APPROXIMATION BY RATIONAL FUNCTIONS
WITH ZEROS ON A JORDAN CURVE

by

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1. Introduction

1.1 The problem to be considered is that of approximating a function $f(z)$ which is holomorphic and not zero in bounded simply-connected domain $D$ by polynomials whose zeros are restricted to lie on the curve $\Gamma$ which is the boundary of $D$. Two aspects of the problem are considered. First, in theorems I, II, and IV the existence of a sequence approximating a given function is asserted and in II and IV the degree of approximation effected by the sequence is given. Second, the question of how good such an approximation can be in general is treated.

We shall state here theorems I and II as they appear in a paper of G. R. MacLane [3]. Section 3 is devoted to the proof of the special case in which the curve $\Gamma$ is a circle. The definition of the approximating polynomials is the same as that for the more general curves and the proof of convergence employs most of the techniques used in the proof of theorem II.

Theorem I. Let $D$ be a bounded simply-connected domain of the $z$-plane bounded by a rectifiable Jordan curve $\Gamma$. Let $f(z)$ be holomorphic and never zero in $D$. There exists a sequence of polynomials $P_n(z), n=1, 2, \ldots$, with all zeros on $\Gamma$, such that $P_n(z) \rightarrow f(z)$ as $n \rightarrow \infty$, uniformly in each closed subset of $D$.

The subscript is not intended to imply anything about the degree of $P_n(z)$.

Theorem II. Let $D$ be a bounded simply-connected domain of the $z$-plane bounded by an analytic curve $\Gamma$. Let $f(z)$ be holomorphic and never zero in $D \cup \Gamma$. There exists a sequence of polynomials $P_n(z), n>n_0, P_n(z)$ of degree $n$ with all its zeros on $\Gamma$, such that
for \( n > \text{Re}(\delta) \), \( 0 < \delta < S_0 \), \( z \in D \), \( \text{dist}(z, \Gamma) \geq \delta \), where \( a > 0 \) is a constant depending only on \( \Gamma \), \( S_0 > 0 \) is a constant depending on \( f(z) \) and \( \Gamma \), and \( \text{Re}(\delta) \uparrow \) as \( \delta \downarrow 0 \), \( \text{Re}(\delta) \) depending on \( f(z) \) and \( \Gamma \).

By a linear transformation of the \( z \)-plane we can get from these theorems an approximation in \( D \) by rational functions with zeros on \( \Gamma \) and a single multiple pole at a point \( z_0 \) in the unbounded region \( D^\circ \) having \( \Gamma \) as a boundary. The second aspect of the problem is approached by determining which functions can be approximated by rational functions of this type with the pole on the curve \( \Gamma \). Section 4 is devoted to a group of theorems of Lindwart and Pólya [2] which show that when there is a circle lying in \( D \cup \Gamma \) and passing through the point \( z_0 \) on \( \Gamma \), only functions which are entire functions of \( \delta = \frac{z}{z-z_0} \) of order less than 2 can be approximated by rational functions \( R(z) \) with \( k \) zeros on \( \Gamma \) and a pole of order \( k \) at \( z_0 \). In section 5 these results are applied to obtain a bound from below on the measure of approximation defined in section 2. The final section exhibits a region \( D \) in which any function holomorphic and not zero there can be approximated by rational functions with all zeros on \( \Gamma \) and a single multiple pole at \( z_0 \) on \( \Gamma \).

1.2 Here we collect together several definitions and theorems which are used later.

(1.2.1) Maximum Modulus Theorem [5, 165]. Suppose \( f(z) \) is holomorphic and uniform in a domain \( D \) and on its boundary \( \Gamma \). If \( |f(z)| \leq M \) on \( \Gamma \) then \( |f(z)| < M \) in \( D \) unless \( f(z) \) is a constant.

(1.2.2) A theorem of Weierstrass [5, 95]. Suppose the
functions $f_n(z)$ are regular and uniform in a domain $D$. Suppose $f_n(z) \rightarrow f(z)$ as $n \rightarrow \infty$, uniformly in each closed subset of $D$. Then $f(z)$ is regular and uniform in $D$; moreover $f_n(z) \rightarrow f(z)$ as $n \rightarrow \infty$, uniformly in each closed subset of $D$.

(1.2.3) A theorem of Hurwitz [5, 115]. Let $f_n(z)$ be a sequence of functions regular and uniform in a domain $D$. Suppose $f_n(z) \rightarrow f(z)$ as $n \rightarrow \infty$, uniformly in every closed subset of $D$. Suppose $f(z) \neq 0$. If $\zeta \in D$ is a zero of $f(z)$ of multiplicity $m$ and $C$ is a circle with center at $\zeta$ such that no other zero of $f(z)$ is in the closed circle $C$, then there is an integer $n_0$ such that for $n > n_0$ $f_n(z)$ has exactly $m$ zeros in $C$.

(1.2.4) Stieltjes-Vitali Convergence Theorem [5, 168]. Let $f_n(z)$ be a sequence of functions regular and uniform in a domain $D$. Let $|f_n(z)| \leq M$ for every $z$ in $D$ and every $n$. Let $f_n(z)$ tend to a limit as $n \rightarrow \infty$ on a set of points having a limit point in $D$. Then $f_n(z)$ converges to a limit uniformly in each closed subset of $D$.

(1.2.5) Normal Family [4, 33]. Suppose the functions $g(z)$ are holomorphic in the domain $D$. The family $G$ of functions $g(z)$ is said to be normal in $D$ if from each infinite sequence of functions $g(z)$ a subsequence can be extracted which converges to a limit function holomorphic in $D$, or to the constant infinity, uniformly in each closed subset of $D$.

(1.2.6) Uniformly bounded family [4, 21]. If the functions $g(z)$ of the family $G$ are holomorphic and uniformly bounded in $D$, $G$ is a normal family in $D$.

(1.2.7) Order of an entire function [5, 248]. The entire function $f(z)$ is said to be of finite order $\rho$ if $\rho$ is the infimum
of numbers $A$ such that $f(z) = O(e^{A})$ as $|z| = r \to \infty$.

(1.2.6) Hadamard's Factorization Theorem [5, 250]. If $f(z)$ is an entire function of order $\rho$ with zeros $z_1, z_2, \cdots, f(0) \neq 0$, then

$$f(z) = e^{Q(z)}P(z)$$

where $P(z)$ is the canonical product of genus $p \leq \rho$ formed with the zeros of $f(z)$ and $Q(z)$ is a polynomial of degree $q$ not greater than $\rho$. The genus of $f(z)$ is the larger of the integers $p$ and $q$.

(1.2.9) Runge's Theorem [1, 296]. Let $f(z)$ be holomorphic in a simply-connected domain $D$ with infinity not an interior point. There exists a sequence of polynomials $P_n(z)$ such that $P_n(z) \to f(z)$ as $n \to \infty$, uniformly in each closed subset of $D$.  

2. The Measure of Approximation.

2.1 Definitions. Let \( D \) be a bounded, simply-connected domain bounded by the Jordan curve \( \Gamma \). For \( d > 0 \) define closed subsets of \( D \) as follows:

\[
A(d) = \{ z : z \in \mathbb{D}, \text{dist}(z, \Gamma) \geq d \},
\]

(2.1.1)

\[
B(d) = \{ z : z \in \mathbb{D}, \text{dist}(z, \Gamma) = d \}.
\]

The sets \( A(d) \) are compact since they are closed. \( B(d) \) is the boundary of \( A(d) \).

Let \( \mathcal{F} \) denote the class of all functions \( f(z) \) such that

1. \( f(z) \) is holomorphic in \( \mathbb{D} \cup \Gamma \),

(2.1.2)

2. \( f(z) \neq 0, z \in \mathbb{D} \cup \Gamma \),

3. \( |f(z)| \leq 1, z \in \mathbb{D} \cup \Gamma \).

Let \( \{ \Gamma, n \} \) denote the class of polynomials \( P(z) \) such that

1. \( P(z) \) is of degree \( n \),

(2.1.3)

2. \( P(z) \) has all zeros on \( \Gamma \).

We are interested in the approximation to \( f(z) \) by polynomials of the classes \( \{ \Gamma, n \} \); therefore we define a measure of approximation, \( \rho_n(d) \), as follows:

\[
M(f, P, d) = \max_{z \in B(d)} |f(z) - P(z)|,
\]

(2.1.4)

\[
\lambda_n(f, d) = \inf_{P \in \{ \Gamma, n \}} M(f, P, d),
\]

\[
\rho_n(d) = \sup_{f \in \mathcal{F}} \lambda_n(f, d).
\]

These definitions will be shown to give a finite, non-negative, decreasing function of \( d \) for \( 0 < d < d_0 \) which is continuous from the left, i.e., \( \rho_n(d^-) = \rho_n(d) \), for any Jordan curve \( \Gamma \) and continuous from the right, i.e., \( \rho_n(d^+) = \rho_n(d) \), when suitable restrictions are put on \( \Gamma \).
2.2 Properties of the maximum modulus function. Let $M(d) = \max_{z \in B(d)} |g(z)|$, where $g(z)$ is holomorphic in $D \cup \Gamma$. By the maximum modulus theorem (1.2.1), $M(d)$ is a decreasing function of $d$, strictly decreasing when $g(z)$ is non-constant.

**Lemma 2.2.1.** $M(d)$ is continuous from the left for each $d > 0$ for which it is defined, i.e., such that $A(d)$ is not void.

Suppose $A(d_0)$ is not void. Take a sequence $d_k < d_0$ such that $d_k \uparrow d_0$ as $k \to \infty$. Let $z_k$ be a point of $B(d_k)$ such that $M(d_k) = |g(z_k)|$. Since $D \cup \Gamma$ is compact we may suppose $z_k$ converges to a point $z_0$ as $k \to \infty$. Further, $z_0 \in B(d_0)$ since $d_k \to d_0$. The uniform continuity of $g(z)$ in $A(d)$, $d < d_0$ ($g(z)$ holomorphic in $D$) gives

\[(2.2.1) \quad |g(z)| > |g(z')| - \varepsilon, \quad |z - z'| < \delta,\]

with $\delta$ depending on $g(z)$ and $\varepsilon$ but not on the particular points $z$ and $z'$. Since $M(d)$ is decreasing

$M(d_k) \geq M(d) \geq M(d_0)$, \quad $d_k < d < d_0$.

