RICE UNIVERSITY

Estimates for Singular Integrals

by

Michael Erle Lord

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
MASTER OF ARTS

Thesis Director's Signature:

Houston, Texas
August, 1964
ABSTRACT

In a paper by Calderón and Zygmund [1] some properties of a certain kind of singular integral are established, and these results are applied to particular fundamental solutions arising in partial differential equations. Jones has considered another class of singular integrals [2] which has application to fundamental solutions of the heat equation. More generally the kernels considered by Jones arise from parabolic differential equations with constant coefficient. This thesis considers a kernel which is a generalization of the kernel treated by Jones, and it has mean value and homogeneity properties analogous to those in Jones's paper [2].
ACKNOWLEDGEMENT

The author is indebted to Professor B. Frank Jones, Jr., for posing the problem considered in this thesis and for his aid in the analysis.
1. **Introduction**

We first define the truncation of a kernel $k(x,t)$ as

$$k_{\varepsilon}(x,t) = \begin{cases} k(x,t) & t \geq \varepsilon, \\ 0 & t < \varepsilon, \end{cases}$$

where $x$ is a point in $n$-dimensional Euclidean space and $t > 0$.

In [2] Jones makes the following assumptions about the kernel $k(x,t)$:

$$k(x,t) = \frac{1}{t\phi(t)^n} \Omega\left(\frac{x}{\phi(t)}\right), \quad t > 0,$$

where $\Omega(x)$ is a measurable complex-valued function on $\mathbb{R}^n$ such that

$$\int_{\mathbb{R}^n} (1+|x|) |\Omega(x)| \, dx < \infty$$

and

$$\int_{\mathbb{R}^n} \Omega(x) \, dx = 0,$$

and $\phi(t)$ is a non-decreasing positive continuous function for $0 < t < \infty$ such that $\lim_{t \to 0} \phi(t) = 0$, $\lim_{t \to \infty} \phi(t) = \infty$.

With certain smoothness properties for $\phi(t)$ and some assumptions on $\Omega(x)$ Jones was able to prove the following theorem:

Let $1 < p < \infty$, and let $f \in L_p(E)$, where $E = \mathbb{R}^n \times (0, \infty)$. Then $f_\varepsilon$, defined by

$$f_\varepsilon(x,t) = \int_E k_\varepsilon(x-\xi, t-\tau) f(\xi, \tau) \, d\xi \, d\tau,$$
belongs to $L^p(E)$, and there exists a constant $A_p$, depending only on $p$ and the kernel $k$, such that $\|f_\varepsilon\|_p \leq A_p\|f\|_p$.

Under the above assumptions it is also proved that $f_\varepsilon$ converges in $L^p(E)$ as $\varepsilon \to 0$.

In this paper we consider a more general kernel $k(x,t)$. Again assuming $\Omega(x)$ is a measurable complex-valued function on $\mathbb{R}^n$, we let

$$k(x,t) = \frac{1}{t \phi_1(t) \cdots \phi_n(t)} \Omega \left( \frac{x_1}{\phi_1(t)}, \ldots, \frac{x_n}{\phi_n(t)} \right)$$

where the assumptions on $\phi_i(t)$, $1 \leq i \leq n$, and the assumptions on $\Omega(x)$ are analogous to those considered by Jones. A complete description of $k(x,t)$ is given in the next section.

The task of this paper is to prove the analog of the main lemma, lemma 4 of [2], using the generalized kernel $k(x,t)$. Once this lemma is established we immediately have the theorem previously stated concerning the bound on the $L^p$ norm of $f_\varepsilon$. The proof may be found in [1].
2. **Description of the Singular Integrals**

We describe a class of kernels with the following assumptions:

I. \( \varphi_i(t) \) \( 1 \leq i \leq n \) is a non-decreasing positive continuous function for \( 0 < t < \infty \) such that
\[
\lim_{t \to 0} \varphi_i(t) = 0, \quad \lim_{t \to \infty} \varphi_i(t) = \infty;
\]

II. \( \Omega(x) \) is a measurable complex-valued function on \( \mathbb{R}^n \) and
\[
k(x,t) = \frac{1}{t^{n-1}} \Omega \left( \frac{x_1}{\varphi_1(t)}, \ldots, \frac{x_n}{\varphi_n(t)} \right), \quad t > 0;
\]

III. \( \int_{\mathbb{R}^n} (1+|x|)|\Omega(x)|dx < \infty \) and \( \int_{\mathbb{R}^n} \Omega(x)dx = 0. \)

The homogeneity of \( k(x,t) \) and mean value properties of \( \Omega(x) \) are given by conditions II and III respectively.

