} \frac{\partial u^-}{\partial x_i} d\Omega = 0 ,$$

we must have

$$a_1(u, -u^-) = a_1(u^-, u^-) .$$

In the same way, we obtain

$$a_3(u, -u^-) = a_3(u^-, u^-) .$$

Furthermore,

$$\begin{aligned} a_2(u, -u^-) &= \alpha \int_{\Gamma_1} |u|^3 u(-u^-) d\Gamma \\ &= \alpha \int_{\Gamma_1} |u|^3 (u^+ - u^-)(-u^-) d\Gamma \\ &= -\alpha \int_{\Gamma_1} |u|^3 (u^+ u^-) d\Gamma + \alpha \int_{\Gamma_1} |u|^3 (u^-)^2 d\Gamma \\ &= \alpha \int_{\Gamma_1} |u|^3 (u^-)^2 d\Gamma . \end{aligned}$$

So, $a_1(u, -u^-) \geq 0$, $a_2(u, -u^-) \geq 0$, $a_3(u, -u^-) \geq 0$ and since L is positive, $L(u^-) \geq 0$. Consequently, (4.1) implies

$$(4.3) \quad \begin{aligned} a_1(u^-, u^-) &= 0 \\ a_2(u, u^-) &= 0 \end{aligned}$$

$$a_3(u^-, u^-) = 0, \quad \text{and}$$

$$L(u^-) = 0.$$

By lemma 4.2, taking $\sigma = \Gamma_2$, we have

$$a_1(u^-, u^-) + \frac{1}{\beta} a_3(u^-, u^-) \geq C \|u^-\|_{H^1}^2, \quad ,$$

so the equations (4.3) imply $u^- = 0$ in Ω . Hence $u = u^+ \geq 0$. This proves the proposition.

5. EXISTENCE AND UNIQUENESS OF THE SOLUTION OF THE VARIATIONAL PROBLEM.

We will need the following two lemmas:

Lemma 5.1. Let σ be an open submanifold of Γ . On $L^p(\sigma)$ ($1 < p < \infty$), the functional

$$\mathcal{J}(u) = \frac{1}{p} \int_{\sigma} |u|^{p-2} u^2 d\sigma$$

is strictly convex and Gâteaux differentiable. Also

$$(\mathcal{J}'(u), v) = \int_{\sigma} |u|^{p-2} uv d\sigma,$$

where (\cdot, \cdot) denotes the duality between L^p and $L^{p'}$.

Proof. Let $f(x)$ be the real function of the real variable x defined by $f(x) = \frac{1}{p} |x|^{p-2} x^2$. The function f is continuous, strictly convex and differentiable with $f'(x) = |x|^{p-2} x$.

Let u and v be two distinct functions of $L^p(\sigma)$. We have

$$\begin{aligned} & \mathcal{J}((1 - \theta)u + \theta v) - ((1 - \theta)\mathcal{J}(u) + \theta\mathcal{J}(v)) \\ &= \int_{\sigma} [f((1 - \theta)u + \theta v) - (1 - \theta)f(u) - \theta f(v)] d\sigma. \end{aligned}$$

However, since f is strictly convex, for $x \neq y$ and $\theta \in]0, 1[$, we have

$$f((1 - \theta)x + \theta y) - (1 - \theta)f(x) - \theta f(y) < 0.$$

By definition $u \neq v$ implies that $(u - v)$ is non zero on a set of non zero Lebesgue measure. Hence

$$\int_{\sigma} [f((1 - \theta)u + \theta v) - (1 - \theta)f(u) - \theta f(v)] d\sigma < 0.$$

Equivalently,

$$\mathcal{J}((1 - \theta) u + \theta v) < (1 - \theta) \mathcal{J}(u) + \theta \mathcal{J}(v)$$

for $u \neq v$ and $\theta \in]0, 1[$. This shows that \mathcal{J} is strictly convex.

Let $\Phi(\lambda)$ be a real functional on $L^p(\sigma)$ defined by

$$\begin{aligned} \Phi(\lambda) &= \frac{1}{\lambda} (\mathcal{J}(u + \lambda v) - \mathcal{J}(u)) \\ &= \int_{\sigma} \frac{1}{\lambda} (f(u + \lambda v) - f(u)) d\sigma . \end{aligned}$$

Since f is continuously differentiable, we have

$$f(x + \lambda y) = f(x) + \lambda y f'(x + \lambda_1 y) ,$$

with $\lambda_1 \in [0, \lambda]$. Hence,

$$\Phi(\lambda) = \int_{\sigma} v f'(u + \lambda_1 v) d\sigma$$

with $0 \leq \lambda_1 \leq \lambda$.

