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Current Algebras and Pion Lagrangians

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ABSTRACT

CURRENT ALGEBRAS AND PION LAGRANGIANS

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The theory of weak interactions of non-strange and strange particles is reviewed. The algebra of currents is discussed briefly and applied to the soft pion calculations. The phenomenological non-linear Lagrangians of pions are developed in detail.

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I. INTRODUCTION

A. Outline

This thesis presents a discussion and, we hope, a lucid derivation of some recent theoretical developments in the theory of weak interactions. Our particular aim is to understand the phenomenological Lagrangian method for calculating meson decays which was first put forward by Weinberg on the basis of Gell-Mann and Levy's σ model, and then elaborated by Schwinger and subsequently by other authors notably Gursey and Chang, Lee, and Brown. Our own work cannot claim to make any original contributions to this fast moving and profound subject, but we do hope that our unified and simplified discussion will have pedagogic value.

Following this outline we review the theory of weak interactions of the non-strange particles (pions and nucleons), and the strange particles (we limit our discussion to the time reversal invariant decays of the K mesons). This review introduces the notions of current-current interactions, conserved vector current (CVC), partially conserved axial vector current (PCAC), the Cabibbo angle, the $\Delta I=1/2$ rule, octet dominance in the K meson weak interaction Lagrangian, and the symmetry group SU(3) x SU(3) which is believed to govern the algebra of currents entering the weak interaction

Lagrangian. We then proceed to Section II to the soft pion calculation using PCAC and the algebra of currents to discuss K meson decays. Following this, in Section III we develop the phenomenological non-linear Lagrangians which are guaranteed to reproduce the soft pion calculations of Section II. Application, discussions and conclusions are presented in Section IV.

B. <u>Weak Interactions of Non-Strange Particles</u>

The story of weak interaction began at the period of discovery of radioactivity (beta decay). The first big breakthrough came when Chadwick discovered the neutron in 1932 and Pauli made the suggestion that another particle (neutrino) is also emitted in the beta decay of the neutron. Then Fermi constructed his famous theory of beta decay in which the electron and neutrino are created at the moment of emission in the same way as a photon is formed at the moment of its emission from an atom. So he supposed that the four fermion interaction Lagrangian density is a Lorentz scalar product of vectors bilinear in the fermion fields without derivatives:

$$L_{int}(\text{Fermi}) = G_V(\bar{\psi}_p\gamma_\alpha\psi_n)(\bar{\psi}_e\gamma^\alpha\psi_\nu)$$

After this important step, people worked in order to determine the interaction form as a scalar of the combination of products of scalar, vector, tensor, axial vector, and pseudoscalar terms (SVTAP). At this time, the meson and pion were discovered. Their comparatively long lifetime led to the conclusion that their decays are the same form as neutron beta decay. The concept of universal weak interaction rose. In 1956 the great revolution in the theory of weak interactions was brought about by the τ-θ puzzle now more familiarly designated as $K_{2\pi}$ and $K_{3\pi}$ decays. All the physical properties of the particles τ and θ seemed to be the same, but τ decayed into two pions whereas θ decayed into three pions. Deep analysis showed that the parities of final states were opposite. If parity is conserved, τ and θ must be different particles in spite of the identical masses and lifetimes. Lee and Yang pointed out the fact that whether the parity is conserved in weak interaction can be determined by measuring some pseudoscalar quantity, such as a spin-angular correlation. The experiment was done by Wu and her collaborators who showed that the weak interaction seemed to be "maximally parity violating" and to contain approximately equal scalar and pseudoscalar parts. The same conclusion was also obtained by Telegdi and coworkers in the $\pi\mu\nu$ decay. A simple two-component theory of the neutrino was soon (1957) suggested independently by Landan, Salam, and Lee and Yang to explain the maximal parity violation. They supposed that the neutrino was fully polarized with its spin opposed to its momentum, i.e. left handed. The wave function of a left handed neutrino can be written as $1/2(1 + \gamma_5)\psi_\nu$. Since the massless Dirac equation is

$$(p-\vec{\alpha}\cdot\vec{p})\psi_\nu = 0$$

then

$$\psi_\nu \sim \begin{pmatrix} 1 \\ \vec{\sigma}\cdot\hat{p} \end{pmatrix} w$$

where w is an arbitrary Pauli spinor. The projection operator $1/2(1 + \gamma_5)$ applied to ψ_ν gives ψ_ν for $\vec{\sigma}\cdot\hat{p}w=-w$, and zero for $\vec{\sigma}\cdot\hat{p}w=w$, i.e. left handed neutrino.

We first consider the decay of the muon:

$$\mu^{\mp} \to e^{\mp} + \nu + \bar{\nu}$$

Because the neutrinos are completely left handed, the most general four-fermion interaction without derivative couplings is

$$L_{int} = (\bar{\psi}_e\gamma_\alpha(C_V-C_A\gamma_5)\psi_\mu)(\bar{\psi}_\nu\gamma^\alpha(1+\gamma_5)\psi_\nu)+h.c. \qquad (1)$$

From the experimental data on longitudinal polarization of the electrons, we get the relation

$$C_A = - C_V \ .$$

The interaction (1) can be then written in the form

$$L_{int} = \frac{G}{\sqrt{2}}(\bar{\psi}_e\gamma_\alpha(1+\gamma_5)\psi_\mu)(\bar{\psi}_\nu\gamma^\alpha(1+\gamma_5)\psi_\nu)+h.c. \qquad (2)$$

By Fierz-Michel reordering theorem, we see that (2) is the same as

$$L_{int} = \frac{G}{\sqrt{2}}(\bar{\psi}_\nu\gamma_\alpha(1+\gamma_5)\psi_\mu)(\bar{\psi}_e\gamma^\alpha(1+\gamma_5)\psi_\nu)+h.c. \qquad (3)$$

This form of interaction suggests that lepton pairs also occur in the bilinear combination, $\bar{\psi}_1\gamma_\alpha(1+\gamma_5)\psi_2$, in leptonic decays of neutron and pion:

$$n \rightarrow p + e^- + \bar{\nu}$$

$$\pi \rightarrow \mu + \nu$$

$$\pi \rightarrow e + \nu$$

Under such assumption, the electron will be almost left handed because of the small rest mass. Therefore the decay process $\pi \rightarrow e\nu$ is almost forbidden by the conservation of angular momentum. In fact, the calculated branching ratio by using this assumption is

$$\frac{\pi \rightarrow e\nu}{\pi \rightarrow \mu\nu} = 1.3 \times 10^{-4}$$

in excellent agreement with the experimental data. In this case, the multiplicity of possible bilinear covariants is reduced from SVTAP to just one, V-A.

In neutron decay, assuming that the nucleonic part of the interaction is also pure V and A, experimental data give the result

$$L_{int} = \frac{G'}{\sqrt{2}} (\overline{\psi}_p \gamma_\alpha (1+1.2\gamma_5)\psi_n)(\overline{\psi}_e \gamma^\alpha (1+\gamma_5)\psi_\nu) + h.c. \quad (4)$$

The experimental value of G and G' are very closed and are assumed to be equal. The factor 1.2 in the axial vector part in eq. (4) is believed to be an effect of strong interaction renormalization of an interaction which is originally of the form

$$L_{int} = \frac{G}{\sqrt{2}} (\overline{\psi}_p \gamma_\alpha (1+\gamma_5)\psi_\mu)(\overline{\psi}_e \gamma^\alpha (1+\gamma_5)\psi_\nu) + h.c. \quad (5)$$

Therefore we get a V-A universal Fermi interaction. Theoretically, it was obtained when expanding the ideal of two component theory of neutrino by Sudarshan and Marshak using a principle of chiral invariance, by Feynman and Gell-Mann using a two component theory of fermions, and by Sakurai using a principle of fermion mass reversal invariance. We note that (5) is invariant under CP, T, and chiral transformations in which the fermion fields are transformed according to

$$\psi \rightarrow i\,\gamma^2{}_{\alpha\beta}\overline{\psi}_\beta \quad (PC)$$

$$\psi \rightarrow i\,\gamma^1\gamma^3\psi \quad (T)$$

$$\psi \rightarrow \gamma_5\psi \quad (chiral)$$

The weak interaction Lagrangian density (3) can be written as

$$L_{int} = \frac{G}{\sqrt{2}} J^{(\ell)\alpha} J_{\alpha}^{(\ell)+} + h.c. \qquad (6)$$

where

$$J_{\alpha}^{(\ell)} = \bar{\psi}_{\nu}\gamma_{\alpha}(1+\gamma_5)\psi_{\mu} + \bar{\psi}_{\nu}\gamma_{\alpha}(1+\gamma_5)\psi_{e} \qquad (7)$$

is the leptonic current. Similarly, eq. (5) can be written as

$$L_{int} = \frac{G}{\sqrt{2}} J^{(N)\alpha} J_{\alpha}^{(\ell)+} + h.c. \qquad (8)$$

where the nucleonic current is

$$J_{\alpha}^{(N)} = \bar{\psi}_{p}\gamma_{\alpha}(1+\gamma_5)\psi_{n} \qquad (9)$$

Thus we describe the weak interaction as a current current interaction form. Each of $J_{\alpha}^{(\ell)}$ and $J_{\alpha}^{(N)}$ contains a vector part and an axial-vector part.