We assume the following additional properties on \( \varphi_i(t) \), \( 1 \leq i \leq n \). There exists a positive constant \( c \) such that

IV. \( \int_{0}^{a} \frac{\varphi_i(t)}{t} dt \leq c \varphi_i(a) , \quad 0 < a < \infty , \)

\[
\int_{0}^{a} \frac{1}{t \varphi_i(t)} dt \leq \frac{c}{\varphi_i(a)} , \quad 0 < a < \infty ;
\]

V. \( \frac{\varphi(2t)}{\varphi(t)} \leq c , \quad 0 < t < \infty ; \)

VI. For \( 0 < t < \infty \) and \( |\delta| \leq \frac{1}{2} \)
\[
\left| \frac{\varphi_i((1+\delta)t)}{\varphi_i(t)} - 1 \right| \leq c |\delta| .
\]
We specify that for some positive $c$, $\Omega(x)$ satisfies the following:

VII. $\int_{\mathbb{R}^n} |\Omega(x) - \Omega(x+y)| \, dx \leq c |y|$, $y \in \mathbb{R}^n$;

VIII. $\int_{\mathbb{R}^n} |\Omega(x) - \Omega((1+\delta_1)x_1, \ldots, (1+\delta_n)x_n)| \, dx \leq c \sum_{i=1}^{n} |\delta_i|$

for $|\delta_i| \leq \frac{1}{2}$, $1 \leq i \leq n$;

IX. $\int_{|x| \geq a} |\Omega(x)| \, dx \leq ca^{-1}$ for $a > 0$.

The above description of the kernel $k(x,t)$ is a generalization of the kernel considered by Jones [2].

We note that the notation $c$ shall stand for a (large) positive generic constant, which depends only on the kernel $k$ and the dimension $n$ of the space.
3. A method of partitioning space

In the following discussion we shall use the word "cell" to describe any set of the form

\[
\{(x,t)|x_i^0 \leq x_i \leq x_i^0 + r_i, \quad i = 1,2,\ldots,n; \quad t^0 \leq t \leq t^0 + s\}
\]

for positive \(r_i, s\) and any numbers \(x_i^0,\ldots,x_n^0, t^0\). Now given the functions \(\psi_1(t),\ldots,\psi_n(t)\) we want to partition the space \(\mathbb{R}^{n+1}\) into finer and finer cells such that at any step the ratio \(\frac{r_i}{\psi_i(s)}\) is bounded above and below by a positive constant for each \(i, 1 \leq i \leq n\).

Suppose initially that \(\mathbb{R}^{n+1}\) is partitioned into non-overlapping cells of dimensions \(\psi_1(a),\ldots,\psi_n(a)\), \(a\). In the \(t\)-direction divide each cell in half and let

\[
k_i^1 = \left\lfloor \frac{\psi_i(a)}{\psi_i(\frac{a}{2})} \right\rfloor, \quad i = 1, 2, \ldots, n,
\]

where for any positive number \(b\), \([b]\) is the largest integer not greater than \(b\). Now divide the \(i\)th side of each cell into \(k_i^1\) equal intervals (each of length \(\frac{\psi_i(a)}{k_i^1}\)). Each cell of dimensions \(\psi_1(a), \ldots, \psi_n(a)\), \(a\) is divided into \(2^n k_i^1\) cells of dimensions \(\frac{\psi_1(a)}{k_1^1}, \ldots, \frac{\psi_n(a)}{k_n^1}\), \(a^2\). This division is well defined if \(k_i^1 \geq 1\), and \(\psi_i(t)\) being non-decreasing assures us of this.

Similarly, let

\[
k_i^2 = \left\lfloor (k_i^1)^{^{-1}} \frac{\psi_i(a)}{\psi_i(2^{-2}a)} \right\rfloor
\]
and then each cell of dimensions $\frac{\varphi_1(a)}{k_1}, \ldots, \frac{\varphi_n(a)}{k_n}$, $\frac{a}{2}$ can be divided into $2^n\frac{k_i^2}{i=1}^n$ cells of dimensions $\frac{\varphi_i(a)}{k_1^2}, \ldots, \frac{\varphi_n(a)}{k_n^2}, \frac{a}{2}$.

Now it is easy to show $k_i^2 \equiv 1$, so this step is well defined.

We proceed by induction. We have the $(v-1)$th subdivision such that $k_i^1, \ldots, k_i^{v-1} \equiv 1$, $i = 1, 2, \ldots, n$ and each cell has dimensions $\frac{\varphi_1(a)}{k_1^{v-1}}, \ldots, \frac{\varphi_n(a)}{k_n^{v-1}}, \frac{a}{2^{v-1}}$. Now define

$$k_i^v = [(k_1^1 \ldots k_i^{v-1})^{-1} \varphi_i(a)/\varphi_i(2^{-v}a)] \quad i = 1, 2, \ldots, n.$$  

But

$$(k_1^1 \ldots k_i^{v-1})^{-1} \varphi_i(a)/\varphi_i(2^{-v}a)$$

$$= (k_i^{v-1})^{-1} [(k_1^1 \ldots k_i^{v-2})^{-1} \varphi_i(a)/\varphi_i(2^{-v+1}a)] \cdot [\varphi_i(2^{-v}a)/\varphi_i(2^{-v+1}a)]$$

$$\equiv (k_i^{v-1})^{-1} [(k_1^1 \ldots k_i^{v-1})^{-1} \varphi_i(a)/\varphi_i(2^{-v+1}a)](\varphi_i(2^{-v+1}a)/\varphi_i(2^{-v}a))$$

$$= \varphi_i(2^{-v+1}a)/\varphi_i(2^{-v}a) \equiv 1,$$

hence $k_i^v \equiv 1$, $i = 1, \ldots, n$, and we can now divide each cell of dimensions $\frac{\varphi_1(a)}{k_1^{v-1}}, \ldots, \frac{\varphi_n(a)}{k_n^{v-1}}, \frac{a}{2^{v}}$ into $2^n\frac{k_i^2}{i=1}^n$ cells of dimensions $\frac{\varphi_i(a)}{k_1^2}, \ldots, \frac{\varphi_n(a)}{k_n^2}, \frac{a}{2^{v+1}}$.