If λ converges to zero, then λ_1 also converges to zero, and since f' is a continuous real function, and the integral with measure $v d\sigma$ a continuous real functional, we have

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \Phi(\lambda) &= \int_{\sigma} v f'(u) d\sigma \\ &= \int_{\sigma} |u|^{p-2} uv d\sigma . \end{aligned}$$

Hence the derivative of \mathcal{J} exists and

$$(\mathcal{J}'(u), v) = \int_{\sigma} |u|^{p-2} uv d\sigma .$$

Lemma 5.2. If $u \mapsto \mathcal{J}(u)$ is a real functional, Gâteaux differentiable on V and strictly convex, then the mapping $u \mapsto \mathcal{J}'(u)$, of V into V' , is strictly monotone and hemicontinuous.

Proof. The monotonicity and hemicontinuity are well known, see [3, p. 158].

We have to prove the strict monotonicity. We have from the monotonicity that

$$(\mathcal{J}'(u), v - u) \leq \mathcal{J}(v) - \mathcal{J}(u).$$

Let φ be the real function of the real variable θ defined by

$$\varphi(\theta) = \mathcal{J}(u + \theta(v - u))$$

for all u, v , such that $u \neq v$, and for θ in the interval $[0, 1]$.

From lemma 5.1, the function φ is continuous, differentiable, strictly convex, and satisfies

$$\varphi'(0) = (\mathcal{J}'(u), v - u)$$

$$\varphi(0) = \mathcal{J}(u), \text{ and}$$

$$\varphi(1) = \mathcal{J}(v).$$

Suppose

$$(\mathcal{J}'(u), v - u) = \mathcal{J}(v) - \mathcal{J}(u).$$

It follows that

$$\varphi'(0) = \varphi(1) - \varphi(0).$$

However, this is impossible for a strictly convex real function. Hence,

for $u \neq v$

$$(\mathcal{J}'(u), v - u) < \mathcal{J}(v) - \mathcal{J}(u).$$

Similarly

$$(\mathcal{J}'(v), u - v) < \mathcal{J}(u) - \mathcal{J}(v).$$

Hence

$$u \neq v \Rightarrow (\mathcal{J}'(u) - \mathcal{J}'(v), u - v) > 0$$

and \mathcal{J}' is strictly monotone. This proves the lemma.

The form $a(u,v)$ defined by (3.2) is defined everywhere on V . Furthermore, it is linear and continuous in v , so it can be written as

$$a(u,v) = (Au,v),$$

where $(\ , \)$ denotes the duality between V and V' , and A is an operator from V into V' .

Proposition 5.1. The operator A is hemicontinuous and strictly monotone.

Proof. The forms $a_1(u,v)$, $a_2(u,v)$, $a_3(u,v)$ are linear and continuous in v , so they can be written as

$$a_1(u,v) = (A_1u,v)$$

$$a_2(u,v) = (A_2u,v)$$

$$a_3(u,v) = (A_3u,v)$$

Clearly

$$A = A_1 + A_2 + A_3 .$$

Since the forms $a_1(u,v)$ and $a_3(u,v)$ are bilinear and bicontinuous, the operators A_1 and A_3 are linear and continuous. Clearly the proposition follows if we can show that A_2 is hemicontinuous and strictly monotone. But this is straightforward, since by setting $\sigma = \Gamma_1$ and $p = 5$ in lemma 5.2 we have

$$a_2(u,v) = \alpha(\mathcal{J}'(u),v).$$

Proposition 5.2. The form $a(u,v)$ is coercive on V .

Proof. A trivial generalization of (4.2) is

$$(5.1) \quad a_1(u,u) + a_3(u,u) \geq C\|u\|_{H^1}^2$$

Hence, by adding $a_2(u,u)$ to (5.1) we have

$$a(u,u) \geq C\|u\|_{H^1}^2 + \alpha\|u\|_{L^5}^5$$

where $\|u\|_{L^5}$ denotes the norm in $L^5(\Gamma_1)$ of the trace of u on Γ_1 .

