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The Peratization Approximation

by

William Arthur Gale

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Thesis Director's signature:

Ian Duck

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William Arthur Gale

ABSTRACT

The peratization program is investigated in the field of singular potential scattering, as an analog to the unrenormalizable field theories. We review some known results on the scattering amplitude, and establish the applicability of the commonly used regularizations. A general argument for peratization is presented. Several examples are considered. The results for weak types of singularities offer an unresolved contradiction.

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INTRODUCTION

1.1 Relation to Field Theories

The motivation to the study of singular potentials which has been taking place recently stems from field theory. In studying unrenormalizable field theories, such as spin-1 electrodynamics and various weak interaction theories, there has been a search for a method of obtaining meaningful finite answers. The ultimate justification for such methods will be their agreement with experiment. The experimental situation is not sufficiently clear in these areas now to accept or reject any of the various proposals. In this situation one turns to a simpler theoretical model in which answers can be obtained by well established theory and will be available to check the new methods. Scattering from a potential is such a simple model; the Schrodinger equation may be solved mechanically if not analytically. The potentials more singular than a logarithm to some power times an inverse square are those that correspond to the unrenormalizable theories; we refer to these as singular potentials.[25] The reverse process of finding methods suitable for singular potentials with the suggestion that they may be applicable to unrenormalizable field theories has also taken place. We shall examine here the peratization approximation originally suggested as a method for solving the Bethe-Salpeter equation for the vector meson theory of weak interactions.

We give an abstract example of the use of peratization: if we had $\exp(-g(\alpha^{-1}+1))$ only available as a power series in g

$$1 - g\frac{(\alpha^{-1}+1)}{1!} + g^2\frac{(\alpha^{-2}+2\alpha^{-1}+1)}{2!} - \dots$$

and were interested in this at $\alpha = 0$, we would note the divergence of each term as a function of α. The peratization approximation is to sum the leading singularities first, then let $\alpha \to 0$. In this case one obtains

$$1 - g\frac{\alpha^{-1}}{1!} + g^2\frac{\alpha^{-2}}{2!} - \dots$$
$$= \exp(-g\alpha^{-1})$$
$$\to 0$$

which is already the correct answer.

1.2 General Restrictions

We define the problem considered further, by a review of some well known results. Case[1] has considered potentials r^{-m}, $m \geqslant 2$, in the attractive case. He found that all solutions of the Schrodinger equation were quadratically integrable. This contrasts with the situation for less singular potentials where this condition specifies the eigenfunctions. The requirement of orthogonality required introducing a new parameter, but a complete, integrable, orthonormal set of solutions could then be selected. The eigenvalue spectrum consists of a positive continuum and a negative point spectrum which extends to minus infinity. Since no physical system has an infinite binding energy, we must require that the leading singularity be repulsive.

This also suffices for us to distinguish a regular solution and a singular solution.

Secondly, for potentials which fall off at infinity only as slowly as an inverse cube, the Jost function and hence scattering parameters will not necessarily be analytic.[2] To avoid any difficulties from this source, we will assume the property $\int_c^\infty dr\ r^2 |V(r)| < \infty$, for any $c > 0$.

Finally, Predazzi and Regge[3] have pointed out that once the zero energy case has been solved, one can obtain a regular Voltera integral equation for the non-zero energy case, leading to a function analytic in k^2. Since such equations can be solved by standard iterative procedures, we often consider only the case of zero energy, for which the scattering will be entirely s-wave.[15] We will examine this in more detail on page 27. The p and higher waves cannot be obtained by this technique.

1.3 Singularity of the Scattering Amplitude

Jabbur[4] has proved the following result: For a repulsive singular potential of the form gr^{-m}, $m > 3$, the scattering amplitude has an essential singularity at the origin as a function of the coupling constant. We shall see by example that for various singular potentials the singularity may be an algebraic branch point, a logarithmic branch point, or an accumulation of poles. It can be viewed physically as due to the infinite number of bound states associated with an arbitrarily small, but negative, coupling constant. We follow Jabbur's proof. The Schrodinger equation

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{g}{r^m}\right] \psi(k,r) = 0 \tag{1.301}$$

is rewritten by scaling r = sx, so that the equation reads

$$\left[\frac{d^2}{dx^2} + s^2k^2 - g s^{-m+2} x^{-m}\right] \psi(k,x) = 0 \tag{1.302}$$

We now choose s so that

$$s^2 k^2 = g s^{-m+2} \equiv \lambda^2 \tag{1.303}$$

and the equation becomes

$$\left[\frac{d^2}{dx^2} + \lambda^2 - \lambda^2 x^{-m}\right] \psi(\lambda,x) = 0 \tag{1.304}$$

Borrowing at this point from a result we prove later as the Intrinsic Cutoff Method, equation (2.101), we can write

$$\psi(\lambda,x) = h(\lambda^2;\lambda,x) + A(\lambda)\, f(\lambda^2;\lambda,x) \tag{1.305}$$

where

$$\begin{aligned} h(\lambda^2;\lambda,x) &= \lambda^2 \int_x^\infty \lambda^{-1} \sin\lambda(\xi - x)\, \xi^{-m}\, h(\lambda^2;\lambda,\xi)\, d\xi \\ &\quad + \lambda^{-1} \sin\lambda x \\ f(\lambda^2;\lambda,x) &= \lambda^2 \int_x^\infty \lambda^{-1} \sin\lambda(\xi - x)\, \xi^{-m}\, f(\lambda^2;\lambda,\xi)\, d\xi \\ &\quad + e^{-i\lambda x} \end{aligned} \tag{1.306}$$

and

$$A(\lambda) = -\lim_{x\to 0} \frac{h(\lambda^2;\lambda,x)}{f(\lambda^2;\lambda,x)} \tag{1.307}$$

The redundant argument results from lumping coupling constant and energy together. We aim to show that $A(\lambda)$, the scattering length, has no power series expansion in λ about $\lambda = 0$, or equivalently that if one is assumed, its radius of convergence is zero.

Now
$$A(\lambda) = \lim_{x\to 0} \frac{-\lambda^2\int_x^\infty d\xi\, \lambda^{-1}\sin\lambda(\xi-x)\,\xi^{-m}h(\lambda,\xi) - \lambda^{-1}\sin\lambda x}{\lambda^2\int_x^\infty d\xi\,\lambda^{-1}\sin\lambda(\xi-x)\,\xi^{-m}f(\lambda,\xi) + e^{-i\lambda x}}$$

$$\equiv \lim_{\epsilon\to 0} A_\epsilon(x) \tag{1.308}$$

$$\doteq \lim_{\epsilon\to 0} \frac{-\lambda^2\int_\epsilon^\infty d\xi\,(\xi-\epsilon)\,\xi^{-m}h(\lambda,\xi)}{1+\lambda^2\int_\epsilon^\infty d\xi\,(\xi-\epsilon)\,\xi^{-m}f(\lambda,\xi)}$$

