RICE UNIVERSITY

SUPERNOVAE LIGHT CURVES

by

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A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
MASTER OF SPACE SCIENCE

Thesis Director's Signature: 

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Houston, Texas

June, 1972
ABSTRACT

The characteristics of the light curves of supernovae, (luminosities as functions of time) are an old problem in Astrophysics. Characteristic half-widths (in the visual spectrum) are of the order of 20 - 50 days, with peak (integrated) luminous energies at least of the order of $10^{50}$ ergs. In addition, spectroscopy reveals Doppler shifts suggesting associated velocities of expansion of the order of $10^3 - 10^4$ km/sec in the outer regions of the exploding star. The problem of constructing a hydrodynamic model of such an object, which duplicates the features of the light curve, especially the energetics and the temporal characteristics, is further complicated by the presence of a variety of distinct types of supernovae, with widely varying light curves.

An attempt is made to explore the hydrodynamic structure required to produce such characteristic luminosities, with emphasis upon the case of radiative diffusion as the dominant
mode of energy transport. The models investigated will be of concentric spherical mass shells expanding radially at velocities of order $10^3 - 10^4$ km/sec. Effects of the outer zones becoming optically thin will be examined, as well as effects of varying masses, compositions, ionization, shock wave heating, and radioactive decay of the (possible) nuclear products of the "explosion". It is hoped that the successful specification of such parameters as mass, composition, spatial size, as well as the mechanism responsible for the supernova outburst, will add to our understanding of the processes of stellar evolution, nucleosynthesis, and galactic evolution.
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I. INTRODUCTION

Supernovae are among the most cataclysmic events in nature, traditionally believed to represent the final disruption of a star at its evolutionary death. The problem of detailing the processes involved in the formation of a supernova outburst is complex, and is subject to wide speculation among astrophysicists. The observational data are at best sketchy, although since 1956 astronomers have added greatly to our knowledge through a systematic and internationally cooperative search.

Zwicky (1965) divides supernovae into five distinct types. We will, however, be primarily concerned with the features of the so-called Type I and Type II supernovae. A discussion of these classes chiefly involves contrasts in the characteristic light curves and spectra, as well as other features, such as the associated velocities of expansion inferred from Doppler shifts, ejected masses as inferred from spectra and the study of supernova remnants, and the like. The spectra of both classes are quite complicated. Type II (SNII) spectra resemble that of ordinary novae (Zwicky, 1956; 1965), with the major difference that the velocities of ejected matter are larger in the case of
the supernovae -- typically on the order of $10^3 - 10^4$ km/sec. Type I (SNI) spectra, on the other hand, are quite confusing; for a complete discussion, see Minkowski (1939). They appear to consist of very broad, overlapping bands which have thus far defied interpretation. SNI are inherently several magnitudes brighter than SNII; velocities of ejection are inferred on the order of or greater than $10^4$ km/sec. Both types seem to be associated with spiral arms in spiral galaxies, while only SNI are observed in elliptical and old spiral (SO) galaxies.

There are major contrasts, too, in the light curves of SNI and SNII. Curves for Type I objects are reasonably uniform in shape, while those of Type II show considerable variation. The essential features of the (visual) light curves of these objects are depicted in Figures I and II. Typical integrated luminosities are on the order of at least $10^{50}$ ergs, and widths of the peaks are typically of the order 20-50 days. The brightest SNI may have an absolute photographic magnitude, $M_p$, as much as $-18$ to $-19$, while those of SNII are less bright, generally from $-13$ to $-16$. Typical values of $\Delta m$ (see Figure I) are near 2.5; the change in magnitude from peak to final luminosity $m_L$ lies generally
in the range $m_{\text{max}} - m_{\text{L}} = 15$ to 19. Type II supernovae display very intense ultraviolet spectra near maximum; observations of Type are incomplete in this regard (Arp, 1961). The decrease in supernovae luminosity is observed to be accompanied by a change in color toward the red (Shklovsky, 1968). For a full discussion of these curves, and supernovae in general, see, for example (Zwicky, 1956; 1965), (Payne-Gaposchkin, 1964), and (Shklovsky, 1968).

Several previous attempts to model the behavior of supernovae should be mentioned briefly. Colgate and White (1966) suggested that the collapse of a stellar core to a neutron star supplied the kinetic and optical energies of supernovae; however, the predicted optical energy was far too small. Even the presence of a large mass ($2M_\odot$) of radioactive material proved inadequate to explain observed luminosities. The essential difficulty was that the thermal energy was converted to kinetic energy (by adiabatic expansion) far too rapidly for it to be radiated by the object. Colgate and McKee (1969) have suggested that the production of a large mass of Ni$^{56}$ may be responsible for the light curve, the radioactive decay of which provides the radiant energy of an object undergoing spherical expansion. Subsequent
decay modes are then invoked to explain the exponential decay of the light curve after maximum. Arnett (1966, 1967) has made calculations similar to those of Colgate and White (1966), and has concluded that the energy transfer mechanism (neutrinos and antineutrinos escaping from a hot core) will probably not result in an explosive event. Arnett (1969) has also suggested that explosive ignition of a stellar core in the $^{12}\text{C} + ^{12}\text{C}$ reaction resulting from high (degenerate) densities in the core may be a possible supernova mechanism. However, this model totally annihilates itself, leaving no remnant. Significant amounts of $^{56}\text{Ni}$ are produced, suggesting the possibility of radioactive decay being responsible for the later part of the light curve. Resulting velocities of expansion of ejected matter are predicted to be of the order of $2 \times 10^4$ km/sec, not too far removed from possible values inferred for SNI.

Morrison and Sartori (1969) have suggested a "fluorescence" model as a mechanism for SNI. Illumination of an extended ($R \sim 10^{17}$ cm) helium ion envelope by pulsed ultraviolet radiation can lead to a fluorescent process of re-emission by the envelope, which might account for the exponential decay of the SNI light curve. This model also attempts to
explain, at least in part, the band-like spectrum. However, Minkowski points out (Brancazio and Cameron, 1968) several major problems with the red part of the spectrum which the model predicts.

Grasberg, Imshennik, and Nadyozhin (1970) studied the propagation of strong shock waves through extended atmospheres in an attempt to explain the light curves of SNII. Shock heating as the wave emerges from optically thick regions may account for the luminosity peak, while the onset of a recombination cooling wave is used to explain the post-maximum "hump" (refer to Figure II).

This summary is by no means complete; however it serves to indicate the extreme complexity of the supernova phenomenon, and mentions most of the approaches currently being taken toward this problem. Only with a suitable hydrodynamic model can we hope to delineate the mechanism of the explosion and assign values to the various parameters important in specifying stellar evolution and nucleosynthetic processes and the possible importance of the supernova in them.
II. THE HYDRODYNAMIC EQUATIONS

Consider a macroscopic (ideal) fluid in which the characteristic distance of interest \( d \) is very much larger than the interparticle spacing, \( d \gg (N)^{-1/3} \), where \( N \) is the number of particles (be they nucleons, ions, atoms, of molecules, etc.). If we construct an area element \( d\mathcal{A} \) where \( d\mathcal{A} \) is the area and \( \mathcal{A} \) is normal to this area in the outward direction, then the mass flux through \( d\mathcal{A} \) is given by \( \rho v \cdot \mathcal{A} \), where \( \rho \) is the mass density and \( v \) the fluid velocity. The mass flow out of volume \( V \) per unit time is then \( \int_S \rho v \cdot d\mathcal{A} \) and the decrease in mass contained in \( V \) is \( \frac{d}{dt} \int_V \rho \, dV \). Equating these equations and transforming via Green's theorem,

\[
\int_S \rho v \cdot d\mathcal{A} = \int \nabla \cdot (\rho v) \, dV = -\frac{d}{dt} \int_V \rho \, dV
\]

or

\[
\int \left[ \frac{d\rho}{dt} + \nabla \cdot (\rho v) \right] \, dV = 0
\]

for the case of a fixed volume \( V \). The integrand must then be zero, and we have the so-called "equation of continuity" which reflects mass conservation:

\[
\frac{d\rho}{dt} + \nabla \cdot (\rho v) = 0 \quad (2-1)
\]
Now if $P$ is the isotropic pressure, the force on the surface of volume $V_0$ is given by

$$ -\oint P \, dA = -\int_{V_0} \nabla P \, dV $$

Using Newton's Law, we may write

$$ \rho \frac{d}{dt} (\mathbf{V}) = -\nabla P $$

as the force per unit area acting on the surface of $V_0$. Now

$$ \frac{d}{dt} \mathbf{v} = \frac{d}{dt} \mathbf{V} + (\mathbf{v} \cdot \nabla) \mathbf{v} $$

so that we have Euler's Equation:

$$ \rho \frac{d}{dt} \mathbf{v} = \rho \left[ \frac{d}{dt} \mathbf{V} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = -\nabla P $$

In the presence of a Newtonian gravitational field we have a second term on the RHS due to the additional force $pg$ on each unit volume and Euler's Equation becomes

$$ \rho \frac{d}{dt} \mathbf{v} = -\nabla P + \rho g \tag{2-2} $$

where, for a self-gravitating fluid mass we have the gravitational potential $\phi$ which satisfies Laplace's Equation

$$ \nabla^2 \phi = 4\pi G \rho $$

and as usual $g = -\nabla \phi$. 
Now define the mass variable $m(r,t)$ as

$$m(r,t) = \int_0^r 4\pi r^2 \rho(r,t) \, dr$$

(2-3)

so that

$$\frac{\partial m(r,t)}{\partial r} = 4\pi \rho r^2$$

(2-3a)