Note also that
\[
\frac{\varphi_i(a)}{k_1 \cdots k_i \varphi_i(2^{-\nu} a)} = (k_1^\nu - 1) \left( (k_1^1 \cdots k_i^1)^{-1} \frac{\varphi_i(a)}{\varphi(2^{-\nu} a)} \right)
\]

\[
< (k_1^\nu - 1) \left[ (k_1^1 \cdots k_i^1)^{-1} \frac{\varphi_i(a)}{\varphi(2^{-\nu} a)} \right] + 1
\]

\[
= (k_1^\nu) \left[ k_i^\nu + 1 \right] \leq 2.
\]

Now the expression on the left is just \( \frac{r_i}{\varphi_i(s)} \) for a cell of dimensions \( r_1, \ldots, r_n, s \), where \( r_i = (k_1^1 \cdots k_i^1) \varphi_i(a) \) and \( s = 2^{-\nu} a \). Therefore, any cell in the process has dimensions \( r_1, \ldots, r_n, s \) satisfying

\[
1 \leq \frac{r_i}{\varphi_i(s)} \leq 2.
\]

We can now find a constant bound on \( k_i^\nu \) which depends only on the function \( \varphi_i(t) \). We have

\[
k_i^{\nu+1} = [(k_1^1 \cdots k_i^1)^{-1} \varphi_i(a)/\varphi_i(2^{-\nu-1} a)]
\]

\[
= [(k_1^1 \cdots k_i^1)^{-1} \varphi_i(a)/\varphi_i(2^{-\nu} a) \cdot \varphi_i(2^{-\nu} a)/\varphi_i(2^{-\nu-1} a)]
\]

\[
\leq [2\varphi_i(2^{-\nu} a)/\varphi_i(2^{-\nu-1} a)] \leq [2c] = c,
\]

where use was made of condition V.

The following notation for the rearrangement of a function was used in Calderón and Zygmund [1] and Jones [2]. If \( f \in L_p(E) \), where \( E = \mathbb{R}^n \times (0, \infty) \), let \( f^*(\lambda) \) be a non-increasing function for \( 0 < \lambda < \infty \) which is equimeasurable with \( |f| \). Let

\[
\beta_f(\lambda) = \lambda^{-1} \int_0^\lambda f^*(\mu) d\mu \quad 0 < \lambda < \infty.
\]
We denote by $\beta_f^*(\lambda)$ a function inverse to $\beta_f(\lambda)$. The properties of $\beta_f(\lambda)$ and $\beta_f^*(\lambda)$ are given in [1], pp. 91-93.

Let $|G|$ denote the Lebesgue measure of a measurable subset $G$ of $\mathbb{R}^{n+1}$.

We have the following lemma analogous to Lemma 3 of Jones [2]:

**Lemma 1:** Let $f \in L^p(E)$ and let $\eta > 0$. Then there is a sequence of non-overlapping cells $I_j^n$ such that

1. *If* $I_j^n$ *has dimensions* $r_1, \ldots, r_n$, *then*
   
   $$1 \equiv \frac{r_i}{\phi_i(s)} \leq 2, \quad i = 1, 2, \ldots, n.$$  

2. $\eta \leq |I_j^n|^{-1} \int_{I_j^n} |f| \leq c\eta$;

3. $|f| \leq \eta$ a.e. outside $D_\eta = \bigcup_{j=1}^\infty I_j^n$;

4. $|D_\eta| \leq \beta_f^*(\eta)$;

5. $\eta \leq |D_\eta|^{-1} \int_{D_\eta} |f| \leq c\eta$.

**Proof:** Since $\lim_{\lambda \to \infty} \beta_f(\lambda) = 0$, it follows that $\beta_f(\lambda) < \eta$ for sufficiently large $\lambda$. Now partition $\mathbb{R}^{n+1}$ into cells of dimension $\phi_1(a), \ldots, \phi_n(a)$, a such that $\phi_1(a) \ldots \phi_n(a)a \equiv \lambda$.

Over any such cell $I$

$$\frac{1}{|I|} \int_{I} |f| \equiv \frac{1}{|I|} \int_{0}^{\infty} f^*(\mu)d\mu = \beta_f(|I|)$$

$$= \beta_f(\phi_1(a) \ldots \phi_n(a)a) \leq \beta_f(\lambda) < \eta.$$
Now partition these cells as described above into cells of dimension \( \frac{\varphi_1(a)}{k_1}, \ldots, \frac{\varphi_n(a)}{k_n}, \frac{a}{2} \) and select for the sequence \( I^j \) those cells over which the average of \(|f|\) is at least \( \eta \).