While for λ , and $x \neq 0$,

$$h(\lambda,x) = \Sigma\,\lambda^n h_n(x)$$

$$f(\lambda,x) = \Sigma\,\lambda^n f_n(x) \tag{1.309}$$

and we can iterate the integral equations for $h_n(x)$, $f_n(x)$, then substitute into $A_\epsilon(x)$. The largest contribution to the integrals in $A_\epsilon(x)$ comes near the origin, so we consider x small. For h, $h_0(x)=0$, $h_1(x)=x$, and $\sin\lambda x \doteq \lambda x$ gives

$$h_{n+1}(x) \underset{0}{\sim} \int_x^\infty h_{n-1}(\xi)\,(\xi-x)\,\xi^{-m}d\xi$$

$$h_3(x) = O(x^{-m+3})$$

$$h_5(x) = O(x^{-2m+5})$$

$$\vdots$$

$$h_{2n}(x) = 0$$

$$h_{2n+1}(x) = O(x^{-nm+2n+1}) \quad . \tag{1.310}$$

For f, we have $f_0(x)=1$, $f_1(x)=x$, and the same iteration gives

$$f_{2n}(x) = O(x^{-mn+2n})$$

$$f_{2n+1}(x) = O(x^{-mn+2n+1}) \quad . \tag{1.311}$$

If we expand $A_\epsilon(\lambda) = \lambda^n a_n$

$$a_n \underset{\epsilon\to 0}{\sim} \int_\epsilon^\infty \xi^{-m+1} h_{n-2}(\xi)\, d\xi = O\left(\epsilon^{-m/2+n}\right) \tag{1.312}$$

The usual theorem for the radius of convergence of a power series gives:

$$R(\epsilon) = \lim_{n\to\infty} |a_n|^{-\frac{1}{n}} = O\left(\epsilon^{(m-2)/2}\right) \tag{1.313}$$

but, m $>$3, so: $\lim_{\epsilon\to 0} R(\epsilon) = 0$. Thus A(λ) is not analytic at $\lambda = 0$. It could still have only a pole, but since

$$\lim_{g\to 0} A(g) = \lim_{g\to 0} \left(\frac{S(g)-1}{2ik}\right) = \frac{1-1}{2ik} = 0 \tag{1.314}$$

where S(g) is the S matrix, A does not have a pole at the origin, and the singularity is essential.

This property accounts for the mathematical difficulties associated with the singular potentials. We note from the proof again that the inverse square potential is the real dividing line for the singular and non-singular behavior. We shall now turn to various means of dealing with singular potentials.

METHODS FOR SINGULAR POTENTIALS

2.1 Intrinsic Cutoff Method

Pais and Wu[5] have proved the following result for the potentials we are considering: the scattering amplitude

$$A(k) = -\lim_{r\to 0} \frac{h(g;k,r)}{f(g;-k,r)} \tag{2.101}$$

where f(g;-k,r) is the Jost function and h(g;k,r) = $(-2ik)^{-1}$(f(g;k,r) - f(g;-k,r)). Since the radial coordinate is used as a natural cutoff, we refer to this as the Intrinsic Cutoff Method.

We start with the radial Schrödinger equation

$$\left[\frac{d^2}{dr^2} + k^2 - g\,V(r)\right]\psi(k,r) = 0 \tag{2.102}$$

with the asymptotic solution

$$\psi(k,r) \underset{\infty}{\sim} k^{-1}\sin kr + A(k)\,e^{ikr}$$

or if $k=0$ $\underset{\infty}{\sim} r + A(0)$ (2.103)

The equation is second order, so that not considering the boundary conditions, we have two solution ψ_r, ψ_s, labeled so that

$$\lim_{r\to 0} \frac{\psi_r(r)}{\psi_s(r)} = 0 \tag{2.105}$$

We can take ψ_r to be real without loss of generality. If we now write

$$\left[\frac{d^2}{dr^2} + k^2\right]\psi_0 = 0$$

$$\left[\frac{d^2}{dr^2} + k^2\right]\psi = g\,V(r)\,\psi(k,r)$$

(2.106)

we can regard $gV\psi$ as a source term and write

$$\psi(k,r) = \psi_0(k,r) + g\int_0^\infty dr'\, G(r,r')\, V(r')\, \psi(k,r')$$

where
$$G(r,r') = \begin{cases} k^{-1} \sin k(r'-r) & r' \geqslant r \\ 0 & r' < r \end{cases} \qquad (2.107)$$

is the appropriate Green's function. The equation for ψ_0 is the asymptotic equation for ψ, hence

$$\psi_0 = k^{-1} \sin kr + A(k)\, e^{ikr} \qquad (2.108)$$

and
$$\psi = k^{-1} \sin kr + A(k)\, e^{ikr} + g\int_r^\infty dr'\, k^{-1} \sin k(r'-r)\, V(r')\, \psi(k,r') . \qquad (2.109)$$

Now define

$$f(g;k,r) = e^{-ikr} + g\int_r^\infty dr'\, k^{-1} \sin k(r'-r)\, V(r')\, f(g;k,r')$$

$$h(g;k,r) = k^{-1} \sin kr + g\int_r^\infty dr'\, k^{-1} \sin k(r'-r)\, V(r')\, h(g;k,r') . \qquad (2.110)$$

We recognize f(g;k,r) as the Jost function and

$$h(g;k,r) = (-2ik)^{-1}\,[f(g;k,r) - f(g;-k,r)] \qquad (2.111)$$

These are entire functions of g since the equations are of Voltera type for all $r>0$.

By construction and linearity of operators:

$$\psi(k,r) = h(g;k,r) + A(k)\, f(g;-k,r) \qquad (2.112)$$

But we already had two independent solutions ψ_r , and ψ_s , so write

$$f(g;-k,r) = \alpha_r \psi_r(k,r) + \alpha_s \psi_s(k,r)$$

$$h(g;k,r) = \beta_r \psi_r(k,r) + \beta_s \psi_s(k,r) \tag{2.113}$$

Then
$$\begin{aligned}\psi &= h + Af \\ &= (\beta_r + A\alpha_r)\psi_r + (\beta_s + A\alpha_s)\psi_s \end{aligned} \tag{2.114}$$

$$= \text{const}\cdot \psi_r \tag{2.105}$$

hence $\beta_s + A\alpha_s = 0$ (2.115)

Now not both β_s and α_s can vanish, or f and h would be linearly dependent, so we have two cases $\alpha_s = 0$ and $\alpha_s \neq 0$.

If $\alpha_s = 0$ then $A = \infty$, and $f = \alpha_r \psi_r$ is real so that e^{-ikr} is also, hence $k = 0$. This is a zero energy bound state, of which an example will be seen later.