The electromagnetic current of nucleons and pions is

$$j_{\alpha}^{e.m.} = \bar{\psi}_{p}\gamma_{\alpha}\psi_{p} + i(\phi\partial_{\alpha}\phi^{+} - \partial_{\alpha}\phi\,\phi^{+}). \qquad (10)$$

Using isospin notation, this current can be written as

$$j_{\alpha}^{e.m.} = j_{\alpha}^{(B)} + j_{\alpha}^{3} \qquad (11)$$

where

$$j_{\alpha}^{(B)} = \frac{1}{2}\bar{\psi}_{N}\gamma_{\alpha}\psi_{N}$$

is the isoscalar current, and j_α^3 is the third component of the isovector current

$$\underset{\sim}{j}_\alpha = \frac{1}{2}(\bar{\psi}_N \gamma_\alpha \underset{\sim}{\tau} \psi_N) + \underset{\sim}{\phi} \times \partial_\alpha \underset{\sim}{\phi} \,. \tag{12}$$

Inspection shows that the vector part of the nucleonic current of eq. (9) is just the first term of $j_\alpha^1 + ij_\alpha^2$ in eq. (12). It was suggested by Feynman and Gell-Mann that we must add a pionic vector current to the nucleonic weak interaction current,

$$\begin{aligned} V_\alpha &= j_\alpha^1 + i\, j_\alpha^2 \\ &= \bar{\psi}_p \gamma_\alpha \psi_n + i\sqrt{2}(\phi_o \partial_\alpha \phi - \partial_\alpha \phi_o \phi) \end{aligned} \tag{13}$$

and the total vector current is conserved:

$$\partial_\alpha V^\alpha = 0. \tag{14}$$

With this assumption of CVC (conserved vector current), one can calculate the decay rate of

$$\pi^+ \rightarrow \pi^o + e^+ + \nu$$

and get 0.39 ± 0.04 sec^{-1}. The experimental value is in agreement with this prediction.

The axial vector current can not be conserved, otherwise the decay

$$\pi \rightarrow \mu + \nu$$

will be forbidden. This decay is governed by the matrix element

$$\langle 0|A^{\alpha}|\pi\rangle = \frac{i}{(2\pi)^{3/2}(2q_o)^{1/2}} f_{\pi}q^{\alpha} \tag{15}$$

where q is the momentum of pion. But

$$\langle 0|\partial_{\alpha}A^{\alpha}|\pi\rangle = \frac{f_{\pi}M_{\pi}^{2}}{(2\pi)^{3/2}(2q_o)^{1/2}} \tag{16}$$

is not zero, therefore $\partial_{\alpha}A^{\alpha} \neq 0$, and the axial vector current is not conserved. The eq. (16) suggests that the divergence of the axial vector current is proportional to the pion field

$$\partial_{\alpha}A^{\alpha} = f_{\pi}M_{\pi}^{2}\,\phi\ . \tag{17}$$

This is the assumption of PCAC (partially conserved axial vector current).

The interaction Lagrangian densities (6) and (8) with the currents (7), (13) and (15) are the realization of the universal Fermi interaction for non-strange particles incorporating the experimental facts of CP, T invariance, left hand neutrinos, CVC, PCAC, and coincidentally by our definition of neutrinos, lepton conservation. We should also differentiate between meun neutrinos and electron neutrinos.

C. Weak Interactions of Strange Particles

The next step is to extend the theory to include the strange particles, the K mesons and the hyperons. By analogy to non-strange particles, we seek an interaction Lagrangian that is product of a vector-axial vector current with its Hermitian conjugate. In doing so, some basic experimental facts must be noted. First, the branching ratio $K \to e\nu / K \to \mu\nu$ is undetectable small, it seems that the leptonic current has the same form (7) and that K leptonic decays are inhibited by the same mechanism that inhibits pion leptonic decays. Second, there are several selection rules for weak interactions:

1. $|\Delta Q|=1$ for leptonic decays. ΔQ is the difference of charges between final and initial states excluding leptons. This rule is concluded from the fact that decay modes like

$$K^+ \to \pi^+ + \nu + \bar{\nu}$$

$$K^+ \to \pi^+ + e^- + e^+$$

$$\Lambda \to n + e^- + e^+$$

have never been seen. Therefore no neutral leptonic currents or doubly charged leptonic currents exist.

2. $|\Delta S| \leq 1$. There is no evidence of the decays

$$K^\circ \to K^- + e^+ + \nu$$

$$\to N + \text{leptons}$$

3. $\Delta Q = \Delta S$ for strangeness violating weak leptonic decays. Thus the following decay modes are forbidden:

$$K^+ \to \pi^+ + \pi^+ + e^- + \bar{\nu}$$

$$K^\circ \to \pi^+ + e^- + \bar{\nu}$$

$$\Sigma^+ \to n + e^+ + \bar{\nu} \quad .$$

Combining this rule and $|\Delta Q|=1$, one concludes that $|\Delta I_3| = 1/2$. But there is a stronger assumption which requires that not only the third component of isospin, but also the value of isospin itself changes by 1/2. $\Delta I = 1/2$ rule works well in many cases, for example, it predicts

$$\frac{\Gamma(K^\circ \to \pi^\pm \mu^\mp \nu)}{\Gamma(K^+ \to \pi^\circ \mu^+ \nu)} = 2 \quad .$$

$$\frac{\Gamma(\Lambda \to p\pi^-)}{\Gamma(\Lambda \to n\pi^\circ)} = 2 \quad .$$

in agreement with observation within experimental errors. The largest departure from $\Delta I = 1/2$ rule is given by the two pion decay mode of K^+. The rule predicts a zero branching ratio $K^+ \to \pi^+\pi^\circ / K^\circ \to \pi^+\pi^-$, but the observed value is some five times larger than can be accommodated by electromagnetic corrections. This rule may be only an approximate selection rule.

To treat the decay of strange particles, we must consider the strangeness changing currents. Because of the selection rule $|\Delta S| \leq 1$, we need to discuss $\Delta S=1$ current $J_\mu^{(1)}$ only. Let $V_\mu^{(1)}$ be its vector part, and $A_\mu^{(1)}$ its axial vector part, then

$$J_\mu^{(1)} = V_\mu^{(1)} + A_\mu^{(1)} \tag{18}$$

The first point to be noted is that the vector current $V_\mu^{(1)}$ cannot be conserved. If it were, then there would exist a conserved quantity

$$Q^{(1)} = \int d^3 x\ V_o^{(1)}$$

which would commute with the total Hamiltonian H. It follows that

$$\langle p|[Q^{(1)},H]|\Lambda\rangle = (M - M_p)\langle p|Q^{(1)}|\Lambda\rangle = 0$$
$$\int d^3x\ \langle p|V_o^{(1)}(x)|\Lambda\rangle = 0 .$$

But the current $V^{(1)}$ leads to Λ-p transitions, it cannot be rigorously conserved. In the SU(3) limit, however, all the baryon masses are equal, the above argument breaks down, therefore it is possible that $V_\mu^{(1)}$ is conserved in that limit. In view of the association of $V_\mu^{(0)}$ with the i-spin current (13), i.e.

$$V_\mu^{(0)} = j_\mu^{\ 1} + ij_\mu^{\ 2} \tag{19}$$

Cabibbo proposed that $V_\mu^{(1)}$ is the SU(3) generating current which transforms as V_+, i.e.

$$V_\mu^{(1)} = j_\mu^{\ 4} + ij_\mu^{\ 5} \tag{20}$$

By the analogy with vector currents, it was assumed that there exists an octet of axial vector currents $j_{\mu 5}^{\ i}$, and

$$A_\mu^{(0)} = j_{\mu 5}^{\ 1} + i\, j_{\mu 5}^{\ 2}$$
$$A_\mu^{(1)} = j_{\mu 5}^{\ 4} + i\, j_{\mu 5}^{\ 5} \qquad (21)$$

