RICE UNIVERSITY

A NUCLEAR STUDY UTILIZING THE FADDEEV EQUATIONS

by

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ABSTRACT

A Nuclear Study Utilizing the Faddeev Equations

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The Faddeev equations with separable two-body scattering amplitudes are used to calculate the binding energy of the three-alpha system. We find a value of the binding energy in reasonable agreement with the experiment value. Using separable potentials of the Yamaguchi, Tabakin, and Barbour-Schult types to generate the S-wave scattering amplitudes, we find the three-body bound state is not extremely sensitive to the form of the two-body interaction. We also find the admixed D-state contributes little to the three-body binding energy. With a view to future studies, a set of separable potentials is presented to describe neutron-alpha scattering.
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CHAPTER I

INTRODUCTION

Since the historic work of L. D. Faddeev (1, 2, 3, 4), the quantum mechanical problem of the interaction of three non-relativistic particles has been systematized and reduced to the problem of solving a set of coupled integral equations for a three-particle T-matrix closely analogous to the two-body t-matrix. The integral equations for T can be solved numerically provided certain simplifying representations of the two-body t-matrix are used which reduce the integral equations to one dimensional equations.

The approximation usually made to simplify the equations is that of approximating the two-particle scattering amplitude by a function separable in the initial and final momenta. This approximation has been given justification by Lovelace (5) in the neighborhood of a bound state or resonance.

At the present time, a large number of three-body problems has been solved with this limitation. Lovelace has derived equations for the rearrangement scattering amplitude for three particles interacting through separable S-wave potentials. These equations have been solved numerically by Phillips (6) for the case of
neutron-deuteron elastic and inelastic scattering taking account of the nucleon-nucleon $^3S_1$ and $^1S_0$ interactions via deuteron and singlet deuteron quasi-particles. Phillips found it necessary to include a repulsive three-body force in the doublet channel to obtain agreement with the triton binding energy and the $n$-$D$ scattering length.

Amado and his co-workers (7, 8, 9, 10, 11), using a model derivable from the Faddeev equations but interpreted in field theoretic terms, have treated some aspects of the three-nucleon problem. They found good agreement with experiment for the cases of $n$-$D$ elastic and inelastic scattering.

Good results have also been obtained for the three-nucleon problem by Mitra and his co-workers (12, 13, 14, 15) who based their analysis directly upon the Schrodinger equation with separable potentials.

Ahmadzadeh and Tjon (16) have applied the Faddeev equations to the study of the pion as a three-particle bound state. Using the separable approximation they reduced the Faddeev equations to one dimensional equations which reduce to one equation for identical particles. With relativistic kinematics and the effective range formula of Chew and Mandelstam (17) to approximate the two-body scattering amplitude, they found the pion as a three-particle bound state with total angular momentum zero, isospin one and odd parity.
The results, however, were found to be strongly dependent on the two-body form factor.

Hetherington and Schick (18, 19, 20) have solved the Faddeev equations with separable S-wave potentials chosen to simulate K-D elastic scattering taking full account of the spin, isospin, and the K meson and n-p mass splitting. They were interested in comparing the results of the impulse approximation and double-scattering approximation with the results of the full multiple scattering including all rescattering corrections. They found significant corrections at the energies where the calculations were performed.

Shanley (21) has recently reported an analysis of Li$^6$ and low energy deuteron-alpha scattering as a three-body problem in which the alpha particle was assumed to be structureless and the Coulomb effects were neglected. The n-p interaction was taken to be S-wave and the n-$\alpha$ interaction was chosen to reproduce the low energy phase shifts in the $1/2^+$, $3/2^-$, and $1/2^-$ states. The problem was solved for the $T = 0$ and $T = 1$ states of Li$^6$ and for d-$\alpha$ scattering up to 10 MEV. The results were in general agreement with experiment and exhibited the $T = 1^+$ ground state of Li$^6$ as well as the low energy D-wave resonance in d-$\alpha$ scattering.

Barbour and Schult (22) have derived the Faddeev equations for the $K \rightarrow 3\pi$ amplitude using non-local S- and P-wave separable
potentials. Numerical solutions were done for the S-wave case. They found the shape of the decay spectrum insensitive to the shape of the potential. Some aspects of three-body states in the presence of a resonance were investigated.

Gale and Duck (23) have investigated the possibility that the pion-nucleon (3, 3) resonance is sufficient to produce a three-particle N-N-π bound state. Only the interactions in the most important N-π channels were included. A T = 2^+, T = 2 bound state was found in the non-relativistic case, but the use of a relativistic propagator moved the bound state to a resonance.

Harrington (24) has treated the three-alpha system using the Faddeev equations with separable S-wave interactions. Using charged spheres as an estimate of the Coulomb interaction energy he found a reasonable value for the binding energy of C^{13}. He did not find any of the excited states of C^{12}, however.

Duck (25), in studying the three-alpha particle system with the Faddeev equations and a separable two-body scattering amplitude, found no bound states and only one resonance for the system if the alpha-particle structure was ignored and the alpha-alpha interaction was taken as an S-wave resonance. This resonance could be driven to negative energies only by using an unreasonable off-shell dependence of the two-particle scattering amplitude. Duck (26) has also
studied this problem in the continuum scattering states using a phenomenological approach with an aim to understanding the experimental results.

Wong, et al. (27), have not used separable potentials, but have instead used a separable function expansion for the two-body t-matrix with local potentials. They have shown that the ground state and the first excited state of a three-body system can be calculated fairly accurately by keeping only the first few terms of the expansion. They have applied these methods to a single attractive Yukawa potential for three identical spinless bosons, the \((e^-, e^-, e^+)\) system with pure Coulomb interaction, and the three-alpha system using a phenomenological potential derived from alpha-alpha scattering data.

One of the main interests in the three-alpha particle problem is that it represents a physical example of a three-body problem with overlapping two-body resonances. This is a fundamental problem which has not, as yet, been fully understood, nor, in fact, investigated in great detail although there are some controversial results due to Schmid (28) and to Aitchison (29).

The problem is also interesting because of the experimental results obtained at Rice University by Bronson, et al. (30, 31). They studied the reaction \(^7\text{B}(p, 2\alpha)^4\text{He}\) at several proton bombarding energies. These energies excite several resonant states of \(^{12}\text{C}\).
Two of the three final alpha particles were detected in coincidence and thus the kinematics of the reaction were uniquely determined. The reaction was found to be a sequential two-body decay involving the ground and first excited states of Be$^8$. The direct three-body decay was found to be less than 5 percent. Bronson, et al., concluded that the decay is a predominantly sequential two-body decay with a cross section of 100 to 200 mb, which is comparable to the geometric cross section. These facts imply that C$^{12}$, in the energy region between 17 and 21 MEV, has a strong cluster configuration of an alpha particle plus a Be$^8$ nucleus in either the ground or first excited state.

In the present work, the three-alpha problem is treated in the absence of the Coulomb interaction. Because of the neglect of this interaction the alpha-alpha interaction is assumed to be dominated by a bound state with the difference between the assumed bound state energy and the experimentally observed resonance energy accounted for by the Coulomb interaction. With this in mind, a set of potentials of the Barbour-Schult and Tabakin (32, 33) types are constructed to describe the alpha-alpha interaction. This problem is carried only to a numerical solution for the binding energy. The numerical work in this calculation is straightforward because the integral equations are non-singular. We investigate the sensitivity of the binding
energy to the shape of the two-body potential and also the sensitivity to an admixed D-state.

With a view to work on d-He\(^4\) (since reported by Shanley) and Be\(^9\) (\(\gamma, n\)), we present a set of neutron-alpha separable potentials.

In the near future we hope to extend these calculations to a search for excited states of C\(^{12}\), and to describe the continuum scattering states. The calculations for these problems are more difficult because of the singularities of the integral equations.

Chapter II, Reduction of the Faddeev Equations, is divided into three sections. In the first section an outline of a derivation of the Faddeev equations is presented along with a reduction to one equation by considering particle identity. The second section presents an angular momentum decomposition of the equation, and the third section presents the reduction to a set of one-dimensional integral equations through use of the separable approximation.

Chapter III contains a discussion of the two-body alpha-alpha potentials used in this study.

Chapter IV outlines the methods used for a numerical solution of the binding energy of the three-alpha system.