This leads to

$$(5.2) \quad \frac{a(u,u)}{\|u\|_V} \geq \frac{C\|u\|_{H^1}^2 + \alpha\|u\|_{L^5}^5}{\|u\|_{H^1} + \|v\|_{L^5}}$$

The right hand side of (5.2) converges to infinity as $\|u\|_V$ goes to infinity, hence the form $a(u,u)$ is coercive. This proves the proposition.

We have already shown that the form L is linear and continuous on V . Hence, it can be written in the form (see [3, p. 171]).

$$L(v) = (L,v) .$$

This implies that (4.1) can be written as

Find $u \in V$ such that

$$(5.3) \quad Au = L \quad \text{in } V'.$$

Theorem 5.1. Problem (5.3) has a unique solution in V .

Proof. The operator A from V into V' is hemicontinuous, strictly monotone and coercive. Since V is a separable reflexive Banach space, the operator A is one to one and onto

By proposition 4.1 we have shown that problem (3.7) has a unique solution, and to the extent specified in proposition (3.4) and (3.5), the original problem, problem (1.1), has a unique solution.

6. A RELATED LINEAR PROBLEM.

Let φ be a positive function defined on Γ_1 , belonging to $L^\infty(\Gamma_1)$.

Let $\tilde{a}_2(u,v)$ be the bilinear form defined on $H^1(\Omega)$ by

$$(6.1) \quad \tilde{a}_2(u,v) = \int_{\Gamma_1} \varphi uv \, d\Gamma .$$

Clearly $\tilde{a}_2(u,v)$ is defined everywhere on H^1 , and is continuous.

To see this observe that, by the trace theorem and the embedding theorem, u and v are in $L^2(\Gamma_1)$; so, the product uv is in L^1 .

Define the bilinear form $\tilde{a}(u,v)$ by

$$(6.2) \quad \tilde{a}(u,v) = a_1(u,v) + \tilde{a}_2(u,v) + a_3(u,v)$$

where $a_1(u,v)$ and $a_3(u,v)$ are given by (3.2) and $\tilde{a}_2(u,v)$ by (6.1).

Let the linear form \tilde{L} be the linear form L in (3.6); however, g_1 and g_2 in (3.6) are required to be in $H^{-\frac{1}{2}}(\Gamma_1)$ and $H^{-\frac{1}{2}}(\Gamma_2)$ respectively.

Consider the following linear problem, in the variational form.

Find u in $H^1(\Omega)$ such that

$$(6.3) \quad \tilde{a}(u,v) = \tilde{L}(v), \quad \text{for all } v \text{ in } H^1(\Omega) .$$

Lemma 6.1. The form $\tilde{a}(u,v)$ given by (6.2) is bilinear, continuous, and coercive on $H^1(\Omega)$.

Proof. The forms a_1 and a_3 are bilinear and continuous, and \tilde{a}_2 is bilinear. From the Cauchy-Schwarz inequality, we obtain

$$|\tilde{a}_2(u,v)| \leq \|\varphi\|_{L^\infty} \|u\|_{L^2} \|v\|_{L^2},$$

where the norms are in L^∞ and L^2 of Γ_1 . Hence, by the continuity of the

trace mapping from $H^1(\Omega)$ into $H^{\frac{1}{2}}(\Gamma_1)$, \tilde{a}_2 is continuous, consequently so is \tilde{a} .

Since the function φ is positive, we have

$$\tilde{a}_2(v,v) = \int_{\Gamma_1} \varphi v^2 d\Gamma \geq 0.$$

Hence, it suffices to prove the coerciveness of $a_1 + a_2$. However, this follows from (4.2).

Lemma 6.2. The linear form \tilde{L} is continuous on $H^1(\Omega)$.

Proof. By the trace theorem, if v is in $H^1(\Omega)$, then v on Γ_1 or Γ_2 is in $H^{\frac{1}{2}}(\Gamma_1)$ or $H^{\frac{1}{2}}(\Gamma_2)$ respectively. Hence, since g_1 and g_2 are in $H^{-\frac{1}{2}}(\Gamma_1)$ and $H^{-\frac{1}{2}}(\Gamma_2)$ respectively,

$$\tilde{L}(v) = (\tilde{g}_1, u_1)_1 + (\tilde{g}_2, u_2)_2$$

where $(\ , \)_1$ and $(\ , \)_2$ represent the duality $[H^{\frac{1}{2}}(\Gamma_1), H^{-\frac{1}{2}}(\Gamma_1)]$ and $[H^{\frac{1}{2}}(\Gamma_2), H^{-\frac{1}{2}}(\Gamma_2)]$ respectively, and \tilde{L} is continuous on $H^1(\Omega)$.