From (2.2.1) it follows that

$M(d_k) < M(d_0) + \varepsilon$, \quad $k > K(\varepsilon)$,

by taking $z = z_0$ and $z' = z_k$ since $|g(z_0)| < M(d_0)$. Then, $M(d) < M(d_0) + \varepsilon$, $0 < d_0 - d < \delta$, or $M(d_0) = M(d_0)$.

It need not be true that $M(d + 0) = M(d)$. This can be made clear by a diagram such as the following.

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**Figure 1**
The component at the right first enters at \( d = d_0 \) as a set with no interior. If \( g(z) \) should have its maximum in this component, then it could be true that \( M(d_0 + O) = M(d_0) \). Likewise, the "finger" at the left first enters as a spike on \( A(d_1) \) whose points do not belong to the closure of the interior of \( A(d_1) \) and it need not be true that \( M(d_1 + O) = M(d_1) \). We can say the following:

**Lemma 2.2.2.** If \( B(d_0) \) is contained in the closure of the interior of \( A(d_0) \), i.e. each point \( z \in B(d_0) \) is a limit point of interior points of \( A(d_0) \), then \( M(d_0 + O) = M(d_0) \).

Let \( z_0 \) be a point of \( B(d_0) \) such that \( M(d_0) = |g(z_0)| \). Given any circle \( C(z_0) \), center at \( z_0 \), there is a point \( z_1 \) in \( C(z_0) \) such that \( d_1 > d_0, d_1 = \text{dist}(z_1, \Gamma) \). In \( A(d_0) \), \( g(z) \) is uniformly continuous; thus for \( z, z' \in A(d_0) \)

\[
|g(z)| < |g(z')| + \epsilon, |z - z'| < \delta.
\]

Choosing \( C(z_0) \) of radius \( \delta \),

\[
M(d_0) = |g(z_0)| < |g(z_1)| + \epsilon < M(d_1) + \epsilon.
\]

Since \( M(d_1) \leq M(d) \leq M(d_0) \), \( d_1 > d > d_0 \), it follows that \( M(d_0 + O) = M(d_0) \).

Requiring only that \( \Gamma \) be a Jordan curve does not insure that there exists \( d_0 \) such that for \( 0 < d < d_0 \), \( M(d + O) = M(d) \). If \( \Gamma \) has an infinite set of "fingers" as shown in figure 1 with width tending to zero, there can be no such \( d_0 \). In the sequel, by \( \Gamma \in \mathcal{R} \), we shall mean that \( \Gamma \) has the following properties:

1. \( \Gamma \) is a simple, closed Jordan curve,

(2.2.2) there exists a number \( R, R > 0 \), such that for \( 0 < d < R \), \( B(d) \subset [A(d)]^o \).
Now we shall give a sufficient condition in order that $\Gamma \in \mathcal{C}$.

By $\Gamma \in \mathcal{C}^k$, $k = 1, 2$, we mean that $\Gamma$ is a simple closed Jordan curve given by

$$z(t) = x(t) + i y(t)$$

where

$$\dot{x}^2 + \dot{y}^2 > 0, \quad \frac{d^k x}{dt^k}, \frac{d^k y}{dt^k} \text{ continuous}$$

with $\dot{x} = \frac{dx}{dt}, \dot{y} = \frac{dy}{dt}$.

Lemma 2.2.3. If $\Gamma \in \mathcal{C}^2$, then $\Gamma \in \mathcal{R}$.

This will be proved by means of lemmas 2.2.4, 2.2.5, and 2.2.6.

Lemma 2.2.4. If $\Gamma \in \mathcal{C}^1$ and $z \in B(d)$ then the circle $C(z)$ with center $z$ and radius $d$ is tangent to $\Gamma$ at each point $z_1$ common to $\Gamma$ and $C(z)$. There is at least one such point.

Since $\Gamma \in \mathcal{C}^1$, one of $x$ and $y$ can be expressed as a function of the other in the neighborhood of $z_1 = x_1 + i y_1$. Suppose $y = f(x)$ in the neighborhood of $x_1$. Also $f'(x)$ exists and is continuous. Any circle passing through $z_1$ not tangent to $\Gamma$ will have in its interior a segment of the tangent to $\Gamma$ at $z_1$. Clearly there will be points of $\Gamma$ in the interior of the circle also. Since the circle $C(z)$ contains in its interior no points of $\Gamma$ it is tangent to $\Gamma$ at each point $z_1$ common to $\Gamma$ and $C(z)$.

Lemma 2.2.5. If $\Gamma \in \mathcal{C}^2$ there exists a constant $M$ such that each point $z_0$ of $\Gamma$ is the midpoint with respect to arclength of an arc $\gamma$ of $\Gamma$ of length $\frac{1}{3M}$ which has only $z_0$ in common with either circle of radius $\frac{1}{2M}$ tangent to $\Gamma$ at $z_0$. $M$ is independent of $z_0$.

From $\dot{x}, \dot{y}$ continuous, $\dot{x}^2 + \dot{y}^2 > 0$, it follows that $\dot{x}^2 + \dot{y}^2 \geq m > 0$. Then, the curvature $k = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$ is bounded, $|k| \leq M$. For convenience we can suppose that $z_0$ is the origin and that the real
axis is the tangent to \( \Gamma \) at \( z_0 \). Moreover we can suppose that \( x \) and \( y \) are expressed as functions of the arclength: \( x(s), y(s) \), with \( s = 0 \) at \( z_0 \). For \( x(s) \) and \( y(s) \)

\[
\dot{x}^2 + \dot{y}^2 = 1, \quad \ddot{x} + \ddot{y} = 0,
\]

and the curvature is given by

\[
k = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2 + \dot{y}^2}, \quad k = -\frac{\ddot{x}}{\dot{x}}, \quad k = \frac{\ddot{y}}{\dot{x}},
\]

and \( \dot{x}^2 + \dot{y}^2 = 1, \ |k| \leq M \) gives \( |\dot{x}| \leq M, \ |\dot{y}| \leq M \). Consider a circle center at \((0, h)\) passing through the origin. It will be shown that \( h \) can be chosen so that a certain arc of \( \Gamma \) touches this circle only at the origin, that is,

\[
x^2 + (y - h)^2 \geq h, \text{ or } x^2 + y^2 - 2hy > 0.
\]

Letting \( \varphi(s) = [x(s)]^2 + [y(s)]^2 - 2hy(s) \) we note that \( \varphi(0) = \varphi'(0) = 0 \) and \( \varphi(s) \) has a continuous second derivative. By Taylor's theorem

\[
\varphi(s) = \frac{S^2}{2} \varphi''(\theta s), \ 0 \leq \theta \leq 1.
\]

Thus it suffices to show that \( \varphi''(s) \) is positive in an interval of \( a \).

The second derivative is

\[
\varphi''(s) = 2(1 + xx + yy - hy).
\]

From \(- (xx + yy - hy) \leq |xx| + |yy| + |hy| \leq M(|x| + |y| + |h|)\) we get

\[
\frac{1}{2} \varphi''(s) \geq 1 - (|x| + |y| + |h|)M.
\]

But since \( s \) is arclength \( |x| \leq |s|, \ |y| \leq |s| \). Choosing \( |h| = \frac{1}{2M} \)
we have \( \frac{1}{2} \varphi''(s) \geq \frac{1}{2} - 2M|s| \). For \( |s| < \frac{1}{4M}, \frac{1}{2} \varphi''(s) > 0 \) and the proof is complete.

Lemma 2.2.6. If \( \Gamma \in C^2 \) there exists a constant \( R \) such that at each point \( z \in \Gamma \) there is a circle of radius \( R \) which is tangent to \( \Gamma \) at \( z \) and lies in \( D \) except for the point \( z \in \Gamma \).

This lemma takes the result of lemma 2.2.5 and extends it to
say that not only can a circle of fixed radius be drawn touching \( \gamma \), an arc of fixed length, only at the point of tangency, but also the fixed radius can be chosen small enough that no point of \( \Gamma \) other than the point of tangency will be a point of the circle.

Suppose the statement is false. Then there are numbers \( r_k, r_k \downarrow 0 \), and points \( z_k \in \Gamma \) so that a circle of radius \( r_k \) tangent to \( \Gamma \) at \( z_k \) intersects \( \Gamma \) at \( z_k' \), distinct from \( z_k \). Since \( \Gamma \) is a closed set, thus compact, we can assume \( z_k \to z \in \Gamma \). For \( k > K \) the points \( z_k \) belong to \( \gamma \), an arc of length \( \frac{1}{3M} \) with midpoint \( z \) and \( z \) belongs to \( \gamma_k \), of length \( \frac{1}{3M} \), midpoint \( z_k \). Thus \( z \) is a limit point of points \( z_k ' \in \Gamma \) which, by Lemma 2.2.5, do not lie on \( \gamma \); hence \( \Gamma \) has the double point \( z \), contrary to hypothesis.

We use Lemmas 2.2.4 and 2.2.6 to prove Lemma 2.2.3. Suppose \( z \in B(d), 0 < d < R \). Let \( z_1 \in \Gamma \) be such that \( |z - z_1| = d \). By Lemma 2.2.4 the normal to \( \Gamma \) at \( z_1 \) passes through \( z \). A circle of radius \( R \) with center on this line and passing through \( z_1 \) will lie in \( D \) except for \( z_1 \) (Lemma 2.2.6). Since \( d < R \), \( z \) is a limit point of interior points of \( A(d) \).

2.3 Properties of the measure of Approximation.

Theorem III. If \( \Gamma \in \mathcal{Q}, (2.2.2) \), then \( \mathcal{C}_n(d) \) is a bounded, decreasing, and continuous function of \( d \) for \( 0 < d < R \). Moreover, if \( \Gamma \in \mathcal{O} \), then \( \mathcal{C}_n(d) \) satisfies a Lipschitz condition of order one in each interval \([a, R] \), a positive.

