RICE UNIVERSITY

A PROBLEM IN HARMONIC CONTINUATION IN A DISK

BY

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This paper treats the problem of describing the behavior of a function \( u \) bounded and harmonic in a disk, when its behavior is known on a portion \( B \) of the disk that contains an open set. If \( |u| \) is bounded by one on the disk and bounded by \( \varepsilon \) on \( B \), then \( |u| \) is bounded at the origin by \( K_1 \varepsilon^{K_2} \), \( K_2 > 0 \). The method is to complexify \( u \), use a lemma of Carleman to bound \( |u| \) in a neighborhood of the origin, and use a three-circle theorem of Miller to bound \( |u| \) in the rest of the disk.
INTRODUCTION

The problem treated in this paper is that of harmonic continuation, in a disk, of information given on a subset, containing an open set, of the disk. This is one of a number of problems in harmonic continuation, among them continuation from information on a circle or at a finite number of points. [2]*

In the problem worked in this paper the region where the information is given contains a part of the boundary of the disk. The same method of solution can be used when the region lies in the interior of the disk. At the end of the paper is a short discussion of this problem.

There is an alternative means of proof for the lemma in Section I, namely, the application of the following theorem of Landis for the case when n=2 and the differential equation (*) is Laplace's equation.

Theorem: Let \( u \) be a solution of

\[
\sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_k} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu = 0
\]

with continuous coefficients such that \( i) a_{ik} \) have continuous first and second derivatives, \( ii) b_i \) have continuous first derivatives, and \( iii) \) all coefficients and all the derivatives indicated in \( i) \) and \( ii) \) are bounded in absolute value by one in the region where the equation is given. Assume also that in this region

\[
\sum_{i=1}^{n} \xi_i^2 = 1 \quad \text{implies} \quad (**\quad \sum_{i,k} a_{ik} \xi_i \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_k} > \infty > 0.
\]

* Numbers in square brackets refer to the references in the bibliography.
Let $G$ be a sphere in an $n$-dimensional Euclidean region with center at the origin and radius one. Let $S$ be its boundary and $\gamma$ be the part of $S$ defined by

$$\sum_{i=1}^{n} \alpha_{i}^{2} = 1, \; \alpha_{1} > 0, \; \sum_{i=2}^{n} \alpha_{i}^{2} \leq \alpha_{1}^{2} < \frac{1}{4}.$$  

Then there exists a positive constant $C$, depending only on the size of the region and on $\alpha$ from (**), such that, for any $\varepsilon > 0$ and any solution $u$ of (*) defined in $G$ and continuous in the closure of $G$, having normal derivative on $\gamma$ and satisfying

$$|u|_{3} \leq 1,$$
$$|u|_{\gamma} \leq \varepsilon,$$
$$|uy|_{\gamma} \leq \varepsilon,$$

the inequality $|u(\partial B \cap \partial \Omega)| \leq C \varepsilon$ is satisfied. [3]

A word about the region $B$ where $|u| \leq \varepsilon$ is in order. No conditions on its shape have been imposed in the statement of the problem, since the result does hold for any $B$ which contains an open set. Nonetheless, the final result depends heavily on the shape of $B$ in two ways.

We will make the following construction: choose $d > 0$ and let $B_{0} = \{ z \in B \mid \text{dist}(z, \partial B) > d \}$.

First, in order to apply the proof as given in the lemma of Section I, we must have $B_{0}$ connected. Thus, areas such as $A \subset B_{0}$ (figure 1) must be neglected.
Second, most appendages to the main body of $B$ (figure 2) will automatically be excluded from $B_0$. In case part of one should remain, unless it is of sizeable area it should be excluded also. The rationale for this is that the result of the lemma will depend directly on the maximum, taken over all $z \in B_0$, length of the shortest possible path, lying in $B_0$, from $z$ to a given fixed point $z_0$ of $B_0$. Therefore it may be desirable to neglect such regions as $A_1$.