If $\alpha_s \neq 0$
$$\begin{aligned} -A(k) &= \beta_s / \alpha_s \\ &= \frac{h - \beta_r \psi_r}{f - \alpha_r \psi_r} \\ &= \lim_{r\to 0} \frac{h/\psi_s - \beta_r \psi_r/\psi_s}{f/\psi_s - \alpha_r \psi_r/\psi_s} \\ &= \lim_{r\to 0} \frac{h(g;k,r)}{f(g;-k,r)} \end{aligned} \tag{2.116}$$

which is the result stated. For non-singular potentials the corresponding result is

$$-A(k) = \frac{\lim_{r\to 0} h(g;k,r)}{\lim_{r\to 0} f(g;-k,r)} \tag{2.117}$$

but in our case the limits do not exist independently. It is most convenient to write as we did in the proof of Jabbur's theorem

$$-A(k) = \lim_{\epsilon \to 0} \frac{g k^{-1} \int_{\epsilon}^{\infty} dr' \, V(r') \, h(g; k, r') \sin kr'}{1 + g k^{-1} \int_{\epsilon}^{\infty} dr' \, V(r') \, f(g; k, r') \sin kr'} \tag{2.118}$$

It would be possible to use this to obtain a non-analytic expansion for the scattering amplitude in the neighborhood of g = 0. Following a slightly different procedure, Delguidice and Galzenati[6] have obtained such expansions for potentials r^{-m}, $m > 2$. In particular for gr^{-4}, they obtained

$$\tan \delta = -\lambda^2 + \frac{\pi}{3}\lambda^4 + \left(\frac{8}{3}\gamma - \frac{28}{9}\right)\lambda^6 + \frac{8}{3}\lambda^6 \ln 2\lambda$$
$$\lambda = k^{1/2} g^{1/4} \tag{2.119}$$

The Intrinsic Cutoff Method has been applied to several examples, all solvable, and its conclusion checked.[7,8] Guttinger, Penzl, and Pfaffelhuber[9] have published a method for unrenormalizable theories which does not depend on a cutoff, and may be a corresponding theory.

2.2 Regularization

A usual method for regular potentials, the iterative solution, or Born series of approximations, fails to work for singular potentials, because each integral diverges. This is of course due to trying to obtain an expansion in the coupling constant, while the scattering amplitude is non-analytic in the coupling constant around zero, so that no series exists. In order to overcome this difficulty, and corresponding problems in field theories, a <u>regulated</u> potential is introduced. A regulated potential $V(g,r,\alpha)$ satisfies

(1) $V(g,r,\alpha)$ is non-singular for $\alpha > 0$

(2) $V(G,r,0) = V(G,r)$ (2.201)

The most common regularizations are

$$V(g;r,\alpha) = \vartheta(r-\alpha)\, V(g,r) \tag{2.202}$$

and
$$V(g;r,\alpha) = V(g, r+\alpha) \tag{2.203}$$

It has been noted that an essential difference holds between a regulated potential and a regulated field theory, namely that the former is still a bona fide potential, while a regulated field theory does not satisfy the same postulates as an unregulated one.[10)] The analogy still continues to be exploited.

Since $V(g,r,\alpha)$ is non-singular for each $\alpha > 0$, there is for each α, an analytic expression for the scattering length $A(\alpha,g) = \Sigma\, a_n(\alpha) g^n$. Of course as $\alpha \to 0$ each $a_n(\alpha) \to \infty$, but it may be possible to sum the series to obtain $A(\alpha,g)$. If now $\lim_{\alpha \to 0} A(\alpha,g) = A(g)$, then the process of regularization has succeeded. The common regularizations have not been known to fail, but there have been no adequate theorems of sufficient conditions or necessary conditions. Some regularizations have been invented which fail.[11)]

The difficulty of the problem is shown by the breakdown of a widely cited proof.[10)] It was proposed to regulate

$$V(g,r) = g\, r^{-m} \qquad m > 3$$

by
$$V(g,r,\alpha) = g\, \alpha^{-m}\, U(r/\alpha) \tag{2.204}$$
$$U(\xi) \underset{\circ}{\to} \xi^{-m}$$

For inverse power potentials, then, this would encompass a variety of regulatizations. From the Schrodinger theory, the three dimensional zero energy wave function Ψ satisfies

$$\Psi(\vec{x};\alpha,U) = 1 - \frac{g}{4\pi}\int \frac{V(y,\alpha)\ \Psi(\vec{y};\alpha,U)}{\vec{x}-\vec{y}}\ d^3y \tag{2.205}$$

Integrating over the angle variables we find that the solution depends only on $x = |\vec{x}|$:

$$\begin{aligned}\psi(x;\alpha,U) = 1 &- \frac{g}{x}\int_0^x dy\ y^2\, V(y,\alpha)\,\psi(y;\alpha,U)\\ &- g\int_x^\infty dy\ y\ V(y,\alpha)\ \psi(y,\alpha,U)\end{aligned} \tag{2.206}$$

We now separate equation (2.206) into two parts, one regular, and one singular as $\alpha \to 0$. Let

$$\begin{aligned}\psi_1(x;\alpha,U) &= -\frac{g}{x}\int_0^\infty dy\, y^2 V(y,\alpha)\,[\psi_1(y;\alpha,U) + \psi_2(y;\alpha,U)]\\ \psi_2(x;\alpha,U) &= 1 - \frac{g}{x}\int_x^\infty dy\ y\ V(y,\alpha)[\psi_1(y;\alpha,U) + \psi_2(y;\alpha,U)]\,(x-y)\end{aligned} \tag{2.207}$$

Then $\psi = \psi_1 + \psi_2$, and ψ_1 can always be solved for, since

$$\psi_1(x;\alpha,U) = \frac{A(\alpha,U)}{x} \tag{2.208}$$

where $A(\alpha,U)$ is clearly a constant, and hence the scattering length since it is the coefficient of $\frac{1}{x}$. We then substitute the result for ψ_1, in the equation for ψ_2 , with the result

$$\begin{aligned}\psi_2(x;\alpha,U) = \{\, 1 &- \frac{g}{x}\int_x^\infty dy\ \ y\,(x-y)\, V(y,\alpha)\ \psi_2(y;\alpha,U)\,\}\\ &+ A(\alpha,U)\,\{\, -\frac{g}{x}\int_x^\infty dy\ V(y,\alpha)\ (x-y)\,\}\end{aligned} \tag{2.209}$$

$$\equiv \psi_2^{(1)}(x;\alpha,U) + A(\alpha,U)\,\psi_2^{(2)}(x;\alpha,U) \tag{2.210}$$

This is a Volterra type integral equation, and non-singular as $\alpha \to 0$.
We now write

$$\chi \varphi_1 = A(\alpha, U)$$
$$= \lim_{\epsilon \to 0} -g \int_\epsilon^\infty dy \; y^2 \; V(y,\alpha)\left[\varphi_1(y;\alpha,U) + \varphi_2(y;\alpha,U)\right] \tag{2.211}$$

which can be substituted into (2.208) and 2.210) to give

$$A(\alpha,U) = \lim_{\epsilon \to 0} \frac{-g\int_\epsilon^\infty dy \; y^2 \; V(y,\alpha)\, \varphi_2^{(1)}(y;\alpha,U)}{1 + g\int_\epsilon^\infty dy \; y^2 \; V\left[\varphi_2^{(1)}(y;\alpha,U) + \frac{1}{y}\right]} \tag{2.212}$$

$$= \lim_{\epsilon \to 0} A_\epsilon(\alpha, U) \tag{2.213}$$

if we compare with our previous notation. The correspondence is
$r\varphi_2^{(1)} = h$, $r\varphi_2^{(1)} + 1 = f$. We would like to calculate

$$A(0,U) = \lim_{\alpha \to 0} \lim_{\epsilon \to 0} A_\epsilon(\alpha, U)$$

but all that can be found is

$$\lim_{\epsilon \to 0} \lim_{\alpha \to 0} A_\epsilon(\alpha,U) = \lim_{\epsilon \to 0} \frac{-g\int_\epsilon^\infty dy\, y^2\, V(y,0)\, \varphi_2^{(1)}(y;0,U)}{1 + g\int_\epsilon^\infty dy\, y^2 V(y,0)\left[\varphi_2^{(1)}(y;0,U) + \frac{1}{y}\right]}$$

because this is exactly the result of Pais and Wu.