The total time derivative of $m$ is

$$\frac{d m(r,t)}{d t} = \frac{\partial m}{\partial t} + \mathbf{v} \cdot \frac{\partial m}{\partial r}$$

Now $\frac{\partial m}{\partial t} = \int_0^r 4\pi r^2 \rho \, dr \, \mathbf{v}$ and in spherical coordinates

$$\frac{\partial \rho}{\partial t} = \frac{\partial (r^2 \rho \mathbf{v})}{r^2 \partial r}$$

so we have $\frac{d m(r,t)}{d t} = 0$ which reflects mass conservation; this same property also allows us to make frequent use of the mass variable as a Lagrangian variable (i.e., one which uses the reference frame of the fluid). We may identify then, for all values of $t$,

$$\frac{d m}{d t} = 4\pi r^2 \rho \, dr$$

(2-4)

and we may write Euler's Equation in the form

$$\rho \frac{d \mathbf{v}}{d t} = - \frac{\partial \mathbf{P}}{\partial r} - \frac{G m(r) \rho}{r^2}$$

(2-5)
where $g = - \frac{Gm(r)}{r^2}$. We may rewrite this as

$$\ddot{r} = - 4\pi r^2 \frac{\partial P}{\partial m} - \frac{Gm(r)}{r^2}$$

(2-6)

where we have used the notation

$$\dot{r} = \frac{dr}{dt} = \dot{v}, \quad \ddot{r} = \frac{d^2r}{dt^2} = \ddot{v}$$

and the relation between $dm$ and $dr$ in Equation (2-4).

We may now consider the second order moment Equation by multiplying the momentum equation (Euler's Equation) by $r$:

$$r \dot{\dot{r}} = - 4\pi r^2 \frac{\partial P}{\partial m} \dot{r} - \frac{Gm(r) \dot{r}}{r^2}$$

Noting that

$$\dot{r} \ddot{r} = \frac{d}{dt} \left( \frac{r^2}{2} \right) = \frac{d}{dt} \left( \frac{v^2}{2} \right)$$

and also that, by the chain rule, $\dot{r}/r^2 = - d(r^2)/dt$

we may rewrite the second order moment equation as

$$\frac{d}{dt} \left( \frac{v^2}{2} \right) = - 4\pi r^2 \frac{\partial P}{\partial m} v + Gm \frac{d}{dt} \left( \frac{1}{r} \right)$$

Now notice

$$\frac{\partial}{\partial m} (4\pi r^2 v P) = 4\pi r^2 v \frac{\partial P}{\partial m} + P \frac{\partial}{\partial m} (4\pi r^2 v)$$

Look at the second term on the right:

$$P \frac{d}{dm} (4\pi r^2 v) = P \frac{\partial}{\partial m} (4\pi r^2 \frac{dr}{dt}) = P \frac{d}{dt} (4\pi r^2 \frac{dr}{dm})$$
and from equation (2-4) we have a relation for $\frac{\partial \rho}{\partial m}$, so that we may write

$$P \frac{\partial}{\partial m} (4\pi r^2 v) = P \frac{d}{dt} \left( \frac{1}{\rho} \right) = P \frac{d}{dt} (V)$$

where $V = \frac{1}{\rho}$ is the specific volume, i.e. the volume of one gram of fluid. Hence we may write the second moment equation as

$$\frac{d}{dt} \left( \frac{v^2}{2} - \frac{Gm}{r} \right) = - \frac{\partial}{\partial m} \left( 4\pi r^2 v P \right) + P \frac{d}{dt} (V) \quad (2-7)$$

We may interpret the LHS of equation (2-7) as the time rate of change of the total energy (kinetic plus gravitational potential), and the RHS as (first term) the divergence of the flux of mechanical energy and (second term) the work done. Recalling the first law of thermodynamics, $dQ = dE + PdV$ where $dQ$, $dE$ are the differentials of heat and internal energy respectively, and $PdV$ represents the mechanical work, we may write equation (2-7) as

$$\frac{d}{dt} \left( \frac{v^2}{2} + E - \frac{Gm}{r} \right) = - \frac{\partial}{\partial m} \left( 4\pi r^2 v P \right) + \frac{dQ}{dt}$$

If we now make the identification

$$\frac{dQ}{dt} = - \frac{dL}{\partial m} - \dot{S}_v + \dot{\varepsilon} \quad (2-7a)$$
where $L$ is the luminosity, i.e. the total flux through the surface, $S_v$ is the energy loss by neutrinos, and $\mathcal{E}$ is the nuclear energy generation rate, we may finally write the complete energy conservation equation as

$$
\frac{d}{dt} \left( \frac{v^2}{2} + E + S_v - \mathcal{E} - \frac{G_m}{r} \right) = - \frac{\partial}{\partial m} \left( 4\pi r^2 \nu P + L \right)
$$

(2-8)

However, for the purposes of most calculations, the combination of (2-7a) and $dQ = dE + PdV$ is most useful;

$$
E + P \dot{V} + \frac{dL}{dm} = 0
$$

(2-9)

and equation (2-8) may then be utilized as an independent check on the accuracy of the calculational method.

---

RADIATIVE TRANSPORT

Assume a distribution function $f(\nu, \Omega, \mathbf{r})$ for photons in the radiation field; the number of photons per cm$^3$ in frequency $d\nu$, solid angle $d\Omega$, and in the direction of $d\mathbf{r}$ is

$$
f(\nu, \Omega, \mathbf{r}) d\nu d\Omega d\mathbf{r};
$$

the number density is given by

$$
f_0(\nu, \mathbf{r}) = \int \Omega f d\Omega
$$

(2-10)
The intensity of radiation is

$$I(\nu, \Omega, r) = h \nu c \int (\nu, \Omega, r) d\Omega = h \nu c f$$  \hspace{1cm} (2-11)$$

where \(c\) is the velocity of light. The total energy density is then given by

$$u(\nu, \Omega) = \int_\Omega I(\nu, \Omega, r) d\Omega = h \nu c f$$ \hspace{1cm} (2-12)$$

the energy flux through a surface normal to \(r\) is

$$j(\nu, \Omega) = \int_\Omega I(\nu, \Omega, r) \cos \Theta d\Omega = \int_\Omega h \nu c f(\nu, \Omega, r) \cos \Theta d\Omega$$ \hspace{1cm} (2-13)$$

and the radiation pressure is

$$P_r = \frac{2}{c} \int_\Omega \int_\Sigma I(\nu, \Omega, r) \cos^2 \Theta d\Theta$$ \hspace{1cm} (2-14)$$

where \(\Theta\) is the angle between the (outward) normal to the surface and the photon direction. In general, stellar material will absorb from, emit into, and scatter photons both into and out of any such radiation flux.

In a condition of local thermodynamic equilibrium (LTE), there will be an energy balance governing the behavior or any specified volume of matter of fixed mass as to (true) absorption and emission:

$$4 \pi j(\nu) = \chi_\nu(\nu) c f(\nu)$$ \hspace{1cm} (2-15)$$
where $\kappa_a(\nu)$ is the opacity due to true absorption, and the factor of $4\pi$ represents an integral over the solid angle $\Omega$ in an isotropic radiation field. We define the source function (Planck Function) $B_\nu(T)$ for the case of LTE by

$$B_\nu(T) = \frac{c u(\nu)}{4\pi} = \frac{2\hbar \nu^3}{c^2} \frac{1}{e^{\hbar \nu/kT} - 1}$$  \hspace{1cm} (2-16)$$

The balance of emission and absorption in LTE, known as Kirchoff's Law, may then be expressed as

$$j(\nu) = \kappa_a(\nu) B_\nu(T)$$  \hspace{1cm} (2-17)$$

In cases where LTE is not strictly valid, however, the emission contribution to Kirchoff's Law must be treated in two separate terms, as induced emission and spontaneous emission. Departures from LTE serve to introduce anisotropies in the radiation field, and hence anisotropies in the induced emission:

$$j(\nu, \theta) = \kappa_a(\nu)[1 - e^{-\hbar \nu/kT}] B_\nu(T) + \kappa_a(\nu) e^{-\hbar \nu/kT} I(\nu, \theta)$$

where the last term represents the (anisotropic) induced emission, and $I(\nu, \theta)$ is not in general equal to $B_\nu(T)$.

Scattering effects, both into and out of a given percent of flux, may be represented by (Clayton, 1968, pp. 178-
and the equation of transfer (for a plane-parallel atmosphere) for the case of LTE is

\[
\frac{1}{\rho} \frac{dI(v)}{dr} \cos \Theta = - \left[ \kappa^s(v) + \kappa_s(v) \right] I(v, r, \Theta) \\
+ \kappa^\star(v) B_v(T) + \kappa_s(v) \frac{1}{4\pi} \int_{\Omega} f(\Theta_1, \Theta_2, \Theta_3) I(v, \Theta_1, \Theta_2, \Theta_3) d\Omega
\]

where \( \kappa_s(v) \) is the scattering opacity, \( f(\Theta_1, \Theta_2, \Theta_3) \) is the scattering phase function which accounts for the angular relation of the percent of flux under consideration and any other such percent which may be involved in the scattering process, and \( \kappa^\star(v) \) is defined to be

\[
\kappa^\star(v) = \kappa_a(v) \left[ 1 - e^{-h\nu/kT} \right]
\]

which is just the opacity due to true absorption multiplied by the ratio of spontaneous emission to total emission. To obtain a relation for the energy flow per unit area in terms of an average opacity and a temperature gradient, we multiply equation (2-18) by \( \cos \Theta \) (to relate the photon flux direction to the radial coordinate) and integrate over \( d\Omega \).
Noting the relations (2-13) and (2-14), and that, for LTE, $B_\nu(T)$ is isotropic (and its integral over $d\Omega$ is zero), the equation of transfer reduces to

$$\frac{c}{\rho} \frac{dP_r}{dr} = - \left( \chi_a^* (\nu) + \chi_s (\nu) \right) j (\nu, r) \tag{2-19}$$

if we assume that the phase function contains only even powers of $\cos \Theta$, (Clayton, 1968, p. 181).