Remark. The functional L defined by (3.6) with g_1 and g_2 in $H^{-\frac{1}{2}}(\Gamma_1) + L^{\frac{5}{4}}(\Gamma_1)$ and $H^{-\frac{1}{2}}(\Gamma_2) + L^{\frac{5}{4}}(\Gamma_2)$ respectively, is not continuous on $H^1(\Omega)$.

Proposition 6.1. Problem (6.3) has a unique solution.

Proof. By lemma (6.1), $\tilde{a}(u,v)$ is bilinear, continuous, coercive, and by lemma (6.2) $\tilde{L}(v)$ is linear continuous, so the variational problem (6.3) has a unique solution [1, ch. 3].

We also have the following maximum principle.

Proposition 6.2. If the functional \tilde{L} in (6.3) is positive, i.e., if \tilde{g}_1 and g_2 in (3.6) are positive, then the solution of problem (6.3) is positive.

Proof. This result is well known and follows easily from the fact that, in H^1 , a function u can be written as

$$u = u^+ - u^- \quad \text{with } u^+, u^- \in H^1, \text{ and } u^+, u^- \geq 0.$$

Hence, if in (6.3) we take $v = -u^-$, then

$$\tilde{a}(u^+ - u^-, -u^-) = -L(u^-)$$

and since $a(u^+, u^-) = 0$, we have

$$\tilde{a}(u^-, u^-) = -\tilde{L}(u^-) \leq 0.$$

Now, since \tilde{a} is coercive, $u^- = 0$ and

$$u = u^+ \geq 0.$$

7. NEWTON'S METHOD AND MONOTONE CONVERGENCE.

An iterative procedure to solve problem (4.1) such that each iteration requires only the solution of a Dirichlet problem would be very desirable. To do this we need a more restrictive hypothesis on g_1 and g_2 in (3.6). Namely we require that

$$g_1 \in H^{-\frac{1}{2}}(\Gamma_1), \quad g_1 \geq 0 \quad \text{and}$$

$$g_2 \in H^{-\frac{1}{2}}(\Gamma_2), \quad g_2 \geq 0 .$$

By the trace theorem, a function in $H^1(\Omega)$ has a trace on Γ_1 . The set W will be the subset of $H^1(\Omega)$ such that this trace is positive and in $L^\infty(\Gamma_1)$, i.e.,

$$W = \{u \mid u \in H^1(\Omega), u|_{\Gamma_1} \in L^\infty(\Gamma_1), u|_{\Gamma_1} \geq 0\} .$$

We have

$$W \subset V \subset H^1(\Omega) .$$

Consider problem (6.3) where

$$\varphi = 4u^3 ,$$

$$(7.1) \quad \tilde{g}_1 = g_1 + 3u^4 , \quad \text{and}$$

$$\tilde{g}_2 = g_2 , \quad \text{with } u \text{ in } W.$$

Since u is in W , the trace of u on Γ_1 is in $L^\infty(\Gamma_1)$, and consequently, by the Sobolev imbedding theorem, in $H^{-\frac{1}{2}}(\Gamma_1)$. Since u is positive on Γ_1 , we have

$$4u^3 \in L^\infty, \quad 4u^3 \geq 0$$

$$g_1 + 3u^4 \in H^{-\frac{1}{2}}(\Gamma_1) \quad ,$$

$$g_2 \in H^{-\frac{1}{2}}(\Gamma_2),$$

and by proposition 6.1, there is a unique solution to the problem (6.3) with the data (7.1). Let v be this solution, the mapping Φ will be the mapping with domain W and range $H^1(\Omega)$ defined by

$$v = \Phi(u) \quad .$$

Remark. If all functions are smooth, problem (6.3) with data (7.1) corresponds to a classical system analogous to the system (3.8), however, the relation on Γ_1 has been changed to

$$(7.2) \quad \frac{\partial v}{\partial n} + 4u^3 v = g_1 + 3u^4 \quad .$$

Equation (7.2) can be rewritten as

$$\frac{\partial u}{\partial n} + u^4 + \frac{\partial(v - u)}{\partial n} + 4u^3(v - u) = g_1 \quad .$$

From this expression we see that the iterates $u_{n+1} = \Phi(u_n)$ are the Newton iterates for (3.8).