The authors did not justify the exchange of limits, and other limit exchange problems in singular potentials suggest that it is not permissible. Calogero adduced examples of regularizations where this interchange could be shown false, realizing though that they lay outside the class (2.204).

Cornille[12)] has attempted to prove that the ϑ regularization works for any singular potential. He introduces the extra parameter in the potential via the ϑ function, but at a certain point ignores the explicit cutoff, and arrives at the result (2.118) of the Implicit Cutoff Method. He continues to use the radial coordinate as a cutoff.

The following argument was conceived as a generalization of some specific examples which we shall come to later, and is based on a simpler calculation. In several cases the reduced radial wave function had a convenient factorization $\psi(r) = r\,\Psi(r)$, leading to an expression for the scattering length in terms of the entire radial wave function. We consider the regularization (2.203) first.

We write the solutions to the unregulated Schrodinger equation

$$\frac{d^2\psi}{dr^2} = V(g,r)\,\psi$$

as $\quad \psi_r(r) = r\,\Psi_r(r) \quad$, the regular solution

and $\quad \psi_s(r) = r\,\Psi_s(r) \quad$, the singular solution

with $$\lim_{r\to 0} \psi_r(r)/\psi_s(r) = 0 \tag{2.214}$$

We know that the asymptotic behavior of the solution is $\psi_r(r) \underset{\infty}{\sim} N(r+A)$, hence there is an expansion about infinity,

$$\Psi_r(r) = \Psi_r(\infty) + \Psi_r'(\infty)\,\frac{1}{r} + O\left(\frac{1}{r^2}\right)$$

and $$A = \Psi_r'(\infty)/\Psi_r(\infty) \tag{2.215}$$

We then turn to the regulated problem

$$\frac{d^2\psi(r,\alpha)}{dr^2} = V(g,r+\alpha)\,\psi(r,\alpha)$$

which is solved by

$$\psi(r,\alpha) = c_r\,\psi_r(r+\alpha) + c_s\,\psi_s(r+\alpha) \tag{2.216}$$

In order that $\psi(0,\alpha) = 0$, we write

$$\psi(r,\alpha) = \psi_s(\alpha)\,\psi_r(r+\alpha) - \psi_r(\alpha)\,\psi_s(r+\alpha) \tag{2.217}$$

$$\underset{\infty}{\sim} (r+\alpha)\,[\psi_s(\alpha)\,\Phi_r(\infty) - \psi_r(\alpha)\,\Phi_s(\infty)] + [\psi_s(\alpha)\,\Phi_r'(\infty) - \psi_r(\alpha)\,\Phi_s'(\infty)] \tag{2.218}$$

Since the coefficient of r must be finite, and $\Phi_r(\infty)$ is defined, $\Phi_s(\infty)$ is also defined; and again since the scattering length exists for the regulated potential, $\Phi_s'(\infty)$ exists. Hence

$$A(\alpha) = \frac{\psi_s(\alpha)\,\Phi_r'(\infty) - \psi_r(\alpha)\,\Phi_s'(\infty)}{\psi_s(\alpha)\,\Phi_r(\infty) - \psi_r(\alpha)\,\Phi_s(\infty)} + \alpha \tag{2.219}$$

$$= \frac{[\psi_s(\alpha)/\psi_r(\alpha)]\,\Phi_r'(\infty) - \Phi_s'(\infty)}{[\psi_s(\alpha)/\psi_r(\alpha)]\,\Phi_r(\infty) - \Phi_s(\infty)} + \alpha \tag{2.220}$$

Then

$$\lim_{\alpha\to 0} A(\alpha) = \frac{\Phi_r'(\infty)}{\Phi_r(\infty)} = A \tag{2.221}$$

Since the correct scattering length is regained, the regularization procedure V(g,r,α) V(g,r+α) has succeeded for any singular potential. In a similar fashion, we obtain for the regularization (2.202)

$$A(\alpha) = \frac{\Phi_s'(\alpha)\,\Phi_r'(\infty) - \Phi_r'(\alpha)\,\Phi_s'(\infty)}{\Phi_s'(\alpha)\,\Phi_r(\infty) - \Phi_r'(\alpha)\,\Phi_s(\infty)} \tag{2.222}$$

Now if one function is more singular than another, its derivative will be more singular than that of the other. Thus the ϑ regularization also works.

Having made a regularization, we can then solve the zero energy problem formally by the Born series.

We have the equation

$$\frac{d^2\psi(r,\alpha)}{dr^2} = V(g,r,\alpha)\,\psi(r,\alpha)$$

$$\psi(0,\alpha) = 0$$

$$\psi(r,\alpha) \underset{\infty}{\sim} r \tag{2.223}$$

equivalent to the integral equation

$$\psi(r,\alpha) = r - \int_0^\infty dr'\, V(g,r',\alpha)\, G(r,r')\, \psi(r',\alpha)$$

where $\frac{d^2 G(r,r')}{dr^2} = \delta(r-r')$

or $$G(r,r') = \min(r,r') \tag{2.224}$$

The scattering length is defined by

$$A(\alpha) = \lim_{r\to 0} (\psi(r,\alpha) - r) \tag{2.225}$$

We can solve formally by iteration

$$\psi_0(r,\alpha) = r$$

$$\psi_{n+1}(r,\alpha) = -\int_0^\infty dr'\, G(r,r')\, V(g,r',\alpha)\, \psi_n(r',\alpha)$$

$$A_0(\alpha) = 0$$

$$A_n(\alpha) = \psi_n(\infty,\alpha) \quad , n \geq 1$$

$$A(\alpha) = \sum_{n=1}^\infty A_n(\alpha) \tag{2.226}$$

For example, regulate r^{-4} by $V(g,r,\alpha) = g\,\vartheta(r-\alpha)\,r^{-4}$

$$\psi_1(r,\alpha) = -g\left(\int_\alpha^r dr'\,r'^{-2} + r\int_r^\infty dr'\,r'^{-3}\right)$$
$$= -g\left(-\tfrac{1}{2}r^{-1} + \alpha^{-1}\right)$$
$$A_1(\alpha) = -g\alpha^{-1}$$
$$A_2(\alpha) = \tfrac{1}{3}g^2\alpha^{-3}$$
$$\vdots$$
$$A(\alpha) = -g\alpha^{-1} + \tfrac{1}{3}g^2\alpha^{-3} - \tfrac{2}{15}g^3\alpha^{-5} + \cdots \tag{2.227}$$

We cite here for reference further examples:

