RICE UNIVERSITY

EXTENSION THEOREMS FOR SOLUTIONS TO
OVERDETERMINED SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

by

Joe Elgin Krueger

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF

MASTER OF ARTS

Thesis Director's signature:

Houston, Texas
May, 1966
ABSTRACT

EXTENSION THEOREMS FOR SOLUTIONS TO
OVERDETERMINED SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

Joe Elgin Krueger

Ehrenpreis has shown that, under suitable conditions, an infinitely differentiable solution to an overdetermined system of linear partial differentiable operators with constant coefficients has an extension across the bounded part of the complement of an annular region. It is shown in this thesis that a similar result holds for distribution solutions. Necessary and sufficient conditions for such extension are discussed and several theorems are also given on the extension problem for solutions to one differential operator.
Acknowledgement

I should like to thank Professor R. O. Wells, Jr., for his helpful suggestions.
# Table of Contents

I. Introduction

II. Preliminary Results

III. Ehrenpreis's Theorem for Distributions

IV. Results for One Operator

Bibliography
I. Introduction

In 1961 Ehrenpreis [3] stated and proved an extension of Hartog's classical theorem on the extension of analytic functions of several complex variables. He proved that under suitable conditions an infinitely differentiable solution to a system of two or more linear partial differential operators with constant coefficients in an annular region of $\mathbb{E}^n$ has an extension across the bounded part of the complement of the region. Earlier Bochner [1] obtained an extension of Hartog's theorem by a different method.

The purpose of this thesis is to prove and discuss Ehrenpreis's result for distributions instead of infinitely differentiable functions. In the discussion several results will be given on the extension of distribution solutions to one partial differential operator.
II. Preliminary Results

Before beginning the main part of this thesis it will be convenient to define some notation and state several theorems from the theory of distributions that will be used in the sequel. This will be done here. The references used in this thesis for distribution theory are Bremermann [1] and Hörmander [4].

As regards to notation we will let $\mathbb{E}^n$ and $\mathbb{C}^n$ denote real $n$-dimensional space and complex $n$-dimensional space, respectively. We will let $D'(V)$ and $E'(V)$ denote, respectively, the set of distributions in $V$ and the set of distributions in $V$ that have compact support in $V$ where $V$ is an open set in $\mathbb{E}^n$. We shall denote the open ball in $\mathbb{E}^n$ with center at the origin and radius $A$ by $S_A$ where $A > 0$.

The following lemma and two theorems shall be relied on very much in this thesis.

Lemma 2.1. Let $L$ be a linear partial differential operator with constant coefficients of order greater than zero. If $f \in E'(\mathbb{E}^n)$ and $Lf = 0$, then $f = 0$.

Proof. Take the Fourier transform of both sides of $Lf = 0$. Then

$$\hat{Lf} = 0.$$ 

Therefore, for each $\varphi \in C_0^\infty(\mathbb{E}^n)$ we have

$$\hat{f}(\hat{L}(A)\varphi(A)) = 0 \quad t \in \mathbb{E}^n.$$
But the Fourier transform of \( f \) is the entire function \( f_x(e^{-i(x,z)}) \), \( z \in \mathbb{C}^n \), restricted to \( \mathbb{E}^n \). Hence, for \( z \in \mathbb{E}^n \), we have

\[
\int_{\mathbb{E}^n} f_x(e^{-i(x,z)}) \hat{\Lambda}(z) \varphi(z) dz = 0.
\]

Since \( \varphi \) is an arbitrary test function, we have

\[
(2.1) \quad f_x(e^{-i(x,z)}) \hat{\Lambda}(z) = 0, \quad z \in \mathbb{E}^n.
\]

Equation (2.1) implies that

\[
f_x(e^{-i(x,z)}) \hat{\Lambda}(z) = 0, \quad z \in \mathbb{C}^n,
\]

because if an entire function in \( \mathbb{C}^n \) is zero on \( \mathbb{E}^n \), then it is zero on \( \mathbb{C}^n \).

Since \( \hat{\Lambda}(z) \neq 0 \) and the ring of entire functions on \( \mathbb{C}^n \) has no divisors of zero it follows that \( \hat{f} = 0 \) and hence that \( f = 0 \) because the Fourier transform is an isomorphism of \( S' \) onto \( S' \). This completes the proof.