Chapter V contains the discussion of the results of the study, and the conclusions are presented in Chapter VI.
There are three appendices. A set of separable potentials to describe neutron-alpha scattering is presented in Appendix A. The details of the angular momentum decomposition are treated in Appendix B, and Appendix C comprises the details of the separable approximation.
CHAPTER II

REDUCTION OF THE FADDEEV EQUATIONS

A. DERIVATION OF THE FADDEEV EQUATIONS

Faddeev (1) begins his derivation by writing the complete Green's function in terms of an operator $T$,

$$ G = G_0 - G_0 T G_0 $$

so that $T$ satisfies

$$ T = V - V G_0 T. $$

By iterating this equation and considering only pair interactions,

$$ T = V_1 + V_2 + V_3 - (V_1 + V_2 + V_3) G_0 (V_1 + V_2 + V_3) + \ldots. $$

Reconstructing the series,

$$ T = V_1 - V_1 G_0 V_1 + V_1 G_0 V_1 G_0 V_1 - \ldots $$

$$ + V_2 - V_2 G_0 V_2 + \ldots + V_3 - V_3 G_0 V_3 + \ldots $$

$$ + (V_1 - V_1 G_0 V_1 + \ldots) G_0 (V_2 - V_2 G_0 V_2 + \ldots) $$

$$ + \ldots. $$

(2.1)

Denoting $t_1$ by the series

$$ t_1 = V_1 - V_1 G_0 V_1 + \ldots, $$

we see that $t_1$ is the solution of the equation

$$ t_1 = V_1 - V_1 G_0 t_1. $$
Likewise, $t_1$ and $t_3$ are similarly introduced. Equation (2.1) can be written in terms of these operators with the result

$$T = t_1 + t_2 + t_3 - t_1 G_0 t_2 - t_1 G_0 t_3 + \ldots .$$

We represent $T$ as a sum of three terms

$$T = T^1 + T^2 + T^3$$

where $T^1$ satisfies the equation

$$T^1 = V_1 - V_1 G_0 T$$

with similar definitions for $T^2$ and $T^3$. These operators satisfy a coupled set of equations, which are conveniently written in matrix form as

$$
\begin{pmatrix}
T^1 \\
T^2 \\
T^3
\end{pmatrix} =
\begin{pmatrix}
t_1 \\
t_2 \\
t_3
\end{pmatrix} -
\begin{pmatrix}
0 & t_1 & t_1 \\
t_2 & 0 & t_2 \\
t_3 & t_3 & 0
\end{pmatrix} G_0
\begin{pmatrix}
T^1 \\
T^2 \\
T^3
\end{pmatrix}.
$$

These equations have also been derived by Faddeev (4) without the use of perturbation theory. $T_1$ is the terms in $T$ corresponding to the pair (1) interacting last, $t_1$ is the two-particle $t$-matrix of the pair (1) in the three-particle Hilbert space, and $G_0$ is the three-particle free Green's function. The pair (1) is labeled, for example, by (1) or equivalently by the pair (2, 3). Figure 1 will make clear the meaning of $T_1$ as the terms in $T$ corresponding to the pair (1) interacting last. The solid lines represent the particles (1, 2, 3), and the connecting dotted lines represent the full off-energy-shell
two-particle scattering amplitude $t$. The blobs represent the full three-particle T-matrix.

In the case of three identical particles, the $T^i$ are related by the symmetry conditions

$$T^3 = (-1)^{e+L} T^1,$$
$$T^3 = (-1)^{e+L'} T^1,$$

where

$$e = 1 \text{ for fermions}$$
$$e = 0 \text{ for bosons}$$

and $L$ and $L'$ are relative angular momenta. Since we are dealing with bosons and all relative angular momenta are even, the Faddeev equations reduce to the single equation

$$T^1 = t_1 - 2 t_1 G_0 T^1. \quad (2.2)$$

B. ANGULAR MOMENTUM DECOMPOSITION

It is convenient to factor out of Equation (2.2) the conservation of total momentum $\delta$ function and work entirely in the system in which the total linear momentum is zero. Therefore, it is necessary to specify only two independent vector quantities in order to determine the state of the system.

Throughout this paper, the convention $m_\alpha = \hbar = c = 1$ is used.
Figure 1

A diagrammatic representation of the Faddeev equations.
Taking \( \mathbf{r}_i \) as the position vector of particle \( i \) in the total center of mass system, it follows that

\[
\mathbf{r}_{3c} = \mathbf{r}_3 - \frac{\mathbf{r}_2}{2},
\]

where

\[
\mathbf{r}_{3c} = \text{vector to particle 3 from the center of mass of the 2, 3 subsystem.}
\]

Corresponding to the \( \mathbf{r}_{3c} \),

\[
\mathbf{p}_{3c} = \frac{\mathbf{p}_3 - \mathbf{p}_2}{2}
\]

\[
\mathbf{p}_{3c} = \mathbf{K}_1,
\]

where \( p \) indicates momentum.

Similar relations among \( \mathbf{p}_{ic} \) may be obtained by cyclic interchange of subscripts.

Take for the final state of the system the states specified by

\[
| \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 > = | \mathbf{p}_1, \mathbf{K}_1 > = | \mathbf{p}_2, \mathbf{K}_2 > = | \mathbf{p}_3, \mathbf{K}_3 >
\]

with the normalization

\[
\langle \mathbf{p}_1', \mathbf{K}_1' | \mathbf{p}_1, \mathbf{K}_1 \rangle = (2\pi)^3 \delta^3 (\mathbf{p}_1' - \mathbf{p}_1) \delta^3 (\mathbf{K}_1' - \mathbf{K}_1).
\]

Specify the initial state of the system by \( | \mathbf{p}'' > \).

Taking matrix elements of the Faddeev equation,

\[
\langle \mathbf{p}_1', \mathbf{K}_1' | T^i(s) | \mathbf{p}_a'' > = \langle \mathbf{p}_1', \mathbf{K}_1' | t_i(s) | \mathbf{p}'' >
\]

\[
- 2 \int \delta^3 \mathbf{p}_a \delta^3 \mathbf{K}_a \langle \mathbf{p}_1', \mathbf{K}_1' | t_i(s) | \mathbf{p}_a, \mathbf{K}_a > (2.3)
\]

\[
G(s, |p|) \langle \mathbf{p}_a, \mathbf{K}_a | T^i(s) | \mathbf{p}'' > ,
\]

where a complete set of states has been inserted.
Since the interaction conserves total angular momentum, a partial wave expansion is made in order to reduce the dimensions of the problem (details are discussed in Appendix B). The result of that partial wave expansion follows:

\[
\langle p', l', k'_i, l'_j, J^i, m^i_j | T'(s) | \tilde{p}'' \rangle = \langle p', l', k'_i, l'_j, J^i, m^i_j | t_i (s) | \tilde{p}'' \rangle 
\]

\[
- \frac{32\sqrt{\pi}}{p'} \int_0^\infty d\rho_2 \int_0^{p' + \frac{1}{2} p^2} d\rho_1 \left[ k_2 G_0 (s \{ p \}) \right] 
\]

\[
\sum_{L, m_1, m_2} \sqrt{\frac{2L+1}{2L-1}} \left( \begin{array}{c} L' \\ m_1 \\ L \\ m_2 \\ J' \\ 0 \\ L \\ m_2 - m_1 \end{array} \right) 
\]

\[
\gamma_{m_1}^{m_2} (\theta_{p, k}, 0) \gamma_{m_2}^{* m_2} (\theta_{p, k}, 0) \gamma_{L'}^{m_1} (\theta_{k, k}, 0) \langle k_i, L' | t_i (s - \frac{2}{p^2} \tilde{p}^2) | k_i, L' \rangle \]

\[
\langle p, l, k, L, J^i, m^i_j | T'(s) | \tilde{p}'' \rangle 
\]

Thus the problem has been reduced to a two dimensional integral equation. Even so, a numerical solution is beyond the capability of present day electronic computers; consequently, further reduction to an equation of one dimension is required.
C. SEPARABLE APPROXIMATION

In order to accomplish this reduction to one dimensional integral equations, it appears necessary to assume a separable form for the two-particle scattering amplitudes. We choose to generate these separable amplitudes by the use of non-local separable potentials. In any case, the two-particle t-matrix is written in the form

\[
\langle \vec{p}'|t(s)|\vec{p}\rangle = \frac{v(\vec{p})}{D(s)} \cdot \frac{v(\vec{p}')}{D(s)}.
\]