First we note that \( 0 \leq \mathcal{C}_n(d) \leq 1 \). \( \mathcal{P}(z) \equiv \{\Gamma, n\} \) can be written as \( \mathcal{P}(z) = A(z - z_1) \cdots (z - z_n), z_k \in \Gamma \). Letting \( N = \text{diam}(D \cup \Gamma) \), we can say that for \( z \in D \cup \Gamma \), \( |\mathcal{P}(z)| \leq |A|^n \). With \( n \) fixed \( |\mathcal{P}(z)| \) can be made arbitrarily small by choice of \( |A| \). But
\[ |f(z) - P(z)| \leq |f(z)| + |P(z)| \leq 1 + |P(z)|, \quad z \in D. \]

Then from (2.1.4) \( 0 \leq \Omega_n(d) \leq 1. \)

Next, both \( \lambda_n(f, \alpha) \) and \( \Omega_n(d) \) are decreasing functions of \( \alpha \) for \( 0 < \alpha < \alpha_0. \) Choose \( \alpha_0 \) such that \( \alpha_0 = \text{Sup}(\alpha). \) Then for \( 0 < \alpha < \alpha_1 < \alpha_0, \) \( A(\alpha_1) \) is not void and by the maximum modulus theorem
\[ M(f, P, \alpha) \geq M(f, P, \alpha_1) \]

Thus
\[ \lambda_n(f, \alpha) = \inf_{P \in \mathcal{P}(\alpha)} M(f, P, \alpha) \geq \inf_{P \in \mathcal{P}(\alpha_1)} M(f, P, \alpha_1) = \lambda_n(f, \alpha_1). \]

Also
\[ \Omega_n(d) \geq \lambda_n(f, \alpha) \geq \lambda_n(f, \alpha_1) \]

for all \( f \in F; \) therefore
\[ \Omega_n(d) = \sup_{f \in F} \lambda_n(f, \alpha_1) = \Omega_n(d_1). \]

Lemma 2.3.1. Let \( \Gamma \) be any simple closed Jordan curve and let \( d < d_0. \) For any \( f \in F \) there is a polynomial \( P_d(z) \in \{g, n\}(or\ P_d(z) \equiv 0) \) such that
\[ (2.3.1) \quad \lambda_n(f, d) = M(f, P_d, d). \]

Since \( \lambda_n(f, d) = \inf_{P \in \mathcal{P}(\alpha)} M(f, P, d) \) we can choose a sequence \( P_k(z) \in \{g, n\} \)
such that \( M(f, P_k, d) \to \lambda_n(f, d) \) as \( k \to \infty. \) Then \( P_k(z) \) is uniformly bounded for \( z \in A(d) \) and all \( k. \) Since \( n \) is fixed and \( A(d) \) has a nonvoid interior \( (d < d_0), \) it follows easily from Cauchy's estimate
that \( |P_k(z)| \) is uniformly bounded on \( D \cup \Gamma \) for all \( k. \) Therefore we may suppose \( P_k(z) \to P(z), \) uniformly in \( A(d) \) (1.2.6). Either \( P(z) \in \{g, n\} \) or \( P(z) \equiv 0. \) But
\[ |\lambda_n(f, d) - M(f, P, d)| \leq |\lambda_n(f, d) - M(f, P_k, d)| + |M(f, P_k, d) - M(f, P, d)| \]

If we show \( M(f, P_k, d) \to M(f, P, d) \) as \( k \to \infty \) then it follows that \( P(z) \)
is the \( P_d(g) \) of the lemma. Thus we show that \( g_k(z) \to g(z), \) uniformly
in $A(d)$, implies $M(g_k) \rightarrow M(g)$ where $M(g_k) = \max_{z \in A(d)} |g_k(z)|$. Let $z_k \in B(d)$ be such that $M(g_k) = |g_k(z_k)|$ and $z_0 \in B(d)$ be such that $M(g) = |g(z_0)|$. From the uniform convergence,

$$|q_k(z_k)| - \varepsilon < |q(z_k)| \leq M(q), \quad k > K \quad \text{and} \quad |q(z_0)| - \varepsilon < |q_k(z_0)| \leq M(q_k), \quad k > K$$

Thus $M(g) - \varepsilon < M(g_k) < M(g) + \varepsilon$, $k > K$.

Lemma 2.3.2. Let $\Gamma$ be any simple closed Jordan curve and let $d < d_0$. There exists a function $g(z)$, depending on $d$ and $n$, holomorphic in $D$, never zero in $D$, such that

\begin{equation}
\rho_n(d) = \lambda_n(g, d).
\end{equation}

Let $f_k(z) \in F$ be a sequence such that $\lambda_n(f_k, d) \rightarrow \rho_n(d)$ as $k \rightarrow \infty$. Since the class $F$ is uniformly bounded in $D$, we may suppose $f_k(z) \rightarrow g(z)$ as $k \rightarrow \infty$, $z \in D$, uniformly in each $A(d)$ $(1, 2, 6)$. Thus $g(z)$ is holomorphic in $D$ and either identically zero or never zero in $D$ $(1, 2, 3)$. Define $\lambda_n(g, d)$ as in $(2, 1, h)$. By Lemma 2.3.1 there are polynomials $f_k(z), P(z)$ such that

$$\lambda_n(f_k, d) = M(f_k, P_k, d), \quad \lambda_n(g, d) = M(g, P, d).$$

The uniform convergence gives $\max_{z \in B(d)} |f_k(z) - g(z)| < \varepsilon$, $k > K$. Now

$$\lambda_n(f_k, d) \leq \max_{z \in B(d)} |f_k(z) - P(z)| \leq \max_{z \in B(d)} |f_k(z) - g(z)| + \max_{z \in B(d)} |g(z) - P(z)|$$

and therefore

$$\lambda_n(f_k, d) < \varepsilon + \lambda_n(g, d), \quad k > K.$$ 

Similarly

$$\lambda_n(g, d) < \varepsilon + \lambda_n(f_k, d), \quad k > K.$$ 

Thus $|\lambda_n(g, d) - \lambda_n(f_k, d)| < \varepsilon$, $k > K$. If $g(z) \equiv 0$, $z \in D$, then $\lambda_n(g, d) = 0$, $0 < d < d_0$. Because of the way the sequence $f_k(z)$ was chosen we conclude that $g(z) \neq 0$ and $\rho_n(d) = \lambda_n(g, d)$.
Lemma 2.3.3. If $\Gamma \subseteq \mathcal{Q}$, then $\lambda_n(f,d)$ is a continuous function of $d$ for $0 < d < R$. Moreover, if $\Gamma \subseteq \mathcal{O}^2$, $\lambda_n(f,d)$ satisfies a Lipschitz condition of order one in each interval $[a,R]$, $a > 0$.

Suppose $P(z) \in \{ \Gamma, \mathcal{N} \}$ satisfies $\lambda_n(f,d) = M(f,P,d)$. Then for $z \in D \cup \Gamma$, $|P(z)| \leq M_n$, where $M_n$ is a constant depending only on $n$ and $\Gamma$. For $z \in A(d)$

$$|P(z)| \leq |f(z)| + |f(z) - P(z)| \leq 2$$

since $f \in F$ and $|f(z) - P(z)| \leq M(f,P,d) = \lambda_n(f,d) \leq 1$. Let $d_0$ be again the least upper bound of numbers $d$ such that $A(d)$ is not void. It follows that $A(d_0)$ is a non-void set with no interior points.

Then, for $z \in A(d_0)$, $z_1, z_2, \ldots, z_n \in \Gamma$,

$$|P(z)| = |A| |z - z_1| |z - z_2| \ldots |z - z_n| \geq |A| \delta_0^n$$

and

$$|A| \delta_0^n \leq \max_{z \in A(d_0)} |P(z)| \leq \max_{z \in A(d_0)} |P(z)| \leq 2.$$

Setting $N = \text{diam}(D \cup \Gamma)$ we have for any $z \in D \cup \Gamma$

$$|P(z)| = |A| |z - z_1| \ldots |z - z_n| \leq |A| |N|^n \leq \frac{2}{d_0^n} N^n = M_n.$$

We shall denote the subclass of $\{ \Gamma, \mathcal{N} \}$ which has the bound $M_n$ by $\{ \Gamma, M_n \}$.

The functions $P(z) \in \{ \Gamma, M_n \}$ and $f(z) \in F$ are equicontinuous in $D$ since they are uniformly bounded in $D [k,2k]$, i.e. given a closed set $D_1 \subset D$, there exists a constant $K$ such that

$$|g(z) - g(z')| < K |z - z'|$$

for $z, z' \in D_1$, and all $g(z) \in \{ \Gamma, M_n \} \cup F$. It follows then that for a given $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|g(z) - g(z')| < \varepsilon$$

for $z, z' \in D_1$, $|z - z'| < \delta$, and all $g(z) \in \{ \Gamma, M_n \} \cup F$. Now it is clear from the proofs of lemmas 2.2.1 and 2.2.2 that when $\Gamma \in \mathcal{Q}$
not only is $M(f, P, d)$ continuous for $0 < d < R$ but also for $f \in F$, $P \in \{ \Gamma, M_n \}$, the functions $M(f, P, d)$ are equicontinuous in $a \leq d < R$ for any positive $a$. Suppose $0 < a \leq d_1 < d_2 < R$. Let $P_1 \in \{ \Gamma, M_n \}$, $i = 1, 2$, be such that $M(f, P_i, d_i) = \lambda_n(f, d_i)$. Then, from the lemmas, there is a $\xi > 0$, depending on $\xi$ and $a$, such that

$$M(f, P_1, d_1) \leq M(f, P_2, d_1) \leq M(f, P_2, d_2) + \varepsilon, \quad |d_1 - d_2| < \xi.$$  

Thus,

(2.3.3) $\lambda_n(f, d_2) \leq \lambda_n(f, d_1) \leq \lambda_n(f, d_2) + \varepsilon, \quad |d_2 - d_1| < \xi.$

Since, by (2.3.2), $\xi$ does not depend on the particular $f$, the $\lambda_n(f, d)$ are equicontinuous as well as continuous.