Proposition 7.1. For any u in W , $\Phi(u)$ is positive.

Proof. Since \tilde{g}_1 and \tilde{g}_2 are positive, this proof is a straightforward application of proposition 6.2.

Lemma 7.1. If u, v in W are such that $v = \Phi(u)$, then $w = \Phi(v)$ is well defined and $w \leq v$. Furthermore, w is in W .

Proof. Let v and w be the unique functions in $H^1(\Omega)$ which satisfy

$$(7.3) \quad a_1(v, \varphi) + \int_{\Gamma_1} 4u^3 v \varphi \, d\Gamma + a_3(v, \varphi) = \int_{\Gamma_1} (g_1 + 3u^4) \varphi \, d\Gamma + \int_{\Gamma_1} g_2 \varphi \, d\Gamma$$

$$(7.4) \quad a_1(w, \varphi) + \int_{\Gamma_1} 4v^3 w \varphi \, d\Gamma + a_3(w, \varphi) = \int_{\Gamma_1} (g_1 + 3v^4) \varphi \, d\Gamma + \int_{\Gamma_2} g_2 \varphi \, d\Gamma$$

for all φ in $H^1(\Omega)$.

Subtracting (7.3) from (7.4) we obtain

$$(7.5) \quad a_1(w - v, \varphi) + \int_{\Gamma_1} 4v^3 (w - v) \varphi \, d\Gamma + a_3(w - v, \varphi) = - \int_{\Gamma_1} p(v, u) \varphi \, d\Gamma$$

where $p(x, y)$ is the polynomial

$$p(x, y) = (x - y)^2 [2y^2 + (x + y)^2] .$$

Since u and v are in $L^\infty(\Gamma_1)$ the function $p(v, u)$ is also in $L^\infty(\Gamma_1)$. Furthermore, since $p(x, y)$ is positive for any value of x and y , the function $p(v, u)$ is positive. So, since v^3 is positive and in $L^\infty(\Gamma_1)$, and since $p(v, u)$ is positive and in $H^{-\frac{1}{2}}(\Gamma_1)$ by the Sobolev embedding theorem, we obtain, using proposition (6.2)

$$w \geq v .$$

However, since v is in $L^\infty(\Gamma_1)$, w is also in $L^\infty(\Gamma_1)$. By definition of $\bar{\Phi}$, w is in $H^1(\Omega)$, so w is in W . This proves the lemma.

We say that a function u of W has property (p) iff $\bar{\Phi}(u)$ is in W .

We will see that, under certain assumptions on the domain Ω , the partition $(\Gamma_1, \Gamma_2, \Gamma_3)$, and the data g_1 and g_2 , such functions exist.

Proposition 7.2. If the function u_0 satisfies property (p), then the sequence

$$(7.6) \quad u_n = \bar{\Phi}^n(u_0), \quad n = 1, 2, \dots .$$

is well defined, remains in W , and is decreasing for $n \geq 2$.

Proof. The proof that $u_n \in W$ for all n is made by induction on n .

Suppose the sequence remains in W for $n \leq n_0$. Then, since $u_{n_0-1}, u_{n_0} \in W$, by lemma 7.1 u_{n_0} is in W .

Now, for any $n \geq 2$,

$$u_{n-1} = \Phi(u_{n-2}) ,$$

$$u_n = \Phi(u_{n-1}), \quad \text{so by lemma 7.1}$$

$$u_n \leq u_{n-1} ,$$

and this proves the proposition.

Proposition 7.3. If u_0 satisfies property (p), then the sequence u_n defined by (7.6) is bounded in $H^1(\Omega)$.

Proof. The sequence u_n is defined by the following equation

$$(7.7) \quad a_1(u_n, v) + \int_{\Gamma_1} 4u_{n-1}^3 u_n v \, d\Gamma + a_3(u_n, v) = \int_{\Gamma_1} (g_1 + 3u_{n-1}^4) v \, d\Gamma \\ + \int_{\Gamma_2} g_2 v \, d\Gamma$$

for all v in $H^1(\Omega)$. So we can choose $v = u_n$. Since $u_{n-1} \geq 0$, we have

$$(7.8) \quad a_1(u_n, u_n) + a_3(u_n, u_n) \leq \int_{\Gamma_1} g_1 u_n \, d\Gamma + \int_{\Gamma_2} g_2 u_n \, d\Gamma .$$