Now, even for the case of slight departures from strict isotropy in the radiation field, the radiation pressure is related to the energy density by

$$P_r (\nu) = \frac{1}{3} u (\nu) \tag{2-20}$$

and further, in the case of LTE we may write

$$\frac{d u (\nu)}{dr} = \frac{d u (\nu)}{dT} \frac{dT}{dr} \tag{2-21}$$

Hence, the total heat flux per unit (surface) area is given by the integral over all frequencies $\nu$ of $j(\nu)$:

$$F = \int_0^\infty j (\nu) d\nu = - \frac{c}{3 \rho} \int_0^\infty \frac{1}{\chi_a^* (\nu) + \chi_s (\nu)} \frac{d u (\nu)}{dT} \frac{dT}{dr} d\nu$$

The temperature gradient may be taken out of the integral over all frequencies and we are left with
The Rosseland mean opacity $\kappa$ is defined by

$$F = -\frac{c}{3\rho} \frac{dT}{dr} \int \frac{1}{\kappa_a^{\infty}(v) + \kappa_s(v)} \frac{d\nu(v)}{dT} d\nu$$  \hspace{1cm} (2-22)$$

The Rosseland mean opacity $\kappa$ is defined by

$$\frac{1}{\kappa} = \frac{\int_0^\infty \frac{1}{\kappa_a^{\infty}(v) + \kappa_s(v)} \frac{dB_v(T)}{dT} d\nu}{\int_0^\infty \frac{dB_v(T)}{dT} d\nu}$$  \hspace{1cm} (2-23)$$

where, again, $B_v$ is the Planck function. If we note that

$$\int_0^\infty \frac{d\nu(v)}{dT} d\nu = \frac{d}{dT} \int_0^\infty \nu(v) d\nu = \frac{d\nu}{dT} = 4a T^3$$

from the black body radiation law, we may then rewrite equation (2-21) in the form

$$F(r) = \frac{4ac}{3\kappa \rho} T^3 \frac{dT}{dr}$$  \hspace{1cm} (2-24)$$

The luminosity is defined as the total flux over the entire surface, so we have, finally, for conditions of LTE,

$$L(r) = -4\pi r^2 \frac{4ac}{3\kappa \rho} T^3 \frac{dT}{dr}$$  \hspace{1cm} (2-25)$$

where the minus sign reflects outflow of energy. (For a more complete discussion of the various opacities involved
above, see Clayton (1968, Ch. 2, 3).

Such relations refer, of course, only to such conditions as are characteristic of local thermodynamic equilibrium, that is, those conditions under which Kirchoff's Law is valid. As mentioned above, one characteristic of LTE is an isotropic radiation field. It is useful to consider conditions under which Kirchoff's Law is no longer valid, and to ask what consequences will arise, most especially with regard to the form of the equation of transfer. However, for the purposes of constructing an initial model, we will use the assumptions of LTE and Kirchoff's Law. Later these simplifications may have to be removed. For a discussion of the equation of transfer in conditions which depart from LTE and/or isotropy in the radiation flux, see, for example, (Frank-Kamenetskii, 1962), (Cox and Guili, 1968), (Campbell, 1965), (Imshennik and Morozov, 1969). One final comment needs to be made concerning the process of energy transfer. For the case in which the temperature gradient is too large, an instability in the pressure balance in the (stellar) fluid will result, and convective processes will set in. The criteria for stability (after Clayton, 1968, pp. 254-255) is that

\[ \left| \frac{d(T)}{dr} \right|_{\kappa} < \left| \frac{d(T)}{dr} \right|_{ad} \]
where \((dT/dr)_{ad}\) is the adiabatic temperature gradient defined by
\[
\frac{dP}{P} + \frac{\Gamma_z}{1-\Gamma_z^2} \frac{dT}{T} = 0
\]
where, for our conditions we can use:
\[
\Gamma_z = \frac{32 - 24 \beta - 3 \beta^2}{2.4 - 18 \beta - 3 \beta^2}
\]
\[
\beta = \frac{P_g}{(P_g + P_r)}
\]
where \(P_g\) is the ideal gas (kinetic) pressure, and \(P_r\) is the radiation pressure. The stability condition may then be written
\[
\left( \frac{dT}{dr} \right)_\ast > \left( 1 - \frac{1}{\Gamma_z^2} \right) \frac{T}{\rho} \left( \frac{dP}{dr} \right)_\ast
\]
We choose to neglect the (possible) effects of convection, however, in our initial models. It is likely that the inaccuracies due to the equilibrium diffusion approximation will far outweigh those which may be expected from ignoring convection; furthermore, such considerations would prove cumbersome and would only serve to complicate the initial models.
BOUNDARY CONDITIONS

The boundary conditions at the stellar center are trivial, namely

\[ m(o,t) = 0, \quad L(o,t) = 0 \]

The simplest choices for outer boundary conditions, that is, at the stellar surface, \( r = R \), are

\[ m(R,t) = M_\ast, \quad P(R,t) = 0, \quad T(R,t) = 0 \]

These conditions are certainly adequate for cases where the region of interest is in the interior; more specifically, in regions of the star where the mean free path for photons is small with respect to the (stellar) characteristic dimension. It is clear that for regions near the surface, where the optical depth is small -- i.e., where the surface lies within several mean free photon paths from the point(s) of interest -- these approximations are no longer valid. The region of interest is "transparent".

For the case of such a calculation, then, a different set of surface conditions must be adopted. Among the simplest of these is the solution of the equation of transport
for a static, gray atmosphere:

\[
\mathcal{T}^4 = \frac{3}{4} \, T_e^4 \left( \tau + q(\tau) \right)
\]

where \( \tau \) is the optical depth, defined by

\[
\tau \equiv - \int_{R}^{r} \rho(x) \, dx
\]

with \( x \) increasing inwardly from the surface, \( T_e \) is the so-called effective (black body) temperature, and \( q(\tau) \) is a numerical value determined from the solution of the integro-differential equation of transfer by the method of solution of a system of 2n corresponding linear equations (after Chandrasekhar, 1960, Ch. III and XI). A close approximation for the time independent diffusive case is given by the Eddington approximation

\[
\mathcal{T}^4 = \frac{3}{4} \, T_e^4 \left( \tau + b \right)
\]

with \( b = 2/3 \), that is,

\[
\left. \frac{d(T^4)}{d\tau} \right|_R = \frac{3}{4} \, T_e^4 = \frac{3}{2} \, T^4 \left|_R \right.
\]

In such a case, the formula for the outer luminosity becomes that of a black body,

\[
L(R) = \sigma \, T_e^4 \, 4\pi r^2
\]
and we have as surface boundary conditions

\[ m(R, t) = M_* \quad P(R + 2/3, t) = 0 \quad T(R + 2/3, t) = 0 \]

where \( R \) in the argument is in units of optical depth. A truly satisfactory prescription for applying these boundary conditions to this problem has not yet been devised.
III. THE EQUATION OF STATE

In the most general treatments it is necessary to include the effects of not only mechanical pressure, radiation pressure, and their associated energy terms, but also effects due to ionization, interparticle interactions, excited states, composition, nuclear reactions, electron degeneracy (both partial and complete), and the relativistic effects in the momentum distribution function. For the case of interest, temperatures (typically of the order of $10^5 - 10^6$ K) and densities (of order unity or less) are such that we may neglect degeneracy and relativistic effects upon the distribution function, and also neglect the effects of nuclear processes and their associated effects (such as energy loss by neutrinos). (See text below; also see Clayton, 1968, Ch. 2).

We may then write the pressure and energy terms for a perfect nonrelativistic, nondegenerate gas, including the radiation field, as

$$P = \mathcal{R} \frac{\mu}{m} T + \frac{a}{3} T^4$$

$$E = \frac{3}{2} \mathcal{R} \frac{\mu}{m} T + \frac{a}{\rho} T^4$$

(3-1)

(3-2)
where \( T \) is the temperature in °K, \( R \) is the gas constant 
\( (R = a_0 k, \text{where } a_0 = 1/M_u = \text{Avagadro's number, and } k \) is 
Boltzmann's constant), \( a \) is the Stephan-Boltzmann constant, 
\( \rho \) is the specific density, and \( \mu \) is the mean weight per 
particle of the gas and is given by

\[
\frac{1}{\mu} = \frac{1}{\mu_e} + \frac{1}{\mu_{ion}} \quad (3-3)
\]

where the subscripts \( e, \text{ion} \) refer to electrons and ions, respectively.