Let $z_1$ be a point of $B(d_1)$. When $r \in C^2$ we can, by lemma 2.2.6, choose a point $z_2 \in B(d_2)$, $d_1 < d_2 < R$, such that $z_2$ is on the normal to $\Gamma$ through $z_1$. Then $|z_1 - z_2| = |d_1 - d_2|$. With $P_2$ chosen such that $M(f, P_2, d_2) = \lambda_n(f, d_2)$ we get, by (2.3.1),

$$\lambda_n(f, d_1) \leq M(f, P_2, d_1) = \max_{z_i \in B(d_i)} |f(z_i) - P_2(z_i)|$$

$$\leq \max_{z_i \in B(d_i)} \left\{ |f(z_1) - P_2(z_1)| + 2K |z_1 - z_2| \right\}$$

$$\leq \lambda_n(f, d_2) + 2K (d_2 - d_1).$$

The result, then is

(2.3.4) $\lambda_n(f, d_2) \leq \lambda_n(f, d_1) \leq \lambda_n(f, d_2) + 2K(d_2 - d_1).$

This is the Lipschitz condition on $\lambda_n(f, d)$. Note again that $K$ does not depend on the particular $f$. This completes the lemma.

From (2.3.3) and (2.3.4) we get, by taking the supremum,

$$\rho_n(d_2) \leq \rho_n(d_1) \leq \rho_n(d_2) + \varepsilon, \quad 0 < d_2 - d_1 < \xi,$$

and

$$\rho_n(d_2) \leq \rho_n(d_1) \leq \rho_n(d_2) + 2K(d_2 - d_1)$$

when $r \in \mathcal{Q}$, $r \in C^2$, respectively, with $a \leq d_1 < d_2 < R$, $\xi$. 

depending on $\epsilon$ and $\alpha$, $K$ depending on $\alpha$. This completes the proof of theorem III.
3. Approximation in a Circle.

3.1 The principal theorem of the section is the following.

Theorem IV. Let \( \Gamma \) be the circle \(|z| = 1\). Let \( D \) be the domain \(|z| < 1\). Suppose \( f(z) \) is holomorphic and never zero in \(|z| < R, R > 1\). There exists a sequence of polynomials \( P_n(z) \) of degree \( n \) with all zeros on \( \Gamma \) such that

\[
|f(z) - P_n(z)| < \left( |z| + \varepsilon \right)^n, \quad \varepsilon > 0,
\]

for \( n > n(\varepsilon, |z|) \), \( \frac{1}{|z|} < |z| < 1 \), where \( n(\varepsilon, |z|) \) is an increasing function of \(|z|\) and a decreasing function of \( \varepsilon \).

3.2 Representation for the polynomials. Denote by \( \{ \Gamma \} \) the class of all polynomials with all zeros on \( \Gamma \) and by \( \{ \Gamma, 1 \} \) the subclass having the value 1 at \( z = 0 \). It will be sufficient to prove the theorem with \( f(0) = 1 \) and \( P_n(z) \in \{ \Gamma, 1 \} \).

Let \( z = re^{it} \). Then \( \psi(t) = e^{it} \) is a parametric representation of \( \Gamma \). Let \( P_n(z) \in \{ \Gamma, 1 \} \),

\[
P_n(z) = \frac{n}{1!} \left( 1 - \frac{z}{e^{it_n}} \right).
\]

Let the associated step function \( S_n(t) \) be defined by

\[
S_n(t) = \begin{cases} 
0 & 0 < t < t_1 \\
k & t_k < t < t_{k+1}, k = 1, 2, \ldots, n-1 \\
n & t_n < t < 2\pi 
\end{cases}
\]

then

\[
\log P_n(z) = \int_0^{2\pi} \log(1 - \frac{z}{e^{it}}) dS_n(t), \quad z \in D.
\]

Here, and in the following, if \( h(z) \) is holomorphic and never zero in \( D, h(0) = 1 \), then \( \log h(z) \) denotes that branch such that \( \log h(0) = 0 \).

Now let \( z = -t, \quad j = \rho e^{i\theta} \). Then

\[
\int_0^{2\pi} \log \left( 1 - \frac{z}{e^{it}} \right) dt = -i \int |z| \frac{\log(1 - z_\theta)}{|z|} d\theta = 0, \quad |z| < 1,
\]

for at \( j = 0, \log(1 - z_\theta) = 0 \), \( \left[ \log (1 - z_j) \right]_j \) is regular in \(|z| < 1\) and
the integral on the right is zero by Cauchy's theorem. The polynomial representation may now be written

\[ \log p_n(z) = \int_0^{2\pi} (1 - \frac{z}{e^i t}) d(s_n(t) + c t), \]

\( z \in D, c \) a constant. Conversely, if \( S_n(t) \) is any unit-jump step function with \( S_n(0) = 0, S_n(z\pi) = n \), then the right hand member of (3.2.1) is the logarithm of a polynomial \( p_n(z) \in \{ r, 1 \}. \)

3.3 Representation for \( f(z) \). The basis for the proof is a similar representation for \( \log f(z) \).

Suppose \( g(z) \) is holomorphic in \( D \cup \Gamma \), \( g(0) = 0 \). Then Cauchy's formula gives

\[ \frac{g(z)}{z} = \frac{1}{2\pi i} \int_{\Gamma} \frac{g(t)}{j(t-z)} \, dt, \quad z \in D. \]

Suppose \( g_1(z) \) is holomorphic in \( D^* \cup \Gamma \), where \( D^* \) is the domain \( |z| > 1 \). Then

\[ 0 = \frac{1}{2\pi i} \int_{\Gamma} \frac{g_1(t)}{j(t-z)} \, dt, \quad z \in D. \]

For \( R > 1 \), Cauchy's theorem gives

\[ \int_{|t|=R} \frac{g_1(t)}{j(t-z)} \, dt = \int_{|t|=R} \frac{g_1(t)}{j(t-z)} \, dt. \]

But \( |g_1(t)| \leq M \) in \( D^* \cup \Gamma \) and \( |z-t| > R - 1, z \in D \), hence

\[ | \int_{|t|=R} \frac{g_1(t)}{j(t-z)} \, dt | \leq M \int_{|t|=R} |\frac{dt}{j(t-z)}| < \frac{2\pi M}{R-1}. \]

Since \( R \) can be chosen arbitrarily large (3.3.2) follows. Combining (3.3.1) and (3.3.2),

\[ g(z) = \int_{\Gamma} \frac{\frac{g(t)}{j(t-z)} - \frac{g_1(t)}{j(t-z)}}{2\pi i} \, dt. \]

Observe that \( \frac{d}{dt} \log (1 - \frac{z}{t}) = \frac{z}{j(t-z)} \). The object is to choose
So that on \( \Gamma \) the function \( \frac{f(z) - g(z)}{2\pi i} \) is a real-valued function of \( t \) which vanishes at \( t = 0 \). Then an integration by parts will give the desired representation.

Since \( g(z) \) is holomorphic in \( |z| \leq R \), \( g(\frac{1}{R}) \) is holomorphic in \( |z| > \frac{1}{R} \) and the difference \( g(z) - g(\frac{1}{R}) \) is pure imaginary on \( \Gamma \).

Moreover, since \( g(1) - g(\frac{1}{R}) = 2i \mathcal{A}(g(1)) \), the function

\[
I(z) = \frac{g(z) - g(\frac{1}{R})}{2\pi i},
\]

where \( g(z) = \overline{g(\frac{1}{z})} + 2i \mathcal{A}(g(1))i \), is holomorphic in an annular region \( \frac{1}{R} < |z| < R \), real-valued for \( z = e^{it} \), and vanishes for \( t = 0 \).

Setting \( g(z) = \log f(z) \) we get

\[
I(z) = \frac{1}{2\pi i} \left[ \log f(z) + i\arg f(z) - \log f(\frac{1}{z}) - i\arg f(\frac{1}{z}) - 2i \arg f(t) \right]
\]

and

\[
I(e^{it}) = \frac{1}{\pi} \arg f(e^{it}) - \frac{1}{\pi} \arg f(t) = \frac{1}{\pi} \arg f(t) f(e^{it}).
\]

Now, substituting in (3.3.3),

\[
\log f(z) = \int_0^{2\pi} I(e^{it}) dQ(t) \log(1 - \frac{z}{e^{it}})
\]

and after integrating by parts, recalling that at \( t = 0, 2\pi, I(e^{it}) = 0 \),

\[
\log f(z) = -\int_0^{2\pi} \log(1 - \frac{z}{e^{it}}) dQ(t).
\]

Setting \( Q(t) = -I(e^{it}) \) we can say \( Q(t) \) is real-valued, of period \( 2\pi \), and analytic for \( t \) real, \( Q(0) = 0 \), and

(3.3.4) \[
\log f(z) = \int_0^{2\pi} \log(1 - \frac{z}{e^{it}}) dQ(t).
\]

3.4 The Approximating Polynomials. Since \( Q(t) \) is analytic there is a constant \( c_1 \) such that for real \( t \) \( |Q'(t)| < c_1 \). Then

\[
\frac{d}{dt} \left( \frac{nt}{2\pi} + Q(t) \right) > \frac{n}{2\pi} - c_1 > 0
\]

for \( n > n_0 \geq 2\pi c_1 + 1 \) and \( \frac{nt}{2\pi} + Q(t) \) increases strictly from 0 to \( n \) as \( t \) increases from 0 to \( 2\pi \). There is exactly one value of \( t \), say \( t_{nk} \), such that \( nt/2\pi + Q(t) = k, k = 0, 1, 2, \ldots, n \). Then,
\[ 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 2\pi \]

and \( S_n(t) = \left[ \frac{nt}{2\pi} + \phi(t) \right] \), where \( \left[ X \right] \) denotes the largest integer not exceeding \( X \), does, by (3.2.1), define a polynomial \( P_n(z) \in [1, 1] \).