Theorem 2.1. (Ehrenpreis-Malgrange). Let \( L \) be a non-zero linear partial differential operator with constant coefficients, let \( c \in E'(\mathbb{E}^n) \) where the support of \( c \) is in \( S_A \) for some \( A > 0 \). Then there exists an \( f \in E'(\mathbb{E}^n) \) with support in \( S_A \) such that \( Lf = c \) if and only if \( \frac{\hat{c}}{\hat{L}} \) is an entire function.

Theorem 2.2. (Ehrenpreis-Malgrange). Let \( L \) be as in Theorem 2.1 and let \( \delta \) denote the Dirac measure at zero. Then there exists an \( f \in D'(\mathbb{E}^n) \) such that \( Lf = \delta \).
Proofs of theorems 2.1 and 2.2 may be found in Hörmander [4].
III. Ehrenpreis's Theorem for Distributions

We want to now consider an over-determined system of partial differential equations

$$Lu = 0$$

where \( L = (L_1, \ldots, L_r) \), \( r > 1 \), \( L_i \) is a linear partial differential operator with constant coefficients for \( i = 1, 2, \ldots, r \), \( u \) is a distribution in some domain of \( \mathbb{E}^n \), and where \( Lu = 0 \) means that \( L_i u = 0 \) for \( i=1,2,\ldots,r \) in said domain. In the sequel we shall let \( V \) denote an open set in \( \mathbb{E}^n \), \( K \) a compact subset of \( V \) and (A), (B), and (C) denote the following statements:

(A) \( \overset{\wedge}{L}_i \) and \( \overset{\wedge}{L}_j \) are relatively prime in the ring of entire functions on \( \mathbb{E}^n \) for \( i \neq j \).

(B) There exists a bounded open set \( M \supset K \) such that the following property holds: If \( Lr = 0 \) outside \( \overline{M} \) and \( r > 0 \) outside \( \overline{S_A} \) where \( A \) is a positive number such that \( M \subset S_A \), then \( r = 0 \) outside \( \overline{M} \).

(C) If \( x_0 \in M-K \), then there exists an open set \( M \) such that (B) holds for \( N, N \subset M \) and \( x_0 \notin N \).

Theorem 3.1. Let \( f \in D'(V-K) \) and suppose \( Lf = 0 \) in \( V-K \). If statements (A), (B), and (C) hold, then there exists one and only one \( f_1 \in D'(V) \) such that \( f_1|_{(V-K)} = f \) and \( Lf_1 = 0 \) in \( V \).

Proof. Let \( M \) be a bounded open subset of \( V \) containing \( K \). Then there exists a \( g \in D'(V) \) such that
\( g \mid (V-N) = f \). Let \( g_j \) be defined by \( g_j = L_j g \), \( j = 1, 2, \ldots, r \). Then for some \( A > 0 \) \( \text{supp} g_j \subset S_A \). Extend \( g_j \) to be zero outside \( \mathbb{M} \) and denote the extension by \( \tilde{g}_j \). Now

\[(3.1) \quad L_1 \tilde{g}_j = L_j g_1.\]

Take the Fourier transform of both sides of (3.1). Then we have

\[(3.2) \quad \hat{L} \hat{g}_j = \hat{L}_j \hat{g}_1.\]

Hence, for each test function \( \varphi \), we have

\[(3.3) \quad \hat{g}_j(L_1(t)\varphi(t)) = \hat{g}_1(L_j(t)\varphi(t))\]

where \( t \in \mathbb{E}^n \). But the Fourier transform of \( g_1 \) is the function \( g_{1x}(e^{-i(x,t)}) \). Therefore for each test function \( \varphi \) we have

\[\int_{\mathbb{E}^n} g_{jx}(e^{-i(x,t)}L_1(t)\varphi(t))dt = \int_{\mathbb{E}^n} g_{1x}(e^{-i(x,t)})L_j(t)\varphi(t)dt.\]