We define a conserved mass density \( \rho_i^* \), which is the 
conserved mass density of nucleons bound up in species \( i \) 
per unit volume, and a total conserved mass density of 
nucleons bound up in all nuclei, \( \rho^* \), by the relations

\[
\rho_i^* = N_i A_i M_u \quad (3-4)
\]

\[
\rho^* = \sum_i \rho_i^* = \sum_i N_i A_i M_u \quad (3-5)
\]

(after W. D. Arnett and K. L. Hainebach, private communication),
where \( N_i \) is the number density of nucleons bound up in species 
\( i \), \( A_i \) is the atomic number of species \( i \), and \( M_u \) is the atomic
mass unit. The term "conserved mass density" is used because it is related to a fundamental conserved quantity, baryon number, not the unconserved quantity, mass. We relate the density to \( \mu_e \) and \( \mu_{ion} \) by

\[
\rho^* = n_{ion} \mu_{ion} M_u \quad , \quad n_{ion} = \sum_i N_i
\]  

\[
\rho^" = n_e \mu_e M_u \quad , \quad n_e = \sum_i N_i Z_i f_i
\]

where \( Z_i \) is the nuclear charge of ion species \( i \), and \( f_i \) is the coefficient of ionization of ion species \( i \). Then, by equation (3-3), \( \mu \) is defined by

\[
\rho^* = N \mu M_u
\]

where \( N \) is the total particle number density,

\[
N = \sum_i N_i (1 + Z_i f_i)
\]

Thus we may finally write

\[
\mu = \frac{\sum_i N_i A_i}{\sum_i N_i (1 + Z_i f_i)}
\]

We now define \( X_i \), the mass fraction of total nucleons bound in species \( i \), as
Clearly, since nucleons are conserved, we must have

$$\sum_i X_i = 1$$  \hspace{1cm} (3-12)

We define also \( Y_i \), which turns out to be the ratio of the number of ions of species \( i \) to the number of nucleons in the system,

$$Y_i = \frac{X_i}{A_i} = \frac{N_i M_u}{\rho^*} = \frac{N_i}{\sum_i N_i A_i}$$  \hspace{1cm} (3-13)

In terms of these new parameters we may now write

$$N_i = \rho^* Y_i a_0$$  \hspace{1cm} (3-14)

$$n_{ion} = \sum_i N_i = \rho^* a_0 \sum_i Y_i$$  \hspace{1cm} (3-15)

We have immediately the relations

$$\frac{1}{\mu_{ion}} = \sum_i Y_i$$  \hspace{1cm} (3-16)

$$\frac{1}{\mu_e} = \sum_i Z_i f_i Y_i$$  \hspace{1cm} (3-17)

$$\frac{1}{\mu} = \sum_i Y_i \left(1 + Z_i f_i \right)$$  \hspace{1cm} (3-18)
It is often useful to write equation (3-18) in a form which involves the species for which \( Z > 2 \) less explicitly:

\[
\frac{1}{\mu} = \left[ Y_H N_H + Y_{He} N_{He} + (1 - x_{H} - x_{He}) \left\langle \frac{N_Z}{A_Z} \right\rangle \right], \tag{3-18a}
\]

\[N_i = 1 + Z_i f_i\]

If now we approximate \( A_Z \) by \( A_Z \simeq 2Z + 2 \) (after Clayton, 1968, p. 83), we have, for solar-like compositions, \( \left\langle \frac{N_Z}{A_Z} \right\rangle \simeq \frac{1}{2} \), and we may then write, for the case of complete ionization,

\[
\mu \simeq 4 / (2 + 6 x_H + x_{He})
\]

Ionization is often an important (temperature dependent) factor in determining \( \mu \), and through it, in affecting the equation of state. It is therefore useful to seek an expression relating densities of ionized species to electrons, for various degrees of ionization. (The following is after K. L. Hainebach, private communication.)

If the particle system is in statistical equilibrium, we have

\[
\sqrt{\frac{n_i}{N_i}} = \sum_r \left\{ \exp \left[ (\epsilon_{i,r} - \mu_i) / kT \right] \pm 1 \right\}^{-1} = N_i \tag{3-19}
\]

where \( V \) is the volume of the system, \( N_i \) is the total number of particles of species \( i \), \( \epsilon_{i,r} \) is the energy of the \( r \)-th
state of species i, and \( \mu_i \), the chemical potential, is defined by

\[
\mu_i = \left( \frac{\partial E}{\partial N_i} \right)_{S, v, N}
\]

(3-20)

where \( E \) is the average internal energy of all particles in volume \( V \), and \( N \) is the total number of particles in \( V \), excepting those of species i. The summation in equation (3-20) is over all \((2J+1)\) spin states of any level \( r \), and the + and - are for the cases of Fermi-Dirac and Bose-Einstein statistics, respectively. If species \( A_i \) and \( B_j \) react as

\[
\sum_i \alpha_i A_i \rightleftharpoons \sum_j \beta_j B_j
\]

where \( \alpha_i, \beta_j \) are the reaction coefficients, then the chemical potentials obey

\[
\sum_i \alpha_i \mu_i = \sum_j \beta_j \mu_j - Q
\]

(3-21)

\[
Q = \sum_i \alpha_i m_i c^2 - \sum_j \beta_j m_j c^2
\]

(3-22)

From (Clayton, 1968, p. 88) we see that degeneracy is dominant when \( \left( \frac{\rho}{\mu_e} \right) > 2.4 \times 10^{-8} T^{3/2} \text{ gm/cm}^3 \).

For our purposes, \( \mu_e \sim 1 \).
and $\rho^* \lesssim 1$, and for $T \lesssim 10^6 \, ^0\text{K}$, we find that we may neglect degeneracy effects. A similar condition (Clayton, 1968, p. 90) applies to the relativistic regime; a ratio 

$$\frac{\rho^*}{\mu_e} < 7.3 \times 10^6 \, \text{gm/cm}^3$$

is required for an acceptable non-relativistic approximation. This condition, too, is clearly satisfied. Hence we may assume the presence of a non-degenerate, nonrelativistic electron gas, and thus assume that the Maxwell-Boltzmann statistics are a good approximation for describing the behavior of our gas particles. Equation (3-19) becomes

$$V n_i = \sum_r e^{-(\epsilon_{ir} - \mu_i)/kT} = e^{\mu_i/kT} \sum_r e^{-\epsilon_{ir}/kT}$$

We may separate the internal and kinetic energies in the sum:

$$V n_i = e^{\mu_i/kT} \sum_r (2J_r+1) e^{-\epsilon_{ir}^{\text{int}}/kT} \frac{4\pi V}{h^3} \int_0^\infty p^2 e^{-p^2/2m_i kT} dp$$

Evaluation of the integral leads to

$$n_i = e^{\mu_i/kT} g_i (2\pi m_i kT / h^2)^{3/2}$$

$$g_i = \sum_r (2J_r+1) e^{-\epsilon_{ir}^{\text{int}}/kT}$$

(3-23)

We may relate the abundances of any two species by
\[ \alpha_i \mu_i + \beta_j \mu_j = \mu_k - Q_k \]  

(3-24)

\[ Q_k \equiv (\alpha_i m_i + \beta_j m_j - m_k) c^2 \]  

(3-25)

This is just

\[ e^{\frac{\alpha_i \mu_i}{kT}} e^{\frac{\beta_j \mu_j}{kT}} = [e^{\frac{\mu_i}{kT}}]^{\alpha_i} [e^{\frac{\mu_j}{kT}}]^{\beta_j} = e^{\frac{\mu_k}{kT}} e^{-\frac{Q_k}{kT}} \]

Equation (3-23) can now be applied to yield

\[ \left[ \frac{n_i h^3}{g_i (2\pi m_i kT)^{3/2}} \right]^{\alpha_i} [\frac{n_j h^3}{g_j (2\pi m_j kT)^{3/2}}]^{\beta_j} = \frac{n_k h^3}{g_k (2\pi m_k kT)^{3/2}} e^{-\frac{Q_k}{kT}} \]

(3-26)

For the case of an ion-neutral pair, or an \( r \)-times-ionized atom, and an \( r+1 \)-times-ionized atom of the same species, we may write

\[ \mu_{i,r} = \mu_{i,r+1} + \mu_e - Q_e \]

\[ Q_e = (m_{i,r} - m_{i,r+1} - m_e) c^2 \equiv \gamma_r \]

Equation (3-26) then becomes

\[ \frac{n_{i,r+1} \gamma_e}{n_{i,r}} = \frac{G_{i,r+1} g_e}{G_{i,r}} \left( \frac{2\pi m_e kT}{h^3} \right)^{3/2} e^{-\frac{\gamma_r}{kT}} \]

(3-27)
where \( \kappa_r \) is the ionization potential of the \( r \)-times-ionized atom, and we have made use of the fact that \( m_{i,r+1} \approx m_{i,r} \).

This equation is one form of the Saha Equation. The partition functions \( G_{i,r} \) of species \( i \), ionization level \( r \), are given in general by

\[
G_{i,r} = \sum_k g_{i,r,k} e^{-\frac{\varepsilon_{i,r,k}}{kT}}
\]

(3-28)

where the (third) subscript \( k \) represents the energy states (some possibly degenerate) of the \( r \)-times-ionized species \( i \).

Through the Boltzmann relation for a given species,

\[
\ln \frac{n_{r,k}}{n_{r,k'}} = \frac{g_{r,k}}{g_{r,k'}} e^{-\frac{(\varepsilon_{r,k} - \varepsilon_{r,k'})}{kT}}
\]

(3-29)

we may finally arrive at a general equation relating the populations of energy levels \( k \) and \( l \) of, respectively, an \( r \)-times-ionized and an \((r+1)\)-times-ionized atom of the same species:

\[
\ln \frac{n_{r+1,l}}{n_{r,k}} = \ln \frac{g_{r+1,l}}{g_{r,k}} \frac{(2\pi m_e kT)^{3/2}}{\hbar^3} e^{-\frac{(\varepsilon_{r+1,l} - \varepsilon_{r,k})}{kT}}
\]

(3-30)

Knowledge of the opacity is critical to any calculation.
of stellar structure, but accurate determinations of opacity are often extremely complicated. The most important effects upon the opacity derive from bound-bound, bound-free, and free-free absorption, and from electron scattering. The calculation of opacity is much too complex for a paper of this nature; for a more complete discussion, see for example, (Clayton, 1968, Ch. 3), (Cox and Guili, 1968, Ch. 16), and (A. Cox, 1965).