By (3.2.1) and (3.3.4),

\[ (3.4.1) \quad \log f(z) - \log P_n(z) = \int_0^{2\pi} \log \left( 1 - \frac{z}{e^{it}} \right) dt \left( \frac{nt}{2\pi} + \phi(t) - \left[ \frac{nt}{2\pi} + \phi(t) \right] \right). \]

### 3.5 The Convergence

Define a new variable \( \theta \) by means of the relation

\[ (3.5.1) \quad \frac{n\theta}{2\pi} = \frac{nt}{\pi} + \phi(t). \]

We need some information about \( e^{it} \) as a function of \( \theta \). Choose \( R > 1 \) such that \( L(z) \) is holomorphic and uniform in \( \frac{1}{2} < |z| < R \). Denote the closed annulus \( \frac{1}{2} < |z| < R - \varepsilon, 0 < \varepsilon < R - 1 \), by \( \Delta \varepsilon \). Define

\[ (3.5.2) \quad \beta_n(z) = \varepsilon e^{-\frac{2\pi i}{n} L(z)} \]

Let \( z = re^{it} \) and \( \gamma = \rho e^{i\theta} \) and

\[ (3.5.3) \quad \gamma = \beta_n(z). \]

Then \( e^{i\theta} = e^{i\theta} e^{2\pi i \phi(t)} \). Choosing the determinations of \( \theta \) and \( t \) properly, \( \frac{n\theta}{2\pi} = \frac{nt}{\pi} + \phi(t) \). Then, (3.5.1) is equivalent to (3.5.3) when \( |z| = 1 \). Also, \( \beta_n(z) \) is holomorphic and uniform in \( \frac{1}{2} < |z| < R \) and

\[ (3.5.4) \quad \lim_{n \to \infty} \beta_n(z) = z, \text{ uniformly in } \Delta \varepsilon. \]

The theorem of Hurwitz (1.2.3) implies that for \( n > n(\varepsilon) \), \( \beta_n(z) \) is schlicht in \( \Delta \varepsilon \). Then \( \beta_n(z) \) has an inverse \( z = \gamma_n(\gamma) \) which is holomorphic and uniform in \( \Delta \varepsilon \) of the \( \gamma \)-plane for \( n > n(\varepsilon) \), and such that

\[ (3.5.5) \quad \lim_{n \to \infty} \gamma_n(\gamma) = \gamma, \text{ uniformly in } \Delta \varepsilon. \]

The details here can be handled as follows: For a given \( \varepsilon > 0 \), consider \( \Delta \varepsilon / 2 \) of the \( z \)-plane. Choose \( n(\varepsilon) \) so that for \( n > n(\varepsilon) \), \( \beta_n(z) \) is schlicht in \( \Delta \varepsilon / 2 \) and

\[ (3.5.6) \quad |z - \beta_n(z)| < \frac{\varepsilon}{2}, \quad n > n(\varepsilon), \text{ uniformly in } \Delta \varepsilon / 2. \]
With this choice the image of $\Delta \varepsilon$ under (3.5.3) contains the region $\Delta \varepsilon$ of the $\gamma$-plane. Then, for $n > n(\varepsilon)$, $\gamma \in \Delta \varepsilon$, $\gamma = \beta_n(z)$ has a holomorphic inverse $z = \gamma_n(\gamma)$. Moreover, $\gamma_n(\gamma)$ is uniform.

If it were not, then distinct points $z, z' \in \Delta \varepsilon$ would satisfy $\beta_n(z) = \beta_n(z')$. But this is impossible since $\beta_n(z)$ is schlicht. For any $z \in \Delta \varepsilon$ let $z_n = \gamma_n(z)$, $n > n(\varepsilon)$. Consider $\beta_k(z_n), k > n(\varepsilon)$.

By (3.5.6), $|\beta_k(z_n) - z_n| < \frac{\varepsilon}{2}$. Choosing the particular value $k = n$, $\beta_n(z_n) = z$,

$$|z - z_n| < \frac{\varepsilon}{2}, n > n(\varepsilon), z \in \Delta \varepsilon.$$ 

This proves (3.5.5).

Now we may write (3.4.1) in the form

$$(3.5.7) \quad \log f(z) - \log \beta_n(z) = \int_0^{2\pi} \log \left(1 - \frac{z}{\beta_n(e^{i\theta})}ight) d\left(\frac{n^\theta - [n^\theta]}{2\pi}ight).$$

The function $1 - \frac{z}{\beta_n(\gamma)}$ is holomorphic and satisfies

$$0 < \eta < \left|1 - \frac{z}{\beta_n(\gamma)}\right| < \frac{1}{\eta},$$

for $\frac{1}{\gamma(z)} < |\gamma| < \gamma(z)$ and $n > n(\varepsilon, \eta, |z|)$ where

$$(3.5.8) \quad \frac{1}{\gamma(z)} = \max \left(\frac{|z| + \eta}{\frac{1}{R} - \varepsilon}\right).$$

This inequality is a consequence of $|\gamma_n(\gamma)| > \frac{|z|}{|1 - \eta|}$. To get the latter take $|\gamma| > |z| + \eta$ and $n > n(\varepsilon, \eta, |z|)$. This is possible by (3.5.5) since if $\eta < 1 - |z|$ then $|z| + \eta > \frac{|z|}{1 - \eta}$. Requiring $\eta < 1 - |z|$ insures a non-vacuous result since then $\frac{1}{\gamma(z)} < 1$. The same restrictions on $\gamma$ and $n$ also give

$$\left|\alpha \gamma(z) \left(1 - \frac{z}{\beta_n(\gamma)}\right)\right| < \frac{\eta}{2}$$

since $1 - \frac{|z|}{1 + \gamma(z)} > \eta$ implies $\alpha \left(1 - \frac{z}{\beta_n(\gamma)}\right) > \eta > 0$. Together

the two say that for $\frac{1}{\gamma(z)} < |\gamma| < \gamma(z), n > n(\varepsilon, \eta, |z|), \log (1 - \frac{z}{\beta_n(\gamma)})$
is holomorphic and satisfies

\[(3.5.9) \quad | \log \left(1 - \frac{z}{\eta_n(z)} \right) | < \frac{\pi}{2} + | \log \eta |.\]

Lemma 3.5.1. Let \( \phi(z) \) be holomorphic and uniform in \( \frac{1}{r} \leq |z| \leq r \), \( r > 1 \), and satisfy \( | \phi(z) | < K \). Then

\[(3.5.10) \quad \left| \int_0^{2\pi} \phi(e^{i\theta}) d\left( \frac{n\theta}{2\pi} - \left[ \frac{n\theta}{2\pi} \right] \right) \right| < 2Kn \frac{r^{-n}}{1 - r^{-n}}.\]

For \( \frac{1}{r} \leq |z| \leq r \), \( \phi(z) \) has the Laurent expansion \( \phi(z) = \sum_{k=-\infty}^{\infty} a_k z^k \). The coefficients satisfy

\[(3.5.11) \quad |a_k| < \frac{K}{\pi^{1+|k|}}, \quad k = 0, 1, 2, \ldots.\]

In

\[a_k = \frac{1}{2\pi i} \int_{C} \frac{\phi(z)}{z^{k+1}} \, dz\]

the path of integration \( C \) can be \( |z| = r \) for \( k > 0 \) and \( |z| = \frac{1}{r} \) for \( k < 0 \). Then (3.5.11) follows. Now

\[\int_0^{2\pi} \phi(e^{i\theta}) d\left( \frac{n\theta}{2\pi} - \left[ \frac{n\theta}{2\pi} \right] \right) = \frac{n}{2\pi} \int_0^{2\pi} \phi(e^{i\theta}) d\theta - \sum_{k=1}^{\infty} \phi(e^{\frac{2\pi k}{n} i})\]

\[= n a_0 - \sum_{k=1}^{\infty} \left( \sum_{j=1}^{\infty} a_j e^{\frac{2\pi k}{n} j i} \right)\]

\[= n a_0 - \sum_{k=1}^{\infty} \left( \sum_{j=1}^{\infty} e^{\frac{2\pi k}{n} j i} a_j \right).\]

For \( j \neq \lambda n \), \( \sum_{k=1}^{\infty} e^{\frac{2\pi k}{n} j i} = 0 \), and for \( j = \lambda n \), \( \sum_{k=1}^{\infty} e^{\frac{2\pi k}{n} j i} = n \).

Thus

\[\int_0^{2\pi} \phi(e^{i\theta}) d\left( \frac{n\theta}{2\pi} - \left[ \frac{n\theta}{2\pi} \right] \right) = -n \sum_{\lambda=1}^{\infty} (a_{\lambda n} + a_{\lambda n}).\]

Applying (3.5.11) we get (3.5.10).

Lemma 3.5.1 combined with (3.5.9) gives

\[(3.5.12) \quad \left| \log f(z) - \log p_n(z) \right| < 2kn \frac{r^{-n}}{1 - r^{-n}}.\]
where \( n > n(\epsilon, \eta, |z|) \), and \( r(z) \) is defined by (3.5.6). Choosing \( \eta > \eta' \) so that \( |z| + \eta > \frac{1}{R-\epsilon} \), we can write

\[
(3.5.13) \quad |\log f(z) - \log P_n(z)| < (|z| + \eta')^n, \quad n > n(\eta, |z|)
\]
since \( n_0 \) can be chosen such that for \( n > n_0 \) and \( r_1 < r_2 < 1, \) \( m_1^n < r_2^n \). Moreover, since \( |\log w| < \epsilon \) implies \( |\log|w|| < \epsilon, \) \( \arg w < \epsilon \), and thence \( |1 - w| < 2\epsilon, \) \( \epsilon < 1 \), it follows that \( |\log A - \log B| < \epsilon \) implies \( |A - B| < 2|B|\epsilon \). Now

\[
(3.5.14) \quad |f(z) - P_n(z)| < (|z| + \eta)^n
\]
for \( n > n(\eta, |z|), \) \( |z| + \eta > \frac{1}{R-\epsilon}, \) \( |z| < 1 \). The number \( n(\eta, |z|) \) is an increasing function of \( |z| \) and a decreasing function of \( \eta \). For a given \( \epsilon < 1 \) and \( \frac{1}{R} < |z| < 1 \), (3.5.14) can readily be put into the form stated in theorem IV.

3.6 Consequences of Theorem IV. In terms of the measure of approximation defined in (2.1.4), the conclusion of theorem IV can be stated as follows:

\[
\langle n(d) \rangle < (1+\epsilon -d)^n, \quad n > n(c,d), \quad \epsilon > 0,
\]
where \( n(c,d) \) is a decreasing function of \( \epsilon \) and \( d \).

Theorem IV has as a consequence the following theorem.