An immediate consequence of this hypothesis is that the strangeness changing currents obey the $\Delta S = \Delta Q$ and $\Delta I = \frac{1}{2}$ rules. This is clear since they transform like an isospinor. The $\Delta S=0$ currents transform like an isovector. Therefore the total weak current is

$$J_\mu = J_\mu^{(\ell)} + J_\mu^{(H)} \qquad (22)$$

where $J_\mu^{(\ell)}$ is the leptonic current defined in (7), and $J_\mu^{(H)}$ the hadronic current which contains two parts, i.e. $\Delta S=0$ and $\Delta S=1$ currents. We can write $J_\mu^{(H)}$ as

$$J_\mu^{(H)} = a\left(V_\mu^{(0)} + A_\mu^{(0)}\right) + b\left(V_\mu^{(1)} + A_\mu^{(1)}\right) \qquad (23)$$

Under the assumption of the universality of weak interactions, the strength of the hadronic current is the same as that of the leptonic current, hence we must have the identity

$$a^2 + b^2 = 1 \ ,$$

or

$$a = \cos\theta \qquad (24)$$
$$b = \sin\theta \ .$$

θ is called Cabibbo angle. If the symmetry SU(3) were exact, the separate strength of the $\Delta S=0$ and $\Delta S=1$ parts of $J_\mu^{(H)}$ would have no invariant meaning. Because SU(3) symmetry is violated by weak interactions, we can determine Cabibbo angle by experiment. Either from the comparing the rates for $K^+ \to \mu^+ + \nu$ and $\pi^+ \to \mu^+ + \nu$ decays or comparing the rates for $K^+ \to \pi^\circ + e^+ + \nu$ and $\pi^+ \to \pi^\circ + e^+ + \nu$, we get

$$\theta \simeq 0.26 .$$

The generalized PCAC relation is also assumed:

$$\partial^\mu j_{\mu 5}{}^i(x) = f_\pi M_\pi{}^2 \phi^i(x) . \tag{25}$$

There is some ambiguity in the extension of PCAC to the strange currents since it might be thought that the K meson mass M_k should appear in (25). The prescription (25) predicts the same Cabibbo angle for the axial vector currents from $K\to\mu\nu/\pi\to\mu\nu$ decay as it does for the vector currents from $K\to\pi e\nu/\pi\to\pi e\nu$, and is preferred for this reason.

By the analogy with non-strange particles, we write the Lagrangian density for the leptonic decays of strange particles as

$$L_{int}^{L} = \frac{G}{\sqrt{2}} \; J_\mu^{(\ell)+} \; J^{(1)\mu} + \text{h.c.} \tag{26}$$

Since $J_\mu{}^{(\ell)}$ does not carry isospin, the fact that $J_\mu{}^{(1)}$ has $I=\frac{1}{2}$ implies the selection rule $\Delta I=\frac{1}{2}$ for leptonic decays of strange particles. For non-leptonic decays,

$$L_{int}^{NL} = \frac{G}{\sqrt{2}} J_\mu{}^{(H)+} J^{(H)\mu} . \tag{27}$$

Because $J_\mu^{(0)}$ is an isovector and $J_\mu^{(1)}$ is an isospinor, L_{int}^{NL} contains both $\Delta I=\frac{1}{2}$ and $\Delta I=1$ parts. The $\Delta I=\frac{1}{2}$ experimental rule $\Delta I=\frac{1}{2}$ cannot be derived from the assumption of current-current interaction. In order to get this rule, one can assume that for some reason, the $\Delta I=\frac{1}{2}$ part in L_{int}^{NL} is more effect than the other one. The natural generalization of this selection rule in the framework of SU(3) symmetry is to assume the octet enhancement, i.e. the octet part in L_{int}^{NL} is more effect. The strange members of an octet are isospinors, so that $\Delta I=\frac{1}{2}$ rule for strangeness changing decays follows. Under such an assumption, we can write the non-leptonic Lagrangian density as

$$L_{int}^{NL} = \frac{G}{\sqrt{2}} \sin 2\theta \; d_{6ij} (j^i + j_5^i)_\mu (j^j + j_5^j)^\mu \quad . \tag{28}$$

This interaction incorporates the desired current-current form, but the property of universality of weak interactions is sacrificed. The neutral currents are introduced into (28) whereas there are no neutral leptonic currents. The problem now is to use (28) to compare its predictions with the decays of strange particles, especially of K mesons in which we are particularly interested. There are two equivalent methods of calculation which we will now discuss for calculating non-leptonic decays. The basic technique employs the current algebra together with the postulate of PCAC to relate a decay process involving a pion to the corresponding process without a pion, for example to relate $K \to \pi\mu\nu$ decay to $K \to \mu\nu$. The second equivalent technique which is preferred for its simplicity and close relation to perturbation calculations is that of phenomenological Lagrangians. The phenomenological Lagrangian is constructed to have the same group theoretic invariances as are used in the algebra of currents, which are violated by symmetry breaking pion mass terms which cause the axial vector current conservation to be broken by a term of just the PCAC form. Lowest order perturbation calculations with such a Lagrangian are guaranteed to give the same predictions (in a more transparent way) as current algebra and PCAC.

II. ALGEBRA OF CURRENTS AND THE SOFT PION LIMIT

We first discuss the algebra of current. The hadronic currents of weak interactions are assumed to be the octet currents of SU(3), the corresponding charges are

$$Q^i(t) = \int d^3x j_o{}^i(x) \tag{29}$$

and

$$Q_5{}^i(t) = \int d^3x j_{o5}{}^i(x) \ . \tag{30}$$

Q^i's are the generators of SU(3), therefore they obey the following equal time commutation relations:

$$[Q^i(t), Q^j(t)] = if_{ijk}Q^k(t). \qquad (31)$$

where f's are the structure constants of SU(3). The axial currents are supposed to have octet transformation properties, so that

$$[Q^i(t), Q_5{}^j(t)] = if_{ijk}Q_5{}^k(t), \qquad (32)$$

and the equal time commutation relations between Q_5's are

$$[Q_5{}^i(t), Q_5{}^j(t)] = if_{ijk}Q^k(t). \qquad (33)$$

These relations follow from the quark model, and are just those of the group SU(3) x SU(3). If SU(3) symmetry is exact, vector currents are conserved, and the Q^i's are independent of time. The equal time commutation relations for currents themselves instead of charges are

$$[j_o{}^i(x), j_o{}^j(y)]_{x_o=y_o} = if_{ijk}\delta^3(\vec{x}-\vec{y})j_o{}^k(x)$$
$$[j_o{}^i(x), j_{o5}{}^j(y)]_{x_o=y_o} = if_{ijk}\delta^3(\vec{x}-\vec{y})j_{o5}{}^k(x) \qquad (34)$$
$$[j_{o5}{}^i(x), j_{o5}{}^j(y)]_{x_o=y_o} = if_{ijk}\delta^3(\vec{x}-\vec{y})j_o{}^k(x) .$$

Many important properties of weak interactions do not depend on the form of the currents but only on the commutation relations between them. Because there are some difficulties in the quark model, Gell-Mann proposed that we can treat these commutation relations as the fundamental assumption instead of assuming the currents are in the form of the quark model.

Now consider a process of the type

$$\alpha \to \beta + \pi^i(k)$$

where α and β are arbitrary states. The amplitude for this process is $\langle \beta, \pi^i | H(0) | \alpha \rangle$. By the reduction formula,

$$\langle \beta, \pi^i | H(0) | \alpha \rangle = \frac{1}{i}(k^2 - M_\pi^{\ 2}) \int \langle \beta | T(\phi^i(x) H(0)) | \alpha \rangle \frac{e^{ikx}}{(2\pi^{3/2}(2k_o)^{1/2}} d^4x .$$

Substituting the PCAC relation (25), one can write

$$\langle \beta, \pi^i | H(0) | \alpha \rangle = \frac{k^2 - M_\pi^{\ 2}}{if_\pi M_\pi^{\ 2} (2\pi)^{3/2} (2k_o)^{1/2}} \int \langle \beta | T(\partial^\mu j_{\mu 5}^{\ i}(x) H(0)) | \alpha \rangle e^{ikx} d^4x .$$

Using the identity

$$k^\mu \int \langle \beta | T(j_{\mu 5}^{\ i}(x) H(0)) | \alpha \rangle e^{ikx} d^4x$$

$$= i \int \langle \beta | [j_{o5}^{\ i}(x), H(0)] | \alpha \rangle \delta(x_o) e^{ikx} d^4x + i \int \langle \beta | T(\partial^\mu j_{\mu 5}^{\ i}(x) H(0)) | \alpha \rangle e^{ikx} d^4x .$$

and taking the soft pion limit $k_\mu \to 0$, assuming there is no pole in the term $\int \langle \beta | T(j_{\mu 5}^{\ i}(x) H(0)) | \alpha \rangle e^{ikx} d^4x$ at $k_\mu = 0$, we get finally the relation:

$$\lim_{k_\mu \to 0} (2\pi)^{3/2} (2k_o)^{1/2} \langle \beta, \pi^i(k) | H(0) | \alpha \rangle = \frac{1}{if_\pi} \langle \beta | [Q_5^{\ i}(0), H(0)] | \alpha \rangle . \quad (35)$$

where $Q_5^{\ i}$ is defined in (30).