Next partition the remaining cells into cells of dimension \( \frac{\varphi_1(a)}{k_1k_2}, \ldots, \frac{\varphi_n(a)}{k_nk_2}, \frac{a}{2} \) and select from these for the sequence \( I^j \) those cells where the average of \(|f|\) is at least \( \eta \). We can continue this. Then we have

\[
\eta \leq |I^j|^{-1} \int_{I^j} |f|.
\]

Now \( I^j \) is contained in a cell \( I \) from the previous partition; therefore, \(|I| = 2k_1 \ldots k_n|I^j|\). Since \( k_1 \leq c \) we have \(|I| \leq c|I^j|\). Then

\[
\frac{1}{|I^j|} \int_{I^j} |f| \leq \frac{c}{|I|} \int_{I^j} |f| \leq \frac{c}{|I|} \int_{I^j} |f| \leq c \eta,
\]

thus proving (2). Condition (1) follows from (3.1). The remaining assertions follow exactly as in [2]. Q.E.D.
4. The main lemma

We now establish the same lemma as Jones [2], lemma 4. When this is done we will then have the results obtained by Jones that the transformation taking $f$ into $f_\varepsilon$ represents a bounded mapping of $L_p$ into $L_p$. The proof of the following is quite similar to the proof given by Jones.

**Lemma 2**: Let $f \in L_p(E)$ for some $p$, $1 \leq p \leq 2$ and let $E_\eta = \{ (x,t) \mid |f_\varepsilon(x,t)| > \eta \}$, $\eta > 0$. Let $|f|^\wedge \eta = \min (|f|, \eta)$. Then $\left| E_\eta \right| \leq c_\eta^{-2} \int_E (|f|^\wedge \eta)^2 \, dx \, dt + c_\eta^2(\eta)$.

**Proof**: We use the sequence of cells $I_j$ as guaranteed by the previous lemma. Let $\varepsilon, \eta$ be fixed positive numbers.

Define

\[
(4.1) \quad h(x,t) = \begin{cases} |I_j|^{-1} \int_{I_j} f, & (x,t) \in I_j \\ f(x,t), & (x,t) \notin D_\eta = \bigcup_{j=1}^\infty I_j \end{cases}
\]

and let $g = f - h$. Then

\[
(4.2) \quad \int_{I_j} g = 0,
\]

$g \equiv 0$ outside $D_\eta$.

Let $E_1 = \{ (x,t) \mid |h_\varepsilon(x,t)| \leq \frac{\eta}{2} \},$ $E_2 = \{ (x,t) \mid |g_\varepsilon(x,t)| \leq \frac{\eta}{2} \}.$

Then $E_\eta \subset E_1 \cup E_2$, since $f_\varepsilon = g_\varepsilon + h_\varepsilon$. (Clearly, both $g$ and $h$ are in $L_p(E)$, and $g_\varepsilon, h_\varepsilon$ exist a.e. by Theorem 1 of [2].) Now
\[ \int (|f| \wedge \eta)^2 \leq \eta^2 \int |f|^2 + \int |f| > \eta \]

so that \(|f| \wedge \eta\) is in \(L_2(E)\). For \((x,t) \not\in D_\eta\), \(|f| \leq \eta\) a.e. by lemma 1, so that \(|h| = |f| = |f| \wedge \eta\) a.e. outside \(D_\eta\) by (4.1).

For \((x,t) \in D_\eta\), \(|h| \leq |I^j|^{-1} \int |f| \leq c \eta\) by lemma 1. Therefore,

\[ \int |h|^2 \leq \int |h|^2 + \int |h|^2 \leq c^2 \eta^2 |D_\eta| + \int (|f| \wedge \eta)^2, \]

and \(h \in L_2(E)\). Hence, by Theorem 2 of [2] we have that

\[ \int |h| \geq c \int |h|^2. \]

Thus, \(c \int |h| \geq \int |h| \geq \frac{1}{2} \eta^2 |E_1|\), and hence

\[ |E_1| \leq c \eta^{-2} \int (|f| \wedge \eta)^2 + cs^f(\eta). \]

Since \(|E_\eta| \leq |E_1| + |E_2|\), we will be done if we can prove (4.4) with \(E_1\) replaced by \(E_2\).

Let \(I^j\) be the cell with center \((x^j, t^j)\) and dimensions \(r_1^j, \ldots, r_n^j, s^j\); i.e.

\[ I^j = \{(x,t) \mid |x-x^j| \leq \frac{1}{2} r_1^j, 1 \leq i \leq n, |t-t^j| \leq s^j\}. \]

We consider now the cell, say \(s^j\), with center \((x^j, t^j)\) and dimension in each direction twice that of \(I^j\); i.e.

\[ s^j = \{(x,t) \mid |x-x^j| \leq r_1^j, 1 \leq i \leq n, |t-t^j| \leq s^j\}. \]
Then clearly $I^j \subseteq \delta^j$ and $|\delta^j| \leq c|I^j|$. Let $\delta_\eta = \bigcup_j \delta^j$; then $D_\eta \subseteq \delta_\eta$ and $|\delta_\eta| \leq c|D_\eta|$.

