RICE UNIVERSITY

ANALYSIS OF OPTICAL COLLIMATING SYSTEMS
FOR USE IN HIGH-ENERGY
ATOMIC-BEAM SPECTROSCOPY EXPERIMENTS

by

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ABSTRACT

Analysis of Optical Collimating Systems for Use in High-Energy Atomic-Beam Spectroscopy Experiments

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I have examined the optical and spectroscopic difficulties inherent in high-energy atomic-beam spectroscopy experiments. The Doppler effects which result from the high velocity of the beams severely limit the intensity and resolution obtainable in spectroscopic studies of the beams. I analyse two possible configurations for collection of light from the beams: the first has the optic axis of the spectrograph at $90^\circ$ to the beam; the second has the optic axis of the spectrograph colinear with the beam direction. The second configuration can lead to greatly enhanced intensity, at a given resolution, over that obtainable with the first. I discuss the experimental verification of the analysis. Appendices compare these configurations with another proposed system, and discuss special problems associated with the vacuum ultraviolet.
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I. Introduction.

For many years the field of classical atomic spectroscopy stagnated due to a lack of new source techniques. The ionic states accessible in conventional arc and spark sources were well known, while the entire range of ionization states above 4 or 5 were almost unknown except for the brilliant work of Edlén\(^1\), who was able to examine some links in states up to Cu XIX.

Some recent progress has been made by the Swedish group at Lund\(^2,3\) using \(\theta\)-pinch and similar plasma techniques, who have observed lines in O VI, Ne VII, and N VI. The limitations of this approach were not discussed in any detail, but long exposure times (1 1/2 hours), impurity lines (eg Cu II) and overlapping lines due to the presence of several ionization states were mentioned. Further, charge state and element assignments were made by comparing the spectra made with different gases and source conditions. This procedure is tedious and requires careful attention to avoid different contaminants in the source gas and to ensure repeatable discharge conditions if good results are to be obtained.

A variation on this procedure has been reported by Fawcett and Irons⁴ who used a giant ruby laser to vaporize and ionize a fiber of the material under study.

A general revival of interest in atomic spectroscopy has resulted from the successful employment of a high velocity (10⁸ to 10⁹ cm/sec) beam of ions from a Van de Graaff type accelerator as a source. Kay⁵ and Bashkin⁶ employed similar techniques where a beam of ions with an energy of a few MeV was incident on a thin (10 μ gm/cm²) carbon foil. The ions were excited by their passage through the foil and emitted optical radiation as they continued downstream of the foil. This procedure has many important advantages. First, the atomic composition of the beam can be established with isotopic purity by using the magnetic analyzer usually associated with the accelerator. Second, the foil has been observed to produce very few (often only 2 or 3) charge states, and all possible states in elements at least as far as completely ionized neon.⁷ These factors facilitate the assignment of lines to the proper charge state and atom, as well as the production of highly ionized states. The nature

of the beam permits the use of special techniques to further simplify this assignment. Malmberg\textsuperscript{8} has successfully employed electrostatic separation of the charge states before examining the beam spectroscopically. Also it has been suggested\textsuperscript{9} that a more accurate assignment of lines to charge states could be made by employing a coincidence counter to register the simultaneous arrival of a photon in the photomultiplier (using pulse height analysis to separate photons from dark noise) and a particle in the corresponding charge states. The analyzing field may then be placed downstream from the region of observation. In this way one may study very short-lived states, and also eliminate the Stark and Zeeman effects associated with a strong electromagnetic field in the region of observation. Another advantage of the Van de Graaff source is that the point of excitation is well defined in both space and time. This facilitates lifetime determinations by time-of-flight techniques.\textsuperscript{10,11,12} A similar technique has been employed using gas targets.\textsuperscript{13} The solid

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\textsuperscript{12} J.A. Jordan, Jr., G.S. Bakken, R.E. Yager, "Radiative Lifetimes of Excited States in HeII," \textit{Journal of the Optical Society of America}, 57, 530 (1967).
foil technique has the advantage of a more sharply defined point of excitation and also appears more effective in generating highly ionized states. Another, similar, technique was used successfully as early as 1919 by Wien.

The various applications for studies of highly ionized atoms have been detailed in review papers by Bashkin and by Jordan. Among the applications noted are the verification of theoretical structure calculations of energy levels and lifetimes for the various isoelectronic sequences. The identification of the numerous unidentified lines observed in stellar spectra is another. Lifetime measurements will assist the determination of the relative abundance of elements in a star, including the sun. Further, it is suggested that the Van de Graaff approach is particularly applicable to the complex spectra of the rare earth elements.

