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Estimates for Singular Integrals

by

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ABSTRACT

In a paper by Calderón and Zygmund [1] some properties of a certain kind of singular integral are established, and these results are applied to particular fundamental solutions arising in partial differential equations. Jones has considered another class of singular integrals [2] which has application to fundamental solutions of the heat equation. More generally the kernels considered by Jones arise from parabolic differential equations with constant coefficient. This thesis considers a kernel which is a generalization of the kernel treated by Jones, and it has mean value and homogenity properties analogous to those in Jones's paper [2].

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ESTIMATES FOR SINGULAR INTEGRALS

1. Introduction

We first define the truncation of a kernel $k(x,t)$ as

$$k_\varepsilon(x,t) = \begin{cases} k(x,t) & t \geq \varepsilon , \\ 0 & t < \varepsilon , \end{cases}$$

where x is a point in n-dimensional Euclidean space and $t > 0$.

In [2] Jones makes the following assumptions about the kernel $k(x,t)$:

$$k(x,t) = \frac{1}{t\varphi(t)^n} \Omega\left(\frac{x}{\varphi(t)}\right) , \qquad t > 0,$$

where $\Omega(x)$ is a measurable complex-valued function on R^n such that

$$\int_{R^n} (1+|x|)|\Omega(x)|dx < \infty$$

and

$$\int_{R^n} \Omega(x)dx = 0,$$

and $\varphi(t)$ is a non-decreasing positive continuous function for $0 < t < \infty$ such that $\lim_{t\to 0} \varphi(t) = 0$, $\lim_{t\to\infty} \varphi(t) = \infty$.

With certain smoothness properties for $\varphi(t)$ and some assumptions on $\Omega(x)$ Jones was able to prove the following theorem:

Let $1 < p < \infty$, _and let_ $f \in L_p(E)$, _where_ $E = R^n \times (0,\infty)$. _Then_ f_ε, _defined by_

$$f_\varepsilon(x,t) = \int_E k_\varepsilon(x-\xi,t-\tau)f(\xi,\tau)d\xi d\tau ,$$

belongs to $L_p(E)$, *and there exists a constant* A_p, *depending only on* p *and the kernel* k, *such that* $\|f_\varepsilon\|_p \leqq A_p\|f\|_p$.

Under the above assumptions it is also proved that f_ε converges in $L_p(E)$ as $\varepsilon \to 0$.

In this paper we consider a more general kernel k(x,t). Again assuming $\Omega(x)$ is a measurable complex-valued function on R^n, we let

$$k(x,t) = \frac{1}{t\varphi_1(t)\ldots\varphi_n(t)}\,\Omega\left(\frac{x_1}{\varphi_1(t)},\ldots,\frac{x_n}{\varphi_n(t)}\right)$$

where the assumptions on $\varphi_i(t)$, $1 \leqq i \leqq n$, and the assumptions on $\Omega(x)$ are analogous to those considered by Jones. A complete description of k(x,t) is given in the next section.

The task of this paper is to prove the analog of the main lemma, lemma 4 of [2], using the generalized kernel k(x,t). Once this lemma is established we immediately have the theorem previously stated concerning the bound on the L_p norm of f_ε. The proof may be found in [1].

2. Description of the Singular Integrals

We describe a class of kernels with the following assumptions:

I. $\varphi_i(t)$ $1 \leqq i \leqq n$ is a non-decreasing positive continuous function for $0 < t < \infty$ such that
$\lim_{t\to 0} \varphi_i(t) = 0, \quad \lim_{t\to\infty} \varphi_i(t) = \infty;$

II. $\Omega(x)$ is a measurable complex-valued function on R^n and
$$k(x,t) = \frac{1}{t\varphi_1(t)\ldots\varphi_n(t)}\Omega\left(\frac{x_1}{\varphi_1(t)},\ldots,\frac{x_n}{\varphi_n(t)}\right), \quad t > 0;$$

III. $\int_{R^n} (1+|x|)|\Omega(x)|dx < \infty$ and $\int_{R^n} \Omega(x)dx = 0.$

The homogenity of $k(x,t)$ and mean value properties of $\Omega(x)$ are given by conditions II and III respectively.

We assume the following additional properties on $\varphi_i(t)$, $1 \leqq i \leqq n$. There exists a positive constant c such that

IV. $$\int_0^a \frac{\varphi_i(t)}{t}dt \leqq c\varphi_i(a) \quad , \quad 0 < a < \infty ,$$

$$\int_0^a \frac{1}{t\varphi_i(t)}dt \leqq \frac{c}{\varphi_i(a)} \quad , \quad 0 < a < \infty ;$$

V. $$\frac{\varphi(2t)}{\varphi(t)} \leqq c , \quad 0 < t < \infty ;$$

VI. For $0 < t < \infty$ and $|\delta| \leqq \frac{1}{2}$
$$\left|\frac{\varphi_i((1+\delta)t)}{\varphi_i(t)} - 1\right| \leqq c|\delta|.$$

We specify that for some positive c, $\Omega(x)$ satisfies the following:

$$\text{VII.} \quad \int_{R^n} |\Omega(x) - \Omega(x+y)|dx \leqq c|y| \ , \quad y \in R^n \ ;$$

$$\text{VIII.} \quad \int_{R^n} |\Omega(x) - \Omega((1+\delta_1)x_1, \ldots, (1+\delta_n)x_n)|dx \leqq c \sum_{i=1}^{n} |\delta_i|$$

$$\text{for } |\delta_i| \leqq \frac{1}{2} \ , \quad 1 \leqq i \leqq n \ ;$$

$$\text{IX.} \quad \int_{|x| \geqq a} |\Omega(x)|dx \leqq ca^{-1} \quad \text{for} \quad a > 0.$$

The above description of the kernel k(x,t) is a generalization of the kernel considered by Jones [2].