Consider the electron scattering opacity. The opacity is given, for a nondegenerate, nonrelativistic gas, by

$$\chi_{es} = \sigma_T n_e$$

where $\sigma_T$ is the Thomson scattering cross section, with the numerical value $\sigma_T = 6.65 \times 10^{-23}$ cm$^{-2}$ per electron, and $n_e$ is the number of free electrons per gram of matter. For the case of a hydrogen gas, $n_e$ is equal to $n_p$, and the Saha Equation (3-27) gives

$$\frac{n_e^2}{n_h} = \left(\frac{2\pi m_e k T}{\hbar^2}\right)^{3/2} e^{-\frac{X_H}{k T}}$$

where $k$ is Boltzmann's constant, and $X_H$ is the ionization potential for hydrogen. Values of $n_e$ as functions of temperature are given in Table I, along with the corresponding values of $\chi_{es}$. The same relationship is shown graphically in Figure III; Figure III shows the relationship of the total
solar opacity to temperature (and density). Values of $\kappa_{es}$ corresponding to a gas of the solar composition are also represented in Figure III, as simply values of $\kappa_{es,H}$ scaled by the factor $X_H$. The electron scattering opacity tends to the value .4 in the limit of complete ionization (pure hydrogen gas). As may be seen from Table I and Figure III, the electron scattering opacity is a reasonable approximation to the total opacity for the temperatures and densities characteristic to the outer regions of our model supernova envelope ($T \sim 10^4 \text{ K, } \rho \lesssim 10^{-8} \text{ gm/cm}^3$).
IV. THE DIFFERENCE EQUATIONS

The equations of structure and state, in principle, allow one to calculate forward in time the physical behavior of a state of specified mass, temperature and pressure. However, the computations are involved and long; it is necessary to make use of computers to carry out numerical integrations of these equations ahead in time. However, to do so the form of the equations must be changed into one more suited to numerical techniques. One of the most powerful forms is the difference equation.

The procedure is to divide the star into spherical shells of specified masses, and to specify the pertinent variables in each shell at time $t_0$. The use of the mass as a coordinate parameter in place of the (Eulerian) radii is a convenient convention, and such coordinates are referred to as "Lagrangian". The structure equations, in "finite-difference" form, are then used to integrate forward in time the various physical parameters of interest.

We divide the star into, for example, JJ zones, and specify initially in each zone the radius and velocity of the zone boundary: $R_k^0$, $u_k^0$, $K = 1$ to JJ where the subscript $k$ refers to the spatial zone number (numbered outward from
the center, where \( R = 0 \) and the superscript \( n \) refers to the time \((n = 0 \text{ referring to } t = t_0)\). The temperature, pressure and mass of each zone are likewise specified, but are centered not on the zone boundary but at the "center" of the zone:

\[
P^0_{k+\frac{1}{2}}, T^0_{k+\frac{1}{2}}, \Delta m_{k+\frac{1}{2}}
\]

The specific volume of each zone may be calculated from

\[
V^0_{k+\frac{1}{2}} = \frac{4\pi}{3} \left[ (R^0_{k+1})^3 - (R^0_{k})^3 \right] \frac{1}{\Delta m_{k+\frac{1}{2}}} \quad (4-1)
\]

The mass increment from \( R_{k-\frac{1}{2}} \) to \( R_{k+\frac{1}{2}} \), i.e. the effective mass at a zone boundary, is simply the arithmetic average

\[
\Delta m_k \equiv \frac{1}{2} \left( \Delta m_{k+\frac{1}{2}} + \Delta m_{k-\frac{1}{2}} \right) \quad (4-2)
\]

(Notice that all the quantities \( \Delta m \) are fixed constant in time.) The total mass inside a given zone boundary is

\[
M_{k+1} = M_k + \Delta m_{k+\frac{1}{2}} = \sum_{i=0}^{k} \Delta m_{i+\frac{1}{2}}
\]

The momentum conservation equation may be written in the form
where \( u \) is the fluid velocity

\[
\frac{dr}{dt} = \frac{du}{dt}
\]

and thus

\[
r_k^{n+1} = r_k^n + u_k^{n+1/2} \Delta t^{n+1/2}
\]

The terms \( Q \) in equation (4-3) are "pseudo-viscosity" terms, and are numerical devices to inhibit artificially the approach of two zone boundaries during an expansion or contraction.

The presence of shocks also greatly complicates the numerical solution of the hydrodynamic equations; mathematically, shocks are discontinuities in pressure, temperature, density and velocity. Boundary conditions are required to connect these parameters across the shock surface. Those conditions, however, are difficult to use in the numerical scheme because the shock itself is also moving relative to the space-time points used in the scheme. The method of pseudo-viscosity, due to Von Neumann and Richtmeyer
(Richtmeyer and Morton, 1967, pp. 318-320), conveniently treats shocks. This method avoids the requirement of prior knowledge of the shock behavior by introducing an artificial dissipative mechanism which effectively spreads the shock over several of the space-time points; the parameters may then vary rapidly, but continuously. Hence, the difference equations preserve their accuracy, as if there were no shocks present at all (i.e., no numerical discontinuities are present).

A common choice for the pseudo-viscosity is given by

\[ Q^{n+\frac{1}{2}}_{k+\frac{1}{2}} = 2 \left( u^{n+\frac{1}{2}}_{k+1} - u^{n+\frac{1}{2}}_{k} \right) \frac{1}{v^{n+\frac{1}{2}}_{k+\frac{1}{2}}} \]

\[
\text{if } V^{n+1} < V^{n} \quad u^{n+\frac{1}{2}}_{k+1} < u^{n+\frac{1}{2}}_{k} \]

\[ = 0 \quad \text{otherwise} \]

(after Richtmeyer and Morton, 1967, p. 320). When zone boundaries approach rapidly, the pseudo-viscosity supplies a large "pressure" which prevents their crossing. In the case of a shock, the term converts kinetic energy of the bulk zone into thermal energy, hence avoiding discontinuities. This explanation for pseudo-viscosity is after W. D. Arnett (1966a); for a more complete discussion, refer to Richtmeyer
Clearly, equation (4-5) enables equation (4-1) to be updated to the value at the next time epoch:

\[
V_{k+\frac{1}{2}}^{n+1} = \frac{4\pi}{3} \left[ (R_{k+1}^{n+1})^3 - (R_k^{n+1})^3 \right] \frac{1}{\Delta m_{k+\frac{1}{2}}} \tag{4-7}
\]

Notice that mass is conserved in each shell, and hence in the entire star.

In order now to continue evaluation ahead to the time \(n+1\), a linear extrapolation of \(T_{k+\frac{1}{2}}^n\) is made: Initially,

\[
T_{k+\frac{1}{2}}^1 = T_{k+\frac{1}{2}}^0
\]

thereafter,

\[
\tilde{T}_{k+\frac{1}{2}}^{n+1} = T_{k+\frac{1}{2}}^n + (T_{k+\frac{1}{2}}^n - T_{k+\frac{1}{2}}^{n-1}) \frac{\Delta t_{k+\frac{1}{2}}^{n+1}}{\Delta t_{k+\frac{1}{2}}^{n-1}} \tag{4-8}
\]

The equation of state may now be evaluated at time \(n+1\)

\[
\tilde{P}_{k+\frac{1}{2}}^{n+1} = P(\tilde{T}_{k+\frac{1}{2}}^{n+1}, V_{k+\frac{1}{2}}^{n+1})
\]

\[
\tilde{E}_{k+\frac{1}{2}}^{n+1} = E(\tilde{T}_{k+\frac{1}{2}}^{n+1}, V_{k+\frac{1}{2}}^{n+1})
\]
and in similar fashion the thermodynamic derivatives $P_T$, $P_V$, $E_T$ and $E_V$ may be evaluated. In general, the thermodynamic quantities exhibit a weak dependence on the composition; however, in the simple case considered, the composition is assumed to be constant, and explicit dependence on it is therefore omitted. The tilde ($\sim$) above $T$, $P$ and $E$ denotes an extrapolation, which will be used as a first guess in a scheme designed to converge on the proper solution for $T_{k+1}^{n+1}$. Energy conservation (the First Law) may then be written in the form

\[
\left( E_{k+1/2}^{n+1} - E_{k+1/2}^n \right) \frac{1}{\Delta t_{k+1/2}} + \frac{1}{2} \left( P_{k+1/2}^{n+1} + P_{k+1/2}^n \right) \left( V_{k+1/2}^{n+1} - V_{k+1/2}^n \right) \frac{1}{\Delta t_{k+1/2}} + \left( L_{k+1}^{n+1} - L_k^{n+1} \right) \frac{1}{\Delta m_{k+1/2}} = 0
\]

(4-10)

The extrapolated values of $T$ will, in general, have an error $\delta T_{k+1/2}^{n+1}$ from the true solution at time $n+1$. The energy equations (one for each zone) must be linearized and expressed in terms of the temperature $T_{k+1/2}^{n+1}$ as

\[
T_{k+1/2}^{n+1} = \tilde{T}_{k+1/2}^{n+1} + \delta T_{k+1/2}^{n+1}
\]

(4-11)

The equations, once linearized, will in general contain
terms of first order in \( \delta T_{k+\frac{3}{2}} \), \( \delta T_{k+1} \), and \( \delta T_{k-\frac{1}{2}} \) as well as a term which is a constant characteristic of the spatial zone at time \( n+1 \); we then make use of a fully implicit ("backwards") difference scheme to solve the coupled equations of the form

\[
A_{k+\frac{1}{2}} \delta T_{k+\frac{1}{2}} + B_{k+\frac{1}{2}} \delta T_{k+\frac{1}{2}} + C_{k+\frac{1}{2}} \delta T_{k-\frac{1}{2}} = D_{k+\frac{1}{2}} \quad (4-12)
\]

This scheme makes use of the space-time points \( k-\frac{1}{2}, k+\frac{1}{2}, \)

\( k+\frac{3}{2} \)

at time \( n+1 \) and spatial position \( k+\frac{1}{2} \) at time \( n \). The scheme solves for the \( \delta T \)'s simultaneously. (The physical significance of such an implicit solution is, according to (Christy, 1964, p. 559), a result of the fact that, in general, for a region of low heat capacity or high conductivity (e.g., near the surface of a star) the heat is in fact able to propagate over regions which may be represented by more than a single spatial zone in one time step. The
implicit formulation allows information to "propagate" from boundary to boundary over each time step. This method may, however, lead to problems in the case of optically thick regions; i.e., the information may be allowed to propagate at velocities which are actually in excess of physical limitations: the velocity of light, for example.