Consequently we have by lemma 2.1

\[g_{jx}(e^{-i(x,t)})L_1(t) = g_{1x}(e^{-i(x,t)})\hat{L}_1(t).\]
However $g_{j x}(e^{-i(x,z)})$ and $\hat{\lambda}_1(z)$ are entire functions for $z \in \mathbb{C}^n$. Therefore $g_{j}(e^{-i(x,z)})\hat{\lambda}_1(z)$ is an entire function. Since equation (3.3) holds for $t \in E^n$, it also holds for $t \in \mathbb{C}^n$ because if an entire analytic function on $\mathbb{C}^n$ is zero on $E^n$, then it is zero on $\mathbb{C}^n$. Therefore for $z \in \mathbb{C}^n$ we have

$$g_{j}(e^{-i(x,z)})\hat{\lambda}_1(z) = g_{i}(e^{-i(x,z)})\hat{\lambda}_i(z).$$

But $\hat{\lambda}_1$ and $\hat{\lambda}_j$ are relatively prime in the ring of entire functions on $\mathbb{C}^n$ by assumption (A). Hence there exists an entire function $R$ on $\mathbb{C}^n$ such that for $z \in \mathbb{C}^n$ we have

$$R(z) = \frac{g_{i}(e^{-i(x,z)})}{\hat{\lambda}_i(z)}, \quad i = 1, 2, \ldots, r.$$  

Theorem 2.1 implies that there exists an $r \in E'(E^n)$ such that $\text{supp } r \subset S_A$ and

$$\tag{3.5} \lambda_{j} r = g_{j}, \quad j = 1, 2, \ldots, r.$$ 

Equation (3.5) implies that $\lambda r = 0$ outside $M$ because each $g_{j}$ vanishes outside $M$. Therefore assumption (B) implies that $r = 0$ outside $M$.

Let $f_{1}$ be defined by $f_{1} = g - r$ in $V$. Then $f_{1} = f$ in $V - M$ because in $r - M$ we have that $g = f$ and $r = 0$. We also have that $\lambda f_{1} = 0$ in $V$ because $L_{j}f_{1} = L_{j}g - L_{j}r = g_{k} - g_{k} = 0$. 
We shall now show that $f_1$ is unique in the sense that if $f_2$ is an extension of $f$ and $f_2(V-M) = f$, then $f_2 = f_1$. Suppose $f_2$ is a distribution on $E^n$ such that $f_2|_{(V-M)} = f$. Then $supp(f_1-f_2) \subseteq \{M\}$ and $L(f_1-f_2) = 0$. Thus by lemma 2.1 we have

$$f_1 - f_2 = 0.$$ 

The proof will be complete when it is shown that $f_1 = f$ in $V-K$. To this end let $x_0 \in M-K$. Then by assumption (C) there exists a bounded open set $N$ such that $K \subseteq N \subseteq M$, $x_0 \notin N$, and assumption (B) holds for $N$. Let $f_2$ be the extension of $f$ obtained by the method used above in the proof. Then $N \subseteq M$. But the uniqueness result proved in the above paragraph implies that $f_2 = f_1$, on $V$ and, in particular, that $f_2(x_0) = f_1(x_0)$. But $x_0 \notin N$ implies that $f(x_0) = f_2(x_0)$. Hence $f(x_0) = f_1(x_0)$. Since $x_0$ was arbitrary we have that $f = f_1$ in $V-K$. Lemma 2.1 implies that $f_1$ is the only distribution in $V$ such that $f_1|_{(V-K)} = f$. This completes the proof.

We shall now give several corollaries to Theorem 3.1.

Corollary 3.1. Let $V = S_B$, $K = \overline{S}_A$ where $A < B \leq \infty$, and let $f \in D'(V-K)$. If $Lf = 0$ in $V-K$ and $L_i$ and $L_j$ are relatively prime in the ring of entire functions for $i \neq j$, then there exists one and only one $g \in D'(V)$ such that $Lg = 0$ in $V$ and $g|_{(V-K)} = f$. 


Proof. Take $M = S_{A+\varepsilon}$ where $\varepsilon > 0$ and $A+\varepsilon < B$. Hence if $Lr = 0$ outside $\bar{M}$ and $r = 0$ outside $\bar{S}_{A+\varepsilon}$, then $r = 0$ outside $\bar{M}$. Clearly assumption (C) is satisfied. Hence Corollary 1 is true by virtue of Theorem 3.1. This completes the proof.

Remark. If $K = \{0\}$ then Corollary 3.1 is a theorem similar to Riemann's theorem on removable singularities. Here, however, it is not assumed that $f$ is bounded in a deleted neighborhood of the origin.