From (4.2), we have,

$$(7.9) \quad C_1 \|u_n\|_{H^1}^2 \leq a_1(u_n, u_n) + a_3(u_n, u_n)$$

where C_1 is a strictly positive real constant. Furthermore, since $g_1 \in H^{\frac{1}{2}}(\Gamma_1)$,

$$\left| \int_{\Gamma_1} g_1 u_n \, d\Gamma \right| \leq C_2 \|g_1\|_{H^{-\frac{1}{2}}(\Gamma_1)} \|u_n\|_{H^{\frac{1}{2}}(\Gamma_1)} .$$

By the continuity of the trace mapping

$$\|u_n\|_{H^{\frac{1}{2}}(\Gamma_1)} \leq C_3 \|u_n\|_{H^1(\Omega)} .$$

And finally,

$$(7.10) \quad \left| \int_{\Gamma_1} g_1 u_n \, d\Gamma \right| \leq C_4 \|u_n\|_{H^1(\Omega)} .$$

In the same way

$$(7.11) \quad \left| \int_{\Gamma_2} g_2 u_n \, d\Gamma \right| \leq C_5 \|u_n\|_{H^1(\Omega)} .$$

Combining (7.7), (7.8), (7.9), and (7.10), we have

$$C_1 \|u_n\|_{H^1}^2 \leq (C_4 + C_5) \|u_n\|_{H^1}$$

and

$$\|u_n\|_{H^1} \leq \frac{C_4 + C_5}{C_1}$$

Remark. Property (7.3) depends on the fact that the measure of Γ_2 is strictly positive and Ω is connected.

Remark. In order to prove convergence of (7.6), we take the limit of the expression (7.7). The nonlinear term contains both u_n and u_{n-1} , hence it is not sufficient to prove that a subsequence of (7.6) converges.

Lemma 7.4. Any positive decreasing sequence in $L^2(\Omega)$ is weakly convergent.

Proof. Denote such a sequence by φ_n , and let ψ be any function of $L^2(\Omega)$. The function ψ can be written as

$$\psi = \psi^+ - \psi^-$$

with

$$\psi^+ \in L^2(\Omega) \quad \psi^+ \geq 0$$

$$\psi^- \in L^2(\Omega) \quad \psi^- \geq 0$$

The sequence of products $(\varphi_n \psi^+)$ and $(\varphi_n \psi^-)$ are decreasing, so the sequences of real numbers $\int_{\Omega} \varphi_n \psi^+ d\Omega$, $\int_{\Omega} \varphi_n \psi^- d\Omega$ are positive and decreasing, and consequently must converge. The lemma now follows, since

$$\int_{\Omega} \varphi_n \psi d\Omega = \int_{\Omega} \varphi_n \psi^+ d\Omega - \int_{\Omega} \varphi_n \psi^- d\Omega .$$

Lemma 7.5. Any bounded sequence in $H^1(\Omega)$, which is weakly convergent in $L^2(\Omega)$, is weakly convergent in $H^1(\Omega)$.

Proof. Let φ_n be such a sequence. Since φ_n is a bounded sequence in $H^1(\Omega)$, there exists a subsequence $\varphi_{j(n)}$ that converges H^1 weakly to $\bar{\varphi}$. Suppose a weak neighborhood of $\bar{\varphi}$ excludes infinitely many elements of φ_n . Since these elements are a bounded sequence in $H^1(\Omega)$, there exists a subsequence $\varphi_{k(n)}$ that converges H^1 weakly to $\tilde{\varphi}$, $\tilde{\varphi} \neq \bar{\varphi}$. But, since the injection of H^1 into L^2 is weakly continuous, $\varphi_{j(n)}$ and $\varphi_{k(n)}$ converges L^2 weakly to $\bar{\varphi}$ and $\tilde{\varphi}$ respectively, with $\bar{\varphi} \neq \tilde{\varphi}$, and this is impossible because the whole sequence φ_n converges L^2 weakly.

Proposition 7.4. If u_0 satisfies property (p), then the sequence u_n defined by (7.5) is weakly convergent in $H^1(\Omega)$.

Using the fact that the nonlinear terms occur only on the boundary, we can show that we actually have strong convergence in $H^1(\Omega)$.

Proposition 7.5. If u_0 satisfies property (p), then (7.6) is strongly convergent in $H^1(\Omega)$.