Applying the approximation to the integral equation, in the case of a one-term separable potential, it follows (details are shown in Appendix C) that

\[
\langle \vec{p}', l', J', m_j' | T(s) | \vec{p}'' \rangle = \frac{\rho'}{V_i(k_f)} \langle \vec{p}', l', k', J', J', m_j' | t(s) | \vec{p}'' \rangle
\]

\[
- \frac{32\sqrt{\pi}}{D_i(s - \frac{1}{4} l^2)} \int_0^\infty d\rho \int d\kappa \[ k_i G_o (s - \rho^2) \]
\]

\[
\sum \sqrt{2L + 1} \left( \begin{array}{ccc} L' & 0 & 0 \\ m_L & m_l & m_m \end{array} \right) \left( \begin{array}{ccc} L & l & 0 \\ 0 & m_l & -m_m \end{array} \right) \nu_{l'}^{m_{j'}} (\theta, k, 0) \nu_{L'}^{m_{j'}} (\theta, k, 0) \nu_{L'}^{m_{j'}} (\theta, k, 0)
\]

\[
V_{L'}(k_i) V_{L'}(k_i) \langle p_L, l, J, m_j | T(s) | \vec{p}'' \rangle.
\]
where

\[ \langle p_1', \ell', K_1' L', J' m_\ell' | T^2(s) | \hat{p}'' \rangle = \frac{V_{i'}(K_1')}{p_1'} \langle p_1', \ell' L' J' m_\ell' | T^2(s) | \hat{p}'' \rangle. \]

Now the problem is reduced to a one dimensional integral equation which can be solved numerically.
CHAPTER III

ALPHA-ALPHA INTERACTIONS

The Schrodinger equation with a non-local potential

\[(E - H_0) \psi(\vec{r}) = \int d^3r' V(r, r') \psi(\vec{r}')\]

reduces to a system of linear algebraic equations if the non-local potential is also separable,

\[V(\vec{r}, \vec{r}') = V(\vec{r}) \lambda V(\vec{r}') .\]

Considering a multi-term separable potential in S-waves only, then

\[V(\vec{r}, \vec{r}') = \sum_i V_i(r) \lambda_i V_i(r') ,\]

where \(i\) labels the terms of the potential. Written in matrix form,

\[V(\vec{r}, \vec{r}') = \tilde{V}(\vec{r}) \lambda V(\vec{r}') ,\]

where \(\lambda\) is a diagonal matrix and the Fourier transform

\[(p|V|p') = \tilde{V}(p) \lambda V(p')\]

with

\[V(p) = \int d^3r e^{-i\vec{p} \cdot \vec{r}} V(r) .\]

The scattering amplitudes are of the form

\[\langle K' \ell | t(s) | K \ell \rangle = \lambda V_\ell(K) \tau_\ell(s) V(K') ,\]

where

\[\tau^{-1}(s) = 1 + \frac{2\lambda}{\pi} \int_0^\infty \frac{q^2 dq V_\ell(q)}{q^2 - s + i\epsilon} .\]
On the energy shell,

\[ \langle K \ell | \ell_i(s) | K \ell \rangle = - \frac{e^{i \delta_k} \sin \delta_k}{k} \]

where \( \delta_k \) is the phase shift for the partial wave with angular momentum \( \ell \).

The simplest separable potential is of the Yamaguchi (34, 35) form

\[ V(K) = \left( K^2 + \beta^2 \right)^{-1} \]

where \( K \) is the incident momentum and \( \beta \) is a range parameter. This potential is very limited and will not generate a resonance.

The Barbour-Schult potential is capable of generating a resonance but it will not reproduce the alpha-alpha S-wave resonance. Because of this, we decided to treat the problem assuming a bound state for \( \text{Be}^8 \). A potential of the Tabakin form is also used to describe the S-wave interaction and the D-wave interaction. The parameters for these potentials are chosen to fit the experimental data (36, 37) with the Coulomb effects removed.

Use is made of Harrington's (38) treatment of two charged particles interacting via a separable potential and a long range Coulomb potential. Harrington's choice of the separable potential was of the Yamaguchi form. The effects of the Coulomb potential are taken care of by multiplying by a charge dependent factor. Thus,
\[ g_c(k) = V(k) C_0(\eta) e^{2\eta \tan^{-1}(k/\beta)} \]

where

\[ g_c(K) = \text{Coulomb corrected separable potential} \]

\[ \eta = \mu e_1 e_2 / K \]

\[ C_0^2(\eta) = \frac{2 \pi \eta}{e^{2\pi \eta} - 1}. \]

The Coulomb corrected phase shifts, \( \delta_{sc} \), are given by

\[ K C_0^2(\eta) \cot \delta_{sc} = -\frac{4\pi}{\lambda \mu} (K^2 + \beta^2) e^{-4\eta \tan^{-1}(K/\beta)} [1 + \lambda \overline{I}(k)] \]

where

\[ \overline{I}(k) = \frac{i}{2 \pi^2} \int_0^\infty \frac{q^2}{\epsilon E(q) - E(k)} \frac{dg_c^2(q)}{dg_c^2(q)} dq \]

Harrington and Monga (39) made an expansion of \( \overline{I}(K) \) in order to obtain a relation between the effective range parameters with the Coulomb interaction and the effective range parameters with the Coulomb effects removed.

Using the effective range parameters determined from experiment by Russell, et al. (36), the corresponding uncorrected parameters were found to be

\[ a_o = 4.1 \text{ F} \]

\[ r_o = 2.27 \text{ F}. \]
These parameters correspond to the potential parameters
\[ \beta = 0.736 \, F^{-1} \]
\[ \lambda = -23.6 \, \pi \beta^3. \]

Figure 2 shows the comparison between the experimentally determined phase shifts, the calculated Coulomb corrected phase shifts, and the calculated uncorrected phase shifts.

This choice of parameters leads to a bound state with 2.91 MEV of binding. Experimentally, the Be\(^8\) nucleus is unstable by about 0.1 MEV. The estimation of 3.01 MEV for the Coulomb repulsion is thought by Harrington to be somewhat high. Nevertheless, these values of effective range parameters and binding energy were adopted for use in this study. This will allow a comparison with Harrington's result.

The Barbour-Schult potential in position space is a separable potential of the form
\[ V(r, r') = \lambda \left( \frac{e^{-ar}}{br} \right) \left( \frac{e^{-a'r'}}{br'} \right) \]
In momentum space it has the form
\[ V(K, K') = \lambda \, V(K) \, V(K'), \]
where
\[ V(K) = \frac{1}{(K^2 + \beta^2)(K'^2 + \beta'^2)} \]
Figure 2

A comparison of S-wave experimental phase shifts (---) and phase shifts calculated with the Yamaguchi potential (--- ---) and the Coulomb corrected Yamaguchi potential (--- - -).
\[ \beta = a + i b. \]

This potential generates a t-matrix of the form

\[
\langle K' \mid t(q^2) \mid K \rangle = \frac{V(K') V(K)}{\frac{1}{\lambda} - \frac{a^2 + 4 i q \text{Re} \beta - \beta^2 - 3 |\beta|^2 - q^2}{\sqrt{6} (\text{Re} \beta)^3 (\beta - \xi q)^2 (\beta + \xi q)^2}}
\]

The parameters

\[ \beta = 0.735 + i 0.526 \text{ F}^{-1} \]

\[ \lambda = -360 (\text{Re} \beta)^7 \]

give the correct binding energy, scattering length, and effective range with the Coulomb effects removed.