With the new value

\[ \tilde{T}^{n+1}_{k+\frac{1}{2}} = \tilde{T}^{n+1}_{k} + \delta T^{n+1}_{k+\frac{1}{2}} \]

we can update our guess for \( \tilde{T}^{n+1}_{k+\frac{1}{2}} \) and arrive at a new value for our thermodynamic functions; we then repeat the entire scheme, in principle forcing the \( \delta T \) to go to zero, and

\[ \tilde{T}^{n+1}_{k+\frac{1}{2}} \rightarrow T^{n+1}_{k+\frac{1}{2}}. \]

The cycle is then reiterated for time \( n+2 \).

This method of solution is known as the tridiagonal scheme and is now outlined: we seek to solve a system of equations of the form

\[ A_j X_{j+1} + B_j X_j + C_j X_{j-1} = D_j \quad (4-12) \]

for \( j=1 \) to \( j=JJ \). If we assume a relation of the form

\[ X_{j-1} = E_j X_j + F_j \quad (4-13) \]
then we may write equation (4-12) as

\[ A_j x_{j+1} + (B_j + C_j F_j) x_j + C_j F_j = D_j \]

Rearranging,

\[ x_j = \frac{D_j - C_j F_j - A_j x_{j+1}}{B_j + C_j E_j} \]

Now, also

\[ x_j = E_j x_{j+1} + F_j \]

Hence, we may identify

\[ E_j = -\frac{A_j}{B_j + C_j E_j} \]

(4-14)

\[ F_j = \frac{D_j - C_j F_j}{B_j + C_j E_j} \]

We solve for the \( E_j \) and \( F_j \) in terms of the coefficients \( A, B, C \) and \( D \) which are specified, and then use equation (4-13) to solve for the \( x_j \).

We may derive expressions for the coefficients \( A, B, C \) and \( D \) for the radiative diffusion model from the following
relations:

\[ L = \frac{ac}{3} \frac{A^2}{\chi} \frac{d(T^4)}{dm} \]  
\hspace{1cm} (4-15)

\[ A_k = 4\pi R_k^2 \]  
\hspace{1cm} (4-16)

\[ V_{R+\frac{1}{2}} = \frac{4\pi}{3} \left( R_{k+1}^3 - R_k^3 \right) \frac{1}{\Delta m_{k+\frac{1}{2}}} \equiv \frac{1}{\rho_{k+\frac{1}{2}}} \]  
\hspace{1cm} (4-17)

\[ \dot{V}_{k+\frac{1}{2}} = 4\pi \left( R_{k+1}^2 u_{k+1} - R_k^2 u_k \right) \frac{1}{\Delta m_{k+\frac{1}{2}}} \]  
\hspace{1cm} (4-18)

\[ u_R = \dot{R}_k \]  
\hspace{1cm} (4-19)

\[ R_{k+1}^n = R_k^n + u_R^{n+\frac{1}{2}} \Delta t_{n+\frac{1}{2}} \]  
\hspace{1cm} (4-20)

\[ \dot{E} + P \dot{V} + \frac{\partial L}{\partial m} = 0 \]  
\hspace{1cm} (4-21)
where equation (4-21) is just the second law, and where we are neglecting the hydrodynamic relations of the u's, P's etc. in favor of using just the energy balance equation. We choose instead to specify arbitrary values for the zone boundary velocities $u_k$ which will be constant in time.

We seek to arrive at the explicit form of the energy balance equation, in terms of the parameters of state $E$, $P$ and their derivatives $E_T \equiv \left( \frac{\partial E}{\partial T} \right)_V$, $E_V \equiv \left( \frac{\partial E}{\partial V} \right)_T$, and similarly for $P$. From the equations of state we have then

\[ P_T = \mathcal{R} \frac{\rho}{\mu} + \frac{4}{3} a T^3 \]  

(4-22)

\[ P_V = -\mathcal{R} \frac{\rho^2}{\mu} T \]  

(4-23)

\[ E_T = \frac{3}{2} \mathcal{R} \frac{1}{\mu} \]  

(4-24)

\[ E_V = a T^4 \]  

(4-25)

From the chain rule, we have
\[
\dot{E} = \frac{\partial E}{\partial T} \frac{dT}{dt} + \frac{\partial E}{\partial V} \frac{dV}{dt} = E_T \Delta T + E_V \Delta V 
\] (4-26)

where the symbol \(\Delta_t\) denotes a time difference. Hence, we find

\[
\dot{E} \to E_T^{n+\frac{1}{2}} \left( \frac{\Delta T_{k+\frac{1}{2}}}{\Delta t^{n+\frac{1}{2}}} + E_V^{n+\frac{1}{2}} \right) \dot{V}_{k+\frac{1}{2}}^{n+\frac{1}{2}} 
\] (4-27)

in difference form. The second term in equation (4-21) becomes

\[
P_{k+\frac{1}{2}}^{n+\frac{1}{2}} \dot{V}_{k+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2} \left( P_{k+\frac{1}{2}}^{n+1} + P_{k+\frac{1}{2}}^{n} \right) \dot{V}_{k+\frac{1}{2}}^{n+\frac{1}{2}} 
\]

We find we must assume a form for \(P_{k+\frac{1}{2}}^{n+1}\) similar to that for \(E_{n+1}^{n+1}\) above; we will use

\[
P_{k+\frac{1}{2}}^{n+1} = P_{k+\frac{1}{2}}^{n} + P_{T_{k+\frac{1}{2}}}^{n+\frac{1}{2}} \Delta T_{k+\frac{1}{2}}^{n+\frac{1}{2}} + P_{V_{k+\frac{1}{2}}}^{n+\frac{1}{2}} \Delta V_{k+\frac{1}{2}}^{n+\frac{1}{2}} 
\] (4-28)

We find, then, in difference form

\[
P \dot{V} \to \frac{1}{2} \left( 2 P_{k+\frac{1}{2}}^{n} + P_{T_{k+\frac{1}{2}}}^{n+\frac{1}{2}} \Delta T_{k+\frac{1}{2}}^{n+\frac{1}{2}} 
\]

\[
+ P_{V_{k+\frac{1}{2}}}^{n+\frac{1}{2}} \dot{V}_{k+\frac{1}{2}}^{n+\frac{1}{2}} \Delta V_{k+\frac{1}{2}}^{n+\frac{1}{2}} \right) \dot{V}_{k+\frac{1}{2}}^{n+\frac{1}{2}} 
\] (4-29)

We consider now the term \(\frac{\partial T}{\partial m}\) in equation (4-21):
Explicitly in the temperatures, equation (4-15) is written as

\[
\begin{align*}
L_k &= -A_k^2 \frac{ac}{3 \chi_k} \frac{\Delta m_{k+1/2}^4 - \Delta m_{k-1/2}^4}{\Delta m_k} \\
\end{align*}
\]  

(4-31)

where \( \Delta m_k \equiv \frac{1}{2} (\Delta m_{k+1/2} + \Delta m_{k-1/2}) \) and henceforth \( \chi_k \) will be considered as a constant throughout the star (only for our preliminary calculations). We have, then, by substitution of equation (4-31) into equation (4-30):

\[
\begin{align*}
\frac{L_{k+1} - L_k}{\Delta m_{k+1/2}} &= -\frac{ac}{3 \chi \Delta m_{k+1/2}} \left[ \frac{A_{k+1}^2}{\Delta m_{k+1}} \frac{\Delta m_{k+1}^4}{\Delta m_{k+1}} \right. \\
&\quad - \left. \left( \frac{A_{k+1}^2}{\Delta m_{k+1}} + \frac{A_k^2}{\Delta m_k} \right) \frac{\Delta m_{k+1}^4}{\Delta m_{k+1}} - \frac{A_k^2}{\Delta m_k} \frac{\Delta m_k^4}{\Delta m_k} \right] \\
\end{align*}
\]  

(4-32)

If we define the quantities

\[
\begin{align*}
\alpha_{k+1/2} &\equiv \frac{ac}{3} \frac{1}{\Delta m_{k+1/2}} \\
\beta_k &\equiv \frac{A_k^2}{\chi_k \Delta m_k} = \frac{A_k^2}{\chi \Delta m_k} \\
\end{align*}
\]

we may write equation (4-32) in the form
\[
\frac{\partial L}{\partial m} \rightarrow -\alpha_{k+1/2} \left[ \beta_{k+1/2} T_{k+1/2} - (\beta_{k+1} + \beta_{k}) T_{k+1/2} + \beta_{k} T_{k-1/2} \right]
\] (4-32a)