Proof. In (7.5) replace u, v, w by u_{n-1}, u_n , and u_{n+1} ; and φ by $(u_{n+1} - u_n)$ to obtain

$$\begin{aligned}
 (7.12) \quad & a_1(u_{n+1} - u_n, u_{n+1} - u_n) + \int_{\Gamma_1} 4u_n^3 (u_{n+1} - u_n)^2 d\Gamma \\
 & + a_3(u_{n+1} - u_n, u_{n+1} - u_n) \\
 & = - \int_{\Gamma_1} p(u_n, u_{n-1})(u_{n+1} - u_n) d\Gamma .
 \end{aligned}$$

The second term of the left hand side of (7.12) is positive; hence

$$\begin{aligned}
 & a_1(u_{n+1} - u_n, u_{n+1} - u_n) + a_3(u_{n+1} - u_n, u_{n+1} - u_n) \\
 & \leq \int_{\Gamma_1} p(u_n, u_{n-1})(u_n - u_{n+1}) d\Gamma,
 \end{aligned}$$

using lemma 4.2 we have

$$(7.13) \quad C_1 \|u_{n+1} - u_n\|_{H^1}^2 \leq \int_{\Gamma_1} p(u_n, u_{n-1})(u_n - u_{n+1}) d\Gamma$$

Now, since u_{n-1}, u_n, u_{n+1} are bounded in $L^\infty(\Gamma_1)$, we have

$$(7.14) \quad \int_{\Gamma_1} p(u_n, u_{n-1})(u_n - u_{n+1}) d\Gamma \leq C_2 \int_{\Gamma_2} (u_n - u_{n-1})^2 d\Gamma .$$

Combining (7.13) and (7.14) leads to

$$(7.15) \quad \|u_{n+1} - u_n\|_{H^1(\Omega)}^2 \leq C_3 \|u_n - u_{n-1}\|_{L^2(\Gamma_1)}^2 .$$

Since the sequence (u_n) is weakly convergent in $H^1(\Omega)$, and the injection of $H^1(\Omega)$ into $H^{1-\epsilon}(\Omega)$ is compact for any $\epsilon > 0$, the sequence (u_n) is strongly convergent in $H^{\frac{1}{2}-\epsilon}(\Gamma_1)$ for any ϵ in $]0, \frac{1}{2}[$. Hence u_n is strongly convergent in $L^2(\Gamma_1)$.

Inequality (7.15) implies that (u_n) is a Cauchy sequence in $H^1(\Omega)$ and since $H^1(\Omega)$ is complete, the proposition follows.

We must now show that the limit \bar{u} of the sequence (u_n) is the solution of problem (4.1).

Lemma 7.6. If u_0 satisfies property (p), then

$$\int_{\Gamma_1} u_{n-1}^3 (4u_n - 3u_{n-1})v \, d\Gamma \rightarrow \int_{\Gamma_1} \bar{u}^4 v \, d\Gamma$$

for every v in $H^1(\Omega)$.

Proof. By the trace theorem, v is in $H^{\frac{1}{2}}(\Gamma_1)$; hence, by the Sobolev embedding theorem v is in $L^2(\Gamma_1)$.

Let us now consider

$$\epsilon_n = \int_{\Gamma_1} (u_{n-1}^3 (4u_n - 3u_{n-1}) - u_{n-1}^4)v \, d\Gamma.$$

Since u_n and u_{n-1} are in $L^\infty(\Gamma_1)$ and v is in $L^2(\Gamma_1)$, ϵ_n is well defined. Clearly ϵ_n can be rewritten as

$$\epsilon_n = \int_{\Gamma_1} 4u_{n-1}^3 (u_n - u_{n-1})v \, d\Gamma.$$

It follows from the fact that u_{n-1} is bounded in $L^\infty(\Gamma_1)$ that

$$|\epsilon_n| \leq C_1 \int_{\Gamma_1} (u_n - u_{n-1})v \, d\Gamma,$$

where C_1 is a positive constant.

Since the sequence (u_n) is weakly convergent in $L^2(\Gamma_1)$, the

sequence $(u_{n+1} - u_n)$ converges weakly to zero, and the numerical sequence ϵ_n converges to zero.