The high velocity beam/solid foil technique is not without disadvantages. The high velocity of the beam produces spectacular Doppler effects which seriously affect resolution. Further the radiation from the beam is weak. The result of

18 In the absence of cross sections for the excitation of atomic levels above the ground state (another topic for
the combined effects is a serious limit on the available flux. This limitation is not easily overcome; the usable ion current from the accelerator is limited to a few microamps in most cases, and greater currents, when available, quickly destroy the thin foil. Fast spectrograph and collimator systems are helpful, but are severely limited by Doppler effects which degrade both resolution and plate illuminance. Only image tubes have been an unmixed success.19

This thesis then addresses itself to the analysis of these limitations on signal strength and resolution, and the proposal of some procedures to minimize them, in survey work of moderate (1:10^4 to 1:10^5) resolution.

The problems of intensity and resolution are intimately intertwined by the Doppler effects resulting from the relativistic beam velocities. This interrelation is the crux of the problem, and is best illustrated by considering the simple collimating system frequently employed in Van de Graaff Spectroscopy, and diagrammed in Figure 1. The beam and the spectrograph slit are parallel and lie in the y-z plane. The simple lens images a portion of the beam onto the slit of the spectrograph. This image will yield an illuminance E' at the spectrograph slit which is determined entirely by the focal

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research) it will suffice to note that photographic exposures of spectra regularly require two to twelve hours. (J.A. Jordan, Jr., G.S. Bakken, A.C.L. Barnard, K.E. Kissell, "Excitation of Ionic Spectra in Oxygen," Bulletin of the American Physical Society, 11, 70.)

ratio of the lens and the luminance $L$ of the beam. Note that there is a spread of wavelengths incident at any single point in the beam because of the Doppler effects. The beam velocity causes a photon emitted from point $A$ at angle $\theta_s$ to have an observed wavelength of $\lambda$, given by

$$\lambda = \lambda_0 \gamma \left(1 - \beta \cos \theta_s\right) \quad (1)$$

where $\lambda_0$ is the wavelength which could have been observed if the emitting ion were at rest, $\gamma = (1 - \beta^2)^{-\frac{1}{2}}$, and $\beta$ is the ratio of the beam velocity $v$ to the velocity of light $c$. The light from point $A$ is collected over a range of $\theta$, from $\theta_0'$ to $\theta_0' + \Delta \theta_0'$, so that the radiation falling on $A'$ will have a Doppler wavelength spread of

$$\Delta \lambda_A = \lambda_0 \gamma \beta \left[ \cos \theta_0' - \cos (\theta_0' + \Delta \theta_0') \right] \quad (2)$$

For a typical system this wavelength spread is $\approx 10 \, \text{Å}$. Furthermore, the central frequency of the radiation incident upon the slit will vary from point to point along the slit. The central ray from point $A$ is at an angle $\theta = \theta_A$ and that from $B$ is at an angle $\theta = \theta_B$, so that there will be a

20 Luminance, Flux and Focal Ratio are defined in the Glossary.

net shift in wavelength between A' and B' of

\[ \Delta \lambda_{AB} = \lambda_0 \gamma \beta \left[ \cos \theta_B - \cos \theta_A \right] \]  

(3)

Typically this is about the same as the spread in any one point and for the example previously cited is about 12 Å.

The result of these two effects, for a stigmatic spectrograph, is an illuminance distribution on the image plane (plate) of the spectrograph such as that shown in Figure 2. The points A" and B" are the images of the points A' and B' as formed by the spectrograph optics. The most obvious feature is the slant of the beam by an angle $\gamma$ from the vertical (x axis). It is also evident that the total flux is spread over a larger area by the Doppler effect. This decreases both resolution and the illuminance of the line. Photoelectric scanning is even more strongly affected unless the slit can be slanted at the same angle, $\gamma$. Clearly then, a detailed analysis of the coupling of resolution and illuminance is necessary. We therefore proceed by analyzing the effect of Doppler spreading of the spectral line on the final line profile and illuminance at the image plane. Next the relevant properties of the beam as a source will be discussed. Finally we will apply the results to the analysis of the resolution, photographic illuminance, and total photoelectric flux given by two simple collimating systems.
II. **Line Shapes**

The Doppler-shifted, and Doppler-broadened spectral lines emitted by the ion beam yield a complex line profile at the image plane of the spectrograph. In general this line profile is dominated by instrumental and Doppler effects; other broadening mechanisms such as collision broadening and Stark broadening can be made negligible. The problem then is that of taking the Doppler-broadened spectral line profile and combining it with the instrumental line profile in order to obtain the line profile detected on the image plane. This profile appears directly on a photographic plate, and is used to find the photoelectric line shape by carrying out one additional calculation. The aim of this section is to derive some general relationships between the resolution and illuminance maxima which can be obtained, given certain Doppler and instrumental profiles.