The S-wave Tabakin potential is a two-term separable potential. The first term is an attractive potential of the Yamaguchi form, and the second term is a repulsive potential simulating the hard core. These two terms are given by

\[
V_1(K) = (K^3 + a^3)^{-1}
\]

\[
V_2(K) = \frac{K^3}{[(K - d)^3 + b^3] [(K + d)^3 + b^3]}
\]

This produces a t-matrix

\[
\langle K' \mid t(s) \mid K \rangle = \sum_{ij} \frac{V_i(K') C_{ij}(s) \lambda_j V_j(K)}{D(s)}
\]

where the \( C_{ij}(s) \) are the elements of the adjoint of the matrix with elements
\[ \chi_{ij}(s) = \delta_{ij} + \frac{2\lambda_i}{\pi} \int_0^\infty \frac{q^2 dq}{q^2 - s + \frac{1}{\epsilon}} V_i(q) V_j(q), \]

and

\[ D(s) = \text{DET}(\chi_{ij}(s)). \]

To fit the data the parameters were found to be

\[
\begin{align*}
   a & = .735 \text{ F}^{-1} \\
   b & = 1.0 \text{ F}^{-1} \\
   d & = 1.0 \text{ F}^{-1} \\
   \lambda_1 & = -41 \text{ (a)} \\
   \lambda_2 & = 200 \text{ (b)}. 
\end{align*}
\]

The phase shifts generated by the Tabakin potential and the Barbour-Schult potential are shown in Figure 3.

In the search for a suitable alpha-alpha potential, Harrington's Coulomb correction was applied to the Tabakin potential. The correction was only applied to the first term since the second term is not important at low energies where the Coulomb effects are most important. Everything is the same as in the Tabakin potential except that \( V_i(K) \) is replaced by \( g_o(K) \). The integrations were performed numerically. The parameters were varied to obtain a fit to the experimental phase shifts. The best fit was obtained with

\[
\begin{align*}
   a & = .695 \text{ F}^{-1} \\
   b & = .834 \text{ F}^{-1} 
\end{align*}
\]
Figure 3

A comparison of S-wave experimental phase shifts (---) and phase shifts calculated with the Tabakin potential (---) and the Barbour-Schult potential (--).
\[
\begin{align*}
    d &= 0.834 \quad \text{F}^{-1} \\
    \lambda_1 &= -179 \quad \text{a}^3 \\
    \lambda_2 &= 433 \quad \text{b}^3
\end{align*}
\]

The phase shifts calculated from this potential are shown in Figure 4 along with the experimental results.

However, the potential with these parameters predicts a binding energy of 5.2 MEV without the Coulomb effects. This was thought to be much too high.

For the D-wave interaction a Tabakin potential of the form

\[
V(K) = \frac{K^2}{(K^2 + a^2)^2}
\]

was used.

This generates a t-matrix

\[
\langle K' | t(q^2) | K \rangle = \lambda \sqrt{V(K')} \sqrt{V(q^2)} \sqrt{V(K)}
\]

where

\[
\zeta^{-1}(q^2) = 1 + \lambda \left[ \frac{\cdot q^2}{(q^2 + a^2) \sqrt{q^2 + a^2}} - \frac{\cdot a^2}{(q^2 + a^2) \sqrt{q^2 + a^2}} \right] \\
\{ 1 - \frac{5}{2} \left( \frac{a^2}{a^2} \right) + \frac{15 (q^2 + a^2)^2}{8 a^2} - \frac{15 (q^2 + a^2)^3}{48 a^2} \}
\]
Figure 4

A comparison of experimental S-wave phase shifts (—) and phase shifts calculated using the Coulomb corrected Tabakin potential (— —).
The parameters $a$ and $\lambda$ were chosen to fit the experimental phase shifts. They are

$$a = 1.67 \text{ F}^{-1}$$

$$\lambda = -87 \text{ a}^3.$$ 

The phase shifts calculated with this potential are compared with the experimental results in Figure 5. The neglect of Coulomb effects is apparent at low energies and the neglect of the hard core is visible at high energies.
Figure 5

A comparison of D-wave experimental phase shifts (---) and phase shifts calculated with a Tabakin potential (— —).
CHAPTER IV

NUMERICAL CALCULATIONS

With the problem reduced to a set of one-dimensional integral equations and the form of the two-body scattering amplitudes specified, we are now in a position to make a numerical solution of the equations.

The equations are now in the form

\[ T_l'(p_1', s) = \Phi_l'(p_1', s) \]

\[ - \int_0^\infty dp_2 \sum K_{l\to l}^{l'}(p_1', p_2, s) T_l(p_2, s) \]

where changing to mathematical notation, the left side of the above equation becomes

\[ T_l'(p_1', s) = \langle p_1' l' L' J M_\nu | T^1(s) | p'' \rangle, \]

the inhomogeneous term becomes

\[ \Phi_l'(p_1', s) = \frac{\rho_{l'}}{L_i(K_f)} \langle p', l', K_i L, J M_\nu | t_1(s) | p'' \rangle \]

and the kernel,

\[ K_{l_i}^{l_i}(\rho_1', \rho_2, s) = \frac{32 \sqrt{\pi}}{D_{l_i}(s - \frac{1}{4} \rho^2)} \int_{\rho_1' - \frac{1}{2} \rho}^{\rho_1' + \frac{1}{2} \rho} dK_i \left[ K_2 G_0(s \{ \rho^3 \}) \right] \]
To obtain a numerical solution, the continuous variables are approximated by discrete ones. Consequently,

$$T_{l_1'}(s) = T_{l'}(p_{11}', s)$$

$$\varphi_{l_1'}(s) = \varphi_{l'}(p_{11}', s)$$

$$K_{l_1'}(s) = K_{l'}(p_{11}', p_{21}', s).$$

The integral is approximated by a finite sum. The set of coupled integral equations becomes the set of coupled algebraic equations

$$T_{l_1'}(s) = \varphi_{l_1'}(s) - \sum_{j=1}^{N} \sum_{L_1=1}^{I} K_{l_1j}(s) T_{l_1j}(s) \Delta p_2$$

where

$$I = \text{number of coupled integral equations, and}$$

$$N = \text{number of points necessary to approximate the integral over } p_2.$$
Now to write the equations in matrix form, let

\[
K_{ij} [I(i - 1) + L'] [I(j - 1) + L] = K_{ij} L' L
\]

Then the coupled algebraic equations, written in matrix form, are

\[
\overrightarrow{T}(s) = \overrightarrow{\phi}(s) - \overrightarrow{K}(s) \overrightarrow{T}(s)
\]

where

\[
\overrightarrow{T}(s) = \text{column matrix with elements } \overrightarrow{T}(s)
\]

\[
\overrightarrow{\phi}(s) = \text{column matrix with elements } \overrightarrow{\phi}(s)
\]

\[
\overrightarrow{K}(s) = \text{matrix with elements } \overrightarrow{K}(s)
\]

The asymptotic behavior of the wave function is given by De Alfaro and Regge (40), equation 5.20, as

\[
\varphi(\lambda, K, \alpha) \sim e^{\frac{1}{2iK}[f(\lambda, k)e^{ik\alpha} - f(\lambda, -k)e^{-ik\alpha}]}
\]

where \( f(\lambda, K) \) is the Jost function. For \( K = iK \) we see that \( f(\lambda, -K) \) must be zero for the function to be square integrable. The S-matrix is given by De Alfaro and Regge, equation 5.21, as

\[
S(\lambda, k) = \frac{f(\lambda, k)}{f(\lambda, -k)} e^{i\pi(\lambda - \frac{k}{2})}
\]
Therefore, $S(\lambda, K)$ has a simple pole at $K = ib$. Bound states correspond to stationary states of integer angular momentum described by the square integrable solutions of the Schrödinger equation. Thus we see the correspondence between a bound state and a pole at negative real energies. Since $T = S - 1$ (Wu and Ohmura [41]), we see that $T$ will also have a pole at negative real energies corresponding to a bound state.

$T(s)$ has a pole at a three-body bound state but $\phi(s)$ does not, so that at the bound state

$$[T(s_b)]_{\text{res}} = -\mathbb{K}(s_b)[T(s_b)]_{\text{res}}.$$ 

Since $T(s_b)$ has a non-zero residue and, thus, a non-trivial solution,

$$\text{DET} (\mathbb{1} + \mathbb{K}(s)) = 0.$$ 

The numerical calculations are straightforward because there are no singularities of the integral equations in this region.
CHAPTER V

RESULTS

It was found, using the Barbour-Schult potential, and considering only S-wave interactions, that the binding energy of the three-alpha system without Coulomb effects was 11.51 MEV. With the S-wave Tabakin potential, a binding energy of 11.48 MEV was found. Thus, it seems that the exact form of the potential is not important as long as the potentials produce essentially the same two-body interactions. Harrington, using an S-wave Yamaguchi potential, found a binding energy of 12.8 MEV. The difference between these results is due to the fact that Harrington was using a much stronger two-body interaction. Examination of the phase shifts demonstrates this.