We note that the notation c shall stand for a (large) positive generic constant, which depends only on the kernel k and the dimension n of the space.

3. A method of partitioning space

In the following discussion we shall use the word "cell" to describe any set of the form

$$\{(x,t) \mid x_i^o \leqq x_i \leqq x_i^o + r_i \ , \quad i = 1,2,\ldots,n; \quad t^o \leqq t \leqq t^o + s\}$$

for positive r_i, s and any numbers $x_1^o,\ldots,x_n^o$, t^o. Now given the functions $\varphi_1(t),\ldots,\varphi_n(t)$ we want to partition the space R^{n+1} into finer and finer cells such that at any step the ratio $\frac{r_i}{\varphi_i(s)}$ is bounded above and below by a positive constant for each i, $1 \leqq i \leqq n$.

Suppose initially that R^{n+1} is partitioned into non-overlapping cells of dimensions $\varphi_1(a),\ldots,\varphi_n(a)$, a. In the t-direction divide each cell in half and let

$$k_i^1 = [\frac{\varphi_i(a)}{\varphi_i(\frac{a}{2})}] \ , \qquad i = 1,\ 2,\ \ldots,\ n \ ,$$

where for any positive number b, [b] is the largest integer not greater than b. Now divide the i^{th} side of each cell into k_i^1 equal intervals (each of length $\frac{\varphi_i(a)}{k^1 i}$). Each cell of dimensions $\varphi_1(a)$, ..., $\varphi_n(a)$, a is divided into $2\pi_{i=1}^{n} k_i^1$ cells of dimensions $\frac{\varphi_1(a)}{k_1^1},\ldots,\frac{\varphi_n(a)}{k_n^1}$, $\frac{a}{2}$. This division is well defined if $k_i^1 \geqq 1$, and $\varphi_i(t)$ being non-decreasing assures us of this.

Similarly, let

$$k_i^2 = [(k_i^1)^{-1} \ \varphi_i(a)/\varphi_i(2^{-2}a)]$$

and then each cell of dimensions $\frac{\varphi_1(a)}{k_1^1}, \ldots, \frac{\varphi_n(a)}{k_n^1}, \frac{a}{2}$ can be divided into $2\prod_{i=1}^{n} k_i^2$ cells of dimensions $\frac{\varphi_i(a)}{k_1^1 k_1^2}, \ldots, \frac{\varphi_n(a)}{k_n^1 k_n^2}, \frac{a}{2^2}$. Now it is easy to show $k_i^2 \geqq 1$, so this step is well defined.

We proceed by induction. We have the $(\nu-1)^{th}$ subdivision such that $k_i^1, \ldots, k_i^{\nu-1} \geqq 1$, $i = 1,2,\ldots,n$ and each cell has dimensions $\frac{\varphi_1(a)}{k_1^1 \ldots k_1^{\nu-1}}, \ldots, \frac{\varphi_n(a)}{k_n^1 \ldots k_n^{\nu-1}}, \frac{a}{2^{\nu-1}}$. Now define

$$k_i^\nu = [(k_i^1 \ldots k_i^{\nu-1})^{-1} \varphi_i(a)/\varphi_i(2^{-\nu}a)] \qquad i = 1,2,\ldots,n.$$

But

$$(k_i^1 \ldots k_i^{\nu-1})^{-1} \varphi_i(a)/\varphi_i(2^{-\nu}a)$$

$$= (k_i^{\nu-1})^{-1}\{(k_i^1 \ldots k_i^{\nu-2})^{-1} \varphi_i(a)/\varphi_i(2^{-\nu+1}a)\} \cdot \{\varphi_i(2^{-\nu}a)/\varphi_i(2^{-\nu+1}a)\}$$

$$\geqq (k_i^{\nu-1})^{-1}[(k_i^1 \ldots k_i^{\nu-1})^{-1} \varphi_i(a)/\varphi_i(2^{-\nu+1}a)](\varphi_i(2^{-\nu+1}a)/\varphi_i(2^{-\nu}a))$$

$$= \varphi_i(2^{-\nu+1}a)/\varphi_i(2^{-\nu}a) \geqq 1 ,$$

hence $k_i^\nu \geqq 1$, $i = 1,\ldots,n$, and we can now divide each cell of dimensions $\frac{\varphi_1(a)}{k_1^1 \ldots k_1^{\nu-1}}, \ldots, \frac{\varphi_n(a)}{k_n^1 \ldots k_n^{\nu-1}}, \frac{a}{2^\nu}$ into $2\prod_{i=1}^{n} k_i^\nu$ cells of dimensions $\frac{\varphi_i(a)}{k_1^1 \ldots k_1^\nu}, \ldots, \frac{\varphi_n(a)}{k_n^1 \ldots k_n^\nu}, \frac{a}{2^{\nu+1}}$.