We make the following definitions concerning the parameters of spectrograph: \( t \) is a coefficient which expresses the effects of transmission losses and detector (plate or photomultiplier) efficiencies. The slit dimensions are width \( b \) and length \( l \). \( M \) is the internal magnification of the spectrograph slit image seen at the image plane. \( E' \) is the illuminance at the slit, and is the number of photons/sec-cm\(^2\) which arrive at the slit. Plate coordinates will be denoted by \( \xi \) perpendicular to the direction of dispersion and \( \eta \) the coordinate parallel to it, as shown in Figure 2. If the
photons illuminating the slit are absolutely monochromatic.

the instrumental profile $G_L(\xi - \xi_c)$ (expressed as a function
of $\xi$, with $\xi_c$ the position of the maximum) would describe
the line shape at the plate. The illuminance at the plate of
a small segment of the image ($d\xi$ by $d\xi$) integrated across the
line must equal the total flux falling on an equivalent ele-
ment of slit, just $E'_{nm}b d\ell$. Thus $G_L$ is normalized by

$$N_L = \frac{t E'_{nm} b}{M \int_{-\infty}^{\infty} G_L(\xi - \xi_c) d\xi} \quad (4)$$

As a result of the beam velocity and different angles of col-
lection for different photons, the slit illumination is not
monochromatic--rather it may be described by the Doppler pro-
file $G_D(\lambda - \lambda_P)$, where $\lambda_P$ refers to the wavelength of the max-
imum. This function is most conveniently normalized to unity
by

$$N_D = \frac{1}{\int_{-\infty}^{\infty} G_D(\lambda - \lambda_P) d\lambda} \quad (5)$$

The two profiles may now be combined by treating them as
statistical distributions. $G_D(\lambda - \lambda_P)$ is then the pro-
bability that a photon will be emitted with a wavelength $\lambda$.
Similarly, $G_L(\xi - \xi_c)$ is the probability that it
will fall a distance $\xi - \xi_c$ from the normal position,
$\xi_c$, of the center of a sharp line of wavelength $\lambda$. 
Then $\lambda$ and $\xi_c$ are related by

$$\xi_c = \lambda/D + \xi_o = \xi(\lambda)$$

(6)

where $\xi_o$ refers the function to an arbitrary origin on the plate. (E.g., $\xi_o = 0$ if the origin is taken to be the center of the $\lambda = 0$ line). The probability that any photon will strike any specific point $\xi'$ on the plate is then found by multiplying the distributions and integrating:

$$E'_{nn'}(\xi(\lambda_p) - \xi') = N_D N_L \int_{\xi_a}^{\xi_b} G_D(\xi_c - \xi(\lambda_p))$$

$$G_L(\xi' - \xi_c) d\xi_c$$

(7)

This probability is, by (4), properly normalized to equal the illuminance at the point. A very useful special case, which we will call the wide slit approximation, results when the slit width is assumed to be considerably greater than that which gives maximum resolution (minimum width for $G_L$). Denoting the position of the edges of the slit on the image plane by $\xi_a$ and $\xi_b$, the instrumental line profile assumes the simple, rectangular form:

$$G_L(\xi_c - \xi') = 1 \ , \ \xi_a \leq \xi' \leq \xi_b$$

$$= 0 \ , \ \xi' < \xi_a \ ; \ \xi' > \xi_b$$

(8)
Then equation (7) reduces to

\[ E_{mn}^\prime (\xi (\lambda r)-\xi') = N_D N_L \int_{\xi_a}^{\xi_b} G_D (\xi_c-\xi (\lambda r)) d\xi_c \]  \hspace{1cm} (7a)

The peak illuminance is undiminished if the integral in (7a) equals the integrals in the normalization factors. Specifically this holds at the peak, \( \xi' = \xi (\lambda_0) \) if \( G_D \) has a full Doppler width \( \Delta \lambda /D \) (i.e. \( G_D \) effectively zero outside two points, \( \lambda a \) and \( \lambda b \) such that \( \lambda b - \lambda a = \Delta \lambda \)) which satisfies

\[ \frac{\Delta \lambda}{D} \leq \xi_b - \xi_a = Mb \]  \hspace{1cm} (9)

The equality represents the condition for optimum resolution without loss of illuminance. The width at half maximum will be \( Mb/D \) for \( \Delta \lambda /D \leq Mb \), and less (the exact value depending on the detailed shape of \( G_D \)) for \( \Delta \lambda /D > Mb \), although not usually less than the width at half maximum of \( G_D \). This discussion holds equally true for \( G_D \) rectangular and arbitrary \( G_L \).