Suppose for the moment that we have proved the inequality

\[(4.7) \quad \int_{E-\delta_\eta} |g_\epsilon| \leq c \int_E |g| = c \int_{D_\eta} |g|.
\]

Then, since

\[\int_{I^j} |h| = \int_{I^j} |f| \leq \int_{I^j} |f|, \]

it follows that

\[\int_{I^j} |g| \leq \int_{I^j} (|f| + |h|) \leq 2 \int_{I^j} |f| ;\]

hence (4.7) implies that

\[(4.8) \quad \int_{E-\delta_\eta} |g_\epsilon| \leq c \sum_j \int_{I^j} |g| \leq c \sum_j 2 \int_{I^j} |f| = 2c \int_{D_\eta} |f|.
\]

Now (4.8) together with lemma 1 implies

\[(4.9) \quad \int_{E-\delta_\eta} |g_\epsilon| \leq c \eta |D_\eta|.
\]

But

\[(4.10) \quad |E_2| \leq |\delta_\eta| + |E_2-\delta_\eta| \leq c |D_\eta| + |E_2-\delta_\eta|.
\]

On $E_2$, $|g_\epsilon| \leq \frac{1}{2} \eta$; therefore, by (4.9)

\[c \eta |D_\eta| \geq \int_{E_2-\delta_\eta} \frac{1}{2} \eta = \frac{1}{2} \eta |E_2-\delta_\eta|.
\]

Hence (4.10) implies $|E_2| \leq c |D_\eta| + c |D_\eta| \leq c |D_\eta|$. Then by
lemma 1, \(|E_2| \leq c_\beta^E(\eta)|, and this together with (4.4) proves the lemma. Therefore, it suffices to prove (4.7).

By Theorem 1 of [2], \(g \in L_\eta(E)\) and \(g_\varepsilon(x, t)\) exists a.e. Since \(g = 0\) outside \(D_\eta,\)

\[
g_\varepsilon(x, t) = \int_{D_\eta} k_\varepsilon(x - \xi, t - \tau)g(\xi, \tau)d\xi d\tau
\]

\[
= \sum_j \int_{I_j} k_\varepsilon(x - \xi, t - \tau)g(\xi, \tau)d\xi d\tau.
\]

Let

\[
(4.11) \quad G_j^\varepsilon(x, t) = \int_{I_j} k_\varepsilon(x - \xi, t - \tau)g(\xi, \tau)d\xi d\tau,
\]

so that \(g_\varepsilon = \sum_j G_j^\varepsilon\) a.e. Now \(E - \delta \subset E - \delta_j\) implies

\[
\int_{E - \delta_j^\varepsilon} |g_\varepsilon| \leq \sum_j \int_{E - \delta_j^\varepsilon} |G_j^\varepsilon| \leq \sum_j \int_{E - \delta_j^\varepsilon} |G_j^\varepsilon|.
\]

Therefore, if we can demonstrate the inequality

\[
(4.12) \quad \int_{E - \delta_j^\varepsilon} |G_j^\varepsilon| \leq c \int_{I_j} |g|,
\]

then (4.7) and hence the lemma will follow.

In proving (4.12) we are considering a fixed cell \(I_j^j\), so that we can delete the super script \(j\) from \(I_j^j, \delta_j, r_j, s_j,\) and \(G_j^\varepsilon\) for simplicity. We will prove (4.12) for the case in which \(s\) is less than or equal to \(2\varepsilon\) and then for the case in which \(s\) is greater than \(2\varepsilon\).

Case 1. \(s \leq 2\varepsilon\). We can write
\[ (4.13) \quad \int_{E-\delta} |G_\varepsilon| \equiv \int_{E} |G_\varepsilon| = J_1 + J_2, \]

where \( J_1 = \int \int_{t \equiv t^j + 2\varepsilon} |G_\varepsilon|; \quad J_2 = \int \int_{t \equiv t^j + 2\varepsilon} |G_\varepsilon| \).

Now \((\xi, \tau) \in I^j\) implies \(|\tau - t^j| \leq \frac{s}{2} \leq \varepsilon\), so for \( t \geq t^j + 2\varepsilon\),
then \( t - \tau \equiv t^j - \tau + 2\varepsilon \equiv -\varepsilon + 2\varepsilon = \varepsilon\). Therefore,
\[
J_1 = \int \int_{R^n} I \int_{t^j + 2\varepsilon} ^\infty \left| \int_{R^n} \int [k(x - \xi, t - \tau) - k(x - x^j, t - t^j)]g(\xi, \tau)d\xi d\tau \right| dx dt.
\]