Note also that

$$\frac{\varphi_i(a)}{k_i^1 \cdots k_i^\nu \varphi_i(2^{-\nu}a)} = (k_i^\nu)^{-1}\{(k_i^1 \cdots k_i^{\nu-1})^{-1}\varphi_i(a)/\varphi(2^{-\nu}a)\}$$

$$< (k_i^\nu)^{-1}\{[(k_i^1 \cdots k_i^{\nu-1})^{-1}\varphi_i(a)/\varphi(2^{-\nu}a)] + 1\}$$

$$= (k_i^\nu)^{-1}\{k_i^\nu + 1\} \leqq 2.$$

Now the expression on the left is just $\frac{r_i}{\varphi_i(s)}$ for a cell of dimensions $r_1,\ldots,r_n,s$, where $r_i = (k_i^1 \ldots k_i^\nu)^{-1}\varphi_i(a)$ and $s = 2^{-\nu}a$. Therefore, any cell in the process has dimensions $r_1,\ldots,r_n,s$ satisfying

(3.1) $$1 \leqq \frac{r_i}{\varphi_i(s)} \leqq 2.$$

We can now find a constant bound on k_i^ν which depends only on the function $\varphi_i(t)$. We have

$$k_i^{\nu+1} = [(k_i^1 \cdots k_i^\nu)^{-1}\varphi_i(a)/\varphi_i(2^{-\nu-1}a)]$$

$$= [\{(k_i^1 \cdots k_i^\nu)^{-1}\varphi_i(a)/\varphi_i(2^{-\nu}a)\} \cdot \varphi_i(2^{-\nu}a)/\varphi_i(2^{-\nu-1}a)]$$

$$\leqq [2\varphi_i(2^{-\nu}a)/\varphi_i(2^{-\nu-1}a)] \leqq [2c] = c \ ,$$

where use was made of condition V.

The following notation for the rearrangement of a function was used in Calderón and Zygmund [1] and Jones [2]. If $f \in L_p(E)$, where $E = R^n \times (0,\infty)$, let $f^*(\lambda)$ be a non-increasing function for $0 < \lambda < \infty$ which is equimeasurable with $|f|$. Let

$$\beta_f(\lambda) = \lambda^{-1}\int_0^\lambda f^*(\mu)\,d\mu \qquad 0 < \lambda < \infty .$$

We denote by $\beta^f(\lambda)$ a function inverse to $\beta_f(\lambda)$. The properties of $\beta_f(\lambda)$ and $\beta^f(\lambda)$ are given in [1], pp. 91-93.

Let $|G|$ denote the Lebesgue measure of a measurable subset G of R^{n+1}.

We have the following lemma analogous to Lemma 3 of Jones [2]:

Lemma 1: Let $f \in L_p(E)$ and let $\eta > 0$. Then there is a sequence of non-overlapping cells I^j such that

(1) if I^j has dimensions $r_1, \ldots, r_n, s$, then

$$1 \leqq \frac{r_i}{\varphi_i(s)} \leqq 2, \quad i = 1,2,\ldots,n.$$

(2) $\eta \leqq |I^j|^{-1} \int_{I^j} |f| \leqq c\eta;$

(3) $|f| \leqq \eta$ a.e. outside $D_\eta = \bigcup_{j=1}^{\infty} I^j$;

(4) $|D_\eta| \leqq \beta^f(\eta);$

(5) $\eta \leqq |D_\eta|^{-1} \int_{D_\eta} |f| \leqq c\eta.$

Proof: Since $\lim_{\lambda\to\infty} \beta_f(\lambda) = 0$, it follows that $\beta_f(\lambda) < \eta$ for sufficiently large λ. Now partition R^{n+1} into cells of dimension $\varphi_1(a), \ldots, \varphi_n(a)$, a such that $\varphi_1(a)\ldots\varphi_n(a)a \geqq \lambda$. Over any such cell I

$$\frac{1}{|I|} \int_I |f| \leqq \frac{1}{|I|} \int_0^{|I|} f^*(\mu)d\mu = \beta_f(|I|)$$

$$= \beta_f(\varphi_1(a)\ldots\varphi_n(a)a) \leqq \beta_f(\lambda) < \eta.$$

Now partition these cells as described above into cells of dimension $\frac{\varphi_1(a)}{k_1^1}, \ldots, \frac{\varphi_n(a)}{k_n^1}, \frac{a}{2}$ and select for the sequence I^j those cells over which the average of $|f|$ is at least η. Next partition the remaining cells into cells of dimension $\frac{\varphi_1(a)}{k_1^1 k_1^2}, \ldots, \frac{\varphi_n(a)}{k_n^1 k_n^2}, \frac{a}{2^2}$ and select from these for the sequence I^j those cells where the average of $|f|$ is at least η. We can continue this. Then we have

$$\eta \leq |I^j|^{-1} \int_{I^j} |f| .$$

Now I^j is contained in a cell I from the previous partition; therefore, $|I| = 2k_1^\nu \ldots k_n^\nu |I^j|$. Since $k_i^\nu \leq c$ we have $|I| \leq c|I^j|$. Then

$$\frac{1}{|I^j|} \int_{I^j} |f| \leq \frac{c}{|I|} \int_{I^j} |f| \leq \frac{c}{|I|} \int_{I} |f| \leq c\eta ,$$

thus proving (2). Condition (1) follows from (3.1). The remaining assertions follow exactly as in [2]. Q.E.D.