From the definitions of \( N_D \) and \( N_L \), it is clear that (7) can assume the peak value \( \frac{t E_{mn}^\prime b}{M} \) only if \( G_L \) or \( G_D \) is rectangular and \( \Delta \lambda /D \leq Mb \). Losses due to the failure of these conditions are not necessarily unacceptably severe, however. For example assume \( G_D \) is given by the gaussian distribution
\[ G_D(\xi - \xi(\lambda)) = e^{-\lambda^2} \] (10)

and \( G_L \) by (8). The gaussian distribution extends to infinity, but may be effectively terminated at twice the width at half maximum, \( \Delta \lambda / 2 \), and retain 96% of the area. Then (9) gives the same 96% of the peak value for \( b = \Delta \lambda g / MD \). If however \( b = \Delta \lambda g / 2MD \), (9) gives about 68% of the peak value, while resolution is improved by a factor of about 2.

It should be noted here that King and Emslie have applied a similar but more graphical technique to the problems of line shape in absorption spectroscopy.

III. Source Characteristics

We will first show the dependence on direction of emission of the number of photons available within a given Doppler spread \( \Delta \lambda \). We use the coordinate systems shown in Figure 3. A rectangular coordinate system \((x,y,z)\) is centered on the foil and a spherical coordinate system \((r,\theta,\phi)\) centered on an arbitrary point \((o,o,z_0)\) in the beam downstream from the foil. The number of photons with a given Doppler spread is a function of \( \theta \) only, as there is complete \( \phi \) symmetry. There will therefore be an element of solid angle \( d\Omega \), of

angular width $d\theta$ and extending over all $\Omega$, ($0 \leq \Omega \leq 2\pi$), which will include all photons emitted with a Doppler-shifted wavelength in the range $\lambda$ to $\lambda + d\lambda$. The angular width $d\theta$ can be expressed in terms of the Doppler spread $d\lambda$ by differentiating equation (1)

$$d\theta(\lambda) = \frac{d\lambda}{\lambda_0 \gamma \beta \sin \theta}$$

Then the solid angle $d\Omega$ becomes

$$d\Omega = 2\pi \sin \theta \, d\theta = 2\pi \frac{d\lambda}{\lambda_0 \gamma \beta} \quad (11)$$

Now we assume the point in the beam to be isotropically emitting $P$ photons/sec into $4\pi$ steradians, as viewed in the rest frame of the emitting ions. Due to the relativistic effects the emission is not isotropic in the laboratory system. Noting that $\theta$ transforms according to the relation

$$\cos \theta_i = \frac{\cos \theta_l - \beta}{1 - \beta \cos \theta_l} \quad (12)$$

when the subscript $i$ denotes the beam inertial frame and $l$ denotes the laboratory frame. Then the solid angles transform as

$$d\Omega_i = \frac{d\Omega_l}{\gamma^2 (1 - \beta \cos \theta_l)^2} \quad (12b)$$

Flux as a function of \( \lambda \) then becomes

\[
F(\lambda) = \frac{F}{\gamma^2 (1-\beta \cos \theta_0)^2} \frac{2\pi d\lambda}{\lambda \coth \beta} \tag{13}
\]

This function peaks in the forward direction \( (\theta = 0) \). However, values of \( \beta \) normally encountered in beam/foil spectroscopy are low, so that this effect is negligible. Therefore it will be disregarded in the remainder of this paper.

To a good approximation, the total flux observed within a Doppler spread \( \Delta \lambda \) depends only on the fraction of the \( 2\pi \) degrees in the \( \Omega \) angular coordinate which the collector sees.

There are two extreme angles of orientation, \( \theta \), shown in Figure 4 and discussed in detail in Sections IV and V, which are the most useful for high-energy atomic-beam spectroscopy. The case most often used has the optic axis of the collimator at \( \Theta = \pi/2 \). In this case there is no first-order Doppler correction. However, a simple lens collimator with an acceptance angle \( Q_a \) can collect only the fraction \( Q_a / 2\pi \) of the available flux \( F(\lambda) \) given by (13). This contrasts with the second case in which a simple lens is oriented with its optic axis coincident with the beam axis. Here an annular ring from \( \theta \) to \( \theta + \Delta \theta \) falls completely on the lens (if \( \theta + \Delta \theta \leq Q_a \)), and thus the total fraction of the flux having the Doppler spread defined by \( F(\lambda) \) is collected.