Since \( \int g = 0 \) by (4.2),
\[
J_1 = \int \int_{R^n} I \int_{t^j + 2\varepsilon} ^\infty \left| \int_{R^n} \int [k(x - \xi, t - \tau) - k(x - x^j, t - t^j)]g(\xi, \tau)d\xi d\tau \right| dx dt
\]
\[ (4.14) \]
\[
\equiv \int \int_{R^n} I \int_{t^j + 2\varepsilon} ^\infty J_3(\xi, \tau)|g(\xi, \tau)|d\xi d\tau,
\]

where \( J_3(\xi, \tau) = \int \int_{R^n} \int_{t^j + 2\varepsilon} ^\infty [k(x - \xi, t - \tau) - k(x - x^j, t - t^j)]dx dt. \)

We can write \( J_3 \equiv J_4 + J_5 \), where
\[
J_4(\xi, \tau) = \int \int_{R^n} \int_{t^j + 2\varepsilon} ^\infty [k(x - \xi, t - \tau) - k(x - x^j, t - t^j)]dx dt,
\]
\[
J_5(\xi, \tau) = \int \int_{R^n} \int_{t^j + 2\varepsilon} ^\infty [k(x - x^j, t - \tau) - k(x - x^j, t - t^j)]dx dt.
\]
Now by II,

\[ J_4(\xi, \tau) = \int \int_{t_j+2\epsilon} \frac{1}{R^n (t-\tau) \omega_1(t-\tau) \ldots \omega_n(t-\tau)} \left[ \Omega(\frac{x_1-\xi}{\omega_1(t-\tau)}, \ldots, \frac{x_n-\xi}{\omega_n(t-\tau)}) \right] dx dt \]

Making the change of variables \( y_i = \frac{x_i-\xi}{\omega_i(t-\tau)} \) yields

\[ J_4(\xi, \tau) = \int_{t_j+2\epsilon}^\infty \frac{dt}{t-\tau} \int_{R^n} |\Omega(y) - \Omega(y+z)| dy \]

where \( z = \left( \frac{\xi_1-x_1}{\omega_1(t-\tau)}, \ldots, \frac{\xi_n-x_n}{\omega_n(t-\tau)} \right) \). Then by VII,

\[ J_4(\xi, \tau) \leq c \int_{t_j+2\epsilon}^\infty \frac{|z|}{t-\tau} dt \]

\[ \leq c \sum_{i=1}^n \int_{t_j+2\epsilon}^\infty \frac{|\xi_i-x_i|}{\omega_i(t-\tau)} \frac{dt}{t-\tau} \leq c \sum_{i=1}^n \int_{t_j-\tau+2\epsilon}^\infty \frac{d\sigma}{\sigma \omega_i(\sigma)} \cdot \]

But \( |\xi_i-x_i| \leq cr_i \) and by IV,

\[ J_4(\xi, \tau) \leq c \sum_{i=1}^n r_i \int_{t_j-\tau+2\epsilon}^\infty \frac{d\sigma}{\sigma \omega_i(\sigma)} \leq c \sum_{i=1}^n \frac{r_i}{\omega_i(t_j-\tau+2\epsilon)} \]
Now $t^{j+2\epsilon} \leq \epsilon \leq \frac{s}{2}$; hence

$$J_4(\xi, \tau) \leq c n \sum_{i=1}^{\frac{r_1}{\phi_1(s)}} \leq c n \sum_{i=1}^{\frac{r_1}{\phi_1(s)}}.$$

From lemma 1 we know $1 \leq \frac{r_1}{\phi_1(s)} \leq 2$, so $J_4(\xi, \tau) \leq c$. Now if we can show that $J_5(\xi, \tau) \leq c$, then we have $J_3(\xi, \tau) \leq c$ which by (4.14) implies $J_1 \leq c \|g\|$.

By II,

$$J_5(\xi, \tau) = \int_{t^{j+2\epsilon}}^{\infty} \int_{R^n} \frac{1}{(t-t^j)\phi_1(t-t^j)\ldots\phi_n(t-t^j)} \Omega(\frac{x_1-x^j_1}{\phi_1(t-t^j)}, \ldots, \frac{x_n-x^j_n}{\phi_n(t-t^j)}) |dxdt,$$

where

$$J_6(\xi, \tau) = \int_{t^{j+2\epsilon}}^{\infty} \int_{R^n} \frac{1}{(t-t^j)\phi_1(t-t^j)\ldots\phi_n(t-t^j)} \Omega(\frac{x_1-x^j_1}{\phi_1(t-t^j)}, \ldots, \frac{x_n-x^j_n}{\phi_n(t-t^j)}) |dxdt,$$

and

$$J_7(\xi, \tau) = \int_{t^{j+2\epsilon}}^{\infty} \int_{R^n} \frac{1}{(t-t^j)\phi_1(t-t^j)\ldots\phi_n(t-t^j)} \Omega(\frac{x_1-x^j_1}{\phi_1(t-t^j)}, \ldots, \frac{x_n-x^j_n}{\phi_n(t-t^j)}) |dxdt.$$
In $J_6$ let $y_i = \frac{x_i - x_j}{\phi_i(t-\tau)}$ and in $J_7$ let $y_i = \frac{x_i - x_j}{\phi_i(t-t^j)}$. Then

\[ J_6(\xi, \tau) = \int_{t_j+2\epsilon}^{t_j+2\epsilon} \int_{R^n} |\Omega(y) - \Omega(y_1^{\phi_1(t-\tau)} \ldots y_n^{\phi_n(t-\tau)})| \, dy, \]

\[ J_7(\xi, \tau) = \int_{t_j+2\epsilon}^{t_j+2\epsilon} \int_{R^n} \frac{1}{\phi_1(t-\tau) \ldots \phi_n(t-\tau)} \cdot \frac{t - t^j}{t - \tau} \cdot |\Omega(y)| \, dy. \]