It is a well-known fact that if $u=0$ on an open set in the disk, then $u=0$ throughout the disk. From the form of the theorem in Section III, it is clear that this result may also be obtained as a corollary to that theorem, simply by letting $\epsilon$ approach 0.
SECTION I

1. Statement of the Problem

We are given a function \( u(x,y) \) which is harmonic in the unit disk \( \mathbb{D} \) and which satisfies \( |u| \leq 1 \) on \( \partial \mathbb{D} \) and \( |u| \leq \epsilon \) on some portion \( B \) of the disk, as in figure 3. \( B \) must contain an open set. We are asked to bound \( |u| \) pointwise in the interior of \( \mathbb{D} \).

This will be done in three steps.

1) Bound \( |u(x,y)| \) in terms of \( \epsilon \). This will be the object of the rest of Section I.

2) Bound \( |u(x,y)| \) where \( \epsilon = \sqrt{\alpha^2 + \beta^2} \), by choosing various circles lying inside \( \mathbb{D} \) and using the method of Section I to bound \( |u| \) at their centers. This will be done in Section II.

3) Apply a three-circle theorem of Miller to obtain the bound in \( \epsilon \leq \epsilon \). This will be done in Section III.

2. Complexification of \( u(x,y) \)

Choose \( 0 < \alpha < 1 \) and adopt the following notation:

\[
\mathbb{D}_\alpha = \{ \sqrt{\alpha^2 + \beta^2} \}, \\
B_\alpha = \{ z \in \mathbb{D} \mid \text{dist } (z, \partial B) > \alpha \}, \\
\gamma = \{ z \in \mathbb{D} \mid x > 0, y^2 \leq a^2 \},
\]

where restrictions on the magnitude of \( \alpha \) are given below,
\( P: (-1,0, 0), \quad Q: (1,0,0) \),

\( S = \) the sector bounded by the arc \( Y \) and the two line segments from \( P \) to the endpoints of \( Y \),

\[ S_* = S \cap T_0, \]

\[ \kappa_* = (\partial T_0 \cap S_*) - P. \]

We must take a small enough to have \( \kappa_* \subseteq \partial B_0 \); since the larger \( a \) is, the better our estimate for \( |\mu(0,0)| \) will be, let \( a \) be as large as possible, subject to the above restraint. The magnitude of \( a \) will depend on the size and shape of \( B_0 \) and on the magnitude of \( d \).

Let \( \varphi(\partial \Omega) \) be that harmonic conjugate of \( \mu \) which satisfies \( \varphi(\partial \Omega) = 0 \). Let \( f = \mu + iv \).

The method, in general, will be to bound the gradient of \( \mu \) in \( T_0 \) and to use this to bound \( \nu \) in \( T_0 \); to bound the gradient of \( \mu \) in terms of \( \epsilon \) in \( B_0 \), and to use this to bound \( \nu \) in terms of \( \epsilon \) there. (As the proof progresses, it will become apparent that it would have sufficed to bound \( \nu \) in terms of \( \epsilon \) along \( Y_0 \); in Section II, however, we will need this bound throughout \( B_0 \).) Then a lemma of Carleman [1] can be applied to \( S_0 \) to bound \( \mu(0,0) \) and hence \( \mu(0,0) \) in the desired way.

Let \( z_0 \in T_0 \) be given and let \( \eta = z - z_0 \). We define \( U(\eta) \) by

\[ u(z) = u(\eta + z_0) = U(\eta) \]

Since \( z_0 \in T_0 \), the circle \( |z-z_0|=\eta_/a \) lies entirely in \( T_0 \). Writing the Poisson integral for
\[ \mathcal{L} = (\beta \Psi) \text{ and } |\mathcal{L}| = d \] we have

\[
U(\rho, \psi) = \frac{1}{2\pi d} \int_{0}^{2\pi d} \bar{U}(\eta) \left[ \frac{d^2 \rho^2}{d^2 \rho^2 - 2d \rho \cos(\theta - \psi)} \right] d\theta
\]

\[
= \frac{1}{2\pi d} \int_{0}^{2\pi d} \bar{U}(\eta) \rho(\rho, \psi) \, d\theta,
\]

where \( \bar{U}(\eta) = U(\eta) \) for \(|\mathcal{L}| = d \). Now,

\[
1 \, DU = \sqrt{\frac{\partial U}{\partial \rho}^2 + \frac{\partial U}{\partial \psi}^2} = \left| \frac{\partial U}{\partial \rho} \right| + \left| \frac{\partial U}{\partial \psi} \right|.
\]