4. The main lemma

We now establish the same lemma as Jones [2], lemma 4. When this is done we will then have the results obtained by Jones that the transformation taking f into f_ϵ represents a bounded mapping of L_p into L_p. The proof of the following is quite similar to the proof given by Jones.

Lemma 2: Let $f \in L_p(E)$ for some p, $1 \leqq p \leqq 2$ and let $E_\eta = \{(x,t) \mid |f_\epsilon(x,t)| > \eta\}$, $\eta > 0$. Let $|f| \wedge \eta = \min(|f|, \eta)$. Then $|E_\eta| \leqq c\eta^{-2} \int_E (|f|\wedge\eta)^2 \, dxdt + c\beta^f(\eta)$.

Proof: We use the sequence of cells I^j as guaranteed by the previous lemma. Let ε, η be fixed positive numbers. Define

$$(4.1) \qquad h(x,t) = \begin{cases} |I^j|^{-1} \int_{I^j} f, & (x,t) \in I^j \\ f(x,t), & (x,t) \notin D_\eta = \bigcup_{j=1}^{\infty} I^j, \end{cases}$$

and let $g = f - h$. Then

$$(4.2) \qquad \int_{I^j} g = 0, \qquad g \equiv 0 \text{ outside } D_\eta.$$

Let
$$E_1 = \{(x,t) \mid |h_\epsilon(x,t)| \geqq \tfrac{\eta}{2}\},$$
$$E_2 = \{(x,t) \mid |g_\epsilon(x,t)| \geqq \tfrac{\eta}{2}\}.$$

Then $E_\eta \subset E_1 \cup E_2$, since $f_\epsilon = g_\epsilon + h_\epsilon$. (Clearly, both g and h are in $L_p(E)$, and g_ϵ, h_ϵ exist a.e. by Theorem 1 of [2].) Now

$$\int_E (|f|\wedge\eta)^2 = \int_{|f|\leq\eta} |f|^2 + \int_{|f|>\eta} \eta^2$$

$$\leq \eta^{2-p} \int_{|f|\leq\eta} |f|^p + \eta^2|\{(x,t)|\ |f| > \eta\}| < \infty ,$$

so that $|f|\wedge\eta$ is in $L_2(E)$. For $(x,t)\notin D_\eta$, $|f| \leq \eta$ a.e. by lemma 1, so that $|h| = |f| = |f|\wedge\eta$ a.e. outside D_η by (4.1). For $(x,t)\in D_\eta$, $|h| \leq |I^j|^{-1} \int_{I^j} |f| \leq c\eta$ by lemma 1. Therefore,

$$(4.3) \qquad \int_E |h|^2 = \int_{D_\eta} |h|^2 + \int_{E-D_\eta} |h|^2 \leq c^2\eta^2|D_\eta| + \int_E (|f|\wedge\eta)^2,$$

and $h\in L_2(E)$. Hence, by Theorem 2 of [2] we have that $\int_E |h_\epsilon|^2 \leq c\int_E |h|^2$. Thus, $c\int_E |h_\epsilon|^2 \geq \int_{E_1} |h_\epsilon|^2 \geq \frac{1}{4}\eta^2|E_1|$, and hence it follows by (4.3) and lemma 1 that

$$(4.4) \qquad |E_1| \leq c\eta^{-2} \int_E (|f|\wedge\eta)^2 + c\beta^f(\eta).$$

Since $|E_\eta| \leq |E_1| + |E_2|$, we will be done if we can prove (4.4) with E_1 replaced by E_2.

Let I^j be the cell with center (x^j,t^j) and dimensions $r_1^j,\ldots,r_n^j$, s^i; i.e.

$$(4.5) \qquad I^j = \{(x,t)|\ |x_i-x_i^j| \leq \tfrac{1}{2}r_i^j,\ 1 \leq i \leq n,\ |t-t^j| \leq \tfrac{1}{2}s^j\}.$$

We consider now the cell, say $\mathcal{I}^j$, with center (x^j,t^j) and dimension in each direction twice that of I^j; i.e.

$$(4.6) \qquad \mathcal{I}^j = \{(x,t)|\ |x_i-x_i^j| \leq r_i^j,\ 1 \leq i \leq n,\ |t-t^j| \leq s^j\}.$$

Then clearly $I^j \subset \mathcal{J}^j$ and $|\mathcal{J}^j| \leq c|I^j|$. Let $\mathcal{D}_\eta = \bigcup_j \mathcal{J}^j$; then $D_\eta \subset \mathcal{D}_\eta$ and $|\mathcal{D}_\eta| \leq c|D_\eta|$.