At first, we apply (35) to the decays

$$\pi^+ \to \pi^\circ + e^+ + \nu \tag{36}$$

and

$$\pi^+ \to e^+ + \nu \, . \tag{37}$$

Using the Lagrangian (26), the amplitude of process (36) is

$$\langle \pi^\circ e^+ \nu | L_{int}^{L} | \pi^+ \rangle$$

$$= \frac{G}{\sqrt{2}} \frac{1}{(2\pi)^3} \frac{\sqrt{M_e}}{\sqrt{E_\mu}} \frac{1}{\sqrt{2E_\nu}} \bar{u}_\nu \gamma^\alpha (1+\gamma_5) v_e \langle \pi^\circ | V_\alpha^{(0)+} | \pi^+ \rangle \, , \tag{38}$$

and that of (37) is

$$\not{o} \, \frac{G}{\sqrt{2}} \frac{1}{(2\pi)^3} \frac{\sqrt{M_e}}{\sqrt{E_\mu}} \frac{1}{\sqrt{2E_\nu}} \bar{u}_\nu \gamma^\alpha (1+\gamma_5) v_e \langle 0 | A_\alpha^{(0)+} | \pi^+ \rangle \, . \tag{39}$$

By Lorentz invariance, we can define the form factors as

$$\langle 0 | A_\alpha^{(0)+} | \pi^+ \rangle = \frac{i}{(2\pi)^{3/2} (2k_o)^{1/2}} \sqrt{2} \, f_\pi k_\alpha \tag{40}$$

$$\langle \pi^\circ | V_\alpha^{(0)+} | \pi^+ \rangle = \frac{1}{(2\pi)^3 (4k_o q_o)^{1/2}} f_+ (k+q)_\alpha \tag{41}$$

where k and q are momenta of π^+ and π° respectively.

Applying (35) to $\langle \pi^\circ | V_\alpha^{(0)+} | \pi^+ \rangle$

$$\lim_{q \to 0} \frac{1}{(2q_o)^{1/2}} \langle \pi^\circ | V_\alpha^{(0)+} | \pi^+ \rangle = \frac{1}{i (2\pi)^{3/2} f_\pi} \langle 0 | [Q_5^3, V_\alpha^{(0)+}] | \pi^+ \rangle . \tag{42}$$

Since $V_\alpha^{(0)+} = j_\alpha^1 - i j_\alpha^2$, and $A_\alpha^{(0)+} = j_{\alpha 5}^1 - i j_{\alpha 5}^2$, the commutation relations (34) give

$$[Q_5^3, V_\alpha^{(0)+}] = -A_\alpha^{(0)+} \, . \tag{43}$$

Combining (40), (41), (42) and (43), we get

$$\lim_{q\to 0} f_+ = \sqrt{2} \tag{44}$$

This result can also be obtained by using CVC, and is in agreement with the experimental value 1.5 ± 0.2.

In $K_{\ell 2}$ and $K_{\ell 3}$ decays, the form factors are

$$\langle 0|A_\alpha^{(1)+}|K^+\rangle = \frac{1}{(2\pi)^{3/2}(2k_o)^{1/2}} F_K M_K k_\alpha \tag{45}$$

$$\langle \pi^\circ|V_\alpha^{(1)+}|K^+\rangle = \frac{1}{(2\pi)^3(4k_o q_o)^{1/2}} [f_{K+}(k+q)_\alpha + f_{K-}(k-q)_\alpha]. \tag{46}$$

where k and q are momenta of K^+ and π°. Since $A_\alpha^{(1)+} = j_{\alpha 5}{}^4 - ij_{\alpha 5}{}^5$, $V_\alpha^{(1)+} = j_\alpha{}^4 - ij_\alpha{}^5$, from (34) and (35),

$$\lim_{q\to 0} \frac{1}{(2q_o)^{1/2}} \langle \pi^\circ|V_\alpha^{(1)+}|K^+\rangle = \frac{1}{i(2\pi)^{3/2} f_\pi} \langle 0|[Q_5{}^3, V_\alpha^{(1)+}]|K^+\rangle$$

$$= -\frac{1}{i(2\pi)^{3/2} f_\pi} \frac{1}{2} \langle 0|A_\alpha^{(1)+}|K^+\rangle \tag{47}$$

Combining (45), (46) and (47), we get

$$|F_K| = \lim_{q\to 0} \left|\frac{2f_\pi}{M_K} (f_{K+} + f_{K-})\right| \tag{48}$$

The experimental values are

$$\frac{2f_\pi}{M_K} = 0.32$$

$$f_{K-}/f_{K+} = 0.46 \pm 0.27$$

$$|f_{k+}| = 0.16 \pm 0.01$$

The right hand of (48) has the value 0.074 ± 0.014. The experimental value of left hand is 0.070 ± 0.001. The agreement is very good.

The $K \rightarrow 3\pi$ decay can also be related to $K \rightarrow 2\pi$ decay by taking one pion to be soft. With octet enhancement, $\Delta I=\frac{1}{2}$ rule is satisfied. Write L_{int}^{NL} as

$$L_{int}^{NL} = (\bar{S}L) = S_1{}^*L_1 + S_2{}^*L_2 \ , \tag{49}$$

where L_1 and L_2 are two components of the isospinor, and S_1 and S_2 are coefficients. Applying (35) to $K \rightarrow 3\pi$ decay, we get

$$\lim_{q_3 \to 0} (2\pi)^{3/2}(2q_3{}^\circ)^{1/2}\langle\pi_1\pi_2\pi_3|(\bar{S}L)|K\rangle$$
$$= \frac{1}{if_\pi}\langle\pi_1\pi_2|[\int d^3x \underset{\sim}{\pi}_3\cdot\underset{\sim}{A}_o,\ (\bar{S}L)]|K\rangle \tag{50}$$

We use k, q_1, q_2 and q_3 to denote momenta of K, π_1, π_2 and π_3 respectively. K, $\underset{\sim}{\pi}_1$, $\underset{\sim}{\pi}_2$ and $\underset{\sim}{\pi}_3$ to be isospin functions of various particles. Because of the V-A structure of L, the commutator in (50) can be replaced by

$$[\int d^3x \underset{\sim}{\pi}_3\cdot\underset{\sim}{A}_o,\ (\bar{S}L)] = [\int d^3x \underset{\sim}{\pi}_3\cdot\underset{\sim}{V}_o,\ (\bar{S}L)] \tag{51}$$

From the fact that $\underset{\sim}{V}_o$ is the generator of isospin symmetry and L is an isospinor, one concludes that

$$[\int d^3x \underset{\sim}{\pi}_3\cdot\underset{\sim}{V}_o,\ (\bar{S}L)] = -\frac{1}{2}(\bar{S}\underset{\sim}{\pi}_3\cdot\underset{\sim}{\tau}L) . \tag{52}$$

Now we concern the matrix element $\langle \pi_1\pi_2\pi_3|(\bar{S}L)|K\rangle$. There are four independent isoscalars:

$$\underset{\sim}{\pi}_1 \cdot \underset{\sim}{\pi}_2 \times \underset{\sim}{\pi}_3 \quad (\bar{S}K)$$
$$\underset{\sim}{\pi}_1 \cdot \underset{\sim}{\pi}_2 \; \underset{\sim}{\pi}_3 \cdot (\bar{S}\underset{\sim}{\tau}K)$$
$$\underset{\sim}{\pi}_2 \cdot \underset{\sim}{\pi}_3 \; \underset{\sim}{\pi}_1 \cdot (\bar{S}\underset{\sim}{\tau}K)$$
$$\underset{\sim}{\pi}_3 \cdot \underset{\sim}{\pi}_1 \; \underset{\sim}{\pi}_2 \cdot (S\underset{\sim}{\tau}K).$$

Because q_1^2, q_2^2 and k^2 are always equal to the squared masses of the appropriate particles, and momentum is conserved, we choose the scalar variables to be the pion energies in the rest system.