Also,

\[
\frac{\partial U}{\partial \rho} = \frac{1}{2\pi d} \int_{0}^{2\pi d} \bar{U}(\eta) \frac{\partial \rho}{\partial \rho} \, d\theta = \frac{1}{2\pi d} \int_{0}^{2\pi d} \bar{U}(\eta) \left[ \frac{\partial \rho}{\partial \rho} \frac{\partial \psi}{\partial \rho} + \frac{\partial \psi}{\partial \psi} \frac{\partial \rho}{\partial \psi} \right] \rho \, d\theta,
\]

\[
\frac{\partial U}{\partial \psi} = \frac{1}{2\pi d} \int_{0}^{2\pi d} \bar{U}(\eta) \frac{\partial \psi}{\partial \psi} \, d\theta = \frac{1}{2\pi d} \int_{0}^{2\pi d} \bar{U}(\eta) \left[ \frac{\partial \psi}{\partial \rho} \frac{\partial \rho}{\partial \rho} + \frac{\partial \psi}{\partial \psi} \frac{\partial \psi}{\partial \psi} \right] \rho \, d\theta,
\]

where we are for the moment letting \( \alpha = \rho \cos \psi \) and \( \beta = \rho \sin \psi \).

We have the following equalities:

\[
\frac{\partial \rho}{\partial \rho} = \frac{\alpha}{\rho} = \cos \psi, \quad \frac{\partial \rho}{\partial \psi} = \frac{\beta}{\rho} = \sin \psi,
\]

\[
\frac{\partial \psi}{\partial \rho} = -\frac{\sin \psi}{\rho}, \quad \frac{\partial \psi}{\partial \psi} = \frac{\cos \psi}{\rho}.
\]

Also,

\[
\frac{\partial \rho}{\partial \rho} = \frac{(2d \cos(\theta - \psi) + \rho)(\rho^2 - \rho^2)}{[d^2 + \rho^2 - 2d \rho \cos(\theta - \psi)]^2} = \lambda(\rho, \psi),
\]

\[
\frac{\partial \rho}{\partial \psi} = \frac{(2d \cos(\theta - \psi) - 2d \rho \sin(\theta - \psi))}{[d^2 + \rho^2 - 2d \rho \cos(\theta - \psi)]^2} = \rho B(\rho, \psi).
\]
Hence, $\frac{\partial P}{\partial x} = \cos \psi \cdot A(p, \psi) - \frac{\sin \psi \cdot \rho B(p, \psi)}{\rho}$

and

$\frac{\partial P}{\partial y} = \sin \psi \cdot A(p, \psi) + \frac{\cos \psi \cdot \rho B(p, \psi)}{\rho}$.

Hence, $\frac{\partial P}{\partial \psi}(0, 0) = \frac{\partial}{\partial \psi} \left[ \cos \psi \cos(\theta - \psi) - \sin \psi \sin(\theta - \psi) \right]$

$= \frac{\partial}{\partial \psi} \cos(2\psi + \theta)$.

Then $|\frac{\partial P}{\partial \psi}(0, 0)| \leq \frac{\partial}{\partial \psi}$ and by a similar calculation,

$|\frac{\partial P}{\partial \theta}(0, 0)| \leq \frac{\partial}{\partial \theta}$.

Therefore,

$|\frac{\partial \psi}{\partial \theta}(0, 0)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} |U(\theta)| \frac{\partial}{\partial \theta} \int_{0}^{2\pi} U(\theta) \frac{\partial P}{\partial \psi}(0, 0) d\theta$

$\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |U(\theta)| \frac{\partial}{\partial \theta} \int_{0}^{2\pi} U(\theta) \frac{2}{\partial} d\theta \leq \frac{2}{\partial}$,

since $f \in \mathcal{F}$ implies $|U(\theta)| \leq \frac{1}{\partial}$. Similarly,

$|\frac{\partial \psi}{\partial \theta}(0, 0)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |U(\theta)| \frac{\partial}{\partial \theta} \int_{0}^{2\pi} U(\theta) \frac{2}{\partial} d\theta \leq \frac{2}{\partial}$.

Thus, finally we have

$|\nabla U(0, 0)| \leq |\frac{\partial P}{\partial \psi}(0, 0)| + |\frac{\partial P}{\partial \theta}(0, 0)| \leq \frac{4}{\partial}$.