Suppose for the moment that we have proved the inequality

(4.7) $$\int_{E-\mathcal{D}_\eta} |g_\varepsilon| \leq c \int_E |g| = c \int_{D_\eta} |g|.$$

Then, since

$$\int_{I^j} |h| = |\int_{I^j} f| \leq \int_{I^j} |f| ,$$

it follows that

$$\int_{I^j} |g| \leq \int_{I^j} (|f| + |h|) \leq 2 \int_{I^j} |f| ;$$

hence (4.7) implies that

(4.8) $$\int_{E-\mathcal{D}_\eta} |g_\varepsilon| \leq c\sum_j \int_{I^j} |g| \leq c\sum_j 2\int_{I^j} |f| = 2c\int_{D_\eta} |f|.$$

Now (4.8) together with lemma 1 implies

(4.9) $$\int_{E-\mathcal{D}_\eta} |g_\varepsilon| \leq c\eta|D_\eta|.$$

But

(4.10) $$|E_2| \leq |\mathcal{D}_\eta| + |E_2-\mathcal{D}_\eta| \leq c|D_\eta| + |E_2-\mathcal{D}_\eta|.$$

On E_2, $|g_\varepsilon| \geq \frac{1}{2}\eta$; therefore, by (4.9)

$$c\eta|D_\eta| \geq \int_{E_2-\mathcal{D}_\eta} \tfrac{1}{2}\eta = \tfrac{1}{2}\eta|E_2-\mathcal{D}_\eta|.$$

Hence (4.10) implies $|E_2| \leq c|D_\eta| + c|D_\eta| \leq c|D_\eta|$. Then by lemma 1, $|E_2| \leqq c\beta^f(\eta)$, and this together with (4.4) proves the lemma. Therefore, it suffices to prove (4.7).

By Theorem 1 of [2], $g \in L_p(E)$ and $g_\varepsilon(x,t)$ exists a.e. Since $g \equiv 0$ outside D_η,

$$g_\varepsilon(x,t) = \int_{D_\eta} k_\varepsilon(x-\xi,\ t-\tau)g(\xi,\tau)d\xi d\tau$$

$$= \sum_j \int_{I^j} k_\varepsilon(x-\xi,\ t-\tau)g(\xi,\tau)d\xi d\tau.$$

Let

$$G_\varepsilon^j(x,t) = \int_{I^j} k_\varepsilon(x-\xi,\ t-\tau)g(\xi,\tau)d\xi d\tau, \tag{4.11}$$

so that $g_\varepsilon = \sum_j G_\varepsilon^j$ a.e. Now $E-\mathcal{D}_\eta \subset E-\mathcal{J}^j$ implies

$$\int_{E-\mathcal{D}_\eta} |g_\varepsilon| \leqq \sum_j \int_{E-\mathcal{D}_\eta} |G_\varepsilon^j| \leqq \sum_j \int_{E-\mathcal{J}^j} |G_\varepsilon^j|.$$

Therefore, if we can demonstrate the inequality

$$\int_{E-\mathcal{J}^j} |G_\varepsilon^j| \leqq c \int_{I^j} |g|, \tag{4.12}$$

then (4.7) and hence the lemma will follow.

In proving (4.12) we are considering a fixed cell I^j, so that we can delete the super script j from I^j, $\mathcal{J}^j$, r^j, s^j, and G_ε^j for simplicity. We will prove (4.12) for the case in which s is less than or equal to 2ε and then for the case in which s is greater than 2ε.

<u>Case 1</u>. $s \leqq 2\varepsilon$. We can write

$$(4.13)\qquad \int_{E-\mathcal{I}^j} |G_\varepsilon| \leq \int_E |G_\varepsilon| = J_1 + J_2 ,$$

where $J_1 = \int_{t \geq t^j+2\varepsilon} |G_\varepsilon|$; $J_2 = \int_{t \leq t^j+2\varepsilon} |G_\varepsilon|$.

Now $(\xi,\tau) \in I^j$ implies $|\tau - t^j| \leq \frac{s}{2} \leq \varepsilon$, so for $t \geq t^j + 2\varepsilon$, then $t-\tau \geq t^j - \tau + 2\varepsilon \geq -\varepsilon + 2\varepsilon = \varepsilon$. Therefore,

$$J_1 = \int_{t^j+2\varepsilon}^{\infty} \int_{R^n} \left|\int_I [k(x-\xi,\ t-\tau)g(\xi,\tau)d\xi d\tau\right| dxdt.$$

Since $\int_I g = 0$ by (4.2),

$$(4.14)\qquad \begin{aligned} J_1 &= \int_{t^j+2\varepsilon}^{\infty} \int_{R^n} \left|\int_I [k(x-\xi,\ t-\tau) - k(x-x^j, t-t^j)]g(\xi,\tau)d\xi d\tau\right| dxdt \\ &\leq \int_I J_3(\xi,\tau)|g(\xi,\tau)|d\xi d\tau, \end{aligned}$$

where

$$(4.15)\qquad J_3(\xi,\tau) = \int_{t^j+2\varepsilon}^{\infty} \int_{R^n} |k(x-\xi,\ t-\tau) - k(x-x^j,\ t-t^j)|dxdt.$$

We can write $J_3 \leq J_4 + J_5$, where

$$J_4(\xi,\tau) = \int_{t^j+2\varepsilon} \int_{R^n} |k(x-\xi,\ t-\tau) - k(x-x^j,\ t-\tau)|dxdt,$$

$$J_5(\xi,\tau) = \int_{t^j+2\varepsilon} \int_{R^n} |k(x-x^j,\ t-\tau) - k(x-x^j,\ t-t^j)|dxdt.$$