Including the D-wave interaction, the binding energy was increased to 11.65 MEV.

Using the formula from De Benedetti (42) to estimate the Coulomb repulsion between the alpha particles, it was found that this energy was 5.44 MEV. This gives a binding energy of 6.21 MEV for the system as compared with the experimental value of 7.28 MEV.
Herndon and Tang (43), using a variational approach, found that a three-body force was necessary to increase the binding energy to the observed value. However, because of the limitations of the potentials used in this study, it is not possible to attribute the lower binding energy to neglect of a three-body force.

Wong, et al., have treated this problem by expanding the two-body amplitudes in a sum of separable functions. These functions are determined by solving the Lippmann-Schwinger equation using a phenomenological potential. He found a binding energy of 2.79 MEV. At the moment, it is not possible to reconcile the results of this study with those of Wong.
CHAPTER VI

CONCLUSIONS

In this study, the separable potential models are not extremely sensitive to the values of the parameters. A single parameter change of 10 percent does not greatly affect the phase shifts generated by the potentials, nor is the three-body binding energy greatly affected by changes in the parameters. Changes on the order of 10 percent in the parameters change the value of the binding energy by an amount less than the accuracy of the numerical calculations. In addition, the forms of the potential model do not have a great affect on the value of the binding energy as long as the two-body results are approximately the same.

There are some limitations of the potentials. In particular, they predict a Be$^8$ bound state whereas, experimentally, no bound state exists. These potentials also predict a binding energy of C$^{12}$ which is lower than the experimental value. Actually, however, there is no way to know the value of the Coulomb energy. A change in the value of the Coulomb radius would change the estimation of the energy and thus change the value of the C$^{12}$ binding energy. The radius predicted by our potentials is larger than the experimental radius.
Another limitation is the neglect of the hard core which seems to be necessary to produce the desired results at high energies.

Further work can be pursued on this project. An extension of the numerical calculations to positive energies would be the first step. Such an extension would permit a search for excited states of the three-alpha system, and a description of the breakup of the system could be made. This would provide a better test for the potential models than the binding energy. A determination of methods of handling the Coulomb interaction also would be in order.

It may be found that the rigid, structureless model of the alpha particle may be too limited. However, calculations thus far indicate promising results, but it is too soon to make a final determination.
APPENDIX A

NEUTRON-ALPHA SEPARABLE POTENTIALS

This appendix presents a set of separable potentials to describe neutron-alpha scattering.

The alpha particle is regarded as an entity and the problem is treated as a two-body problem. This assumption is valid for low energies.

For the S-wave interaction, a potential of the Barbour-Schult form is used. This is described under alpha-alpha scattering. This is fitted to the experimentally determined phase shifts (44). The scattering length and effective range agree with the values calculated by Pisent and Villi (45). In order to fit the phase shifts, it is necessary to use an attractive potential. This predicts an S-wave bound state of He$^5$. A similar effect is found by other authors (46, 47, 48) when treating the neutron-alpha interaction as a two-body problem. An examination of the many-body problem shows that the incident neutron experiences a repulsive potential due to the exclusion principle. However, using a repulsive potential, the best fit to the phase shifts is obtained with

$$\beta = 0.667 + i \cdot 0.556$$
$$\lambda = 495 (\text{Re } \beta)^7.$$
The calculated phase shifts are compared with the experimental phase shifts in Figure 6.

For the \( P_{1/2} \) and \( P_{3/2} \) interactions, a \( P \)-wave Barbour-Schult potential is used. This is of the form

\[
V(K, K') = 3 \lambda V(K) P_1(\hat{K} \cdot \hat{K}) V(K'),
\]

where

\[
V(K) = \frac{K}{(K^2 + \beta^2) (K^2 + \beta^* \beta)}.
\]

The corresponding \( t \)-matrix

\[
\langle K' | t(q) | K \rangle = 3 V(K') P_1(\hat{K'} \cdot \hat{K}) \tau(q) V(K),
\]

where

\[
\tau^{-1}(q) = \frac{1}{\lambda} - \frac{q^2 + 4 i q \Re \beta - \beta \beta^*}{16 (\Re \beta)^3 (\beta - i q)^2 (\beta^* - i q)^2}.
\]

The best fit to the experimental phase shifts is obtained with the parameters

\[
\beta = 0.5 + i 0.5 F^{-1}
\]

\[
\lambda = -419 (\Re \beta)^7
\]

for the \( P_{1/2} \) interaction and

\[
\beta = 0.667 + i 0.654 F^{-1}
\]

\[
\lambda = -379 (\Re \beta)^7
\]

for the \( P_{3/2} \) interaction.

The comparison between the calculated and experimental phase shifts is shown for the \( P_{1/2} \) interaction in Figure 7 and for
the $P_{3/2}$ interaction in Figure 8. Again the scattering length and effective range agree with the values obtained by Pisent and Villi.
A comparison of experimental phase shifts (——) and calculated phase shifts (---) for the $S_{1/2}$ interaction.
Figure 7

A comparison of experimental phase shifts (---) and calculated phase shifts (---) for the $P_{1/2}$ interaction.
A comparison of experimental phase shifts (---) and calculated phase shifts (----) for the $P_{3/2}$ interaction.
APPENDIX B

ANGULAR MOMENTUM DECOMPOSITION

The purpose of this appendix is to show explicitly the angular momentum decomposition of the Faddeev equations.

Recalling Equation (2.3),

\[
\langle \hat{p}_i, \hat{R}_i' | T(s) | \hat{p}'' \rangle = \langle \hat{p}_i, \hat{R}_i' | t_1(s) | \hat{p}'' \rangle
- 2 \int d^3 p_2 d^3 k_2 \langle \hat{p}_i, \hat{R}_i' | t_1(s) | \hat{p}_2 \hat{k}_2 \rangle \delta_{ij}(s \frac{d \hat{p}}{d \hat{k}}) \langle \hat{\hat{R}}_i | T'(s) | \hat{p}'' \rangle.
\]

The expansions,

\[
| p l m_j, k l m_L > = \int d \Omega \rho d \Omega_{\hat{k}} \ Y^{* m_j}_{\hat{k}} (\hat{\rho}) Y^{m_j}_{l m_L} (\hat{k}) | \hat{p} \hat{R} >
\]

\[
| p k l j m_J > = \sum_{m_L} \left( \begin{array}{c} m_L \m L \m J \end{array} \right) | p l m_j, k l m_L >
\]

where

\[
\left( \begin{array}{c} m_L \m L \m J \end{array} \right)
\]

is the Clebsch-Gordon coefficient, are used throughout this appendix.
Applying these to Equation (2. 3),

\[
\langle \hat{\rho} ', \hat{K}' | \hat{T}'(s) | \hat{\rho}'' \rangle = \sum_{\ell', m_{\ell'}, \ell''} Y_{\ell'}^{m_{\ell'}}(\hat{\rho}') Y_{\ell''}^{m_{\ell''}}(\hat{K}') \left( m_{\ell'} \quad m_{\ell''} \quad | \quad m_J \right)
\]

\[
\langle \hat{\rho} ', \hat{L}', \hat{K}', \hat{J}' | \hat{T}'(s) | \hat{\rho}'' \rangle
\]

\[
= \sum_{\ell', m_{\ell'}, \ell''} Y_{\ell'}^{m_{\ell'}}(\hat{\rho}') Y_{\ell''}^{m_{\ell''}}(\hat{K}') \left( m_{\ell'} \quad m_{\ell''} \quad | \quad m_J \right)
\]

\[
-2 \int d^3 \hat{\rho}_2 d^3 \hat{k}_2 \ G_0(s \rho_3) \left\{ \sum_{\ell, m_{\ell}} \sum_{\ell', m_{\ell'}} \ \gamma_{\ell}^{m_{\ell'}}(\hat{\rho}_2) Y_{\ell'}^{m_{\ell'}}(\hat{k}_2) \left( m_{\ell} \quad m_{\ell'} \quad | \quad m_{\rho_3} \right)
\]

\[
\gamma_{\ell}^{m_{\ell}}(\hat{\rho}_2) Y_{\ell}^{m_{\ell}}(\hat{k}_2) \left( m_{\ell} \quad m_{\ell} \quad | \quad m_{\rho} \right)
\]