$$E_1 = q_1 \cdot k/M_K$$
$$E_2 = q_2 \cdot k/M_K$$
$$E_3 = q_3 \cdot k/M_K$$

Using Bose statistics and the assumption of linear dependence made by Hara and Nambu, and Elias and Taylor, we have

$$\langle \pi_1\pi_2\pi_3|(\bar{S}L)|K\rangle = (2\pi)^{-6}(16q_1^\circ q_2^\circ q_3^\circ k^\circ)^{-1/2}$$
$$\times[\underset{\sim}{\pi}_1 \cdot \underset{\sim}{\pi}_2\underset{\sim}{\pi}_3 \cdot (\bar{S}\underset{\sim}{\tau}K)(\alpha+\beta E_3) + \underset{\sim}{\pi}_2 \cdot \underset{\sim}{\pi}_3\underset{\sim}{\pi}_1 \cdot (\bar{S}\underset{\sim}{\tau}K) \quad (\alpha + \beta E_1)$$
$$+ \underset{\sim}{\pi}_3 \cdot \underset{\sim}{\pi}_1\underset{\sim}{\pi}_2 \cdot (\bar{S}\underset{\sim}{\tau}K) \quad (\alpha + \beta E_2)] \tag{53}$$

with constants α and β. Under the limit $q_3 \to 0$, $E_3=0$, $E_1=E_2=\frac{1}{2}M_K$. From (50), (51), (52) and (53),

$$(2\pi)^{-9/2}(8q_1{}^\circ q_2{}^\circ k^\circ)^{-1/2}[\underset{\sim}{\pi}_1\cdot\underset{\sim}{\pi}_2\underset{\sim}{\pi}_3\cdot(\bar{S}\underset{\sim}{\tau}K)\alpha$$

$$+(\underset{\sim}{\pi}_2\cdot\underset{\sim}{\pi}_3\underset{\sim}{\pi}_1+\underset{\sim}{\pi}_3\cdot\underset{\sim}{\pi}_1\underset{\sim}{\pi}_2)\cdot(\bar{S}\underset{\sim}{\tau}K)(\alpha+\frac{1}{2}\beta M_K)]$$

$$= -\frac{1}{2if_\pi}\langle\pi_1\pi_2|(\bar{S}\underset{\sim}{\pi}_3\cdot\underset{\sim}{\tau}L)|K\rangle$$

$$= -\frac{1}{2f_\pi}(2\pi)^{-9/2}(8q_1{}^\circ q_2{}^\circ k^\circ)^{-1/2}\gamma\underset{\sim}{\pi}_1\cdot\underset{\sim}{\pi}_2(\bar{S}\underset{\sim}{\pi}_3\cdot\underset{\sim}{\tau}K) \quad (54)$$

where γ is defined through the $K\to 2\pi$ amplitude by

$$\langle\pi_1\pi_2|(\bar{S}L)|K\rangle=i(2\pi)^{-9/2}(8q_1{}^\circ q_2{}^\circ k^\circ)^{-1/2}\gamma\underset{\sim}{\pi}_1\cdot\underset{\sim}{\pi}_2(\bar{S}K) \quad (55)$$

Comparing both sides of (54),

$$\alpha = -\frac{\gamma}{2f_\pi}, \qquad \alpha+\frac{1}{2}\beta M_K = 0 \quad (56)$$

Therefore the amplitudes of $K\to 3\pi$ decays are related with those of $K\to 2\pi$ by eqs. (53), (55) and (56).

III. PHENOMENOLOGICAL LAGRANGIAN

We have seen in the last section that many physical predictions can be obtained by using current algebra and the PCAC hypothesis. But there is another method suggested by S. Weinberg, the phenomenological Lagrangian, which allows us to derive the same predictions. The idea of this method is to choose a Lagrangian which satisfies the proper current commutation relations and is invariant under isospin transformation so that the vector currents are conserved (i.e. CVC) and invariant under chiral transformation except a broken symmetry term which will give the correct PCAC. In the soft pion limit we treated, there is a factor g, the pion-nucleon coupling constant, appearing in the matrix element when each pion is taken to be soft (f_π is inversely proportional to g), hence the same result will be obtained if we calculate the soft pion matrix elements to the lowest order in g by using the Lagrangian.

Consider a Lagrangian density L containing the fields ψ_i and their derivatives $\partial_\mu \psi_i$, and the infinitesimal gauge transformation under which the fields change according to

$$\psi_i(x) \rightarrow \psi_i(x) + \chi(x) F_i[\psi_1(x), \psi_2(x), \ldots] \qquad (57)$$

where $\chi(x)$ is called a gauge function. The variation of L is

$$\delta L = \frac{\delta L}{\delta \chi} \chi + \frac{\delta L}{\delta(\partial_\mu \chi)} \partial_\mu \chi \qquad (58)$$

$$\partial_\mu \psi_i \rightarrow \partial_\mu \psi_i + \partial_\mu \chi F_i + \chi \partial_\mu F_i$$

we have

$$\frac{\delta L}{\delta \chi} = \frac{\delta L}{\delta \psi_i} F_i + \frac{\delta L}{\delta(\partial_\mu \psi_i)} \partial_\mu F_i$$

$$\frac{\delta L}{\delta(\partial_\mu \chi)} = \frac{\delta L}{\delta(\partial_\mu \psi_i)} F_i \quad .$$

The current j^μ generated by the gauge transformation (57) is defined by

$$j^{\mu} = \frac{\delta L}{\delta(\partial_{\mu}\chi)} \tag{59}$$

The divergence of the current is

$$\partial_{\mu} j^{\mu} = \partial_{\mu}\left(\frac{\delta L}{\delta(\partial_{\mu}\chi)}\right)$$

$$= \partial_{\mu}\left(\frac{\delta L}{\delta(\partial_{\mu}\psi_i)}\right) F_i + \frac{\delta L}{\delta(\partial_{\mu}\psi_i)} \partial_{\mu} F_i \ .$$

Using Euler-Lagrange equation of fields

$$\partial_{\mu}\left(\frac{\delta L}{\delta(\partial_{\mu}\psi)}\right) = \frac{\delta L}{\delta \psi_i}$$

we get

$$\partial_{\mu} j^{\mu} = \frac{\delta L}{\delta \chi} \tag{60}$$

If L is invariant under (57) with a constant gauge transformation, or gauge transformation of the first kind, then the current j^{μ} is conserved.

$$\partial_{\mu} j^{\mu} = 0 \tag{61}$$

We first consider the gauge transformation

$$\psi \rightarrow e^{-ie\chi}\, \psi, \qquad \bar{\psi} \rightarrow \bar{\psi} e^{ie\chi} \tag{62}$$

for electrons whose Lagrangian density is

$$L = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi \tag{63}$$

The infinitesimal form of (62)

$$\psi \rightarrow (1-ie\chi)\psi, \quad \bar{\psi} \rightarrow \bar{\psi}(1+ie\chi) \tag{64}$$

generates a current

$$j^{\mu} = \frac{\delta L}{\delta(\partial_{\mu}\chi)} = e\bar{\psi}\gamma^{\mu}\psi \tag{65}$$

Because (63) is invariant under the transformation (64) with constant χ, the current (65) is conserved.

The Lagrangian density for the nucleon system

$$L_N = \bar{\psi}(i\gamma^{\mu}\partial_{\mu}-M)\psi \tag{66}$$

is invariant under the first kind isospin gauge transformation in which ψ transforms as

$$\psi \rightarrow e^{-i\frac{g}{2}\omega}\psi \simeq (1-i\tfrac{g}{2}\omega)\psi\ . \tag{67}$$

where

$$\psi = \begin{pmatrix} p \\ n \end{pmatrix}, \quad \omega = \underset{\sim}{\omega} \circ \underset{\sim}{\tau}\ .$$

The conserved isovector vector current generated by (67) is

$$\underset{\sim}{j}_{v}{}^{\mu} = \frac{\delta L_N}{\delta(\partial_{\mu}\underset{\sim}{\omega})} = \frac{1}{2}\, g\bar{\psi}\gamma^{\mu}\underset{\sim}{\tau}\psi \tag{68}$$

For the pion field

$$L_{\pi} = \frac{1}{4}\,\mathrm{tr}(\partial_{\mu}\phi\partial^{\mu}\phi) - \frac{1}{4}\, M_{\pi}{}^{2}\mathrm{tr}(\phi\phi) \tag{69}$$

where $\phi = \underset{\sim}{\phi} \cdot \underset{\sim}{\tau}$. Since $\underset{\sim}{\phi}$ is an isovector, the transformation law is

$$\phi \rightarrow e^{-i\frac{g}{2}\omega}\phi e^{i\frac{g}{2}\omega} \simeq \phi - i\frac{g}{2}\,[\omega,\phi] \tag{70}$$

under which L_π is invariant, and the conserved isovector vector current

$$\underset{\sim}{j}_V^{\ \mu} = g\, \underset{\sim}{\phi} \times \partial^\mu \underset{\sim}{\phi} \tag{71}$$

is generated.