Now by II,

$$J_4(\xi,\tau) = \int_{t^j+2\epsilon} \int_{R^n} \left| \frac{1}{(t-\tau)\varphi_1(t-\tau)\ldots\varphi_n(t-\tau)} \left[\Omega\left(\frac{x_1-\xi_1}{\varphi_1(t-\tau)},\ldots,\frac{x_n-\xi_n}{\varphi_n(t-\tau)}\right) - \Omega\left(\frac{x_1-x_1^j}{\varphi_1(t-\tau)},\ldots,\frac{x_n-x_n^j}{\varphi_n(t-\tau)}\right)\right]\right| dxdt.$$

Making the change of variables $y_i = \dfrac{x_i-\xi_i}{\varphi_i(t-\tau)}$ yields

$$J_4(\xi,\tau) = \int_{t^j+2\epsilon}^{\infty} \frac{dt}{t-\tau} \int_{R^n} |\Omega(y) - \Omega(y+z)|dy$$

where $z = \left(\dfrac{\xi_1-x_1^j}{\varphi_1(t-\tau)},\ldots,\dfrac{\xi_n-x_n^j}{\varphi_n(t-\tau)}\right)$. Then by VII,

$$J_4(\xi,\tau) \leq c \int_{t^j+2\epsilon}^{\infty} \frac{|z|}{t-\tau}dt$$

$$\leq c \sum_{i=1}^{n} \int_{t^j+2\epsilon}^{\infty} \frac{|\xi_i-x_i^j|}{\varphi_i(t-\tau)} \frac{dt}{t-\tau}$$

$$\leq c \sum_{i=1}^{n} |\xi_i-x_i^j| \int_{t^j-\tau+2\epsilon}^{\infty} \frac{d\sigma}{\sigma\varphi_i(\sigma)} .$$

But $|\xi_i-x_i^j| \leq cr_i$ and by IV,

$$J_4(\xi,\tau) \leq c \sum_{i=1}^{n} r_i \int_{t^j-\tau+2\epsilon}^{\infty} \frac{d\sigma}{\sigma\varphi_i(\sigma)} \leq c \sum_{i=1}^{n} \frac{r_i}{\varphi_i(t^j-\tau+2\epsilon)} .$$

Now $t^j-\tau+2\epsilon \geqq \epsilon \geqq \frac{s}{2}$; hence

$$J_4(\xi,\tau) \leqq c\sum_{i=1}^{n} \frac{r_i}{\varphi_i(\frac{s}{2})} \leqq c\sum_{i=1}^{n} \frac{r_i}{\varphi_i(s)} .$$

From lemma 1 we know $1 \leqq \frac{r_i}{\varphi_i(s)} \leqq 2$, so $J_4(\xi,\tau) \leqq c$. Now if we can show that $J_5(\xi,\tau) \leqq c$, then we have $J_3(\xi,\tau) \leqq c$ which by (4.14) implies $J_1 \leqq c\int_I |g|$.

By II,

$$J_5(\xi,\tau) = \int_{t^j+2\epsilon}^{\infty} \int_{R^n} \Big| \frac{1}{(t-\tau)\varphi_1(t-\tau)\dots\varphi_n(t-\tau)} \Omega\Big(\frac{x_i-x_1^j}{\varphi_1(t-t^j)},\dots,\frac{x_n-x_n^j}{\varphi_n(t-t^j)}\Big)$$

$$- \frac{1}{(t-t^j)\varphi_1(t-t^j)\dots\varphi_n(t-t^j)} \Omega\Big(\frac{x_1-x_1^j}{\varphi_1(t-t^j)},\dots,\frac{x_n-x_n^j}{\varphi_n(t-t^j)}\Big) \Big| dxdt$$

$$\leqq J_6(\xi,\tau) + J_7(\xi,\tau),$$

where

$$J_6(\xi,\tau) = \int_{t^j+2\epsilon}^{\infty} \int_{R^n} \frac{1}{(t-\tau)\varphi_1(t-\tau)\dots\varphi_n(t-\tau)} \Big| \Omega\Big(\frac{x_1-x_1^j}{\varphi_1(t-\tau)},\dots,\frac{x_n-x_n^j}{\varphi_n(t-\tau)}\Big)$$

$$- \Omega\Big(\frac{x-x_1^j}{\varphi_1(t-t^j)},\dots,\frac{x_n-x_n^j}{\varphi_n(t-t^j)}\Big) \Big| dxdt,$$

$$J_7(\xi,\tau) = \int_{t^j+2\epsilon}^{\infty} \int_{R^n} \Big| \frac{1}{(t-\tau)\varphi_1(t-\tau)\dots\varphi_n(t-\tau)} - \frac{1}{(t-t^j)\varphi_1(t-t^j)\dots\varphi_n(t-t^j)} \Big|$$

$$\Big| \Omega\Big(\frac{x_1-x_1^j}{\varphi_1(t-t^j)},\dots,\frac{x_n-x_n^j}{\varphi_n(t-t^j)}\Big) \Big| dxdt.$$