Theorem V. Let \( a_n, n = 1, 2, \ldots \), be an infinite sequence of complex numbers such that

\[
\sum_{k=1}^{\infty} a_k z^k
\]
represents a function holomorphic in \( |z| \leq R, \) \( R > 1 \). There exist real numbers \( t_{n,k}, n > n_0, 1 \leq k \leq n \), such that

\[
|a_p - \sum_{k=1}^{n} e^{-ip\frac{t_{n,k}}{R}}| < pR^p \left( \frac{1}{R} + \epsilon \right)^n
\]
for \( p = 1, 2, \ldots, n > n(\epsilon) \), where \( n(\epsilon) \) depends on \( \epsilon \) and the sequence \( a_p \).
For a finite sequence $a_1, a_2, \ldots, a_q$ the stronger inequality

$$|a_p - \sum_{k=1}^{n} e^{-i p t_{nk}}| < \frac{1}{R^n}, \quad 1 \leq p \leq q,$$

holds for any $R > 1$ and $n > n(R)$.

Let $F(z) = \sum_{k=1}^{\infty} \frac{a_k}{k^z}$. Then, $F(z)$ is holomorphic in $|z| < R$ and $F(0) = 0$. Let $f(z) = e^{-F(z)}$. Theorem IV can be applied to $f(z)$. From (3.5.13) it follows that

$$|\log f(z) - \log P_n(z)| < (|z| + \varepsilon)^n$$

for $\varepsilon > 0$, $\frac{1}{R} < |z| < 1$, $n > n(\varepsilon, |z|)$. Also $P_n(z) \in \mathbb{C}[i, 1]$; thus $P_n(z)$ is of the form

$$\sum_{k=1}^{n} \left(1 - z e^{-i t_{nk}}\right)$$

Expanding $\log P_n(z) - \log f(z) = \log P_n(z) + F(z)$ in Taylor's series about the origin,

$$\left|\sum_{p=1}^{\infty} \frac{a_p}{p} \left(a_p - \sum_{k=1}^{n} e^{-i p t_{nk}}\right)\right| < (|z| + \varepsilon)^n$$

Cauchy's formula for derivatives gives

$$\left|a_p - \sum_{k=1}^{n} e^{-i p t_{nk}}\right| < \frac{1}{2\pi} \int_{|z| = \frac{1}{R}} \frac{(|z| + \varepsilon)^n}{|z|^{p+1}} |dz| = pR^p \left(\frac{1}{R} + \varepsilon\right)^n$$

for $p = 1, 2, \ldots, n > n(\varepsilon)$.

For the finite sequence $a_1, a_2, \ldots, a_q$, $F(z)$ is holomorphic in $|z| \leq R$, for any $R$. Then, since $1 \leq p \leq q$, it follows from (3.6.1) that

$$\left|a_p - \sum_{k=1}^{n} e^{-i p t_{nk}}\right| < \frac{1}{R^n}, \quad R > 1$$

$p = 1, 2, \ldots, q$, $n > n(R)$. 
4. Lindwart-Polya Theorems.

4.1 Statement of the general theorem. The theorems of this section give a connection between a restraint on the zeros of a convergent sequence of rational functions and the nature of the limit function. Though they are stated for rational functions, the theorems can be just as easily stated for polynomials.

Theorem VI. Consider a sequence of rational functions of the form

\[ R_n(z) = \prod_{k=1}^{m} \left(1 - \frac{x}{r_{nk}e^{\theta_{nk}}} \right), \]

satisfying the following conditions:

(4.1.2) \( R_n(z) \) converges uniformly in some circle, center at \( z = 0 \),

and

(4.1.3) \[ \sum_{k=1}^{m} \left|1 - \frac{x}{r_{nk}e^{\theta_{nk}}} \right|^n \leq M \text{ for all } n=1, 2, \ldots, N \text{ and } m \text{ fixed.} \]

Then \( R_n(z) \) converges for every \( z \) of the extended plane except \( z = 1 \), uniformly in each region \( G \) which excludes a neighborhood of \( z = 1 \), to a function \( F(z) \) such that if \( j = \frac{z}{z-1} \), \( \phi(j) = F(z) \), then \( \phi(j) \) is an entire function of order not greater than \( m \). If \( m \) is an integer, \( \phi(j) \) is the product of an entire function of genus not greater than \( m - 1 \) and the function \( e^{\gamma j^m} \), where \( \gamma \) is a constant.

4.2 Proof of Theorem VI. Again denote by \( G \) a region consisting of the extended plane except for the interior of a circle with center at \( z = 1 \). The method of proof is to get a uniform bound on \( R_n(z) \) in \( G \) and then to apply the Stieltjes-Vitali convergence theorem (1.2.4).

From (4.1.2) it follows by a theorem of Weierstrass (1.2.2) that \( F(z) \) is holomorphic in a circle \( C \) about \( z = 0 \). Moreover, since
$R_n(0) = 1$, we can pick $C$ so that $F(z) \neq 0$ in $C$. From a theorem of Hurwitz (1.2.3) it is possible to say that $R_n(z)$ has no zeros in $C$. Now in $C \log R_n(z)$ is holomorphic and $\log R_n(z)$ converges uniformly to $\log F(z)$ as $n \to \infty$. If $C$ is chosen small enough, then

\[(4.2.1) \quad R_n(z) = \prod_{k=1}^{n} \left[ 1 - \frac{z}{z-1} \left( 1 - \frac{1}{\eta_k e^{i\theta_k}} \right) \right] \]

and

\[\log R_n(z) = - \sum_{k=1}^{n} \sum_{p=1}^{\infty} \frac{1}{p} \left( 1 - \frac{1}{\eta_k e^{i\theta_k}} \right)^p \]

or

\[(4.2.2) \quad \log R_n(z) = - \sum_{p=1}^{\infty} \left( \frac{1}{p} \right)^p \left[ \sum_{k=1}^{n} \left( 1 - \frac{1}{\eta_k e^{i\theta_k}} \right)^p \right], \quad z \in C.\]

Since $\log R_n(z)$ tends uniformly to a limit in $C$ it follows that for each $p$ the coefficients of $z^p$ in (4.2.2) tends to a limit as $n \to \infty$.

Then, there exist constants $M_p$, independent of $n$, such that

\[(4.2.3) \quad \left| \sum_{k=1}^{n} \left( 1 - \frac{1}{\eta_k e^{i\theta_k}} \right)^p \right| < M_p, \quad n = 1, 2, \ldots.\]

Set

\[h = \{ \left[ m \right] i f \left[ m \right] < m \]

\[m-1 \quad i f \left[ m \right] = m.\]

For Weierstrass' primary factors the following inequality is known [6, 233].

\[|1 - u| \leq \exp \left[ - \mathcal{R}(u + \frac{u^2}{2} + \cdots + \frac{u^h}{h}) + c |u|^m \right] \]

where $c$ is a constant independent of $u$. Applying this to each factor in (4.2.1)

\[|R_n(z)| \leq \exp \left\{ - \mathcal{R} \left[ \frac{z}{z-1} \sum_{k=1}^{n} \left( 1 - \frac{1}{\eta_k e^{i\theta_k}} \right)^2 + \frac{1}{2} \left( \frac{z}{z-1} \right)^2 \sum_{k=1}^{n} \left( 1 - \frac{1}{\eta_k e^{i\theta_k}} \right)^2 \right] \right. \]

\[+ \cdots + \frac{1}{h} \left( \frac{z}{z-1} \right)^h \sum_{k=1}^{n} \left( 1 - \frac{1}{\eta_k e^{i\theta_k}} \right)^h \right\} + c \left| \frac{z}{z-1} \right|^{m} \sum_{k=1}^{n} \left| 1 - \frac{1}{\eta_k e^{i\theta_k}} \right|^m \]
Since $Q(u) < |u|$ we get

$$|R_n(z)| \leq \exp \left\{ \sum_{k=1}^{p} \left| \frac{z}{z-1} \right|^n + \frac{1}{2} \sum_{k=1}^{p} \left| \frac{z}{z-1} \right|^n \left( 1 - \frac{1}{r_k e^{\theta_k}} \right)^n \right\} + \cdots + \frac{1}{h} \sum_{k=1}^{p} \left| \frac{z}{z-1} \right|^n + c \left| \frac{z}{z-1} \right|^n \right\}.$$

Applying (4.1.3) and (4.2.3)

$$(4.2.4) \quad |R_n(z)| \leq \exp \left\{ \sum_{k=1}^{p} \left| \frac{z}{z-1} \right|^n + \frac{1}{2} \sum_{k=1}^{p} \left| \frac{z}{z-1} \right|^n \left( 1 - \frac{1}{r_k e^{\theta_k}} \right)^n \right\} + c \left| \frac{z}{z-1} \right|^n.$$

From (4.2.4) the uniform bound on $R_n(z)$ in $G$ follows. This uniform bound in $G$ and uniform convergence in a circle in $G$ implies that $R_n(z)$ converges uniformly in $G$ (1.2.4). The limit function $F(z)$ also satisfies (4.2.4). Hence $\phi'(z)$ is an entire function of order not greater than $m$.

From (4.1.3) it follows a fortiori that

$$\sum_{k=1}^{p} \left| 1 - \frac{1}{r_k e^{\theta_k}} \right|^m \leq \mathcal{M}, \quad n = p, p+1, p+2, \ldots.$$

Moreover, if we write $R_n(z) = \sum_{n=0}^{\infty} a_n z^n$, then $a_n(z)$ is a polynomial of the form

$$\prod_{k=1}^{p} \left( 1 - \frac{1}{a_{n_k}} \right), \quad a_{n_k} = \left[ 1 - \frac{1}{r_k e^{\theta_k}} \right]^{-1}, \quad \text{and}$$

$$\sum_{k=1}^{p} \left| \frac{1}{a_{n_k}} \right|^m \leq \mathcal{M}, \quad n = p, p+1, p+2, \ldots.$$

The theorem of Hurwitz (1.2.3) implies that any $p$ zeroes of the limit function, $\alpha_1, \alpha_2, \ldots, \alpha_p$, must satisfy the relation $\sum_{k=1}^{p} \frac{1}{|\alpha_k|^m} \leq \mathcal{M}$. This follows from the fact that if $a_q$ is a zero of $\phi'(z)$ of order $\nu$ and if $C_q$ is a neighborhood of $a_q$ containing no other zero of $\phi'(z)$, then for $n > n_0$, each $a_n(z)$ has zeroes of total multiplicity $\nu$ in $C_q$. For a given $\epsilon > 0$, $C_q$ can be chosen with radius $\epsilon$. 