The chiral transformation of the nucleon field

$$\psi \to e^{+i\frac{g}{2}\gamma_5\lambda}\psi \simeq (1+i\tfrac{g}{2}\gamma_5\lambda)\psi, \tag{72}$$

where $\lambda = \underset{\sim}{\lambda} \cdot \underset{\sim}{\tau}$, generates an isovector axial vector current

$$\underset{\sim}{j}_A^{\ \mu} = \frac{1}{2}\, g\bar{\psi}\gamma^\mu\gamma_5\underset{\sim}{\tau}\psi \tag{73}$$

which is not conserved because the Lagrangian density (66) is not invariant under the chiral transformation (72) due to the term $M\bar{\psi}\psi$. The PCAC hypothesis demands that the divergence of the axial vector current is proportional to the pion mass instead of the nucleon mass, therefore we replace M by a function $M(i\gamma_5\underset{\sim}{\tau}\cdot\underset{\sim}{\phi})$ whose transformation properties are defined by

$$M(i\gamma_5\underset{\sim}{\tau}\cdot\underset{\sim}{\phi}) \to M(i\gamma_5\underset{\sim}{\tau}\cdot\underset{\sim}{\phi}') = e^{-i\frac{g}{2}\omega}M(i\gamma_5\underset{\sim}{\tau}\cdot\underset{\sim}{\phi})e^{i\frac{g}{2}\omega} \tag{74}$$

under isospin transformation, and

$$M(i\gamma_5\underset{\sim}{\tau}\cdot\underset{\sim}{\phi}) \to M(i\gamma_5\underset{\sim}{\tau}\cdot\underset{\sim}{\phi}') = e^{-i\frac{g}{2}\gamma_5\lambda}M(i\gamma_5\underset{\sim}{\tau}\cdot\underset{\sim}{\phi})e^{-i\frac{g}{2}\gamma_5\lambda} \tag{75}$$

under chiral transformation, so that the Lagrangian density

$$L_N = \bar{\psi}[i\gamma^\mu\partial_\mu - M(i\gamma_5\underset{\sim}{\tau}\cdot\underset{\sim}{\phi})]\psi \tag{76}$$

is invariant under both transformations. Within an eigenstate of γ_5 with eigenvalue 1, (74) and (75) become

$$M(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi}') = e^{-i\frac{g}{2}\omega} M(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi}) e^{i\frac{g}{2}\omega} \tag{77}$$

$$M(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi}') = e^{-i\frac{g}{2}\lambda} M(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi}) e^{-i\frac{g}{2}\lambda} \tag{78}$$

respectively. The Lagrangian density (76) must be Hermitian, so that

$$\gamma^\circ M^+ \gamma^\circ = M \tag{79}$$

Expanding M in power series,

$$M(i\gamma_5\underset{\sim}{\tau}\cdot\underset{\sim}{\phi}) = a(\phi^2) + i\gamma_5\tau\cdot\phi\, b(\phi^2) \ , \tag{80}$$

(79) implies a*=a, b*=b. Therefore M is a real function:

$$M^*(z^*) = M(z) \tag{81}$$

We define a real unitary function U as

$$U(z) = [M(z)/M(-z)]^{\frac{1}{2}} \tag{82}$$

so that

$$M(z) = U(z)H(z^2)$$

with

$$H(z^2) = [M(z)M(-z)]^{\frac{1}{2}} \ .$$

Under chiral transformation

$$M(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi}')M^+(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi}') = e^{-i\frac{g}{2}\lambda} M(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi})\ M^+(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi}) e^{i\frac{g}{2}\lambda}$$

as can be derived from (78). This equation implies that

$$H(-\phi'^2) = H(-\phi^2).$$

Since only isotopic spin rotation can preserve the length of the meson field, H must be a constant m.

$$M(i\gamma_5 \underset{\sim}{\tau}\cdot\underset{\sim}{\phi}) = mU(i\gamma_5 \underset{\sim}{\tau}\cdot\underset{\sim}{\phi}) \quad (83)$$

Then the Lagrangian density (76) takes the form

$$L = \bar{\psi}[i\gamma^{\mu}\partial_{\mu} - mU(i\gamma_5 \underset{\sim}{\tau}\cdot\underset{\sim}{\phi})]\psi . \quad (84)$$

Together with the replacement of the nucleon mass term $m\bar{\psi}\psi$ by $m\bar{\psi}U\psi$, we also replace the kinetic energy term of pions by

$$\frac{f\pi^2}{4} \text{ tr } [\partial_{\mu}U(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi})\partial^{\mu}U(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi})] \quad (85)$$

with some constant $f\pi$. From (83) we conclude U has the same transformation properties as M, under which (85) is invariant. Rewriting (80) as

$$U(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi}) = c(\phi^2) + i\tau\cdot\phi d(\phi^2) .$$

choosing $c = \frac{1}{f\pi}\sigma$, $d = \frac{1}{f\pi}$, and defining $\Phi = f\pi U$, we get

$$\Phi = \sigma + i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi} . \quad (86)$$

(85) becomes

$$\frac{1}{4} \text{ tr } (\partial_{\mu}\Phi\partial_{\mu}\Phi) . \quad (87)$$

From (77) and (78), the infinitesimal transformation laws for σ and ϕ are obtained immediately:

$$\delta\sigma = 0 \quad (88)$$

$$\delta\underset{\sim}{\phi} = g\underset{\sim}{\omega} \times \underset{\sim}{\phi}$$

under isospin transformation, and

$$\delta\sigma = g\underset{\sim}{\lambda}\cdot\underset{\sim}{\phi} \tag{89}$$

$$\delta\underset{\sim}{\phi} = -g\underset{\sim}{\lambda}\sigma$$

under chiral transformation. Since U is unitary, we have

$$\Phi\Phi^+ = \Phi^+\Phi = \sigma^2 + \phi^2 = f\pi^2 \tag{90}$$

therefore

$$\sigma = \sqrt{f\pi^2-\phi^2} = f\pi - \frac{1}{2f_\pi}\phi^2 + \ldots \tag{91}$$

To get the correct PCAC, we must add a broken symmetry term to the Lagrangian. Choosing it to be $f_\pi M_\pi^2 \sigma^1$, where

$$\sigma^1 = \sigma - f\pi \tag{92}$$

The Lagrangian density for the pion is then

$$L_\pi = \frac{1}{4}\,\mathrm{tr}(\partial_\mu\Phi\partial^\mu\Phi) + f_\pi M_\pi^2\sigma^1$$

$$= \frac{1}{2}(\partial_\mu\sigma\partial^\mu\sigma + \partial_\mu\underset{\sim}{\phi}\cdot\partial^\mu\underset{\sim}{\phi}) + f_\pi M_\pi^2\sigma^1 \tag{93}$$

Substituting (91) and (92) into (93),

$$L_\pi = \frac{1}{2}\partial_\mu\underset{\sim}{\phi}\cdot\partial^\mu\underset{\sim}{\phi} - \frac{1}{2}M_\pi^2\underset{\sim}{\phi}^2 + 0(\underset{\sim}{\phi}^4) ,$$

which is just the free pion Lagrangian density plus terms representing π-π scattering. L_π is invariant under transformations (88) and (89) except the term $f_\pi M_\pi^2\sigma^1$ which breaks the chiral symmetry, therefore it gives the conserved vector current j_V^{μ}. The divergence of the axial vector current (without coupling constant g) is

$$\partial_{\mu} j_{\underset{\sim}{A}}^{\mu} = \frac{1}{g} \frac{\delta L_{\pi}}{\delta \underset{\sim}{\lambda}} = f_{\pi} M_{\pi}^{2} \underset{\sim}{\phi} . \tag{94}$$

We see that the constant f_{π} defined in (85) is just the pion decay constant in (25).