(Note that this also leads to the elimination of the slanted-
line signature characteristic of the $\bar{\theta} = 90^\circ$ case.) The $\bar{\theta} = 0$ case therefore appears to have definite advantages, and this will be confirmed in a more detailed analysis of specific systems.

The major limiting factor on the use of the $\bar{\theta} = 0$ configuration results from $\beta$ not being well defined. This is because any accelerator yields a spread in the velocities of the accelerated particles; the effect of that spread can be seen by differentiating (1):

$$\frac{\Delta \lambda}{\lambda_0} = \gamma \beta \cos \theta \frac{d\beta}{\beta}$$

For a $\beta = 0.05$, the accelerators used in our experiments typically give a $d\beta/\beta$ on the order of $1:10^3$. Thus the spread in the peak wavelength is on the order of $1:10^4$. Careful attention to detail in operating the accelerator can improve on this somewhat, but $d\beta/\beta$ is also limited to the velocity profile of the ions emerging from the foil. (Typically, ions lose about $1:10^3$ in velocity in the foil and the spread in energies is of that same order of magnitude.) For $\bar{\theta} = 0$ the effect is clearly significant. For $\bar{\theta} = \pi/2$, (13) appears negligible. However Rutherford scattering and imperfect beam optics produce a comparable $\rho$ spread perpendicular to the beam axis. Thus the $\beta$ spread is the ultimate limit on

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resolution in both cases.

The best approach to including this correction is to generate a suitable $\beta$ profile or distribution, $G_{\beta} (\lambda_o - \lambda_p')$, where $\lambda_p'$ is the laboratory wavelength corresponding to the most probable $\beta$. It may then be normalized to unity, multiplied by the Doppler profile resulting from the collimating and integrated to the final $G_D (\lambda - \lambda_p)$ used earlier.

We have considered a point $(0,0,z_o)$ emitting $F$ photon/sec. A more realistic approach is to define a "volume luminance" $L_{nn'}^V$ for the ion beam. This luminance is the flux in photons/sec emitted isotropically per unit volume of the beam in the transition from the $n^{th}$ state to the $n'^{th}$ state. Neglecting cascade effects, this volume luminance is a function of beam current $J$, the velocity of the beam $v$, the beam cross-sectional area $A$, the cross section (per unit beam current) $\sigma_n$ for excitation from the ground state to the $n^{th}$ state, and the Einstein $A$-coefficients for spontaneous emission from the $n^{th}$ state. The volume luminance takes the form:

$$L_{nn'}^V = C \exp \left[ - \sum_m A_{nm} \frac{Z_m}{v} \right]$$

$$C = \frac{J}{vA} \sigma_n A_{nn'}$$

(14)

where $m$ is summed over all possible transitions from the $n^{th}$ state. This expression assumes that the beam has uniform density and that the beam is optically thin. The former
condition is realistic (indeed it can be produced if it does not already exist\textsuperscript{25}) and the latter is distressingly valid. The volume luminance can be replaced by the conventional (area) illuminance \(L_{nn}'\) by integrating the expression for \(L_{nn}'\) over the beam thickness and multiplying by \(1/2\) to correct for the fact that the beam radiates into \(4\pi\) steradians rather than the \(2\pi\) assumed in the usual definition of luminance. For a collimator situated in the \(yz\) plane, as defined by Figure 3, and oriented with the optic axis at an angle with respect to the beam axis \(z\), \(L_{nn}'\) is given by

\[
L_{nn}' = \frac{C}{2} \int_{y_{min}}^{y_{max}} \exp \left[ -\sum \frac{A_{nm}}{1} \left( z_0 + \frac{y}{\tan \varphi} \right) \right] \frac{dy}{\sin \varphi} \quad (15)
\]

The evaluation of this equation requires explicit knowledge of the beam geometry and the transition under study. The expression simplifies somewhat for the \(\varphi = 0\) and \(\varphi = \pi/2\) cases considered in the next sections.