For $V(g,r,\alpha) = g\,\vartheta(r-\alpha)\,\ln^2 r^{-4}$

$$\begin{aligned} A(\alpha) = &-g\alpha^{-1}\left(\ln^2\alpha + 2\ln\alpha + 2\right) \\ &+ \tfrac{1}{3}g\alpha^{-3}\left(\ln^4\alpha + \tfrac{7}{3}\ln^3\alpha + \tfrac{17}{6}\ln^2\alpha + \tfrac{17}{9}\ln\alpha + \tfrac{17}{27}\right) \\ &- \tfrac{2}{15}g\alpha^{-5}\left(\ln^6\alpha + \cdots\right) \\ &+ \cdots \end{aligned} \tag{2.228}$$

For $V(g,r,\alpha) = g\,\vartheta(r-\alpha)\left(r^{-5} + \varphi r^{-4}\right)$

$$\begin{aligned} A(\alpha) = &-g\left(\tfrac{1}{2}\alpha^{-2} + \varphi\alpha^{-1}\right) \\ &+ g^2\left(\tfrac{2}{15}\alpha^{-5} + \tfrac{5}{12}\varphi\alpha^{-4} + \tfrac{1}{3}\varphi^2\alpha^{-3}\right) \\ &- g^3\left(\tfrac{11}{9\cdot 32}\alpha^{-8} + \cdots + \tfrac{2}{15}\varphi^3\alpha^{-5}\right) \\ &+ \cdots \end{aligned} \tag{2.229}$$

We make a few remarks about the potential $-g\ln r/r^2$ at this point. If truncated at some point before infinity in order to define a scattering length, this scattering length has an accumulation of poles for $g \to -0$.[24] When the untruncated potential is regulated, the wave function is calculated to be

$$\psi(r) = \left\{ 1 + \frac{g}{4\pi} \ln^2\alpha + g^2\left(\frac{1}{2}\ln^4\alpha + \frac{1}{3}\ln^3\alpha - \frac{1}{4}\ln^2\alpha + \cdots \right) + \cdots \right\} \times$$
$$\left\{ r - gr\left(\frac{1}{2}\ln^2 r - \ln r + 1 \right) + g^2 r\left(\frac{1}{8}\ln^4 r \right.\right.$$
$$\left.\left. - \frac{1}{12}\ln^3 r + \cdots \right) + \cdots \right\} \qquad (2.230)$$

In other words, the cutoff dependent portions can be factorized and hence absorbed into the normalization constant. The potential corresponds to a renormalizable interaction. We shall not discuss the renormalizable potentials further.

PERATIZATION

3.1 Definition and Examples

We have noted in general and in several examples that the Born series of the scattering length of a regulated singular potential is of the form

$$A_n(\alpha) = \Sigma\, a_n(\alpha)\, g^n \tag{3.101}$$

where each $a_n(\alpha)$ is divergent as $\alpha \to 0$. The process of peratization, suggested originally by Feinberg and Pais[14)] is a process designed to attach meaning to such a series. (The word was coined by them from Greek roots meaning "that in which one is entangled past escape.")

We can define peratization for such series regardless of their origin. The first peratization approximation is made by summing the leading singularity as a function of α in each order of g, then taking the limit of the resulting expression. The second peratization approximation is made by summing the two greatest singularities, etc. For each order of approximation, there are two questions: (1) Is the limit of the sum finite? (2) If it is finite, is it a good approximation? We will term a series, each of whose terms diverges as a function of a parameter, peratizable if in each order of approximation, the sum is finite, and the approximation is improved.

We should always consider that the peratization process is built upon the regularization process, and is subject to the prior failure of the latter. We shall content ourselves with the two common regularizations previously studied.

As an example, we check the finiteness of the first approximation in the three examples given on page 17. The leading singularities of the first and second, summed, give $-g^{1/2}\tanh(g^{1/2}\alpha^{-1})$ and $-g^{1/2}\ln\alpha\tanh(g^{1/2}\alpha^{-1}/\ln\alpha)$ respectively. The first is finite as $\alpha\to 0$, while the second is not. We shall see that the leading term of the third is also finite.

3.2 A General Argument for Peratization

Let us investigate some series, each of whose terms is divergent, obtained by other means. In particular, if we have an analytic function given by some power series, and multiply it by a peratizable series with value one, or add to it a peratizable series with value zero, would we have a peratizable series for the original function?

Consider
$$C(\alpha,g) = \sum_{m=0}^{\infty} c_m(\alpha)\, g^m$$
$$c_m(\alpha) = \sum_{j=0}^{m} c_{jm}(\alpha)$$

$c_{\ell,m}(\alpha)$ more singular than $c_{\ell+1,m}(\alpha)$

with the peratization approximations to $C(0,g)$
$$C_1(0,g) = \lim_{\alpha\to 0}\sum_{m=0}^{\infty} c_{0m}(\alpha)\, g^m$$
$$C_2(0,g) = C_1(0,g) + \lim_{\alpha\to 0}\sum_{m=1}^{\infty} c_{1m}(\alpha)\, g^m \tag{3.201}$$

Then for $A(g)=\sum a_n g^n$, $A(g)+C(g)$ has the first approximation $C_1(0,g)$, the second approximation, $a_0+C_2(0,g)$, third $a_0+a_1 g+C_3(0,g)$, etc. We see that the sum A+C is well approximated. The approximations obtained for a product depend on the relative order of singularities among the coefficients $c_{jm}(\alpha)$, but as long as each order of g has only a finite number of terms in α multiplying it, and if this number is non-decreasing in order, the sum or product of an analytic function with the peratizable representation of another function will again be peratizable.

The same result no longer applies in general if both functions are in a peratizable representation. Conditions on the orders of the singularities can be set up which will allow this: in particular if the singularity of $b_{jm}(\alpha)$ is the same as the singularity of $c_{jm}(\alpha)$ then the sums and products of B and C will be peratizable. We now extend the argument given previously for regularization.

We recall equation (2.220)

$$A(\alpha) = \frac{[\psi_s(\alpha)/\psi_r(\alpha)]\,\Phi_r'(\infty) - \Phi_s'(\infty)}{[\psi_s(\alpha)/\psi_r(\alpha)]\,\Phi_r(\infty) - \Phi_s(\infty)} + \alpha$$

and suppose that the leading singularities in each order for $[\psi_s(\alpha)/\psi_r(\alpha)]$ diverge (which considering its behavior as $\alpha \to 0$ means that it can be peratized). Taking singularities before any multiplication of addition, it is apparent that the correct answer is regained with the first approximation. If we perform all the multiplications and additions before taking leading singularities, then some of the non-singular terms of Φ_r' or Φ_r may be neglected in their order of g as we have seen. As long as only a finite number of singular terms occurs in each order of g, we will recover this information, in successive approximations. We expect to be able to divide the two paratizable functions in the numerator and denominator because of their similarity of singularities, though after multiplication they may no longer be associated with the same order of g.