If the gauge function is taken to be coordinates dependence, i.e. we consider the second kind gauge transformation, then the various Lagrangian densities are not invariant. To make them invariant, we must adopt the Yang-Mills trick. First consider the electron's Lagrangian density (63), a new vector field $A_{\mu}(x)$ is introduced to couple to the electron field, under the gauge transformation it transforms according to

$$A_{\mu} \rightarrow A_{\mu} + \partial_{\mu} \chi. \tag{95}$$

The vector field A_{μ} must be coupled to ψ field only through the replacement of $\partial_{\mu}\psi$ by $D_{\mu}\psi$, where

$$D_{\mu}\psi = (\partial_{\mu} + ieA_{\mu})\psi, \tag{96}$$

in the free Lagrangian density. This is called the minimal coupling. If the current (65) is taken to be the electric current of the electrons, then A_{μ} is just the electromagnetic field. The total Lagrangian density becomes

$$L = \bar{\psi}(i\gamma^{\mu}D_{\mu}-m)\psi - \frac{1}{4}F^{\mu\upsilon}F_{\mu\upsilon} \qquad (97)$$

where

$$F^{\mu\upsilon} = \partial^{\mu}A^{\upsilon} - \partial^{\upsilon}A^{\mu} \qquad (98)$$

L is invariant under the second kind gauge transformation. Note that the replacement of $\partial_{\mu}\psi$ by $D_{\mu}\psi$ is just the same as the replacement of P_{μ} by $P_{\mu}-eA_{\mu}$ in classical electrodynamics, and the invariance will be broken by a mass term $\frac{1}{2}mA_{\mu}A^{\mu}$ of the vector field. Representing the two parts of (97) as L_o and L_A respectively,

$$L = L_o + L_A$$
$$L_o = \bar{\psi}(i\gamma^{\mu}D_{\mu}-m)\psi \qquad (99)$$
$$L_A = \frac{-1}{4} F^{\mu\upsilon}F_{\mu\upsilon}$$

Using the Ealer-Lagrange equation for the field A_{μ},

$$\partial_{\upsilon}\frac{\partial L}{\partial A_{\mu,\upsilon}} = \frac{\partial L}{\partial A_{\mu}}$$

Because

$$\partial_{\upsilon}\frac{\partial L}{\partial A_{\mu,\upsilon}} = \partial_{\upsilon}F^{\mu\nu} = -\Box A^{\mu} + \partial_{\upsilon}\partial^{\mu}A^{\upsilon} \; ,$$

and defining

$$j^{\mu} = -\frac{\partial L}{\partial A_{\mu}} = -\frac{\partial L_o}{\partial A_{\mu}} = e\bar{\psi}\gamma^{\mu}\psi \; . \qquad (100)$$

we get

$$\Box A^{\mu} - \partial_{\upsilon}\gamma^{\mu}A^{\upsilon} = j^{\mu} \qquad (101)$$

Since L_o is invariant under the second kind gauge transformation,

$$\delta L_o = \frac{\partial L_o}{\partial A_\mu} \delta A_\mu + \left.\frac{\partial L_o}{\partial(\partial_\mu \chi)}\right|_{\delta A=0} \partial_\mu \chi + \left.\frac{\partial L_o}{\partial \chi}\right|_{\delta A=0} \chi$$

but

$$\delta A_\mu = \partial_\mu \chi$$

and

$$\left.\frac{\partial L_o}{\partial \chi}\right|_{\delta A=0} = 0$$

in virtue of the fact that L_o is also invariant under the first kind gauge transformation. Therefore we get

$$\frac{\partial L_o}{\partial A_\mu} = - \left.\frac{\partial L_o}{\partial(\partial_\mu \chi)}\right|_{\delta A=0} \tag{102}$$

Combining (100) and (102),

$$j^\mu = \left.\frac{\partial L_o}{\partial(\partial_\mu \chi)}\right|_{\delta A=0} \tag{103}$$

Equation (60) implies that

$$\partial_\mu j^\mu = \frac{\partial L_o}{\partial \chi} = 0, \tag{104}$$

the current is conserved. Looking back at (101), we see that the supplementary condition $\partial_\mu A^\mu$ can be imposed. If the mass of the vector field is not zero, the broken symmetry term $\frac{1}{2}M^2 A_\mu A^\mu$ will appear in the Lagrangian density (97), but the expression (103) for current and the conservation law (104) remain unchanged. We have a vector meson coupled to a conserved current in a partially gauge-invariant theory.

The Yang-Mills trick can be easily extended to the case of isospin gauge transformation to make the Lagrangian density (66) and (69) invariant. The derivatives are replaced by the gauge invariant derivative D_μ where

$$D_\mu \psi = (\partial_\mu + i\frac{g}{2} V_\mu)\psi \tag{105}$$

$$D_\mu \phi = \partial_\mu \phi + i\frac{g}{2} [V_\mu, \phi]$$

where $V_\mu = \underset{\sim}{\tau} \cdot \underset{\sim}{V}_\mu$ is a vector meson field coupling to the nucleon-pion field. Because of the isovector character of $\underset{\sim}{V}_\mu$, under the isospin gauge transformation,

$$V_\mu \rightarrow V_\mu + \partial_\mu \omega - i\frac{g}{2} [\omega, V_\mu] \tag{106}$$

where the term $\partial_\mu \omega$ is the usual gauge transformation, and the other term $-i\frac{g}{2} [\omega, V_\mu]$ is the isospin transformation. Combining (64), (70) and (106), we find that

$$D_\mu \psi \rightarrow (1 - i\frac{g}{2} \omega) D_\mu \psi$$

and

$$D_\mu \phi \rightarrow D_\mu \phi - i\frac{g}{2} [\omega \; D_\mu \phi]$$

just the same way as ψ and ϕ transform, so that the Lagrangian density

$$L = \bar{\psi}(i\gamma^{\mu}D_{\mu}-M)\psi + \frac{1}{4}tr(D_{\mu}\phi D^{\mu}\phi) - \frac{1}{4}M_{\pi}^{2}tr(\phi\phi)$$

is invariant. The Lagrangian density of the vector meson is

$$L_V = -\frac{1}{8}tr(G_{\mu\upsilon}G^{\mu\upsilon}) + \frac{1}{4}M_V^{2}tr(V_{\mu}V^{\mu})$$

$$= -\frac{1}{4}\underset{\sim}{G}_{\mu\upsilon}\cdot\underset{\sim}{G}^{\mu\upsilon} + \frac{1}{2}M_V^{2}\underset{\sim}{V}_{\mu}\cdot\underset{\sim}{V}^{\mu} \tag{107}$$

where

$$G_{\mu\upsilon} = \partial_{\mu}V_{\upsilon} - \partial_{\upsilon}V_{\mu} + i\frac{g}{2}[V_{\mu}, V_{\upsilon}] \tag{108}$$

which transforms as

$$\underset{\sim}{G}_{\mu\upsilon} \to \underset{\sim}{G}_{\mu\upsilon} + g\underset{\sim}{\omega} \times \underset{\sim}{G}_{\mu\upsilon} \tag{109}$$

The first term of (107) is invariant, but the second term is not except ω is coordinates independent. If we write the total Lagrangian density as

$$L = \bar{\psi}(i\gamma^{\mu}D_{\mu}-M)\psi + \frac{1}{4}tr(D_{\mu}\phi D^{\mu}\phi) - \frac{1}{4}M_{\pi}^{2}tr(\phi\phi)$$

$$- \frac{1}{8}tr(G_{\mu\upsilon}G^{\mu\upsilon}) + \frac{1}{4}M_V^{2}tr(V_{\mu}V^{\mu})$$

and define the vector current

$$\underset{\sim}{j}^{\mu} = \frac{\partial L}{\partial(\partial_{\mu}\underset{\sim}{\omega})} ,$$

then $\underset{\sim}{j}^{\mu}$ is conserved because L is invariant under constant isospin gauge transformation.

Now we will add the Yang-Mills fields to the Lagrangian density (93). From (77) and (78), we write the combined isospin and chiral transformation as

$$\Phi \rightarrow \Phi' = e^{-i\frac{g}{2}\chi^{+}} \Phi e^{i\frac{g}{2}\chi^{-}} \tag{110}$$

where $\chi^{\pm} = \underset{\sim}{\chi}^{\pm} \cdot \underset{\sim}{\tau} = \omega \pm \lambda$. The infinitesimal transformations are

$$\delta\Phi = -i\frac{g}{2}(\chi^{+}\Phi - \Phi\chi^{-}) \tag{111}$$

$$\delta\Phi^{+} = -i\frac{g}{2}(\chi^{-}\Phi^{+} - \Phi^{+}\chi^{+})$$

Now let $\chi^{\pm}$ be functions of coordinates, we must introduce the vector meson fields to couple to the field Φ

$$V_{\mu}^{\pm} = \rho_{\mu} \pm a_{\mu} \tag{112}$$

where $V_{\mu}^{\pm} = \underset{\sim}{\tau} \cdot \underset{\sim}{V}_{\mu}^{\pm}$, and $\rho_{\mu} = \underset{\sim}{\tau} \cdot \underset{\sim}{\rho}_{\mu}$ is the vector meson field, $a_{\mu} = \underset{\sim}{\tau} \cdot \underset{\sim}{a}_{\mu}$ the axial vector meson field. The fields $V_{\mu}^{\pm}$ transform as

$$\delta V_{\mu}^{\pm} = \partial_{\mu}\chi^{\pm} - i\frac{g}{2}[\chi^{\pm}, V_{\mu}^{\pm}] \tag{113}$$