IV. Analysis of Collecting Systems Perpendicular to the Beam Axis

Having examined the general line shape characteristics of fast ion beam experiments, we turn now to the analysis of specific systems. We will first look at the simple-lens

collecting system diagrammed in Figure 5. The beam is assumed to have a rectangular cross section, d wide by h high. The simple lens is effectively stopped to a rectangular aperture by the grating. We will use small angle approximations which are consistent with our aim of achieving a small wavelength spread, and hence a high resolution. The effective aperture of the lens then subtends a solid angle approximately $\theta_\alpha'$ by $\alpha'$ as seen from Figure 5. For small $\beta$,

$$G_D (\lambda - \lambda_P) = \begin{cases} 1 & 1\lambda - \lambda_P \leq \frac{\Delta \lambda}{\alpha} \\ 0 & 1\lambda - \lambda_P > \frac{\Delta \lambda}{\alpha} \end{cases} \tag{16a}$$

By (1)

$$\Delta \lambda = 2\lambda_0 \gamma \beta \sin (\theta_\alpha' / 2) \tag{16b}$$

Marquet gives a more precise evaluation of $G_D$ which may be used here if greater accuracy is necessary.

Using this Doppler profile in the "wide slit" expression for the illuminance at the plate, (14a), we obtain:

$$E_{\text{ill}}'' = \frac{t E_{\text{ill}}'}{M^2 \Delta \lambda} \int_{\xi_{\text{min}}}^{\xi_{\text{max}}} d \xi_0$$

The illuminance at the center of the line is then given by:

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We will use (17a) since we wish to examine the effects of \( \Delta \lambda \), \( D \), and \( b \) on the peak illuminance.

By a theorem of classical optics the illuminance of the slit due to a spectral line is the product of the source luminance for that line, \( L_{nn'} \), and the solid angle subtended by the exit pupil at the image (for a spectrograph slower than \( f/6 \)). Making the same restriction, this solid angle can be approximated by the angular dimensions \( \theta_a \) by \( \delta_a \), so that integration of \( \sin \theta \, d\theta \, d\lambda \) gives for the illuminance at the slit:

\[
E_{nn'} = L_{nn'} \Delta \lambda \approx L_{nn'} \delta_a \sin \frac{\theta_a}{2} \tag{18}
\]

Using this equation, (15), and (16b) in the expression for the illuminance at the plate, (17a), we obtain:

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for the peak illuminance at a point on the plate corresponding to the point \( z_0 \) in the beam. The first term in parentheses involves only constants related to the spectrograph. The second term is a function of the beam parameters and the transition under study. The last term is the one of interest. By Figure 6, it is equal to \( \frac{f_0'}{f_0} \) for all values of \( \theta_a \) and \( \theta'_a \). This implies that the Doppler spread may be decreased (without affecting the peak intensity) by judiciously masking the lens to reduce \( \theta'_a \) but not \( \theta'_a \). Note that as soon as \( \Delta \lambda / D < b \), (17b) obtains and (19) reduces to (18). Further decrease of \( \theta'_a \) then results in a decrease of \( E_{nn'}'' \), but the line width is not decreased.

The ratio \( \frac{f_0'}{f_0} \) may be increased by using a shorter focal length lens of smaller diameter, or by increasing the distance from the spectrograph to the beam. This increase in \( \frac{f_0'}{f_0} \) results in a reduced beam image size, which in turn limits the usable value of \( \frac{f_0'}{f_0} \) in two ways: First, the image on the plate must be large enough to be identifiable. Second, the width of the image of the beam must be larger than the slit width which gives the minimum instrumental line profile width. The first condition makes apparent the need
to avoid internal demagnification in the spectrograph, since such demagnification reduces the visibility of the image without yielding a corresponding reduction in the Doppler spread. Note that an ultimate limit is set by the Rutherford-scattering induced spread in $\lambda$ (Section III).

The flux seen by a photoelectric detector can now be calculated by integrating the illuminance at the image plane over the exit slit. As noted earlier, the line on the image plane is slanted, and optimum signal strength and resolution requires that the slit be slanted at an angle equal to $\psi$ to follow the line. We will therefore assume the exit slit to be of width $M_b \cos \psi$ at an angle $\psi$ to the $\xi$ axis. The slit therefore has a width of $M_b$ along the $\xi$ axis and corresponds to the entrance slit width. Further, the luminance of the beam varies, as given in equation (15). In this case (15) produces a variation of $L_{nn'}$ along the $\xi$ direction of

$$L_{nn'} = \frac{c d}{\lambda} \exp \left( -\sum_{m} \frac{A_{nm}}{\nu} \frac{\xi}{MM'} \right)$$  \hspace{1cm} (20)

where $M'$ is the magnification of the collimator, so that $Z_0 = \frac{5_0}{MM'}$. The available photoelectric flux is calculated by