In J_6 let $y_i = \frac{x_i - x_i^j}{\varphi_i(t-\tau)}$ and in J_7 let $y_i = \frac{x_i - x_i^j}{\varphi_i(t-t^j)}$. Then

$$J_6(\xi,\tau) = \int_{t^j+2\varepsilon}^{\infty} \frac{dt}{t-\tau} \int_{R^n} |\Omega(y) - \Omega(y_1\frac{\varphi_1(t-\tau)}{\varphi_1(t-t^j)},\ldots,y_n\frac{\varphi_n(t-\tau)}{\varphi_n(t-t^j)})|dy,$$

$$J_7(\xi,\tau) = \int_{t^j+2\varepsilon}^{\infty} \int_{R^n} \frac{1}{t-t^j}\left|\frac{\varphi_1(t-t^j)\ldots\varphi_n(t-t^j)}{\varphi_1(t-\tau)\ldots\varphi_n(t-\tau)} \cdot \frac{t-t^j}{t-\tau} -1\right|dt\int_{R^n} |\Omega(y)|dy.$$

Now for $(\xi,\tau)\in I$ and $t \geq t^j + 2\varepsilon$, we have

$$\left|\frac{t-\tau}{t-t^j} -1\right| = \frac{|\tau-t^j|}{t-t^j} \leq \frac{\frac{1}{2}s}{2\varepsilon} \leq \tfrac{1}{2};$$

hence, $\frac{1}{2} \leq \frac{t-\tau}{t-t^j} \leq \frac{3}{2}$. Therefore, it follows from V that the ratio $\frac{\varphi_i(t-\tau)}{\varphi_i(t-t^j)}$ is bounded above and below by positive constants depending only on the functions φ_i. Therefore VIII implies

(4.16)

$$\int_{R^n} |\Omega(y) - \Omega(y_1\frac{\varphi_1(t-\tau)}{\varphi_1(t-t^j)},\ldots,y_n\frac{\varphi_n(t-\tau)}{\varphi_n(t-t^j)})|dy \leq c \sum_{i=1}^{n} \left|\frac{\varphi_1(t-\tau)}{\varphi_i(t-t^j)} -1\right|.$$

Condition VI implies $\left|\frac{\varphi_i(t-\tau)}{\varphi_i(t-t^j)} -1\right| \leq c \frac{|\tau-t^j|}{t-t^j}$, and combining this with (4.16) we have

$$(4.17) \qquad J_6(\xi,\tau) \leq c \int_{t^j+2\varepsilon}^{\infty} \frac{|\tau-t^j|}{(t-t^j)} \cdot \frac{1}{t-\tau}dt$$

$$\leq c|\tau-t^j| \int_{t^j+2\varepsilon}^{\infty} \frac{dt}{(t-t^j-\frac{1}{2}s)^2}$$

$$= c\,\frac{|\tau-t^j|}{2\varepsilon-\frac{1}{2}s} \leq \frac{c\frac{1}{2}s}{s-\frac{1}{2}s} = c.$$

Using the inequality

$$|a_1\ldots a_n-1| \leq c\sum_{i=1}^{n}|a_i-1|,\text{ where } 0 < a_i \leq c_i,$$

we have

$$\left|\frac{\varphi_1(t-t^j)\ldots\varphi_n(t-t^j)}{\varphi_1(t-\tau)\ldots\varphi_n(t-\tau)}\cdot\frac{t-t^j}{t-\tau}-1\right|$$

$$\leq \frac{t-t^j}{t-\tau}\left|\frac{\varphi_1(t-t^j)\ldots\varphi_n(t-t^j)}{\varphi_1(t-\tau)\ldots\varphi_n(t-\tau)}-1\right| + \left|\frac{t-t^j}{t-\tau}-1\right|$$

$$\leq c\sum_{i=1}^{n}\left|\frac{\varphi_i(t-t^j)}{\varphi_i(t-\tau)}-1\right| + \left|\frac{\tau-t^j}{t-\tau}\right|$$

$$\leq c\sum_{i=1}^{n}\frac{|\tau-t^j|}{t-\tau} + \frac{|\tau-t^j|}{t-\tau}$$

$$= c\,\frac{|\tau-t^j|}{t-\tau}.$$

Therefore, this inequality and III imply

$$J_7(\xi,\tau) \leq c\int_{t^j+2\varepsilon}^{\infty}\frac{|\tau-t^j|}{(t-\tau)(t-\tau^j)}dt \leq c,\text{ by (4.17).}$$

We have now shown that $J_5(\xi,\tau) \leq c$ and hence

$$(4.18) \qquad J_1 \leq c\int_I |g(\xi,\tau)|d\xi d\tau .$$

Next we estimate J_2.

$$J_2 = \int_0^{t^j+2\varepsilon} \int_{R^n} |G_\varepsilon^j(x,t)|dxdt$$

$$\leq \int_0^{t^j+2\varepsilon} \int_{R^n} (\int_{I^j} |k_\varepsilon(x-\xi,\ t-\tau)||g(\xi,\tau)|d\xi d\tau)dxdt.$$

But

$$\int_0^{t^j+2\varepsilon} \int_{R^n} |k_\varepsilon(x-\xi,\ t-\tau)|dxdt = \int_{\tau+\varepsilon}^{t^j+2\varepsilon} \int_{R^n} |k(x-\xi,\ t-\tau)|dxdt$$

$$= \int_{\tau+\varepsilon}^{t^j+2\varepsilon} \frac{dt}{t-\tau} \int_{R^n} |\Omega(x)|dx$$

$$\leq c \log \frac{t^j+2\varepsilon-\tau}{\varepsilon} \leq c \log \frac{\frac{1}{2}s+2\varepsilon}{\varepsilon}$$

$$\leq c \log 3 = c.$$

Hence, $J_2 \leq c \int_I |g(\xi,\tau)|d\xi d\tau$ and combining this estimate with (4.18) and using (4.13) gives

$$\int_{E-\mathcal{J}} |G_\varepsilon| \leq c \int_I |g|.$$

We have now shown that (4.12) holds for all cells I^j satisfying $s^j \leq 2\varepsilon$. We now consider the other cells.