For each \( n > n(\leq) \) choose one zero \( \zeta_n \) in \( C \). The relation
\[
\sum_{k=1}^{N} \frac{1}{|\zeta_k|^{n}} \leq N
\]
follows directly from this choice and (4.2.5). Then \( \sum_{k=1}^{N} \frac{1}{|\zeta_k|^{m}} \) is convergent. Supposing \( m \) to be an integer, it follows that the corresponding canonical product \( P(\zeta) \) has genus \( \leq m-1 \). The Hadamard factorization (1.2.8) gives
\[
\phi(\zeta) = \varepsilon \psi(\zeta) P(\zeta)
\]
where \( \psi(\zeta) \) is a polynomial of degree not greater than the order of \( \phi(\zeta) \); therefore not greater than \( m \). The factorization \( \phi(\zeta) = \varepsilon \psi(\zeta) \), where \( \psi(\zeta) \) has genus not greater than \( m-1 \) follows directly.

4.3 Corollaries of Theorem VI. Among the several corollaries of theorem VI, [2], only those concerning sequences of rational functions with zeros on and outside a circle will be proved here.

Theorem VII. Let \( R_n(z) \) be of the form
\[
(4.3.1) \quad R_n(z) = A_n \prod_{k=1}^{n} \left( 1 - \frac{z}{\zeta_k} \right)
\]
with \( r_k > 1 \). If \( R_n(z) \) converges uniformly in some circle interior to \( |z| \leq 1 \) to a function not identically zero, then \( R_n(z) \) converges uniformly in each region \( G \), \( 1 \notin G \), to a function \( F(z) = \phi(\zeta) \), where \( \phi(\zeta) \) is an entire function of order not more than 2.

Suppose first that \( \zeta \) is a value of \( z \) such that \( |\zeta| < 1 \) and \( R_n(\zeta) \) tends to a non-zero limit. The linear transformation
\[
\omega = \frac{1 - \zeta}{1 - \alpha} \left( \frac{z - \zeta}{1 - \alpha z} \right)
\]
maps the interior of the unit circle onto itself so that \( \zeta \) goes into zero. Thus we may assume that \( R_n(z) \) converges uniformly to a non-zero limit in \( C \), a circle with center at the origin. Then \( A_n \rightarrow A, A \neq 0 \). It is enough to consider only \( R_n(z) \) with \( A_n = 1 \).
The hypotheses now include (4.1.2), hence we have (4.2.3). From 
\[ \Re(w) \leq |w|, \] and (4.2.3) it follows that 
\[ \sum_{k=1}^{\infty} \left( 1 - \frac{1}{r_{nk}} \cos \theta_{nk} \right) \leq \left| \sum_{k=1}^{n} \left( 1 - \frac{1}{r_{nk} e^{i\theta_{nk}}} \right) \right| < M_1. \]

But also, 
\[ \left| 1 - \frac{1}{r_{nk} e^{i\theta_{nk}}} \right|^2 = \left( 1 - \frac{1}{r_{nk}} \cos \theta_{nk} \right)^2 + \frac{1}{r_{nk}^2} \sin^2 \theta_{nk} \]
\[ = 1 + \frac{1}{r_{nk}^2} - \frac{2}{r_{nk}} \cos \theta_{nk}. \]

The hypothesis \( r_{nk} \geq 1 \) gives \( \left| 1 - \frac{1}{r_{nk} e^{i\theta_{nk}}} \right|^2 \leq 2 \left( 1 - \frac{1}{r_{nk}} \cos \theta_{nk} \right) \) and 
\[ \sum_{k=1}^{\infty} \left| 1 - \frac{1}{r_{nk} e^{i\theta_{nk}}} \right|^2 < 2 M_1. \] Now we can apply theorem VI with \( m = 2 \) to complete the proof of theorem VII.

The proof above gives at once the following:

**Corollary 4.3.1.** If \( R_n(z) \), (4.3.1) has all its zeros on \(|z|=1\) and the sequence converges uniformly to a non-zero limit in some circle in the plane, then it converges uniformly in every region \( G, 1 \notin G \), to a limit function \( F(z) = \phi(j) \), where \( \phi(j) \) is an entire function of order not more than 2.

One should note that the requirement that the limit function be not identically zero is essential. The example \( R_n(z) = \left( \frac{z+1}{z-1} \right)^n \) gives a sequence which converges uniformly to zero in each closed set in the half-plane \( \Re(z) < 0 \) but tends to infinity uniformly in each closed set in \( \Re(z) > 0 \). All zeros are at \( z = -1 \). This requirement was used in the proof in securing the bound on the zeros given in (4.2.3).

Instead of requiring that all zeros lie in \(|z| \geq 1\), we can allow a bounded number of exceptions.

**Corollary 4.3.2.** Suppose \( R_n(z) \) has the form
where \( T \) is fixed, \( r_{nk} \geq 1 \), \( \beta_{nk} \) is unrestricted. If the sequence converges uniformly to a non-zero limit \( R(z) \) in a circle \( C \) about \( z = 0 \), then it converges uniformly in every region \( G \), \( 1 \notin G \). The limit function \( R(z) = \phi(z) \) is an entire function of \( z \) of order at most 2.

The uniform convergence to a non-zero limit in \( C \) implies that for \( n > n_0 \), \( |\beta_{nk}| > S > 0 \). Then \(|1 - \frac{1}{\beta_{nk}}|^2 < (1 + \frac{1}{S})^2\) and

\[
\sum_{k=1}^{T} \left|1 - \frac{1}{\beta_{nk}}\right|^2 < T(1 + \frac{1}{S})^2.
\]

The remainder of the proof is unchanged.

Two further corollaries are most conveniently stated for sequences of polynomials \( P_n(z) \). (1) If \( P_n(z) \) has all zeros in the angular region \( \Omega : |\arg z| \leq \alpha < \frac{\pi}{2} \) and converges uniformly in a circle outside \( \Omega \), the limit function is entire of order at most 1. (2) If \( P_n(z) \) has all zeros in alternate ones of the \( 2\pi \) angular regions with vertex at the origin and opening \( \frac{\pi}{r} \) and converges uniformly in a circle about the origin, the limit function is entire of order at most \( 2r \).
5. Estimate from below on the Measure of Approximation.

5.1 Approximation in the unit circle. The purpose of section 5 is to apply theorem VII to obtain an estimate from below on the measure of approximation $\phi_n(d)$. It will be convenient to define $\phi_n(d)$, a measure of approximation for a class of polynomials larger than $\{r,n\}$. The new class of polynomials will be those with zeros on or outside $\Gamma$. Denote by $\{r,n\}$ this class. The measures $\phi_n(d)$ are defined by the relations of (2.1.4) with $\{r,n\}$ replaced by $\{r,n\}$. The functions $\phi_n(d)$ are clearly decreasing functions of $d$.

Theorem 5.1.1. Let $\Gamma$ be the circle $|z|=1$ and let $\phi_n(d)$ be measure of approximation by polynomials of $\{r,n\}$. There exists a constant $K$, positive, such that

(5.1.1) \[ \phi_n\left(\frac{1}{n}\right) > K, \quad n > n_0. \]

From the definition of $\phi_n(d)$, (2.1.4), it follows that given $f(z) \in F$, there exists a polynomial $P_n(z) \in \{r,n\}$ such that

\[ |f(z) - P_n(z)| \leq \phi_n(1 - |z|). \]

Let $g(z) \in F$ be fixed. For $0 < a < 1$, consider $\gamma = \frac{z - a}{1 - a z}$, $z = \frac{1 + a \gamma}{1 - a \gamma}$. This transformation maps the unit circle onto itself so that $z = a$ goes into $\gamma = 0$, $z = 1$ goes into $\gamma = 1$. Let $F(\gamma, a) = g(z)$. Then $f(\gamma, a) \in F$. Thus

\[ |f(\gamma, a) - P_n, a(\gamma)| \leq \phi_n(1 - |\gamma|), \]

Let $P_n, a(\gamma) = Q_{n,a}(\gamma)/(1 - az)^n$. Since $\phi_n(d)$ is decreasing and

\[ \frac{z - a}{1 - a z} \leq \frac{|z| + a}{1 + a |z|}, \quad 0 \leq |z| \leq 1, \]

(5.1.2) \[ |g(z) - \frac{Q_{n,a}(z)}{(1 - az)^n}| \leq \phi_n(1 - \frac{|z - a|}{1 - az}) \leq \phi_n(1 - \frac{|z| + a}{1 + a |z|}) \]
where \( Q_{n,a}(z) \in \{1,n\} \). Choose \( a = a_n = \frac{-t}{n} \), \( t > 0 \), \( t \) fixed, and set \( Q_{n,a}(z) = Q(z) \). Then, \( \left( \frac{1-a_n}{1-z} \right)^n = \left( 1 + \frac{t}{1-z} / n \right)^n \) and
\[
\lim_{n \to \infty} \left( \frac{1-a_n}{1-z} \right)^n = e^{\frac{t}{1-z}}. \text{ Multiplying (5.1.2) by } \left| \frac{1-a_n}{1-z} \right|^n,
\]
\[
\left( \frac{1-a_n}{1-z} \right)^n g(z) - \frac{Q_n(z)}{(1-z)^n} \leq \left| \frac{1-a_n}{1-z} \right|^n \phi_n(1 - \frac{1}{1+a_n}),
\]

For \(|z| < r < 1\),
\[
(5.1.3) \quad \left| e^{\frac{t}{1-z}} q(z) - \frac{Q_n(z)}{(1-z)^n} \right| \leq C(r,t) \phi_n(1 - \frac{a_n+r}{1+a_n}) + \varepsilon_n
\]

where \( C(r,t) \) is independent of \( n \), and \( \varepsilon_n \) is defined by
\[
\varepsilon_n = \left| q(z) \in \frac{t}{1-z} - q(z) \left( \frac{1-a_n}{1-z} \right)^n \right|.
\]

For fixed \( r \), \(|z| < r \), \( \varepsilon_n \) clearly tends uniformly to zero as \( n \to \infty \).