$$F_{p1}(\xi, \xi') = \int \left[ \frac{M_b \xi}{M \xi} \right] E(\xi, \xi') \, d\xi'$$  \hspace{1cm} (21)
where $\xi(\lambda)$ denotes the position of the center of the slit in the image plane. The parameter of interest is the peak flux, approximately $F_p(0)$. We may define a collection factor $K$

$$K = \frac{1}{E_c(\xi) M b} \int_{\xi(\lambda_p) - \frac{M b}{2}}^{\xi(\lambda_p) + \frac{M b}{2}} E(\xi' - \xi(\lambda_p), \xi') d\xi' \quad (22)$$

where $E_c(\xi)$ is the peak illuminance of the line, and occurs at $\xi = \xi(\lambda_p)$. Then

$$F_{p,1}(0) = k M b \int_0^{M b} E_c(\xi) d\xi \quad (23)$$

In the wide slit approximation, (19) and (20) give

$$F_{p,1}(0) = k \frac{b^2 t D \delta}{\lambda_0 \xi(\beta)} \frac{\delta_n}{\delta_n \frac{\delta}{2}} \frac{c d}{2} \int_0^{M b} \exp\left(-\sum_{m} \frac{A m}{V} \frac{\xi}{M m} \right) d\xi \quad (24)$$

when the slit is adjusted to the optimum resolution illuminance condition, $k = 3/4$, and transforming the integral to $z$,

$$F_{p,1}(0) = \frac{3}{8} \left(\frac{4 \pi}{D M}\right) t \left(2 \delta_n \frac{\delta}{2}\right) \frac{c d M'}{2} \int_0^{M'} \exp\left(-\sum_{m} \frac{A m}{V} z'\right) d\xi' \quad (25)$$
Using small angle approximations and (16b)

\[ F_{\perp}(0) = \frac{3}{4} \left( \frac{\Delta \lambda}{DM} \right) t \lambda_o \xi_c d \frac{\Delta \lambda}{\lambda_o \gamma} \]

\[ \int_0^{\frac{l}{M'}} e^{-\frac{\sum}{\frac{Am}{v} z'}} dz' \]  (25a)

This equation is the photoelectric version of (19). Note that the M dependence is eliminated, as in general D and M are conjugate quantities. The flux may therefore be increased only by increasing \( \Delta \lambda \), the acceptance angle, or the beam thickness \( d \). The integral may be increased by using small \( M' \) and hence imaging a greater length of beam onto the slit. This is limited by the requirement that the beam height \( h \) be such that \( M'h \geq b \). For a fixed beam current this can only be accomplished by decreasing \( d \), which may result in a net loss. The detailed optimization thus depends on limitations in ion beam optics, chamber window sizes, and allowable exit slit slant, and need not concern us here.

V. Collecting Systems Colinear with the Beam Axis.

As we stated in Section III, we may expect gains in the total flux and perhaps in the illuminance by placing the optic axis of the simple lens along the beam axis. This was inferred from two effects: The increase in \( \xi_c' \) to \( 2\pi \), and the small relativistic effect favoring emission in the forward direction. An operational collecting system is shown in Figure 7a. The results of equation (13) show that the value
of the flux $F(\Delta \lambda)$ is essentially constant over all $\theta$. Thus the cone of light from one point $r_s$ on the optic axis gives a Doppler profile $G_D(\lambda) = 1$, for $\lambda_{\text{min}} \leq \lambda \leq \lambda_{\text{max}}$, where $\lambda_{\text{min}}$ is the wavelength cutoff at the Faraday Cup, and $\lambda_{\text{max}}$ is the wavelength corresponding to the largest angle at which the cone of light falls entirely on the grating. (The light falling on the corners of the grating contributes only a "tail" to the basic line shape and may be masked off without loss of peak intensity.) However, the beam is of finite diameter and length, so that the Doppler profile comes from a superposition of single points lying off the optic axis, and the following more complex analysis must be used.

Figure 8 defines the collimator dimensions. The emitting point is located a distance $r_s$ from the beam axis and a distance $z$ from the foil (which corresponds to the distance $f_0 - z$ from the lens). The angle of emission, $\theta_s$, of a given photon is then $\tan^{-1} \left( \frac{r'}{(f_0 - z)} \right)$ where the photon strikes the lens a distance $r'$ from the axis. (For simplicity, we will calculate the Doppler profile as a function of $\theta_s$ and $r_s$, converting this to a function of $\lambda$ and $r_s$ when necessary, using equation (1)). The $r_s$ dependence results naturally from the derivation and implies that the Doppler profile at the slit is a function of position within the image at the slit. The cylindrical symmetry of the apparatus permits the use of the special coordinate system shown in Figure 8b.