If we further define
\[
\gamma_{k+1/2}^{n+1/2} \equiv E_{x_{k+1/2}} \frac{1}{\Delta t} + \frac{1}{2} P_{x_{k+1/2}} \nabla_{x_{k+1/2}}^{n+1/2} \\
\Delta_{k+1/2}^{n+1/2} \equiv \nabla_{x_{k+1/2}}^{n+1/2} \left[ E_{x_{k+1/2}} + P_{k+1/2} + \frac{1}{2} P_{k+1/2} \nabla_{x_{k+1/2}}^{n+1/2} \Delta_{x_{k+1/2}}^{n+1/2} \right]
\]

we may write
\[
\mathbf{E} + \mathbf{P} \nabla \rightarrow \gamma_{k+1/2}^{n+1/2} T_{k+1/2}^{n+1} \\
- \gamma_{k+1/2}^{n+1/2} T_{k+1/2}^{n} + \Delta_{k+1/2}^{n+1/2} 
\] (4-33)

We may now combine equations (4-32a) and (4-33) as provided
by equation (4-21) to give
\[
\alpha_{k+1/2}^{n+1/2} \left[ \beta_{k+1} T_{k+1/2}^{n+1} - (\beta_{k+1} + \beta_{k}) T_{k+1/2}^{n+1} + \beta_{k} T_{k-1/2}^{n+1} \right] \\
+ \beta_{k} T_{k-1/2}^{n+1} - \gamma_{k+1/2}^{n+1/2} T_{k+1/2}^{n} = \Delta_{k+1/2}^{n+1/2} - \gamma_{k+1/2}^{n+1/2} T_{k+1/2}^{n} 
\] (4-34)

Equation (4-34), which is the temperature-explicit
form of the energy balance equation (4-21), must now be linearized to be solved by the tridiagonal method. We define \( T_{k+\frac{1}{2}}^{n+1} \), the linearly extrapolated guess for \( T_{k+\frac{1}{2}}^{n+1} \), by

\[
T_{k+\frac{1}{2}}^{n+1} = \tilde{T}_{k+\frac{1}{2}}^{n+1} + \delta T_{k+\frac{1}{2}}^{n+1}
\]  
\[\text{(4-35)}\]

\[
\tilde{T}_{k+\frac{1}{2}}^{n+1} \equiv T_k^n + \Delta t T_{k+\frac{1}{2}}^{n-\frac{1}{2}} \frac{\Delta t_{n+\frac{1}{2}}}{\Delta t_{n-\frac{1}{2}}}
\]  
\[\text{(4-36)}\]

We then have

\[
\left( T_{k+\frac{1}{2}}^{n+1} \right)^4 = \left( \tilde{T}_{k+\frac{1}{2}}^{n+1} + \delta T_{k+\frac{1}{2}}^{n+1} \right)^4
\]

\[
= \left( \tilde{T}_{k+\frac{1}{2}}^{n+1} \right)^4 + 4 \left( \tilde{T}_{k+\frac{1}{2}}^{n+1} \right)^3 \delta T_{k+\frac{1}{2}}^{n+1} + 6 \left( \tilde{T}_{k+\frac{1}{2}}^{n+1} \right)^2 \delta T_{k+\frac{1}{2}}^{n+1}^2 + 4 \left( \tilde{T}_{k+\frac{1}{2}}^{n+1} \right) \delta T_{k+\frac{1}{2}}^{n+1}^3 + \mathcal{O} \left[ \delta T_{k+\frac{1}{2}}^{n+1}^4 \right]
\]  
\[\text{(4-37)}\]

Substitution of equation (4-37) into (4-34) and rearrangement into the suitable tridiagonal form (4-12) gives finally

\[
\left[ \alpha_{k+\frac{1}{2}}^{n+\frac{1}{2}} \beta_{k+1}^{n+\frac{1}{2}} \tilde{T}_{k+\frac{1}{2}}^{n+1} \right] \delta T_{k+\frac{1}{2}}^{n+1} - \left[ \gamma_{k+\frac{1}{2}}^{n+\frac{1}{2}} + \alpha_{k+\frac{1}{2}}^{n+\frac{1}{2}} \beta_{k+1}^{n+\frac{1}{2}} \tilde{T}_{k+\frac{1}{2}}^{n+1} \right] \delta T_{k+\frac{1}{2}}^{n+1} - \left[ \alpha_{k+\frac{1}{2}}^{n+\frac{1}{2}} \beta_{k+1}^{n+\frac{1}{2}} \tilde{T}_{k+\frac{1}{2}}^{n+1} \right] \delta T_{k+\frac{1}{2}}^{n+1} \\
+ \left[ \alpha_{k+\frac{1}{2}}^{n+\frac{1}{2}} \beta_{k+1}^{n+\frac{1}{2}} \tilde{T}_{k+\frac{1}{2}}^{n+1} \right] \delta T_{k+\frac{1}{2}}^{n+1} = \Delta_{k+\frac{1}{2}}^{n+\frac{1}{2}} - \gamma_{k+\frac{1}{2}}^{n+\frac{1}{2}} \left( T_k^n - \tilde{T}_{k+\frac{1}{2}}^{n+1} \right) - \alpha_{k+\frac{1}{2}}^{n+\frac{1}{2}} \left[ \beta_{k+1}^{n+\frac{1}{2}} \tilde{T}_{k+\frac{1}{2}}^{n+1} \right] \\
- \left( \beta_{k+1}^{n+\frac{1}{2}} + \beta_{k+1}^{n+\frac{1}{2}} \right) \tilde{T}_{k+\frac{1}{2}}^{n+1} + \beta_{k+1}^{n+\frac{1}{2}} \tilde{T}_{k+\frac{1}{2}}^{n+1} \tilde{T}_{k+\frac{1}{2}}^{n+1}
\]  
\[\text{(4-38)}\]
We may then identify the coefficients A, B, C and D by

\[
A_{k+\frac{1}{2}}^{n+1} = \alpha_{k+\frac{1}{2}}^{n+\frac{1}{2}} \beta_{k+1} \tilde{T}_{k+\frac{1}{2}}^{n+1} \\
B_{k+\frac{1}{2}}^{n+1} = -\alpha_{k+\frac{1}{2}}^{n+\frac{1}{2}} (\beta_{k+1}^{n+\frac{1}{2}} + \beta_k^{n+\frac{1}{2}}) \tilde{T}_{k+\frac{1}{2}}^{n+1} - \gamma_{k+\frac{1}{2}}^{n+\frac{1}{2}} \\
C_{k+\frac{1}{2}}^{n+1} = \alpha_{k+\frac{1}{2}}^{n+\frac{1}{2}} \beta_k^{n+\frac{1}{2}} \tilde{T}_{k-\frac{1}{2}}^{n+1} \\
D_{k+\frac{1}{2}}^{n+1} = \Delta_{k+\frac{1}{2}}^{n+\frac{1}{2}} - \gamma_{k+\frac{1}{2}}^{n+\frac{1}{2}} (\tilde{T}_k^{n+\frac{1}{2}} - \tilde{T}_{k+\frac{1}{2}}^{n+\frac{1}{2}}) - \alpha_{k+\frac{1}{2}}^{n+\frac{1}{2}} \left[ \beta_{k+1}^{n+\frac{1}{2}} \tilde{T}_{k+\frac{1}{2}}^{n+1} - \left( \beta_{k+1}^{n+\frac{1}{2}} + \beta_k^{n+\frac{1}{2}} \right) \tilde{T}_{k+\frac{1}{2}}^{n+1} + \beta_k^{n+\frac{1}{2}} \tilde{T}_{k-\frac{1}{2}}^{n+1} \right]
\]

(4-39)

For index \( j=1 \), we choose \( x_{j-1} = x_1 = x_0 \) by symmetry, so the flux through the origin is zero and we have \( C_1 \equiv 0 \); similarly, \( A_{JJ} \equiv 0 \) (because the universe isn't "shining into the star").

(Richtmeyer and Morton, 1967, Chapters 1 and 8) discuss both the stability and accuracy of various difference schemes. The tridiagonal method outlined here is a fully implicit, backwards-centered scheme, and is always stable numerically. The introduction of special boundary conditions (see below) may alter slightly the nature of the scheme, however, and it will therefore be necessary to examine this possibility in further detail in the future. The errors typical to this scheme are usually of the order of \((\Delta t) + (\Delta x)^2\),
since it solves a diffusion equation of the form

\[ \frac{\partial \phi}{\partial t} = \sigma \frac{\partial^2 \phi}{\partial x^2} \]

which is, in difference form, approximated by

\[ \frac{\Delta_t \phi}{\Delta t} = \sigma \frac{\Delta^2 \phi}{(\Delta x)^2} \]

Suitable criteria for both accuracy and efficiency were found to be

\[ \Delta t \leq 0.5 \ t_{ch} \]

where \( t_{ch} \) represents the minimum characteristic timescale of the system; i.e., the minimum of the Kelvin-Helmholtz time, the diffusion time and the expansion time. These criteria are in fact used in conjunction with a procedure which scales \( \Delta t \) as the minimum of the relative changes of the temperature in the various zones of the model to choose the new value for \( \Delta t \).