Now for $(\xi, \tau) \in I$ and $t \equiv t_j + 2\epsilon$, we have

\[ \left| \frac{t - \tau}{t - t^j} - 1 \right| = \left| \frac{t - t^j}{t - t^j} - 1 \right| \leq \frac{3\epsilon}{2\epsilon} \leq \frac{3}{2}; \]

hence, \( \frac{3}{2} \leq \frac{t - \tau}{t - t^j} \leq \frac{3}{2} \). Therefore, it follows from V that the ratio \( \frac{\phi_1(t-\tau)}{\phi_i(t-t^j)} \) is bounded above and below by positive constants depending only on the functions \( \phi_i \). Therefore VIII implies

\[ (4.16) \]

\[ \int_{R^n} |\Omega(y) - \Omega(y_1^{\phi_1(t-\tau)} \ldots y_n^{\phi_n(t-\tau)})| \, dy \leq c \sum_{i=1}^{n} \frac{\phi_1(t-\tau)}{\phi_i(t-t^j)} - 1. \]

Condition VI implies \( \left| \frac{\phi_i(t-\tau)}{\phi_i(t-t^j)} - 1 \right| \leq c \frac{|t - t^j|}{t - t^j} \), and combining this with (4.16) we have

\[ (4.17) \]

\[ J_6(\xi, \tau) \leq c \int_{t_j+2\epsilon}^{t_j+2\epsilon} \frac{|t - t^j|}{t - \tau} \cdot \frac{1}{t - \tau} \, dt. \]
Using the inequality

\[ |a_1 \ldots a_n - 1| \leq c \sum_{i=1}^{n} |a_i - 1|, \text{ where } 0 < a_i \leq c_i, \]

we have

\[
\frac{\varphi_1(t-t^j) \ldots \varphi_n(t-t^j)}{\varphi_1(t-\tau) \ldots \varphi_n(t-\tau)} \cdot \frac{t-t^j}{t-\tau} - 1
\]

\[
\leq \frac{t-t^j}{t-\tau} \cdot \frac{\varphi_1(t-t^j) \ldots \varphi_n(t-t^j)}{\varphi_1(t-\tau) \ldots \varphi_n(t-\tau)} - 1 + \left| \frac{t-t^j}{t-\tau} - 1 \right|
\]

\[
\leq c \sum_{i=1}^{n} \left| \frac{\varphi_i(t-t^j)}{\varphi_i(t-\tau)} - 1 \right| + \left| \frac{t-t^j}{t-\tau} - 1 \right|
\]

\[
= c \sum_{i=1}^{n} \left| \frac{t-t^j}{t-\tau} \right| + \left| \frac{t-t^j}{t-\tau} - 1 \right|
\]

Therefore, this inequality and III imply

\[
J_7(\xi, \tau) \leq c \int_{t^j+2\varepsilon}^{\infty} \frac{|t-t^j|}{(t-\tau)(t-t^j)} dt \leq c, \text{ by (4.17).}
\]
We have now shown that $J_5(\xi, \tau) \leq c$ and hence

(4.18) \quad J_1 \leq c \int_{I} |g(\xi, \tau)| \, d\xi \, d\tau.

Next we estimate $J_2$.

\begin{align*}
J_2 &= \int_{t^j}^{t^{j+2\epsilon}} \int_{\mathbb{R}^n} |G_\epsilon^j(x, t)| \, dx \, dt \\
&= \int_{t^j}^{t^{j+2\epsilon}} \int_{\mathbb{R}^n} |k_\epsilon(x-\xi, t-\tau)| \, |g(\xi, \tau)| \, d\xi \, d\tau \, dx \, dt.
\end{align*}

But

\begin{align*}
\int_{t^j}^{t^{j+2\epsilon}} \int_{\mathbb{R}^n} |k_\epsilon(x-\xi, t-\tau)| \, dx \, dt &= \int_{t^j}^{t^{j+2\epsilon}} \int_{\mathbb{R}^n} |k(x-\xi, t-\tau)| \, dx \, dt \\
&= \int_{t^j}^{t^{j+2\epsilon}} \int_{\mathbb{R}^n} \frac{dt}{t-\tau} |\Omega(x)| \, dx \\
&\leq c \log \frac{t^{j+2\epsilon} - \tau}{\epsilon} \leq c \log \frac{2s+2\epsilon}{\epsilon} \\
&\leq c \log 3 = c.
\end{align*}

Hence, $J_2 \leq c \int_{I} |g(\xi, \tau)| \, d\xi \, d\tau$ and combining this estimate with (4.18) and using (4.13) gives

\begin{align*}
\int_{E-J} |G_\epsilon| \leq c \int_{I} |g|.
\end{align*}

We have now shown that (4.12) holds for all cells $I^j$ satisfying $s^j \leq 2\epsilon$. We now consider the other cells.
Case 2. \( s > 2\varepsilon \).