3.3 Power Potentials

We turn to specific examples for the insight associated with concrete problems. The simple inverse power potentials gr^{-m}, $m > 3$, are well known.[10,15] The radial Schrodinger equation can be solved in terms of the Bessel functions of imaginary argument: (For a discussion of these functions see page 375 of reference 16.)

$$\psi_r(r) = r^{1/2} K_\nu (2\nu g^{1/2} r^{-\frac{1}{2\nu}}) \qquad \nu = (m-2)^{-1} \tag{3.301}$$

Since ν is not an integer,

$$K_\nu(z) = C \cdot (I_{-\nu}(z) - I_\nu(z)) \underset{0}{\sim} C \cdot ((\tfrac{1}{2}z)^{-\nu}/\Gamma(1-\nu) - (\tfrac{1}{2}z)^{\nu}/\Gamma(1+\nu)) \tag{3.302}$$

and $$\psi(r) \underset{\infty}{\sim} C r^{1/2} \left(\frac{(\nu g^{1/2} r^{-\frac{1}{2\nu}})^{-\nu}}{\Gamma(1-\nu)} - \frac{(\nu g^{1/2} r^{-\frac{1}{2\nu}})^{\nu}}{\Gamma(1+\nu)} \right) \tag{3.303}$$

resulting in $$A = -(\nu g^{1/2})^{2\nu} \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)} \tag{3.304}$$

Adopting the regularization $V(g, r, \alpha) = g(r+\alpha)^{-m}$ we obtain the solution $$\psi(r,\alpha) = (r+\alpha)^{1/2} \{ B\, I_\nu(z) + D\, I_{-\nu}(z) \}$$

with $$B = -I_{-\nu}(z_\alpha), \qquad D = I_\nu(z_\alpha)$$

where $$z = 2\nu g^{1/2} r^{-\frac{1}{2\nu}} \qquad z_\alpha = 2\nu g^{1/2} \alpha^{-\frac{1}{2\nu}} \tag{3.305}$$

By the same asymptotic formula for I_ν, $I_{-\nu}$ as before

$$A(\alpha) = \alpha - \frac{I_{-\nu}(z_\alpha)(\nu g^{1/2})^{\nu}\, \Gamma(1-\nu)}{I_\nu(z_\alpha)(\nu g^{1/2})^{-\nu}\, \Gamma(1+\nu)} \tag{3.306}$$

Another application of the asymptotic formulas shows that the regularization is successful. We can obtain the Born series directly, or by expanding (3.306) in powers of g. In the case r^{-4} the Born series is the same as before, even though the regularization is different. For any m the main feature is that there is only one singular term multiplying each power of g. When these are summed, we necessarily obtain $A(\alpha)$, which gives a finite sum as $\alpha \to 0$ and leads to the correct scattering length. In this case, the first peratization approximation is exact, which is only to be expected, since we have discarded nothing.

Let us consider the potential $\left(g r^{-5} + f r^{-4}\right)$. For this to be repulsive near the origin we must have $g > 0$, but there is now possible $f < 0$. The radial Schrodinger equation is solved by

$$\psi_r(r) = r\, Ai(y) \qquad y = g^{1/3}/r + f/g^{2/3} \tag{3.307}$$

where Ai, and Bi later, are independent solutions of the Airy equation

$$w'' - y w = 0 \tag{3.308}$$

This was discovered by consideration of possible solutions $r^{\beta} Ai(y(r))$, leading to a non-linear differential equation for r^{β}, solvable by $\beta = 1/2,\ 1$; and a linear differential equation for y(r), solvable as shown. (The $\beta = 1/2$ root leads to a solution of the potential $\frac{g \ln r}{r^2}$.) The Airy functions and their derivatives are well discussed in reference 16. When we expand Ai(y) in powers of 1/r,

$$\psi(r) = r\, Ai(g^{-2/3} f) + Ai'(g^{-2/3} f)\, g^{1/3} + O\left(\tfrac{1}{r}\right) \tag{3.309}$$

resulting in

$$A(g,f) = g^{1/3}\,\frac{Ai'(g^{-2/3}f)}{Ai(g^{-2/3}f)} \equiv \gamma\,\frac{Ai'(\gamma^{-2}f)}{Ai(\gamma^{-2}f)} \qquad (3.310)$$

This gives for $f \to 0$

$$A(g,0) = g^{1/3}\,\frac{Ai'(0)}{Ai(0)} = -3^{1/3}\,g^{1/3}\,\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})} \qquad (3.311)$$

and for $g \to 0$, $f > 0$, using asymptotic formulae for Ai', Ai

$$\begin{aligned} A(0,f^{+}) &= \lim_{\gamma\to 0}\ \gamma\,\frac{Ai'(\gamma^{-2}f^{+})}{Ai(\gamma^{-2}f^{+})} \\ &= \lim_{\gamma\to 0}\ \gamma\,\frac{-\frac{1}{2}\,\pi^{-1/2}\,(\gamma^{-2}f^{+})^{1/4}\,e^{-\gamma^{-2}f^{+}}}{\frac{1}{2}\,\pi^{-1/2}\,(\gamma^{-2}f^{+})^{-1/4}\,e^{-\gamma^{-2}f^{+}}} \\ &= -f^{+\,1/2} \end{aligned} \qquad (3.312)$$

We check that is necessary that $f > 0$ if $g \to 0$, for now

$$\zeta = \tfrac{2}{3}\,(\gamma^{-2}f)^{3/2}$$

$$\begin{aligned} A(0,-f^{-}) &= \lim_{\gamma\to 0}\ \gamma\,\frac{-\pi^{-1/2}\,(\gamma^{-2}f)^{1/4}\cos(\zeta+\frac{\pi}{4})}{\pi^{-1/2}\,(\gamma^{-2}f)^{-1/4}\sin(\zeta+\frac{\pi}{4})} \\ &= \lim_{\zeta\to\infty}\ -f^{1/2}\cot(\zeta+\tfrac{\pi}{4}) \end{aligned} \qquad (3.313)$$

which does not exist. The formulas indicate the behavior of A as a function of f. In particular, for $f > 0$, A is always negative, and for large f approaches the scattering length of an inverse fourth power repulsive potential. For f negative, the scattering length becomes positive, then infinite, corresponding to a bound state at zero energy. A succession of bound states occurs for f large with respect to g. The first of these occurs at the first zero of the Ai(-x), i.e. x = 2.338... .

To study the peratization properties, we write $V = g(r^{-5} + \varphi r^{-4})$, regarding φ as some fixed number. We use the ϑ regularization, so that

$$V(g, r, \alpha) = \begin{cases} 0 & r \leq \alpha \\ g\,(r^{-5} + \varphi r^{-4}) & r > \alpha \end{cases} \tag{3.314}$$

Then the Schrodinger equation will be solved by

$$\psi(r, \alpha) = \begin{cases} cr + d & r \leq \alpha \\ a r\, A_i(y) + b r\, B_i(y) & r \geq \alpha \end{cases}$$

$$y = \gamma r^{-1} + \gamma\varphi \tag{3.315}$$

To obtain $\psi(0, \alpha) = 0$, d must equal zero. Equating the logarithmic derivatives at $r = \alpha$ we obtain

$$a = B_i'(y_\alpha) \qquad b = -A_i'(y_\alpha) \tag{3.316}$$

if we choose c, the normalization as π^{-1} .