Several tests we made to test the accuracy of the code, most notable of which was a calculation designed to reproduce the characteristics of an adiabatic expansion. The initial
model was a $1M_\odot$ object of nine equal-mass zones with outer radius equal to one solar radius. Initial temperatures varied linearly by zone from $1 \times 10^6 \, ^0\text{K}$ to $1.11 \times 10^5 \, ^0\text{K}$ in the last zone, with $0^0\text{K}$ outside the object. Zone velocities were linearly scaled with radius, $R_i = u_{JJ} (R_i/R_{JJ})$ with $u_{JJ} = 6.96 \times 10^3 \, \text{cm \, sec}^{-1}$. Expansion velocities were constant in time, and the change in temperature resulting from the (adiabatic) cooling were compared to analytical results, based on the equation

$$T_o V_o^{Y-1} = T_f V_f^{Y-1} = \text{constant}$$

The results are reflected in Table II, which shows the values arrived at by each independent method. As is apparent, accuracy was within 4% over a time period of $6.2 \times 10^8 \, \text{sec}$, which we deem more than sufficient for our purposes.

The final item for consideration with regard to the difference equations is the form taken by the boundary conditions of the outer zone. The Eddington approximation discussed above transforms the expression for the (outer zone) luminosity to the form

$$L_{JJ} = \frac{ac}{3} A_{JJ} (\tau_{JJ}^{2/3})^{-1} \tau_{JJ}^{4}$$

$$\tau_{JJ} = \gamma_{JJ} \Delta m_{JJ}/A_{JJ}$$
This approximation clearly assumes that the point $\tau = 2/3$ is within the outer zone. However, as the star expands, the optical thickness of the inner zones also decreases, until at some point in time $\tau = 2/3$ occur within the next-to-outer zone. It is not clear at this time what consequences such an effect will have on the form and validity of the Eddington approximation. Certainly such problems will form a considerable part of any future calculations.
V. PRELIMINARY RESULTS AND DISCUSSION

Our initial model was an extended study of $5 \, M_\odot$, with radius $R = 4 \times 10^{14} \, \text{cm}$. The aim was to expand the model homologously and to watch what happened to the luminosity as the outer zones became optically thin. The star was divided into 10 zones of equal mass with constant density throughout the entire object. Each boundary was given a velocity suitably scaled to the constant velocity of expansion of the outer boundary, $u_i = u_{JJ}(R_i/R_{JJ})$, $u_{JJ} = 10^9 \, \text{cm/sec}$. Composition was assumed, for simplicity, to be pure hydrogen, completely ionized, and the opacity was given the constant value .4, corresponding to electron scattering opacity alone. The equation of state was that of a perfect, nondegenerate, nonrelativistic gas with radiation pressure. The initial temperatures varied linearly by zone from $T_1 = 2 \times 10^5 \, \text{K}$ to $T_{10} = 1.5 \times 10^5 \, \text{K}$, with the boundary value defined by $T(R + (2/3), t) \equiv 0$. The calculations represent approximately 45 days of expansion, corresponding to a final radius of $4.637 \times 10^{15} \, \text{cm}$; this represents a decrease of optical depth in the outer zone from approximately $\tau = .50$ to $\tau = .37$, and in the next zone of from $\tau \sim 100$ to $\tau \sim 8$.

The resulting luminosities in the various zones for
selected intervals of time are shown in Figure IV. The initial temperature distribution clearly leads to a sharp discontinuity in L for the outer zone. However, the figure also shows that the model rapidly adjusts itself toward a more physically realistic condition. The constant decrease in the level of the luminosity curves represents the loss of energy through radiation, unbalanced by the pressure of an energy source.

Figure V shows the (bolometric) luminosity of the outer zone as a function of time. Especially noteworthy is the virtually constant exponential decay of the luminosity. This may be qualitatively understood in the diffusion approximation by realizing that the luminosity varies as

\[ L \propto e^{\exp (-t/\tau_d)} \]

This behavior of the luminosity suggests that the exponential decline of supernova light curves may be in part due to the characteristic expansion of the envelope itself.

Observations of supernova light curves are in general confined to the photovisual region of the spectrum. The calculations of this model, are, of course, bolometric values which must be "uncorrected" to photovisual magnitudes for
purposes of comparison. We use the standard formulae (see Clayton, 1968)

\[ M_{b^\nu} = M_b - B.C. \]

\[ M_b = -2.5 \log_{10} L_b + 88.7 \]

for \( L_b \) in ergs/sec. The bolometric correction \( BC \) is a function of the effective temperature \( T_e \), where \( T_e \) is given by the Eddington approximation

\[ T_e^4 = \frac{L_b}{4\pi R^2 \sigma} \]

Values of \( T_e \) are listed in Table IV and shown graphically in Figure VII. Strictly speaking, as the "photospheric boundary", represented by the point \( \tau = 2/3 \), becomes interior to the outer zone, the model should be rezoned to correctly allow for the radiative flux from the entire photosphere to be included in \( L_b \). However, \( L_b \) is still given to good order by \( L_{10} \). Values resulting from correction of \( M_b \) to \( M_{b^\nu} \) are listed in Table III, and the "light curves" are shown graphically in Figure VI. The broad "peak" in \( M_{b^\nu} \) is a result of the bolometric correction alone, which was determined according to the tables given in Table V. These values are,
of course, crude and subject to refinement, but they serve
to relate the qualitative features of the two light curves
reasonably well. The peak in the photovisual light curve
suggests the possibility that the shape of the observed peaks
may be due in part to an expansion effect in the supernova
envelope. The peak in the bolometric curve is obviously
a product of the (arbitrary) initial conditions of the model,
and reflects the need for future models to explicitly include
hydrodynamic effects; the emergence of a strong shock, for
example, would provide rapid heating of the envelope and
reproduce qualitatively the initial conditions of this model.

As the star expands, the photospheric boundary becomes
interior to more and more mass; that is, it works its way
inward through the outer zones of the star (Table VI). This
is evident from the definition of \( \tau \):

\[
\tau_{\text{photosphere}} = \int_R^{r_{\text{photo}}} -\kappa \rho \, dx
\]

and the fact that \( \rho \propto R^{-3} \). The motion of this point is shown
in Figure VIII, and reflects movement at a nearly constant
velocity. The resultant average values of this velocity are
given in Table IV, and are graphically represented in Figure IX.
The scatter of these points is almost certainly due to round
off errors and the like in the averaging of these values. The scatter, however, actually represents error of the order of only 4%. The discontinuity at \( t \approx 2.0 \times 10^6 \) sec. represents the transition of the point \( \tau = 2/3 \) from zone 10 to zone 9 (refer to Figure VII for values of \( \tau_{10} \) vs \( t \)), and points up the need for finer zoning in order to keep numerical values from changing abruptly at zone boundaries.

We may reasonably conclude that the point \( \tau = 2/3 \) moves "inward" at essentially a constant velocity.

In conclusion, then, we see that the exponential decline of the photovisual luminosity (post maximum) may be due to the effects of hydrodynamic expansion of the envelope. In the diffusion approximation, with \( L \propto \alpha e^{-t/\tau_d} \), \( \tau_d \) is given to first order by

\[
\tau_d = \frac{R^2}{\lambda c} = \tau \rho \frac{R^2}{c} \propto \frac{M}{R c}
\]

which is just the diffusion time scale, where \( \lambda \) is the mean free path of photons. It may therefore be possible to affect the "slope" of the decline by treating variations of, for example, the opacity and/or the mass. Future models will certainly emphasize inclusion of explicit ionization calculations; explicit hydrodynamics must be included; the
effects of decay of residual radioactive elements needs to be explored and incorporated quantitatively; variations in composition will certainly affect the form of the results through its effect on the opacity. An important consideration will certainly be the treatment of the non-equilibrium diffusion case; the form of the boundary conditions as the equilibrium diffusion approximation breaks down needs to be investigated, as well as their form in regions of optical thinness. Certainly the possibility of successful modeling of supernova behavior based on a diffusion model is encouraged by the qualitative results of this initial model.
I would like to take this opportunity to express my appreciation for the help, encouragement and patience shown me by all my friends in the department: Thank you Dave Arnett, Ray Talbot, Don Clayton, Hugh Anderson, Bob Spiger, Hal Goldwire, Ruth Allston, Sarah Dickson, Bryan Zwan, Bob Lindeman, Kem Hainebach and anyone else who can spell both my names correctly. And not least of all, thanks to Drs. Wolf and Stebbings for their continued cooperation and encouragement, both at Rice and at "The University." I sure miss the basketball.
BIBLIOGRAPHY


TABLE AND FIGURE CAPTIONS

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Table III.  Bolometric Correction of Light Curve

Table IV.  Temperatures and Optical Depths in Outer Zones

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Figure II.  Type II Supernovae Light Curves

Figure III.  Electron Scattering Opacities, Solar Composition

Figure IV.  Luminosity Structure at Various Times

Figure V.  Bolometric Light Curve

Figure VI.  Bolometric and Photovisual Light Curves

Figure VII.  Temperatures and Optical Depths in Outer Zones
Figure VIII. Photospheric Radius Behavior

Figure IX. Photospheric Velocity Behavior
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$v = 6.96 \times 10^3 \text{ cm/sec} \quad R_o = 6.96 \times 10^{10} \text{ cm} \quad t_{tot} = 6.2 \times 10^8 \text{ sec}

All temperatures in units of $1^\circ \text{K}$

TABLE II
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**TABLE IV**
### TABLE V

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**TABLE VI**
Schematic characteristics of the light curves of Type I Supernovae

Fig. 1 Photographic light curves of type I supernovae.
FIGURE II
$L_b$  
(erg/sec)

$10^{45}$  
$10^{44}$  
$10^{43}$

$0$ $1$ $2$ $3$ $4$ $t(x10^6$ sec)

FIGURE V