\[
\langle \hat{\rho}_2, \hat{\rho}, \hat{K}_2, \hat{J}_2 | \hat{T}_2(s) | \hat{\rho}_3 \rangle \left( \sum_{\lambda, \mu} \gamma_{\lambda}^{\mu}(\hat{\rho}_2) Y_{\lambda}^{\mu}(\hat{k}_2) \left( m_J \quad \lambda \quad | \quad m_J \right)\right)
\]

\[
\langle \hat{\rho}_2, \hat{\lambda}, \hat{K}_2, \hat{J}_2 | \hat{T}_2(s) | \hat{\rho}_3 \rangle \rangle.
\]
Since the spherical harmonics are orthogonal, the integration over the angular parts of $p_e$ and $K_2$ results in the Kronecker deltas $\delta_{\ell\lambda} \delta_{m_\ell m_\lambda}$ and $\delta_{\ell_1\lambda} \delta_{m_{\ell_1} m_{\lambda_1}}$. The relations among the Clebsch-Gordan coefficients (Edmonds [49], 3.5.4) produces $\delta_{\ell_2 \ell_1} \delta_{m_2 m_1}$ when the coefficients are summed over $m_\ell m_\lambda$. The delta functions are used to perform the summations with the result

$$
\langle \rho', k', l', L', J', m_J | T'(s) | \tilde{p}' \rangle = \langle \rho', k', l', l', J', m_J, | t_1(s) | \tilde{p}' \rangle
$$

$$
-2 \int \frac{d^2 \vec{k}_2}{(2\pi)^2} \frac{d^2 \vec{k}_2}{(2\pi)^2} G_0 (s \{ \rho_2 \})
$$

$$
\sum_{j \neq \ell} \langle \rho', k', l', L', J', m_J | t_1(s) | \rho_2, k_2, L, J, m_J \rangle
$$

Applying the expansions to the matrix elements of $t_1(s)$,

$$
\langle \rho', k', l', L', J', m_J | T_1(s) | \tilde{p}_2, \tilde{k}_2, l, L, J, m_J \rangle
$$

$$
= \sum_{m_\ell \, m_{\lambda_1} \, m_{\ell_1} \, m_{\lambda_1}} \left( \begin{array}{ccc} m_\ell & m_\lambda & m_J \\ \ell & L & J \end{array} \right) \left( \begin{array}{ccc} m_{\ell_1} & m_{\lambda_1} & m_{J_1} \\ \ell' & L' & J' \end{array} \right)
$$

$$
\langle \rho, l, m_\ell, k, l, m_\lambda | t_1(s) | \rho, \ell, m_\ell, k \rangle
$$
\[ \left< p', k', l', l, J', m_f', | t_1(s) | p, k, l, J, m_f \right> = \sum_{m_{k_2} m_{l_2}} \left( \begin{array}{c} m_k \\ m_l \end{array} \right) \left( \begin{array}{c} m_{l_2} \\ m_{l_2}' \end{array} \right) \left< p, k, l, J, m_f, m_{f_2} | p', k', l, J, m_f \right> \]

\[
\begin{align*}
&= \int d\Omega_{p'} d\Omega_{k'} d\Omega_{p_2} d\Omega_{k_2} \gamma^{m_{k_2}'}(\theta') \\
&\quad \gamma_{l_2}^{m_{l_2}'}(k_2') \gamma_{l_2}^{m_{l_2}}(p_2) \gamma_{l_2}^{m_{l_2}}(k_2) \\
&\quad \left< p, k, l, J, m_f | t_1(s) | p, k, l, J, m_f \right> \\
&= \frac{1}{3/4 p_1^2 + K_1^2 - s} \]

or

\[ G(s,\{p\}) = \frac{1}{K_1^2 - (s - 3/4 p_1^2)} . \]

This is now in the form of a two-particle Green's function with energy

\[ E = s - \frac{3}{4} p_1^2 . \]

Accordingly,

\[ \left< p', k', l', l, J', m_f', | t_1(s) | p, k, l, J, m_f \right> = (2 \eta)^3 \delta^3(p_1' - p) \left< k_1' | t_1(s - 3/4 p_1'^2) | k_1 \right> . \]
Making a partial wave expansion,

\[
\langle \hat p', \hat k' | \hat t_i(s) | \hat p, \hat k \rangle = (2\pi)^3 \delta^3(\hat p' - \hat p) \, 4\pi \sum_{\lambda} (2\lambda + 1) \delta_\lambda \langle \hat k', \lambda | \hat t_i(s - \frac{3}{4} \hat p^{'2}) | \hat k, \lambda \rangle
\]

\[
(2\pi)^3(4\pi)^3 \delta^3(\hat p' - \hat p) \sum_{\lambda \mu} Y^\mu_\lambda(\hat k') Y^\mu_\lambda(\hat k) \langle \hat k', \lambda | \hat t_i(s - \frac{3}{4} \hat p^{'2}) | \hat k, \lambda \rangle
\]

Putting this into Equation (B.2),

\[
\langle \hat p', l', k', l', J' m_J' | \hat t_i(s) | \hat p, l, k, l, J m_J \rangle_2 =
\]

\[
= \frac{256 \pi^2}{\rho^2} \sum_{m_0 m'_l} \sum_{m_l m'_l} \delta (\rho^{'2} - \rho^2) \delta (\Omega_{\rho'} - \Omega_\rho) \delta (\Omega_{k'} - \Omega_k) \delta (\Omega_{l'} - \Omega_l) \delta (\Omega_{J'} - \Omega_J)
\]

\[
\left( \begin{array}{c} m_l \\ l \\ \lambda \end{array} \right) \left( \begin{array}{c} m'_l \\ l' \\ \lambda \end{array} \right) \left( \begin{array}{c} m_J \\ J \\ \mu \end{array} \right) \left( \begin{array}{c} m'_J \\ J' \\ \mu' \end{array} \right) Y^{m'_l}_{l'}(\hat p') Y^{m_l}_l(\hat p) Y^{m'_J}_{J'}(\hat k') Y^{m_J}_J(\hat k) \delta (\rho^{'2} - \rho^2) \delta (\Omega_{\rho'} - \Omega_\rho) \delta (\Omega_{k'} - \Omega_k) \delta (\Omega_{l'} - \Omega_l) \delta (\Omega_{J'} - \Omega_J)
\]

\[
\langle \hat k', \lambda | \hat t_i(s - \frac{3}{4} \hat p^{'2}) | \hat k, \lambda \rangle.
\]
Integration over $d\Omega_{\mathbf{r}_1'}$ is performed using the delta function, and over $d\Omega_{\mathbf{k}_1'}$ using the orthogonality of the spherical harmonics with the result

$$\langle \mathbf{r}_1', l', k', l', J' m_{J'} | t_i(s) | \mathbf{r}_2, l', k_2, l_2, J m_J \rangle_2$$

$$= \frac{2 \pi^5 \rho_1'}{\rho_1} \sum_{m_2', m_2} \int d\Omega_{\mathbf{r}_2} d\Omega_{\mathbf{k}_2} \left( \begin{array}{cc} m_2 & m_2' \\ m_2 & m_2' \end{array} \right) \left( \begin{array}{cc} m_j & m_j' \\ l & l' \end{array} \right) Y_j^{m_j'}(\hat{r}_1') Y_j^{* m_j}(\hat{r}_1) Y_l^{* m_l}(\hat{k}_1')$$

$$Y_l^{m_l'}(\hat{k}_1') \delta(r_i^2 - r_i'^2) \langle k_1' \lambda | t_i(s - \frac{z}{\rho_1^2}) | k, \lambda \rangle$$

Rotation operators are used to express the spherical harmonics in terms of different angles. Explicitly,

$$Y_{\lambda}^{m}(\theta, \varphi) = \sum_{a} D_{a}^{\lambda'}(a, \theta_{k_2}, \varphi_{k_2}) Y_{\lambda}^{m'}(\theta_{k_2}, 0).$$