The associated scattering length is

$$A(\alpha) = \gamma \; \frac{B_i'(\gamma(\alpha^{-1}+\varphi))\; A_i'(\gamma\varphi) - A_i'(\gamma(\alpha^{-1}+\varphi))\, B_i'(\gamma\varphi)}{B_i'(\gamma(\alpha^{-1}+\varphi))\; A_i(\gamma\varphi) - A_i'(\gamma(\alpha^{-1}+\varphi))\, B_i(\gamma\varphi)} \tag{3.317}$$

For small enough α , γ/α dominates $\gamma\varphi$, so that the arguments of Bi', Ai' approach positive infinity. The functions themselves approach infinity and zero, respectively. Thus we have checked explicitly that the regularization is successful. The summation of the leading singularities of (2.229), the directly obtained Born series expansion does give

$$\gamma \frac{Bi'(\gamma/\lambda)Ai'(o) - Ai'(\gamma/\lambda)\,Bi'(o)}{Bi'(\gamma/\lambda)\,Ai(o) - Ai'(\gamma/\lambda)\,Bi(o)} \tag{3.318}$$

as can most readily be checked by equations 10.4.2 of reference 16. This gives a limit of $\gamma \frac{Ai'(o)}{Ai(o)}$, the scattering length of gr^{-5}. The leading singularities are in fact exactly the terms one obtains in making a Born series expansion of the regulated version of the most singular part of the potential. The answer is no longer exact; however, for small φ , positive or negative, it is an excellent approximation. The summation of the second terms gives a term linear in φ , or the slope of A near $\varphi = 0$, which improves the approximation. It is clear that we are generating a power series in about its origin, and we can use our analytic solution to find the region in which we can expect such a process to converge.

The region in which the power series in φ for A will converge will be the circle about the origin with the radius equal to the distance of the nearest singularity. We have seen that A has no singularity for positive φ, but has singularities for the negative values of φ for which $Ai(\gamma\varphi) = 0$. The nearest of these is $\gamma\varphi = -2.338\ldots$, so that the process will converge if $|\varphi| < g^{1/3}2.338$. By comparison with earlier remarks we see that this is exactly the nearest zero energy bound state. We could interpret the non-existence of a convergent power series in g in the same fashion since there are zero energy bound states for any small negative g.

It is clear that any finite sum of inverse power potentials (each more singular than r^{-3}) will also display similar behavior: the leading singularities will give the behavior as if the leading singularity alone were considered. The following singularities will sum to a power series which will converge out to the nearest bound state.

Once the zero energy problem is solved, then we treat the energy as a source term. As is well known,[3,17] if the solutions of

$$\left[\frac{d^2}{dx^2} + f(x)\right]\varphi = 0$$

are $\quad \varphi_1(x) \quad , \quad \varphi_2(x)$ (3.320)

then the differential equation

$$\left[\frac{d^2}{dx^2} + f(x)\right]\varphi = h(\lambda;x)\varphi \tag{3.321}$$

is equivalent to the integral equation

$$\varphi(x) = \varphi_{1,2}(x) + \frac{1}{W(\varphi_1,\varphi_2)}\int_{x_1}^{x}\left[\varphi_1(x')\varphi_2(x) - \varphi_1(x)\varphi_2(x')\right] \times h(\lambda;x')\varphi(x')\,dx' \tag{3.322}$$

where $\varphi_{1,2}$ is the appropriate combination of φ_1 , φ_2 which satisfies the boundary conditions. This is the zero energy wave function in this case, and h = $-k^2$. From the theory of integral equations,[18] the successive approximations for this equation converge for all values of k^2 if only $\varphi_{1,2}$ is absolutely integrable on the range of normalization. Since   is a wave function and square integrable on the range of normalization, it satisfies the condition cited, and the equation is iterable.

3.4 Other Singularities

An exponentially singular potential.

A class of exponentially singular potentials has been solved exactly and the peratization properties investigated explicitly. We shall report this work here. The potential considered was[7,19)]

$$V(g,x) = g x^{-4}[\exp(\tfrac{2}{x}) + \nu^2] \tag{3.401}$$

From the Schrodinger equation

$$\frac{d^2\psi}{dx^2} - g x^{-4}[\exp(\tfrac{2}{x}) + \nu^2]\,\psi = 0 \tag{3.402}$$

one substitutes $x = z^{-1}$ $\quad \psi(z) = z^{-1}\varphi(z)$ $\quad t = -g^{1/2} e^{z}$

obtaining

$$t^2 \frac{d^2\varphi}{dt^2} + t\frac{d\varphi}{dt} - (t^2+\nu^2)\varphi = 0 \tag{3.403}$$

which one recognizes as the Bessel equation for functions of complex argument. The solution regular at the origin is

$$\psi_r(x) = x K_\nu(-g^{1/2} e^{1/x}) \tag{3.404}$$

Hence

$$\begin{aligned}\psi(x) &\underset{\infty}{\sim} x K_\nu(-g^{1/2}(1+\tfrac{1}{x})) \\ &\underset{\infty}{\sim} K_\nu(-g^{1/2})\{ x - g^{1/2}\frac{K_\nu'(-g^{1/2})}{K_\nu(-g^{1/2})}\end{aligned} \tag{3.405}$$

Thus

$$A_\nu = -g^{1/2}\frac{K_\nu'(-g^{1/2})}{K_\nu(-g^{1/2})} \tag{3.406}$$

For $\nu = 1/2$, for example

$$A_{1/2} = -g^{1/2} - \tfrac{1}{2} \tag{3.407}$$

Using the regularization $V(g,r,\alpha) = V(g,r+\alpha)$, one obtains

$$\begin{aligned}\psi(r+\alpha) = (r+\alpha)\{ & K_\nu(g^{1/2}\exp(\tfrac{1}{\alpha}))\, I_\nu(g^{1/2}\exp(\tfrac{1}{r+\alpha})) \\ & - I_\nu(g^{1/2}\exp(\tfrac{1}{\alpha}))\, K_\nu(g^{1/2}\exp(\tfrac{1}{r+\alpha}))\}\end{aligned} \tag{3.408}$$

and

$$A_\nu(\alpha) = g^{1/2}\,\frac{I_\nu(g^{1/2}e^{1/\alpha})\,K_\nu'(g^{1/2}) - K_\nu(g^{1/2}e^{1/\alpha})\,I_\nu'(g^{1/2})}{I_\nu(g^{1/2}e^{1/\alpha})\,K_\nu(g^{1/2}) - K_\nu(g^{1/2}e^{1/\alpha})\,I_\nu(g^{1/2})} \tag{3.409}$$

For $\nu = 1/2$.

$$A_{\frac{1}{2}}(\alpha) = -\tfrac{1}{2} - g^{1/2}\coth(g^{1/2}(e^{1/\alpha}-1)) \tag{3.410}$$

so that the regularization succeeds. To peratize we can obtain a series expansion for coth, than keep the leading singularity in each order of g. If $\coth x = \sum_0^\infty c_n x^{2n+1}$, then

$$\begin{aligned}A_{1/2}(\alpha) &= -\tfrac{1}{2} - g^{1/2}\Sigma\, c_n\,(g^{1/2}(e^{1/\alpha}-1))^{2n+1} \\ &= -\tfrac{1}{2} - g^{1/2}\,\Sigma\, c_n\, g^{n+\frac{1}{2}}(e^{1/\alpha}-1)^{2n+1} \\ &\underset{P_1}{=} -\tfrac{1}{2} - g^{1/2}\,\Sigma\, c_1\, g^{n+\frac{1}{2}}(e^{1/\alpha})^{2n+1} \\ &= -\tfrac{1}{2} - g^{1/2}\coth(g^{1/2}e^{1/\alpha}) \\ &\rightarrow -g^{1/2} - \tfrac{1}{2}\end{aligned} \tag{3.411}$$

Calogero also investigated explicitly the $\nu = 0$ case and found that two singularities had to be summed before the correct answer was obtained.