The covariant derivative D_{μ} should be defined by

$$\begin{aligned} D_{\mu}\Phi &= \partial_{\mu}\Phi + i\frac{g}{2}[V_{\mu}^{+}\Phi - \Phi V_{\mu}^{-}] \\ D_{\mu}\Phi^{+} &= (D_{\mu}\Phi)^{+} = \partial_{\mu}\Phi^{+} + i\frac{g}{2}[V_{\mu}^{-}\Phi^{+} - \Phi^{+}V_{\mu}^{+}] \end{aligned} \tag{114}$$

so that $D_\mu\Phi$ transforms in the same way as the field Φ itself, i.e.

$$\delta(D_\mu\Phi) = -i\frac{g}{2}[\chi^+(D_\mu\Phi) - (D_\mu\Phi)\chi^-] \tag{115}$$

$$\delta(D_\mu\Phi^+) = -i\frac{g}{2}[\chi^-(D_\mu\Phi^+) - (D_\mu\Phi^+)\chi^+].$$

With the vector meson field tensor defined by

$$G_{\mu\upsilon}^{\pm} = \partial_\mu V_\upsilon^{\pm} - \partial_\upsilon V_\mu^{\pm} + i\frac{g}{2}[V_\mu^{\pm}, V_\upsilon^{\pm}], \tag{116}$$

the total Lagrangian density without nucleons is

$$L = -\frac{1}{16}\sum_{i=\pm} tr(G_{\mu\upsilon}^{i}G^{i\mu\upsilon}) + \frac{1}{8}M_V^2\sum_{i=\pm} tr(V_\mu^{i}V^{i\mu})$$
$$+ \frac{1}{4}\frac{1}{1-\beta^2} tr(D_\mu\Phi D^\mu\Phi) + f_\pi M_\pi^2\sigma^1 \tag{117}$$

The factor $\frac{1}{1-\beta^2}$ is included to make the kinetic energy term of physical pion field be of the form $\frac{1}{2}\partial^\mu\underset{\sim}{\phi}\cdot\partial_\mu\underset{\sim}{\phi}$.
Substituting (86), (112), (114) and (116) into (117), we get

$$L = -\frac{1}{4}(\partial_\mu\underset{\sim}{\rho}_\upsilon - \partial_\upsilon\underset{\sim}{\rho}_\mu - g\underset{\sim}{\rho}_\mu\times\underset{\sim}{\rho}_\upsilon - g\underset{\sim}{a}_\mu\times\underset{\sim}{a}_\upsilon)^2$$
$$-\frac{1}{4}(D_\mu\underset{\sim}{a}_\upsilon - D_\upsilon\underset{\sim}{a}_\mu)^2 + \frac{M_V^2}{2}(\underset{\sim}{\rho}_\mu^2 + \underset{\sim}{a}_\mu^2)$$
$$+\frac{1}{2}\frac{1}{1-\beta^2}\left\{(\partial_\mu\sigma - g\underset{\sim}{\phi}\cdot\underset{\sim}{a}_\mu)^2 + (D_\mu\underset{\sim}{\phi} + g\sigma\underset{\sim}{a}_\mu)^2\right\}$$
$$+ f_\pi M_\pi^2\sigma^1 \tag{118}$$

where
$$D_\mu = \partial_\mu - \underset{\sim}{\rho}_\mu \times \tag{119}$$

The vector and axial vector currents are

$$\underset{\sim}{j}_V^{\ \mu} = \frac{1}{g}\frac{\partial L}{\partial(\partial_\mu \underset{\sim}{\omega})} = \frac{M_V^{\ 2}}{g}\underset{\sim}{\rho}^{\mu}$$

$$\partial_\mu \underset{\sim}{j}_V^{\ \mu} = \frac{1}{g}\frac{\partial L}{\partial \underset{\sim}{\omega}} = 0 \tag{120}$$

and

$$\underset{\sim}{j}_A^{\ \mu} = \frac{1}{g}\frac{\partial L}{\partial(\partial_\mu \underset{\sim}{\lambda})} = \frac{M_V^{\ 2}}{g}\underset{\sim}{a}^{\mu}$$

$$\partial_\mu \underset{\sim}{j}_A^{\ \mu} = \frac{1}{g}\frac{\partial L}{\partial \underset{\sim}{\lambda}} = f_\pi M_\pi^{\ 2}\underset{\sim}{\phi} \tag{121}$$

Since the interaction $gf_\pi \underset{\sim}{a}_\mu \cdot \partial^\mu \underset{\sim}{\phi}$ should be eliminated, we make the transformation

$$\underset{\sim}{a}_\mu \rightarrow \underset{\sim}{a}_\mu - \frac{\beta}{M_V} D_\mu \underset{\sim}{\phi} , \tag{122}$$

and β must be chosen as

$$\beta = gf_\pi / M_V \tag{123}$$

so that the coefficient of $(D_\mu \underset{\sim}{\phi})^2$ is just 1/2. Thus the Lagrangian density becomes

$$L = -\frac{1}{4}[\partial_\mu \underset{\sim}{\rho}_\upsilon - \partial_\upsilon \underset{\sim}{\rho}_\mu - g\underset{\sim}{\rho}_\mu x \underset{\sim}{\rho}_\upsilon - g\underset{\sim}{a}_\mu x \underset{\sim}{a}_\upsilon + \frac{g\beta}{M_V}(\underset{\sim}{a}_\mu x D_\mu \underset{\sim}{\phi}$$

$$+ D_\mu \underset{\sim}{\phi} x \underset{\sim}{a}_\upsilon) - g(\frac{\beta}{M_V})^2 D_\mu \underset{\sim}{\phi} x D_\upsilon \underset{\sim}{\phi}]^2 + \frac{1}{2} M_V^{\ 2} \underset{\sim}{\rho}_\mu^{\ 2}$$

$$- \frac{1}{4}[D_\mu \underset{\sim}{a}_\upsilon - D_\upsilon \underset{\sim}{a}_\mu - \frac{\beta}{M_V}(D_\mu D_\upsilon - D_\upsilon D_\mu)\underset{\sim}{\phi}]^2 + \frac{1}{2}\frac{M_V^{\ 2}}{1-\beta^2}\underset{\sim}{a}_\mu^{\ 2}$$

$$+ \frac{1}{2}(D_\mu \underset{\sim}{\phi})^2 + \frac{1}{2}\frac{1}{1-\beta^2}\left\{(\partial_\mu \sigma - g\underset{\sim}{\phi}\cdot(\underset{\sim}{a}_\mu - \frac{\beta}{M_V} D_\mu \underset{\sim}{\phi})\right\}^2$$

$$+ \frac{1}{2} \frac{g^2}{1-\beta^2} \sigma'^2 (\underset{\sim}{a}_\mu - \frac{\beta}{M_V} D_\mu \underset{\sim}{\phi})^2 + g\sigma' (\underset{\sim}{a}_\mu - \frac{\beta}{M} D_\mu \underset{\sim}{\phi}) (D^\mu \underset{\sim}{\phi} + \frac{\beta M_V}{1-\beta^2} \underset{\sim}{a}^\mu)$$

$$+ M_\pi^2 \frac{\beta M_V}{g} \sigma' \tag{124}$$

The coefficient of the term $\frac{1}{2} \underset{\sim}{a}_\mu^2$ should be the square of the mass of the axial vector meson.

$$\frac{M_V^2}{1-\beta^2} = M_A^2$$

since

$$M_A = \sqrt{2}\, M_V ,$$

we get

$$\beta = \frac{1}{\sqrt{2}} \tag{125}$$

Because of the transformation (122), the axial vector current (122) becomes

$$\underset{\sim}{j}_A^\mu = \frac{M_V^2}{g} (\underset{\sim}{a}^\mu - \frac{1}{\sqrt{2} M_V} D^\mu \underset{\sim}{\phi}) \tag{126}$$

IV. CONCLUSION

Now we get a non-linear Lagrangian density consisting of pions, isovector vector mesons and isovector axial vector mesons, from which the vector and axial vector currents are constructed. These currents satisfy the current algebras and CVC, PCAC relations. In pertubative calculation, we expand the Lagrangian density in powers of the coupling constant, apply the Feynman rules to the lowest order diagrams, then the current algebras results will be obtained. When kaon decays are concerned, the generalization to SU(3) x SU(3) is needed. The Lagrangian density should contain the octet of pseudoscalar mesons which couple to an octet of vector mesons and an octet of axial vector mesons. But we will not do the generalization in this thesis.

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