Since the change of variables $z \rightarrow \zeta$ is just a translation, $\nabla U(0, 0) = \nabla v(z_0)$. So for any point $z \in \Omega$, $|\nabla v(z)| \leq \frac{4}{\partial}$.

We may make the same construction for any point $z_0 \in B_0$.

Since now the circle $|z - z_0| = d < B_0$, we have $|U(\theta)| \leq \frac{4}{\partial}$. Hence

$|\nabla v(z)| \leq \frac{4\epsilon}{\partial}$ for $z \in B_0$.

Let $z_0 \in \Omega$ be given and let $L$ be the line segment from $Q$ to $z_0$. ( $\zeta \subset \Omega$.)

$\nu(z_0) = \int_{L} \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = \int_{L} -\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$. 
\[
\left\langle \begin{pmatrix} -\frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix}, \begin{pmatrix} \frac{dx}{ds} \\ \frac{dy}{ds} \end{pmatrix} \right\rangle ds,
\]

where \( \cdot \) denotes the dot product and \( s \) denotes arc length. The length of the second vector is one. Therefore, as \( R(1-d) \) is the diameter of \( T \),

\[
|w(z_0)| \leq \int_{\gamma} |\nabla u(z)| ds \leq \frac{\pi}{2} R(1-d) = \frac{\pi}{2} (1-d).
\]

Therefore, for \( z \in T \),

\[
|f(z)| \leq |u(z)| + |w(z)| \leq 1 + \frac{\pi}{2} (1-d)
\]

\[
\leq \frac{\pi}{2} - 7.
\]

In the same way, if \( z_0 \in B \), join \( z_0 \) to \( Q \) by an arc \( \lambda \). If \( B \) is convex, we may use line segments; at any rate, for each \( z_0 \in B \) we choose \( \lambda \) to be as short as possible. If \( h(z) \) denotes the arc length of \( \lambda \) from \( z_0 \) to \( Q \), then let \( L = \max_{z_0 \in B} \{ h(z) \} \).

Then

\[
|w(z_0)| \leq \int_{\lambda} |\nabla u| dx + |\frac{dr}{ds}| dy
\]

\[
\leq \int_{\lambda} |\nabla u(z)| ds \leq \frac{\pi L}{2} .
\]

Therefore,

\[
|f(z)| \leq \epsilon \left( 1 + \frac{\pi L}{2} \right), \quad z \in B.
\]

In particular, we have

\[
|f(z)| \leq \frac{\pi}{2} \left( \frac{\pi}{2} - 7 \right) \quad \text{on the sides of } S_0,
\]

and

\[
|f(z)| \leq \epsilon \left( 1 + \frac{\pi L}{2} \right) \quad \text{on } \gamma_0 \subset B_0.
\]

3. **A Bound for \( |u(0,0)| \)**

We are now in a position to state and prove the following lemma:

**Lemma:** Let \( u \) be harmonic in the unit disk \( \Gamma \) and satisfy

\[
|u| \leq 1 \quad \text{on } \partial \Gamma, \quad u \text{ is } \frac{\pi}{2} \text{ in } B.
\]

Then \( |u(0,0)| \leq K(d, \pi) \epsilon \frac{\mu(d, \pi)}{\pi} \).
where \( d, \lambda, \) and \( a \) have been defined and
\[
K(d, \lambda, a) = \frac{(\frac{\pi}{d} - 7)(\frac{d + 4\lambda^2}{\pi - 7d})}{\alpha}\left(\frac{\pi}{2\tan^{-1}a}\right)
\]
\[
\mu(d, a) = \frac{\pi}{2\tan^{-1}a}
\]
\[
A = \frac{a}{1 - d + \sqrt{1 - a^2}}.
\]

Proof: The result is a direct application of the following lemma, due to Carleman [1]:

Lemma: Let \( g(z) \) be an analytic function regular in the interior of a domain \( OABO \) bounded by two line segments \( OA \) and \( OB \) which end in a fixed point \( \xi \) and by an arc \( AB \) of a Jordan curve. Let \( \xi \) be a point on the bisector of the angle \( AOB = \alpha \pi \). Suppose further that \( |g(z)| \leq M \) on \( OA \) and \( OB \), \( |g(z)| \leq m \) on \( AB \). Let \( r \) denote the distance from \( \xi \) to \( \xi \) and let \( R \) designate the greatest distance from \( AB \) to the point \( O \). Then
\[
|g(\xi)| \leq M \left(\frac{\pi}{2R}\right)^{\frac{1}{2\alpha}} \left(m\frac{\pi}{R}\right)^{\frac{1}{\alpha}}.
\]