The angles $\alpha, \theta_{k_2}, \varphi_{k_2}$ define the solid angle $\omega$. The plane $\phi = \pi$ is defined by the cross product $\hat{\mathbf{k}}_2 \times \hat{\mathbf{p}}_2$ and the plane $\phi = 0$ by the cross products $\hat{\mathbf{k}}_2 \times \hat{\mathbf{p}}_1$ and $\hat{\mathbf{k}}_2 \times \hat{\mathbf{k}}_1$. Using this in Equation (B.3),
\[ \langle p', l', k', l', j | t, (s) | p, l, k, l, j \rangle \]

\[ = \frac{256 \pi^5}{\rho_i} \sum_{m_\rho, m_\mu} \int d\mu D_{\rho \mu} \left( \begin{array}{c} m_l \\ l \end{array} | j \right) \left( \begin{array}{c} m_{l'} \\ l' \end{array} | j' \right) S \left( \rho_i^2 - \rho_i^2 \right) \]

\[ \langle k', l' | t, (s - \frac{3}{4} \rho_i^2) | k, l' \rangle \]

\[ \sum_{\{n\}} D_{n_3}^{x^*} \left( \begin{array}{c} n_3 \\ n_3 \end{array} | 0, 0 \right) D_{n_3}^{*x} \left( \begin{array}{c} n_3 \\ n_3 \end{array} | \theta_3, \theta_3, \alpha, \beta \right) D_{n_2}^{xL} \left( \begin{array}{c} n_2 \\ n_2 \end{array} | 0, 0 \right) D_{n_2}^{*x} \left( \begin{array}{c} n_2 \\ n_2 \end{array} | \theta_2, \theta_2 \right) \]

The summation over \( n \) is performed using the fact that

\[ Y_{n_2}^{*n_2} (0, 0) = \delta_{n_2} \left( \frac{2 L + 1}{4 \pi} \right)^{1/2}. \]

For convenience, call

\[ A_{n_3} = \sum_{m_3} \left( \begin{array}{c} m_3 \\ l \end{array} | j \right) \left( \begin{array}{c} m_{3'} \\ l' \end{array} | j' \right) \int d\mu D_{n_3}^{x} \left( \begin{array}{c} n_3 \\ n_3 \end{array} | \theta_3, \theta_3 \right) D_{n_3}^{*x} \left( \begin{array}{c} n_3 \\ n_3 \end{array} | \theta_3, \theta_3 \right) D_{n_2}^{xL} \left( \begin{array}{c} n_2 \\ n_2 \end{array} | 0, 0 \right) D_{n_2}^{*x} \left( \begin{array}{c} n_2 \\ n_2 \end{array} | \theta_2, \theta_2 \right). \]
Using relations between the rotation operators (Edmonds, 4.3.2), $A_{\{n\}}$ is expressed as

$$A_{\{n\}} = \sum_{[m_j]} \left( \begin{array}{ccc} m_l & m_l & m_j \\ l & l' & j \end{array} \right) \left( \begin{array}{ccc} m_{l'} & m_{l'} & m_{j'} \\ l' & l' & j' \end{array} \right)$$

$$\int d\omega \sum_{j'\mu'} (2j'+1) \left( \begin{array}{ccc} l' & l' & j' \\ m_{l'} & m_{l'} & \mu' \end{array} \right) \left( \begin{array}{ccc} l & l & j' \\ \eta_2 & \eta_2 & \nu' \end{array} \right) D_{\mu'}^{\mu}$$

$$\sum_{j'\mu'\nu'} (2j'+1) \left( \begin{array}{ccc} l & l & j' \\ m_{l'} & m_{l'} & \mu' \end{array} \right) \left( \begin{array}{ccc} l & l & j' \\ \eta_2 & \eta_2 & \nu' \end{array} \right) D_{\mu'}^{\mu}$$

where $\left( \begin{array}{ccc} l & l & j \end{array} \right)$ is a $3-j$ coefficient.

Using another relation (Edmonds, 4.6.1), the integration over $d\omega$ can be performed. This produces delta functions which are used to perform some of the summations. The resulting expression is

$$A_{\{n\}} = \sum_{j\mu} \sum_{[m_j]} \left( \begin{array}{ccc} m_l & m_l & m_j \\ l & l & j \end{array} \right) \left( \begin{array}{ccc} m_{l'} & m_{l'} & m_{j'} \\ l' & l' & j' \end{array} \right)$$

$$8\pi^2 (2J'+1) \left( \begin{array}{ccc} l' & l' & j' \\ m_{l'} & m_{l'} & \mu \end{array} \right) \left( \begin{array}{ccc} l & l & j \\ \eta_2 & \eta_2 & \nu \end{array} \right) \left( \begin{array}{ccc} l' & l' & j' \\ m_{l'} & m_{l'} & \mu \end{array} \right) \left( \begin{array}{ccc} l & l & j \\ \eta_2 & \eta_2 & \nu \end{array} \right).$$
The Clebsch-Gordon coefficients expressed in terms of the $3 - j$ coefficients and relations between the $3 - j$ coefficients (Edmonds, 3.7.8, 3.7.6, 3.6.11) are used to reduce Equation (B.5) to

$$ A_{\ell_1} = 8\pi^2 \delta_{J' J} \delta_{M' M} \left( \begin{array}{ccc} L' & L & J' \\ n_{\ell'} & n_{\ell} & n_{J'} \end{array} \right) \left( \begin{array}{ccc} L & L & J \\ 0 & 0 & 0 \end{array} \right) $$

Replacing this in Equation (B.4) yields

$$ \langle k, L', n_{\ell'} | \ell_i (s) | p, L, J, m J \rangle = \frac{2S6 \pi^2}{\rho^2} \left( \frac{2L + 1}{4\pi} \right)^{1/2} 8\pi^2 \sum_{\ell_1} \delta_{J' J} \delta_{M' M} \left( \begin{array}{ccc} L' & L & J' \\ n_{\ell'} & n_{\ell} & n_{J'} \end{array} \right) \left( \begin{array}{ccc} L & L & J \\ 0 & 0 & 0 \end{array} \right) dM_k \delta_{\rho^2_{\rho', \rho}} \langle k, L' | \ell_i (s - \rho^2_{\rho', \rho}) | k', L' \rangle \gamma^M_{\ell', k, \ell} \gamma^M_{\ell, k', \ell} \gamma^M_{\ell' \ell, \ell'} (\Theta_{\rho, \rho}, 0) \gamma^M_{\ell, \ell, \ell} (\Theta_{\rho', \rho}, 0) \gamma^M_{\ell', \ell, \ell} (\Theta_{\rho', \rho}, 0).$$

Replacing this in Equation (B.1) and performing the summations over $J$ and $m_J$ using the delta function gives
\[ \langle \rho, 1, J', m_1' \mid T_i(s) \mid \hat{p}'' \rangle = \langle \rho, 1, k, l, J, m_1 \mid T_i(s) \mid \hat{p}'' \rangle - \frac{2 \sqrt{s \pi}}{s} \int \rho \ k_z^2 \ d\rho \ d\k_z \ G_0(s \{ p \}) \]
\[ \sum_{l_1 \text{even}} \sqrt{2l_1+1} \begin{pmatrix} l' & l & J' \\ m_1' & m_1 & m_2 \end{pmatrix} \begin{pmatrix} l & l & J \\ m_1 & m_1 & -m_2 \end{pmatrix} \]
\[ \int \ d\mu \delta (p_1'^2 - p_2'^2) \langle k, l' \mid T_i(s - \frac{2m_2}{s}) \mid k, l \rangle \]
\[ \chi_{m_1}(\beta_{p_1 k_1}, 0) \chi^*_{m_2}(\beta_{p_2 k_2}, \pi) \chi_{m_1}(\beta_{m_2}, 0) \]
\[ \langle p, l, k, l, J, m_1 \mid T_i(s) \mid \hat{p}'' \rangle \]

The delta function is used to evaluate the integral over \( d\mu \).