Now \( \int g = 0 \) and the definition of \( G_\varepsilon \) implies

\[
\int |G_\varepsilon| = \int |\int k_\varepsilon(x-\xi, t-\tau)g(\xi, \tau)d\xi d\tau| \, dx \, dt
\]

\[
= \int |\int k_\varepsilon(x-\xi, t-\tau)g(\xi, \tau)d\xi d\tau| \, dx \, dt
\]

\[
E-J \quad I
\]

\[
t \geq t^j + s
\]

\[
\leq \int [J_8(\xi, \tau) + J_9(\xi, \tau)] |g(\xi, \tau)| \, d\xi d\tau,
\]

where

\[
J_8(\xi, \tau) = \int |k_\varepsilon(x-\xi, t-\tau)| \, dx \, dt,
\]

\[
E-J \quad t \leq t^j + s
\]

\[
J_9(\xi, \tau) = \int |k_\varepsilon(x-\xi, t-\tau) - k_\varepsilon(x-x^j, t-t^j)| \, dx \, dt.
\]

\[
E-J \quad t > t^j + s
\]

We can show \( J_8 \) and \( J_9 \) are bounded by positive constants.

Consider \( J_8 \) first. Since \( k_\varepsilon(x-\xi, t-\tau) = 0 \) for \( t < \tau + \varepsilon \), and since \((\xi, \tau) \in I\) implies \( \tau + \varepsilon \leq t^j + \frac{s}{2} + \varepsilon < t^j + s \),

\[
J_8(\xi, \tau) = \int_{\tau+\varepsilon}^{t^j+s} \int_{\mathbb{R}^n} |k(x-\xi, t-\tau)| \, dx \, dt.
\]

\( (x, t) \notin J \)
Now \((x,t) \notin S\) implies \(|x_k - x^j_k| > r^j_k\) for some \(k, l \leq k \leq n\). But \(|s_k - x^j_k| \leq \frac{1}{2} r^j_k\) implies

\(|x_k - s_k| \geq |x_k - x^j_k| - |s_k - x^j_k| > r^j_k - \frac{1}{2} r^j_k = \frac{1}{2} r^j_k;\)

hence,

\[
J_8(\xi, \tau) \equiv \sum_{k=1}^{n} \int_{t+\varepsilon}^{t+s} \int_{\tau+\varepsilon}^{\tau+\varepsilon} |k(x-\xi, t-\tau)| \, dx \, dt.
\]

In each of these \(n\) spatial integrals make the change of variable \(x_i - \xi_i = \phi_i(t-\tau)y_i\), and replace \(t-\tau\) by \(t\) to obtain

\[
J_8(\xi, \tau) \equiv \sum_{k=1}^{n} \frac{t^j_s - \tau + \varepsilon}{t} \int_{\tau+\varepsilon}^{\tau+\varepsilon} |\Omega(y)| \, dy \int_{0}^{t^j_s} |y_k| > r^j_k / 2\varphi_k(t)
\]

\[
\equiv \sum_{k=1}^{n} \frac{3}{2t} \int_{0}^{\varphi_k(t)} \frac{1}{t} \int_{0}^{\varphi_k(t)} |\Omega(y)| \, dy \int_{0}^{t^j_s} |y| > r^j_k / 2\varphi_k(t)
\]

by IX. Then, using IV, V, and lemma 1,

\[
J_8(\xi, \tau) \leq c \sum_{k=1}^{n} \frac{\varphi_k(\varphi_k(t))}{r^j_k} \leq c.
\]

It remains to show \(J_9\) is bounded; but since

\[
t-\tau = (t-t^j) + (t^j - \tau) \geq s - \frac{1}{2}s = \frac{1}{2}s > \varepsilon
\]
clearly

(4.20) \[ J^g(\xi,\tau) \leq \int_{t^j+s}^{\infty} \int_{\mathbb{R}^n} |k(x-\xi, t-\tau) - k(x-x^j, t-t^j)| \, dx \, dt. \]

This inequality is the same as the estimate on \( J^g_3(\xi,\tau) \), (4.15), with \( 2\varepsilon \) replaced by \( s \). The fact that \( J^g(\xi,\tau) \leq c \) can be proved in exactly the same way as the estimate \( J^g_3(\xi,\tau) \leq c \) was proved. Thus \( J^g(\xi,\tau) \leq c \), which proves (4.12):

\[ \int_{E-\delta} |G_\varepsilon| \leq c \int_{I} g(\xi,\tau) \, dx \, d\tau, \]

which completes the proof for all cells in the sequence \( I^j \).

Q.E.D.
REFERENCES