In our case the domain in question is the sector \( S_0 \). We set
\[
M = \left(\frac{\pi}{d} - 7\right), \quad m = \epsilon \left(1 + \frac{4\lambda^2}{d}\right),
\]
\[
r = 1 - d, \quad R = 2(1 - d), \quad \xi = \rho, \quad \xi = (0, 0),
\]
\[
\beta = \alpha \pi, \quad \frac{1}{\alpha} = \frac{\pi}{\beta} = \frac{\pi}{2\tan^{-1}A},
\]
\[
\tan\left(\frac{\beta}{2}\right) = A = \frac{a}{1 - d + \sqrt{1 - a^2}}.
\]

Thus,
\[
|f(0, 0)| \leq \left(\frac{\pi}{d} - 7\right) \left[1 - \frac{\pi}{2\tan^{-1}a}\right] \left\{\epsilon \left(1 + \frac{4\lambda^2}{d}\right)\right\}.
\]
\[
\leq \left( \frac{g}{d} - 7 \right) \left( \frac{d + 4z}{8 - 7d} \right) \left[ \frac{1}{2} \frac{\gamma}{\text{diam} H} \right] - \left[ \frac{1}{2} \frac{\gamma}{\text{diam} H} \right] \leq K(d, L, a) \cdot \varepsilon \mu(d, a)
\]

Hence, \( |\mu(x) - f(x) - \varepsilon \gamma| \leq K(d, L, a) \cdot \varepsilon \mu(d, a) \).

q.e.d.
SECTION II

For any \( 0 \leq \theta \leq 2\pi \) let \( \gamma = \theta + (\omega, \delta) \) where \( \delta > 0 \) is a small number (independent of \( \theta \)) which will be defined below. Holding \( \theta \) fixed, take \( 0 < R(\theta) < 1 \) such that

1) \( \{ |z| = R(\theta) \} \subset \Gamma_0 \) and

ii) \( \{ |z| = R(\theta) \} \cap B_0 \supset \omega(\theta) \) where we require that the length \( L(\theta) \) of the arc \( \omega(\theta) \) satisfy \( L(\theta) > L_0 > 0 \)

for some fixed \( L_0 \) independent of \( \theta \). We choose \( \delta \) as large as possible, subject to allowing us to find such an \( R(\theta) \) for each \( \theta \). The parameter \( \delta \) depends directly on the size of \( B \) and inversely on the magnitude of \( L \) and \( L_0 \).

We have the following situation: choose \( R(\theta) \) so that \( z = (\delta, \theta) \) lies on the bisector of the angle in the sector defined by \( \rho(\theta) \) and \( \omega(\theta) \). Since \( \{ |z| = R(\theta) \} \subset \Gamma_0 \) and \( \omega(\theta) \subset B_0 \), we have all the bounds for \( \mathcal{L} \) that were computed in Section I. Applying Carleman's lemma again we have

\[
\mu(\delta, \alpha(\theta)) \leq \mathcal{L}(\delta, \theta) \leq \kappa(\delta, \alpha(\theta)) \in \mathcal{M}(\delta, \alpha(\theta))
\]

where

\[
\mu(\delta, \alpha(\theta)) = \frac{\pi}{\beta(\theta)} = \frac{\pi}{2} \left( \frac{\sqrt{a(\theta)}}{R(\theta) + \sqrt{R(\theta)^2 - a(\theta)^2}} \right)
\]

and \( \kappa(\delta, \alpha(\theta)) \) is obtained from \( \kappa(\delta, \alpha) \) by replacing \( a, l \), and \( l - \alpha \) in the exponent by \( \alpha(\theta) \) and \( R(\theta) \).
It remains simply to find a bound which holds for all $\theta$

Let $\beta_0 = \sup_{0 \leq \theta \leq 2\pi} \{ \beta(\theta) \}, \ (\beta_0 < 2\pi).$