We now give an application of Theorem 3.1 for the case where the distributions are $C^\infty$ functions. Let $L_1, \ldots, L_r$, $V$, and $K$ be as in Theorem 3.1 and suppose assumption (A) holds. Then we have

**Theorem 3.2.** If $L_1$ is the Laplacian operator and $V - K$ is connected, then any solution to $L$ in $V - K$ has one and only one extension across $K$.

Proof. Assumption (B) is satisfied here because if $r = 0$ outside $\bar{S}_A$ and $L_1r = 0$ outside $\bar{M}$, then $r = 0$ outside $\bar{M}$ by virtue of the unique continuation property for harmonic functions because $V - \bar{S}_A$ is a subdomain of $V - \bar{M}$.

The proof will be complete when it is shown that assumption (C) is valid. To this end, let $x_0 \in M - K$ and let $y_0$ be a point in $V - \bar{M}$. Since $V - K$ is connected, there exists an arc in $V - K$ connecting $x_0$ to $y_0$. Since $K$ is compact, there exists for each point $x$ on the arc an open ball $U(x) \ni x$ such that $U(x) \cap K$ is empty. Thus, the Heine-Borel theorem implies that there exists a finite
subcover, say \( U(x_1), \ldots, U(x_n) \). Let \( N = M \cap (- \cup_{k=1}^{m} U(x_k)) \). Then \( N \) is open, \( N \subset M \), and \( N \) does not contain \( x_0 \). This completes the proof.

We shall now examine assumption (A) more closely by means of an example.

**Theorem 3.3.** Let \( \hat{A} \) be a polynomial that is not a constant and suppose \( \hat{B}_j \) is a polynomial for \( j = 1, 2, \ldots, r \). If \( \hat{L}_j = \hat{A} \hat{B}_j \) for \( j = 1, 2, \ldots, r \); then there exists an \( f \in D'(E^r-S^1) \) such that \( \hat{L}f = 0 \) in \( E^r-S^1 \) and \( f \) does not have an extension across \( S^1 \).

**Proof.** Let \( \hat{A} \) be the differential operator corresponding to \( \hat{A} \) and \( \hat{B}_j \) be the differential operator corresponding to \( \hat{B}_j \). Theorem 2.2 implies that there exists an \( h \in D'(E^n) \) such that \( \hat{A}h = \delta \). Suppose \( f \) has an extension \( g \) across \( S^1 \) such that \( \hat{L}g = 0 \) on \( E^n \). Then \( \hat{L}_j(h-g) = \hat{B}_j \delta \). Since \( h-g \in E'(E^n) \) we have

\[
\hat{A} \hat{B}_j r = \hat{B}_j r
\]

where \( r \) denotes \( h-g \). Hence

(3.6) \[
\hat{A} \hat{r} = 1.
\]

Equation (3.6) implies that \( \hat{A} \) is a constant. But \( \hat{A} \) is not a constant by assumption. Thus \( f \) does not have an extension across \( S^1 \). This completes the proof.
Corollary 3.2. A distribution solution to $L_1$ and $L_2$ in $V - \mathcal{S}_1$ has an extension across $\mathcal{S}_1$ if and only if $\hat{L}_1$ and $\hat{L}_2$ are relatively prime in the ring of entire functions.

Proof. Corollary 3.2 follows from Theorem 3.1 and Theorem 3.2.
IV. Results for One Operator

We will now consider the extension problem for the case of only one operator. The first theorem in this section is due to Ehrenpreis [3].

Theorem 4.1. Let $K = \{x \in E^n \mid |x| \leq 1\}$ and let $L$ be a linear partial differential operator with constant coefficients. Then there exists an $f \in D'(E^n - K)$ such that $Lf = 0$ in $E^n - K$ and with the property that there does not exist an $g \in D'(E^n)$ such that $g|(E^n - K) = f$ and $Lg = 0$. 

Proof. Theorem 2.2. implies that there exists an $f \in D'(E^n)$ such that

\[(4.1) \quad Lf = \delta\]

where $\delta$ denotes the Dirac measure at zero.