Case 2. $s > 2\varepsilon$.

Now $\int_I g = 0$ and the definition of G_ε implies

$$(4.19)\qquad \int_{E-\mathcal{S}} |G_\varepsilon| = \int_{E-\mathcal{S}} \left|\int_I k_\varepsilon(x-\xi,\ t-\tau)g(\xi,\tau)d\xi d\tau\right|dxdt$$

$$= \int_{\substack{E-\mathcal{S}\\ t\leq t^j+s}} \left|\int_I k_\varepsilon(x-\xi,t-\tau)g(\xi,\tau)d\xi d\tau\right|dxdt$$

$$+ \int_{\substack{E-\mathcal{S}\\ t>t^j+s}} \left|\int_I [k_\varepsilon(x-\xi,t-\tau)-k_\varepsilon(x-x^j,t-t^j)]g(\xi,\tau)d\xi d\tau\right|dxdt$$

$$\leq \int_I [J_8(\xi,\tau) + J_9(\xi,\tau)]|g(\xi,\tau)|d\xi d\tau\ ,$$

where

$$J_8(\xi,\tau) = \int_{\substack{E-\mathcal{S}\\ t\leq t^j+s}} |k_\varepsilon(x-\xi,\ t-\tau)|dxdt,$$

$$J_9(\xi,\tau) = \int_{\substack{E-\mathcal{S}\\ t>t^j+s}} |k_\varepsilon(x-\xi,t-\tau)-k_\varepsilon(x-x^j,t-t^j)|dxdt\ .$$

We can show J_8 and J_9 are bounded by positive constants. Consider J_8 first. Since $k_\varepsilon(x-\xi,\ t-\tau) = 0$ for $t < \tau + \varepsilon$, and since $(\xi,\tau)\in I$ implies $\tau + \varepsilon \leq t^j + \frac{s}{2} + \varepsilon < t^j + s$,

$$J_8(\xi,\tau) = \int_{\tau+\varepsilon}^{t^j+s} \int_{\substack{R^n\\ (x,t)\notin\mathcal{S}}} |k(x-\xi,\ t-\tau)|dxdt.$$

Now $(x,t) \notin \mathcal{J}$ implies $|x_k - x_k^j| > r_k^j$ for some k, $1 \leq k \leq n$. But $|\xi_k - x_k^j| \leq \frac{1}{2} r_k^j$ implies

$$|x_k - \xi_k| \geq |x_k - x_k^j| - |\xi_k - x_k^j| > r_k^j - \tfrac{1}{2} r_k^j = \tfrac{1}{2} r_k^j \ ;$$

hence,

$$J_8(\xi,\tau) \leq \int_{\tau+\varepsilon}^{t^j+s} \sum_{k=1}^{n} \int_{|x_k-\xi_k|>\frac{1}{2}r_k^j} |k(x-\xi,\ t-\tau)|\,dxdt.$$

In each of these n spatial integrals make the change of variable $x_i - \xi_i = \varphi_i(t-\tau)y_i$, and replace $t-\tau$ by t to obtain

$$J_8(\xi,\tau) \leq \sum_{k=1}^{n} \int_{\varepsilon}^{t^j-\tau+s} \frac{dt}{t} \int_{|y_k|>r_k^j/2\varphi_k(t)} |\Omega(y)|\,dy$$

$$\leq \sum_{k=1}^{n} \int_{0}^{\frac{3}{2}s} \frac{dt}{t} \int_{|y|>r_k^j/2\varphi_k(t)} |\Omega(y)|\,dy$$

$$\leq \sum_{k=1}^{n} \int_{0}^{\frac{3}{2}s} \frac{1}{t} \frac{c\varphi_k(t)}{r_k^j} dt$$

by IX. Then, using IV, V, and lemma 1,

$$J_8(\xi,\tau) \leq c \sum_{k=1}^{n} \frac{\varphi_k(\frac{3}{2}s)}{r_k^j} \leq c.$$

It remains to show J_9 is bounded; but since

$$t-\tau = (t-t^j) + (t^j-\tau) \geq s - \tfrac{1}{2}s = \tfrac{1}{2}s > \varepsilon$$

clearly

$$(4.20)\qquad J_9(\xi,\tau) \leqq \int_{t^j+s}^{\infty}\int_{R^n} |k(x-\xi,\ t-\tau) - k(x-x^j,\ t-t^j)|dxdt.$$

This inequality is the same as the estimate on $J_3(\xi,\tau)$, (4.15), with 2ε replaced by s. The fact that $J_9(\xi,\tau) \leqq c$ can be proved in exactly the same way as the estimate $J_3(\xi,\tau) \leqq c$ was proved. Thus $J_9(\xi,\tau) \leqq c$, which proves (4.12):

$$\int_{E-\mathcal{J}} |G_\varepsilon| \leqq c\int_I |g(\xi,\tau)|d\xi d\tau,$$

which completes the proof for all cells in the sequence I^j.
Q.E.D.

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