Then $K(d, x) = \left( \frac{\pi}{d} - 7 \right) \left( \frac{d + \sqrt{d^2 + 4\pi^2}}{d - \sqrt{d^2 + 4\pi^2}} \right) \frac{1}{2} \left( \frac{\pi}{\beta_0} \right)$ implies $K(d, x) \geq K(d, \gamma, a(\theta)).$

Let $\beta_1 = \inf_{0 \leq \theta \leq 2\pi} \{ \beta(\theta) \}.$

Since the length $l(\theta)$ of the arc $u(\theta)$ satisfies $l(\theta) > l_0 > 0$ for all $\theta$, we will have $\beta_1 > 0$. We see that $\beta(\theta) \geq \beta_1 > 0$
implies that $\frac{1}{2} \left( \frac{\pi}{\beta(\theta)} \right) \geq \frac{1}{2} \left( \frac{\pi}{\beta_1} \right) > 0.$

Thus, for any $\theta$, $\mu(d, \gamma, a(\theta)) > \tilde{\mu}(d, a) > 0,$ where we have set $\tilde{\mu}(d, a) = \frac{1}{2} \left[ \frac{\pi}{\beta_1} \right].$ It follows immediately that $\epsilon \mu(d, \gamma, a(\theta)) \leq \epsilon \tilde{\mu}(d, a) < 1.$

Consequently, $|z| = d$ implies that $|\mu(z)| \leq K(d, x) \epsilon \tilde{\mu}(d, a)$. 

Now $\tilde{\mu}(d, a)$ formally depends on $d$ and $K(d, x)$ formally depends on $d$ and $x$. Both of the last two quantities, however, are constant in any given problem, so we will write $\tilde{K}$ for $K(d, x)$ and $\tilde{\mu}(a)$ for $\tilde{\mu}(d, a) = \frac{1}{2} \left[ \frac{\pi}{\beta_1} \right].$ With this notation we have, for $|z| \leq d$, the uniform bound

$$|\mu(z)| \leq \tilde{K} \epsilon \tilde{\mu}(a), \quad \tilde{\mu}(a) > 0.$$
SECTION III

We are now ready to apply the following three-circle theorem, due to Miller [4]. It is an analogue, for harmonic functions, of Hadamard's three-circle theorem for analytic functions.

**Lemma:** If $u$ is harmonic on the unit disk and if $|u|_\rho \leq m = \rho^0$ and $|u|_1 \leq 1$, then for $\rho < r < 1$,

$$|u|_r \leq m \frac{\log r}{\log \rho} = r^0.$$

In our case $\rho = \delta$ and $m = \tilde{\gamma} \in \tilde{\mu}(a)$. Therefore, for $\delta < |z| \leq 1$, we have

$$|u(z)| \leq \left[ \tilde{\gamma} \in \tilde{\mu}(a) \right] \frac{\log |z|}{\log \delta}.$$

We have proven the following theorem:

**Theorem:** Let $u$ be harmonic in the unit disk and bounded in modulus by one there. Let $u$ be on a region of the disk which contains an open set. Then

$$|u(z)| \leq \begin{cases} \tilde{\gamma} \in \tilde{\mu}(a), & 1|z| = \delta \\ \left[ \tilde{\gamma} \in \tilde{\mu}(a) \right] \frac{\log |z|}{\log \delta}, & \delta < |z| \leq 1. \end{cases}$$

where $\delta \tilde{\gamma} \in \tilde{\mu}(a)$ are constants defined in Section II, and $\tilde{\mu}(a) > 0$. 
SECTION IV

This method can be used to obtain bounds on $u$ when $B$ lies in the interior of $\Omega$. Since it does not give as good an estimate as the usual method of working the problem, its chief value is that it is not too difficult to calculate $\delta$ for any reasonably shaped $B$. By the usual method of treating this problem, we would choose some point $P \in \text{int } B$, take the Moebius transformation $f$ of the unit disk onto itself which sends $P$ onto the origin, and consider $\Re(f(P))$. We would then let $\delta$ be the minimum distance from the origin to $\{f(P)\}$. $\delta$ will depend on the choice of $P$; and it may not be clear which $P$ will give the largest $\delta$. Moreover, $\delta$ may be difficult to calculate explicitly.
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