From the kinematics,
\[ p_1 a = K_2^a + \frac{1}{4} p_2 a - K_2 p_2 \cos \theta_{2p_2} = p_1' a. \]

Therefore,
\[ \delta (p_1'^2 - p_1^2) = \delta \left( p_1'^2 - K_2^a - \frac{1}{4} p_2 a - K_2 p_2 \mu_{2p_2} \right) \]
\[ = \frac{1}{2K_2 p_2} \delta \left( \frac{2p_1'^2 - 2K_2^a - 1/2p_2^2 - \mu_{2p_2}}{2K_2 p_2} \right). \]
This is true if

\[-1 \leq \frac{2 p_{1}^{2} - 2 K_{3}^{2} - 1/2 p_{2}^{2}}{2 K_{2}p_{2}} \leq 1.\]

It follows that

\[|p_{1} - \frac{1}{2} p_{2}| \leq K_{3} \leq \frac{1}{2} p_{2} + p_{1}'.\]

In order for the wave function to have the proper symmetry,

\[\langle p'_{1}, l', k', l, J', m_{3} | T'(s) | \tilde{\rho}'' \rangle =
= (-1)^{l} \langle p'_{2}, l', k', l, J', m_{2} | T'(s) | \tilde{\rho}'' \rangle\]

Applying these to Equation (B.5) yields

\[\langle p', l', k', l, J, m_{3} | T'(s) | \tilde{\rho}'' \rangle =
= \langle p', l', k', l, J', m_{3} | T(s) | \rho'' \rangle
- \frac{32}{\rho_{1}^{l}} \int_{0}^{\infty} d \rho_{2} \int \frac{\rho_{1}^{l} + \frac{1}{2} \rho_{2}}{|p_{1}' - \frac{1}{2} r_{1}|}
\sum_{k} \sqrt{2L+1} \left( \begin{array}{ccc}
L & l' & J' \\
m_{L} & m_{l} & m_{J}
\end{array} \right) \left( \begin{array}{ccc}
L & l & J' \\
m_{L} & -m_{l} & -m_{J}
\end{array} \right)
\frac{Y_{L}^{m_{L}}(\theta_{P}, k_{L}, \phi_{L})}{Y_{l}^{m_{l}}(\theta_{P}, k_{L}, \phi_{L})} \frac{Y_{L}^{m_{L}'}(\theta_{P}, k_{L}, \phi_{L})}{Y_{l}^{m_{l}'}(\theta_{P}, k_{L}, \phi_{L})}
\langle K', L' | T_{s} (s - \frac{2}{\gamma} \rho_{1}^{l}) | K, L \rangle
\langle p_{1}, l, k, L, J', m_{3} | T'(s) | \tilde{\rho}'' \rangle\]

This is now a two dimensional integral equation.
APPENDIX C

SEPARABLE APPROXIMATION

The purpose of this appendix is to set forth explicitly the reduction of the integral equation using the approximation of the two-body scattering amplitude as a function separable in the initial and final momenta. These amplitudes are generated by non-local separable potentials.

Suppose the scattering amplitude can be approximated by using a one-term separable potential

\[ \langle K', L' | t_i (s - \frac{3}{4} P'^2) | K, L \rangle = \frac{V_{L'}(K', s') V_{L}(K, s)}{D_{L'}(s - \frac{3}{4} P'^2)} \]

Putting this into the partial wave Faddeev equation (Equation [2.4]) produces

\[ \langle \rho', l', k', l', J', m_J', | T(s) | \rho'' \rangle = \langle \rho', l', k', l', J', m_J' | t_i (s) | \rho'' \rangle \]

\[ - \frac{32\sqrt{\pi}}{\rho'} \int \rho' \, dp' \int \rho' \, dp'_2 \int dK_2 \, g_0 (s \, \rho^2) \]

\[ \frac{1}{|p' - 2|} \]

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\[
\sum_{\frac{1}{2L+1}} \sqrt{2L+1} \left( \begin{array}{ccc}
L' & L' & J' \\
m_l' & m_l' & m_2
\end{array} \right) \left( \begin{array}{ccc}
L & \ell & J \\
m_2 & -m_2 & 0
\end{array} \right)
\]

\[
\sqrt{2m_2} \langle \Theta_{\ell K_2, 0} \rangle \sqrt{2m_2} \langle \Theta_{\ell K_2, \pi} \rangle \sqrt{2m_2} \langle \Theta_{\ell K_2, 0} \rangle
\]

\[
\frac{\mathcal{V}_{L'}(K_1') \mathcal{V}_{L'}(K_1)}{D_L'(s - 3/4 p_1')} \langle p_{L'} K_1' L', J' m_2', T'(s) | \bar{p}' \rangle
\]

The factors \( \mathcal{V}_{L'}(K_1') \) and \( D_L'(s - 3/4 p_1') \) are not functions of the variable of integration and can be taken out of the integral. It can be seen that

\[
\langle p_{L'}, K_1' L', J' m_2', T'(s) | \bar{p}' \rangle
\]

must involve \( \mathcal{V}_{L'}(K_1') \) and \( p_{L'} \) as multiplicative factors only. Thus, we take

\[
\langle p_1', J', K_1', L', J' m_2', T'(s) | \bar{p}' \rangle = \frac{\mathcal{V}_{L'}(K_1)}{p_1'} \langle p_1', J', K_1', L', J' m_2', T'(s) | \bar{p}' \rangle
\]

This reduces the final equation to

\[
\langle p_1', L', L', J' m_2', T'(s) | \bar{p}' \rangle = \frac{\rho_1'}{\mathcal{V}_{L'}(K_1')} \langle p_1', L', L', J' m_2', T'(s) | \bar{p}' \rangle
\]

\[
- \frac{32 \sqrt{\pi}}{D_L'(s - 3/4 p_1')} \int_0^\infty \frac{d \rho_2}{\rho_2} \int \frac{d k_z [ k_z G_0 (s \xi \rho_3) ]}{|\rho'_1 - \frac{1}{2} \rho_2|}
\]
This is now a set of coupled integral equations in one dimension which may be solved numerically.

The reduction considering only S-wave interactions and a multi-term separable potential is now shown.

The scattering amplitude is given by

\[
\langle K', 0 | t_1(s - \frac{3}{4} \rho_i^2) | K, 0 \rangle = \sum_{ij} \frac{C_{ij}(s - \frac{3}{4} \rho_i^2) V_i(K_i) V_j(K_i)}{D(s - \frac{3}{4} \rho_i^2)}
\]

Applying this to the S-wave Faddeev equation,

\[
\langle p_i', 0, K_i', 0, 0 | T'(s) | p' \rangle = \langle p_i', 0, K_i', 0, 0 | t_1(s) | p' \rangle - \frac{y}{\pi \rho_i'} \int_0^\infty \int d\rho_i \int dk_i \int d\rho_i \left[ k_2 G_0(s \{ p \}) \right] |p_i' - \frac{1}{2} \rho_i| \sum_{ij} \frac{C_{ij}(s - \frac{3}{4} \rho_i^2)}{D(s - \frac{3}{4} \rho_i^2)} V_i(K_i') V_j(K_i) \langle p_i, K_i, 0, 0, 0 | T'(s) | p' \rangle
\]
We take
\[
\langle \pi', 0, k', 0, 0 0 0 | T'(s) | \bar{\pi}'' \rangle = \langle \pi', 0, k', 0, 0 0 0 | t'(s) | \bar{\pi}'' \rangle + \sum_i \frac{V_i(k'_i)}{\rho'_i} X_i(\pi'_i, s).
\]

This reduces the equation to the form
\[
X_i(\pi'_i, s) = \varphi_i(\pi'_i, s) - \frac{4}{\pi} \int_0^\infty d\rho_2 \int_{|\pi'_i - \frac{1}{2} \rho_2|}^{\rho'_i + \frac{1}{2} \rho_2} dk_2 \left[ k_2 \left( \frac{C_{ij}(s - \frac{3}{4} \rho_2^2)}{D(s - \frac{3}{4} \rho_2^2)} V_j(k_i) \right) \frac{V_j(k_2)}{V_2(k_2)} X_2(\rho_2, s) \right]
\]

where
\[
\varphi_i(\pi'_i, s) = \frac{2}{\pi} \int_0^\infty d\rho_2 \int_{|\pi'_i - \frac{1}{2} \rho_2|}^{\rho'_i + \frac{1}{2} \rho_2} dk_2 \left[ k_2 \left( \frac{C_{ij}(s - \frac{3}{4} \rho_2^2)}{D(s - \frac{3}{4} \rho_2^2)} V_j(k_i) \langle \pi', 0, k, 0, 0 0 0 | \bar{\pi}'' \rangle \right) \right].
\]
Thus, the equation is reduced to a set of coupled one-dimensional equations.
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REFERENCES

21. P. Shanley, to be published.
27. D. Wong, J. Ball, and J. Chen, to be published.
34. Y. Yamaguchi, Phys. Rev. 95 (1954) 1628.
44. J. Seagrave, Phys. Rev. 92 (1953) 122.