The conclusion clearly follows once we show that

$$(7.16) \quad \int_{\Gamma_1} u_n^4 v \, d\Gamma \rightarrow \int_{\Gamma_1} \bar{u}^4 v \, d\Gamma .$$

To prove (7.16) we compute the difference

$$\int_{\Gamma_1} (u_n^4 - \bar{u}^4) v \, d\Gamma = \int_{\Gamma_1} u_n^3 + u_n^2 \bar{u} + u_n \bar{u}^2 + \bar{u}^3)(u_n - \bar{u}) v \, d\Gamma .$$

Notice that, since (u_n) is bounded in $L^\infty(\Gamma_1)$,

$$\left| \int_{\Gamma_1} (u_n^4 - \bar{u}^4) v \, d\Gamma \right| = C_3 \int_{\Gamma_1} |u_n - \bar{u}| |v| \, d\Gamma .$$

and by the Cauchy-Schwarz inequality

$$(7.17) \quad \left| \int_{\Gamma_1} (u_n^4 - \bar{u}^4) v \, d\Gamma \right| \leq C_4 \|u_n - \bar{u}\|_{L^2(\Gamma_1)} \|v\|_{L^2(\Gamma_1)} .$$

This proves the lemma since (7.17) clearly implies (7.16) .

Theorem 7.1. If u_0 satisfies property (p), then the sequence (u_n) defined by (7.6) is well defined. It is also decreasing for $n \geq 2$, and converges strongly in $H^1(\Omega)$ to the unique solution of problem (4.1).

Proof. Equation (7.6) can be rewritten as

$$a_1(u_n, v) + \alpha \int_{\Gamma_1} u_{n-1}^3 (4u_n - 3u_{n-1}) v \, d\Gamma + a_3(u_n, v) = L(v) .$$

The forms $a_1(u, v)$ and $a_3(u, v)$ are continuous, hence

$$a_1(u_n, v) \rightarrow a_1(\bar{u}, v), \quad \text{and}$$

$$a_3(u_n, v) \rightarrow a_3(\bar{u}, v) .$$

From lemma (7.6), since $\bar{u} \geq 0$, we have

$$\alpha \int_{\Gamma_1} u^{3n-1} (4u_n - 3u_{n-1}) v \, d\Gamma \rightarrow a_2(\bar{u}, v) \quad .$$

So,

$$a_1(\bar{u}, v) + a_2(\bar{u}, v) + a_3(\bar{u}, v) = L(v)$$

for any v in V , and \bar{u} is the solution of problem (4.1) .

Remark. Since the sequence (u_n) is decreasing, \bar{u} is in W . Hence Newton's method always gives a smoother solution, i.e., the solution is in $L^\infty(\Gamma_1)$ instead of $L^5(\Gamma_1)$. This is due to the fact that g_1 and g_2 are chosen in $H^{-\frac{1}{2}}(\Gamma_1)$ and $H^{-\frac{1}{2}}(\Gamma_2)$ respectively instead of $(H^{-\frac{1}{2}}(\Gamma_1) + L^{\frac{5}{4}}(\Gamma_1))$ and $(H^{-\frac{1}{2}}(\Gamma_2) + L^{\frac{5}{4}}(\Gamma_2))$ respectively. This gives the following corollary on the regularity of problem (4.1).

Corollary. If, in problem (4.1), the data g_1 and g_2 are in $H^{-\frac{1}{2}}(\Gamma_1)$ and $H^{-\frac{1}{2}}(\Gamma_2)$ respectively, and if there exists a function having property (p), then the solution on Γ_1 is L^∞ .

8. A NECESSARY CONDITION FOR THE EXISTENCE OF A FUNCTION SATISFYING
PROPERTY (P).

If the manifold Γ is C^∞ , g_1 is in $H^\epsilon(\Gamma_1)$ for some $\epsilon > 0$, and Γ_1 is surrounded by Γ_3 or Γ_2 , we take $u_0 = 0$ or $u_0 = k$ respectively. Then, the solution is $H^{\frac{3}{2}+\epsilon}$ in a neighborhood of Γ_1 (Lions l. ch. VII), so, by the trace theorem in $H^{1+\epsilon}(\Gamma_1)$, and by Sobolev embedding theorem, in $L^\infty(\Gamma_1)$.

If the manifold Γ has coins, as in our original problem, we cannot conclude in the general case. But in that particular case, we can extend all data by symmetry over the two planes containing Γ_3 , and the solution u_1 is also extended by symmetry. So, if g_1 is in H^ϵ , and $u_0 = 0$, the solution u_1 is in $H^{1+\epsilon} \subset L^\infty(\Gamma_1)$.

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