Weak singularities.

The evidence is not so complete, nor so convincing for the weak singularities multiplied by an inverse power. We examine the Born series of $g r^{-4} \varphi(r)$ where $\varphi(+\epsilon) > 0$, $r^{\pm 1}\varphi(r) \underset{\infty}{\sim} r^{\pm 1}$, and $r\varphi'(r)$ is less singular than $\varphi(r)$. Examples of such functions are $(\ln \frac{1}{r})^{\beta} \beta \geq 1$; ln(ln(r+1)); ln(ln(ln(r+e)));... giving a series of extremely weak singularities. The leading singularities of the ϑ regulated Born series is

$$-A(\alpha) = \frac{g\varphi(\alpha)}{\alpha} - \frac{1}{3}\frac{g^2\varphi^2(\alpha)}{\alpha^3} + \frac{2}{15}\frac{g^3\varphi^3(\alpha)}{\alpha^5} - \dots \tag{3.412}$$

This is established by one integration by parts, the residual integral being less singular than the product removed. Thus

$$A(\alpha) =_{P_1} -g^{1/2}\varphi^{1/2}(\alpha) \tanh[(g\,\varphi(\alpha)\alpha^{-1})^{1/2}] \tag{3.413}$$
$$\underset{0}{\sim} -g^{1/2}\varphi^{1/2}(\alpha)$$

which lacks a limit in the cases cited. Cornille[12] verifies the same result when peratization is used with equation (2.118) of the Implicit Cutoff Method.

Wu[20] has found however, in the case $g r^{-4} \ln \frac{1}{r}$, that if the asymptotic expressions for all peratization orders are summed, an approximate result is duplicated. She also makes the following argument (generalized here) to obtain an approximation. Since $\varphi(r)$ is a slowly varying function (with respect to r), approximate the potential $gr^{-4}\varphi$ (r) by $g'r^{-4}$, whose scattering length is $-(g')^{1/2}$. Determine g' by requiring the two potentials to be equal at $r = (g')^{1/2}$. Then $g' = \tau^2$

where

$$g\,\varphi(\tau) = \tau^2 \tag{3.414}$$

is a transcendental equation for , and the scattering length is approximated by

$$g^{1/2}\varphi^{1/2}(\tau) \tag{3.415}$$

We note with her also that if the cutoff parameter were reinterpreted as, τ , the first peratization approximation would give this approximation.

In one case, a potential with a logarithmic leading singularity has been solved[8,21] and the investigation of regularization and peratization could be carried out using the analytic solution. The potential is

$$V(g,r) = (g\,r^{-4}\ln^2 r - g^{1/2}r^{-3})\,\vartheta(r_0 - r) \tag{3.416}$$

The equation

$$\frac{d^2\psi}{dr^2} = (g\,r^{-4}\ln^2 r - g^{1/2}\,r^{-3})\,\psi \tag{3.417}$$

has the two general solutions

$$y_1 = r\,\exp(F(r))$$
$$y_2 = y_1\,L(r)$$

where
$$F(r) = g^{1/2}\,r^{-1}\,(\ln r + 1)$$
$$L(r) = \int_{r_0}^{r} x^{-2}\exp(-2F(x))\,dx \tag{3.418}$$

For the potential V(g,r), the regular solution of the Schrodinger equation is $y_1(r)$ for $r \leq r_0$, $br + bA$ for $r \geq r_0$. Equating logarithmic derivatives at $r = r_0$ we find

$$A = g^{1/2}\ln r_0 / (1 - g^{1/2}\,r_0^{-1}\ln r_0) \tag{3.419}$$

The ϑ regularized scattering length is found similarly to be

$$A(\alpha) = \frac{-C(\alpha)\, r_0^2 F'(r_0) + D(\alpha)}{C(\alpha)(1 + r_0 F'(r_0)) + D(\alpha)} \tag{3.420}$$

where $C(\alpha) = 1 + \alpha^2 F'(\alpha) \exp(2F(\alpha)) L(\alpha) \xrightarrow[\alpha\to 0]{} 1$

$$D(\alpha) = \alpha^2 F'(\alpha) \exp(2F(\alpha) - 2F(r_0)) \xrightarrow[\alpha\to 0]{} 0$$

Then since

$$\frac{-r_0^2 F'(r_0)}{1 + r_0 F'(r_0)} = \frac{g^{1/2} \ln r_0}{1 - g^{1/2} r_0^{-1} \ln r_0} = A \tag{3.421}$$

we see that the regularization has succeeded. When we expand the numerator and denominator in powers of g in order to peratize, the only loss of information in the first approximation is the neglect in the denominator of $g^{1/2} r_0^{-1} \ln r_0$, resulting in the first approximation

$$A_1 = g^{1/2} \ln r_0 \tag{3.422}$$

The second approximation is

$$A_2 = \frac{g^{1/2} \ln r_0}{1 - g^{1/2} r_0^{-1} \ln r_0} = A \tag{3.423}$$

that is, we have recovered the scattering length and the peratization has been successful.

The situation is thus far from clear for the weak types of singularities. We consider that the regularization works, and when we can solve, the lines of our general argument are followed. For the ϑ regularization on the other hand, a divergent result is obtained. The $(r+\alpha)$ regularization was also tried in the case $g r^{-4} \ln r^{-1}$, with the result for the leading singularities

$$A_1(\alpha) = -\tfrac{1}{3} g\, \alpha^{-1} \ln \alpha - \tfrac{1}{18} g^2 \alpha^{-3} \ln^2 \alpha + \cdots \tag{3.424}$$

It was not possible to obtain a recursion formula due to the multiplicity of terms to integrate, but: the regularization has changed the coefficients while leaving the leading singularities of the same order; the coefficients may be decreasing faster, but the power counting argument will give a logarithmically divergent dependence on the cutoff again.

There are enough weak points in the general argument presented that it cannot be called a proof, but it may amount to a proof that if the peratization process gives a finite answer, it will be a good approximation, while not guaranteeing that a finite answer will be obtained.

CONCLUSION

Together with the recently published arguments of Montvay[22,23] that peratization is successful in the field theoretic case for zero momentum transfer, we consider that the argument presented here justifies further consideration of the peratization process. In particular, the weak singularities could be investigated to find the nature of the difficulty that they offer to the peratization program.

We claim also to have established the validity of two commonly used regularization processes, which puts these on firmer footing.

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