But \( \frac{Q_n(z)}{(1-z)^n} \) is a rational function of the type considered in theorem VII. Now \( g(z) \) can surely be chosen so that \( g(z)e^{\frac{t}{1-z}} \) is not an entire function of \( \frac{x}{z-1} \). Therefore, by theorem VII, there exists \( K_1(r,t) \), independent of \( n \), such that
\[
0 < K_1(r,t) \leq \frac{M_{\max}}{|z|^{2n}} \left| e^{\frac{t}{1-z}} q(z) - \frac{Q_n(z)}{(1-z)^n} \right|
\]

From (5.1.3) there exists \( K(r,t) \) such that
\[
(5.1.4) \quad 0 < K(r,t) \leq \phi_n(1 - \frac{a_n+r}{1+a_n}), n > n(r,t).
\]

Only the simplification of the form of the result remains.

Consider \( F(x) = \frac{a+x}{1+a} \), \( 0 < a < 1 \), \( 0 \leq x \). Then \( F'(x) = \frac{1-a^2}{(1+a)^2} \). Clearly the graph of \( F(x) \) is concave down, \( F(0) = a \), \( F(1) = 1 \), \( F'(1) = \frac{1-a}{1+a} \). For \( x \geq 0 \) the graph of \( F(x) \) will lie under the tangent at the point \( x = 1 \). In other words, \( F(x) \leq |1 - (1-x) \frac{1-a}{1+a}| \)

for \( x > 0 \), \( 0 < a < 1 \). It is clear that this holds also for \( 0 \leq a \leq 1 \).

Substituting \( a_n \) for \( x \) and \( r \) for \( a \),
or \[
\frac{a_n + r}{1 + a_n r} \leq 1 - (1 - a_n) \left( \frac{1 - r}{1 + r} \right), \quad 0 \leq r \leq 1,
\]
\[
(1 - a_n) \left( \frac{1 - r}{1 + r} \right) \leq 1 - \frac{a_n + r}{1 + a_n r}, \quad 0 \leq r \leq 1.
\]
Since \( 1 - a_n = \frac{t}{n} \) and \( \phi_n(d) \) is decreasing, the choice \( r = \frac{1}{3} \), \( t = 2 \) in (5.1.4) gives
\[
0 < \mathcal{K} \leq \phi_n \left( \frac{1}{n} \right), \quad n > n_0.
\]

5.2 More general regions.

Theorem VIII. Suppose \( \Gamma \) is a closed Jordan curve with interior \( D \) and \( z_0 \) is a point of \( \Gamma \) such that there exists a circle \( C \) passing through \( z_0 \) and lying in \( D \cup \Gamma \). Then there exists \( \mathcal{K} > 0 \) and \( n_0 \) such that \( \phi_n \left( \frac{a}{n} \right) \geq \mathcal{K} \) for \( n > n_0 \), where \( a \) is the radius of \( C \). The function \( \phi_n(d) \) is the measure of approximation defined by (2.1.4) for the class \( \{ \Gamma, n \} \).

For convenience suppose for the moment that \( a = 1 \). Denote by \( p_n(z) \) polynomials of \( \{ \Gamma, n \} \), and by \( q_n(z) \) polynomials of \( \{ C_1, n \} \), i.e. polynomials with zeros on or outside \( C \). Since all points of \( \Gamma \) lie either on or outside \( C \), it follows that \( \{ \Gamma, n \} \subset \{ C_1, n \} \). Let \( A(d) \) and \( \phi_n(d) \) be defined for the region \( D \) bounded by \( \Gamma \) as in (2.1.1) and (2.1.4). Let \( C^0 \) denote the interior of \( C \). Define for \( d > 0 \)
\[
\mathcal{S}(d) = \left\{ z : z \in C^0, \text{dist} (z, C) \geq d \right\}.
\]
Taking \( F \) to be the class of functions holomorphic, never zero, and bounded by one in \( D \cup \Gamma \), we can define a measure of approximation for this class in the sets \( \mathcal{S}(d) \). In fact we define two such functions. Let \( \psi_n(d) \) be the measure of approximation to the class \( F \) in \( \mathcal{S}(d) \) by polynomials \( p_n(z) \in \{ \Gamma, n \} \). Let \( \phi_n(d) \) be the measure of approximation to the class \( F \) in \( \mathcal{S}(d) \) by polynomials \( q_n(z) \in \{ C_1, n \} \). Since
\[ \{g_{1,n}\} \supset \{\Gamma, n\} \text{ it follows that} \]

(5.2.1) \[ \phi_n(\delta) \leq \psi_n(\delta). \]

But also if \( z \in C \) and if \( \text{dist}(z,C) \geq \delta \), then \( z \in D \) and \( \text{dist}(z,\Gamma) \geq \delta \) since \( C \) lies in \( D \cup \Gamma \). In other words \( S(\delta) \subset A(\delta) \). Since \( \psi_n(\delta) \) and \( \rho_n(\delta) \) are both defined for polynomials \( P_n(z) \in \{ \Gamma, n \} \), it follows that

(5.2.2) \[ \psi_n(\delta) \leq \rho_n(\delta). \]

Hence from (5.1.1), (5.2.1) and (5.2.2) we obtain

\[ 0 < K \leq \phi_n(\frac{a}{n}) \leq \psi_n(\frac{a}{n}) \leq \rho_n(\frac{a}{n}). \]

When \( C \) has radius \( a \neq 1 \), the transformation \( J = \frac{1}{a}(z-a) \), \( a \) the center of \( C \), shows that \( \phi_n(\frac{a}{n}) \geq K \) and \( \rho_n(\frac{a}{n}) \geq K \) for \( n > n_0 \).
6. A Special Region.

Theorem I asserts for curves $\Gamma$ which are simple, closed, rectifiable Jordan curves the existence of a sequence of rational functions $R_n(z)$ with $\lambda_n$ zeros on $\Gamma$ and a pole of order $\lambda_n$ at $z_0$ outside $D \cup \Gamma$ which tends uniformly in any compact subset of $D$ to a given function $f(z)$, holomorphic and never zero in $D$. Theorem VII shows that if $z_0$ is a point of $\Gamma$ such that a circle $C$ through $z_0$ lies in $D \cup \Gamma$, sequences of rational functions with $k$ zeros on $\Gamma$ and a pole of order $k$ at $z_0$ on $\Gamma$ can converge to only a restricted class of functions which are entire in the variable $\frac{1}{z-z_0}$. How we shall prove the following.

Theorem IX. There exists a Jordan domain $D$ and a point $z_0$ on its boundary $\Gamma$ with the following property: Given a function $f(z)$ holomorphic and never zero in $D$, there exists a sequence $R_n(z)$ of rational functions with $\lambda_n$ zeros on $\Gamma$ and a pole of order $\lambda_n$ at $z_0$ on $\Gamma$ such that $R_n(z)$ converges to $f(z)$ in $D$, uniformly in each compact subset of $D$.

For convenience we take for $z_0$ the point at infinity; the rational functions are then polynomials.

Theorem I and the theorem of Runge (1.2.9) are the basis of the construction. Runge's theorem can be modified to say that the approximating polynomials have rational coefficients since any polynomial can be approximated by polynomials with rational coefficients. Denote by $\{Q_n(z)\}$ a sequential arrangement of all polynomials with rational coefficients.

Let $D_k$ be the domain interior to the Jordan curve $\Gamma_k$ and let $A_k$ be the set $A_k = \{z : z \in D_k, \text{ dist}(z, \Gamma_k) \geq \frac{1}{k}\}$. 
Let \( \Gamma_1 \) be the circle \(|z| = 2\). Since \( e^{Q_1(z)} \) is holomorphic and never zero in \( D_1 \), \(|z| < 2\), there is a polynomial \( P_1(z) \) with all zeros on \( \Gamma_1 \) (theorem 1) such that
\[
|Q_1(z) - \log P_1(z)| < 1, \quad z \in \Omega_1.
\]
Since \( P_1(z) \) has only a finite number of zeros there is a domain \( D_2 = D_1 \cup R_1 \), where \( R_1 \) is a rectangle \( 0 < x < 3, 0 < y < y_1 \), such that all zeros of \( P_1(z) \) also lie on \( \Gamma_2 \), the boundary of \( D_2 \). This is indicated in figure 2.

![Figure 2](image)

Figure 2

Now there is a polynomial \( P_2(z) \) with all zeros on \( \Gamma_2 \) such that
\[
|Q_2(z) - \log P_2(z)| < \frac{1}{2}, \quad z \in \Omega_2.
\]
Since \( P_2(z) \) has only a finite set of zeros there is a domain \( D_3 = D_2 \cup R_2 \), where \( R_2 \) is a rectangle \( 0 < x < 4, 0 < y < y_2, y_2 < y_1 \), such that all zeros of \( P_2(z) \) also lie on \( \Gamma_3 \). This is shown in figure 3.

![Figure 3](image)

Figure 3

In this way we get a sequence of domains \( D_k \) and a sequence of polynomials \( P_k(z) \) such that
\[
(6.1.1) \quad |Q_k(z) - \log P_k(z)| < \frac{1}{k}, \quad z \in \Omega_k.
\]

The limiting domain \( D \) is bounded by a Jordan curve on the sphere,
has infinity as a boundary point, and, moreover, all zeros of the polynomials \( P_k(z) \) are on the boundary \( \Gamma \) of \( D \).

Suppose \( f(z) \) is holomorphic and never zero in \( D \). By Runge's theorem there is a sequence \( Q_n(z) \) which tends to \( G(z) = \log f(z) \) uniformly in each compact subset of \( D \). Now,

\[
|G(z) - \log P_n(z)| \leq |\log P_n - Q_n(z)| + |Q_n(z) - G(z)|
\]

Since \( D = \bigcup_{k=1}^{\infty} A_k \), it follows from (6.1.1) that the sequence \( P_n(z) \) converges to \( f(z) \), uniformly in each compact subset of \( D \). The polynomials \( P_n(z) \) have all their zeros on \( \Gamma \).

Still unanswered is the question of what is true if, for example, \( \Gamma \) has tangents at \( z_0 \) which make an interior angle less than \( \pi \).
Bibliography