Now equation (4.1) implies that $Lf = 0$ in $E^n - K$ because $\delta = 0$ in $E^n - K$. Suppose there exists a $g \in D'(E^n)$ such that $g|(E^n - K) = f$ and $Lg = 0$ on $E^n$. Then $f - g \in E'(E^n)$ and

\[(4.2) \quad L(f - g) = \delta.\]

Equation (4.2) contradicts Theorem 2.1 because \(\frac{\hat{\delta}}{\hat{L}} = \frac{1}{\hat{L}}\) is not an entire function. This completes the proof.
Theorem 4.1 is not generally valid for $C^\infty$ functions. For example, take $n = 2$, $L = \frac{\partial^2}{\partial x_2^2}$ and $K = \{x \in \mathbb{R}^2 \mid |x| < 1\}$. Then $f \in C^\infty(\mathbb{R}^2 - K)$ and $Lf = 0$ imply that there exists an $h \in C^\infty(E')$ such that $f(x_1, x_2) = h(x_1)$ where $|x| > 1$ and $x = (x_1, x_2)$. However if $g$ is defined by $g(x) = h(x_1)$, then $g$ is an extension of $f$.

On the other hand, if $L = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ and $f(x) = \ln |x|$ for $|x| > 1$, then $Lf = 0$ for $|x| > 1$, but $f$ does not have an extension to $\mathbb{R}^n$.

We shall now consider the extension problem for an inhomogeneous equation. We shall let $L$ denote a linear partial differential operator with constant coefficients with order greater than or equal to 1, $K$ denote a compact subset of $\mathbb{R}^n$ and $c$ denote a distribution in $E'(\mathbb{R}^n)$.

Theorem 4.2. Let $f \in D'(\mathbb{R}^n - K)$ have the property that there exists a $p > 0$ such that $f = 0$ for $|x| > p$ and let $Lf = c$ in $\mathbb{R}^n = K$. If conditions (A), (B), and (C) hold and if $\frac{c}{L}$ is an entire function on $\mathbb{C}^n$, then there exists one and only one $h \in E'(\mathbb{R}^n)$ such that $h|(\mathbb{R}^n - K) = f$ and $Lh = c$ in $\mathbb{R}^n$.

Proof. Let $M$ be a bounded open set containing $K$. Then there exists a $g \in D'(\mathbb{R}^n)$ such that $g = f$ outside $M$. Let $g_1$ be defined by

\[ g_1 = Lg - c. \]
Then supp \( g_1 \subset M \).

Take the Fourier transform of both sides of (4.3). Then
\[
\hat{g}_1 = \hat{Lg} - \hat{c}.
\]
Hence
\[
\left(4.4\right)
\frac{\hat{g}_1}{\hat{L}} = \hat{g} - \frac{\hat{c}}{\hat{L}}.
\]
Equation (4.4) implies that \( \frac{\hat{g}_1}{\hat{L}} \) is an entire function on \( \mathbb{C}^n \) because \( \hat{g} \) and \( \frac{\hat{c}}{\hat{L}} \) are entire functions on \( \mathbb{C}^n \).

Let \( A > 0 \) such that \( M \subset S_A \) and supp \( c \subset S_A \). Then Theorem 2.1 implies that there exists an \( S \in E'(\mathbb{R}^n) \) with support in \( S_A \) such that \( \hat{S} = \frac{\hat{c}}{\hat{L}} \). Thus \( \frac{\hat{g}_1}{\hat{L}} = r \) where
\[
r = g - S.
\]
Hence we have
\[
\left(4.5\right)
Lr = g_1.
\]
Since \( g_1 = 0 \) outside \( M \) and \( r = 0 \) outside \( S_A \) assumption (B) together with equation (4.5) imply that \( r = 0 \) outside \( M \). Let \( h = g - r \). Then \( h = f \) outside \( M \) because \( g = f \) outside \( M \) and \( r = 0 \) outside \( M \). \( h \) is a solution on \( \mathbb{R}^n \) because \( Lh = Lg - Lr = c + g_1 - Lr = c \). Suppose \( h_1 \) is a distribution on \( \mathbb{R}^n \) such that \( h_1(E^n - M) = f \) and \( Lh_1 = c \).
Then \( h - h_1 \) has compact support and \( L(h - h_1) = 0 \). Therefore \( h - h_1 = 0 \) by lemma 2.1. The uniqueness argument is the same as that given in Theorem 3.1. This completes the proof.
Bibliography