To calculate $G_D(\theta_s, r_s)$ we may apply equation (18) to
find the illuminance at the point on the slit corresponding to \( r_s \) as a function of \( \theta_s \). Proceeding in this fashion leaves \( G_D(\theta_s, r_s) \) normalized to equal \( E'_{nn'}(\theta_s, r_s) \), the slit illuminance as a function of wavelength. Thus we may write immediately

\[
N_D = \frac{1}{E_{nn'}} \quad (26)
\]

when \( E'_{nn'} \) is the total entrance slit illuminance. Now if we consider figure 8b, we see that this calculation requires that we consider separately three regions. This is a result of the presence of emitting ions a distance \( r_s \) from the optic axis, i.e. finite beam size. As a result of being off axis, the differential elements of solid angle, \( d\Omega = \sin \theta d\theta \) corresponding to a particular Doppler shift \( (\lambda' - \lambda_p) \) from the central wavelength, may be partially obscured by the Faraday Cup and chamber window. Then \( \Omega \) assumes a value

\[ 0 \leq \Omega' \leq 2\pi \]

depending on \( r_s \) and \( \theta_s \).

We first consider region B. \( \Omega' \) can always assume a value of \( 2\pi \) here. If we note from Figure 7 that

\[ \tan \theta' = \frac{f_0}{f_1} \tan \theta_s, \quad r' = (f_0 - z)^2 \tan^2 \theta_s \]

and use (18), we obtain:

\[
G_B(\theta, r_s) = \int_{z_{\text{min}}}^{z_{\text{max}}} L_{nn'} \, \Omega' \, \sin \theta
\]
\[ G_B(\theta, r_s) = \int_{z_{\text{min}}}^{z_{\text{max}}} L_{n,m}^* \left( \frac{2\pi}{\sqrt{1 + (f_0 - z)^2 \tan^2 \theta_s}} \right) \frac{(f_0 - z) \tan \theta_s}{f_0 - f_{0,m,n}} \, dz \]  

where the limits \( z_{\text{max}} \) and \( z_{\text{min}} \) for a given \( r_s \) and \( \theta_s \) are given by figure 9:

\[ z_{\text{max}} = L \left[ \left( f_0 - \frac{r + r_s}{\tan \theta_s} \right), f_0 - f_{0,m,n} \right] \]

\[ z_{\text{min}} = G \left[ \left( f_0 - \frac{R - r_s}{\tan \theta_s} \right), 0 \right] \]  

(28)

The notation \( L[A,B] \) implies that the lesser of \( A \) and \( B \) is to be used, while \( G[A,B] \) implies that the greater of \( A \) and \( B \) is to be used. \( f_{0,\text{min}} \) is an arbitrary cutoff point dictated largely by depth of field and convenience. (This cutoff eliminates a long wavelength tail that is generated on \( G_D(\lambda) \) by ions emitting to the lens at large \( \theta_s \) from points farthest from the foil. Because of the distance, the beam has largely decayed, and the image on the slit is out of focus, so that the amplitude of the contribution is small. It therefore expedites computation to cut off the computation a distance of 2 or 3 lifetimes downstream, i.e., \( f_0 - f_{0,m,n} \approx 2 \sum \frac{A_{n,m}}{v} \).
FOIL

\( Z = f_0 \frac{R-r_s}{\tan \theta_s} \)

\( Z = f_0 \frac{r+r_s}{\tan \theta_s} \)

\( Z = f_0 \frac{R+r_s}{\tan \theta_s} \)

FIGURE 9
In regions A and C, as noted, $\phi'_a$ is restricted to values less than $2\pi$ by the chamber window. A straightforward application of the cosine law to the triangles $r'_1$, $r_s$, $r$ and $r'_1$, $r_s$, $R$ gives:

$$\phi_a(A) = 2 \cos^{-1}\left[\frac{r^2 + r_s^2 - (f_0 - z)^2 \tan^2 \theta_s}{2rr_s}\right]$$

for $A$

$$\phi_a(C) = 2 \cos^{-1}\left[\frac{(f_0 - z)^2 \tan^2 \theta_s - R^2 - r_s^2}{2rr_s}\right]$$

for $C$ (29)

From equation 27 and again taking the limits from figure 9, we obtain:

$$G_A(\theta, r_s, Z) = \int L_{nn} \phi_a(A) \left(\frac{f_0 - z}{\sqrt{f_0^2 + (f_0 - z)^2 \tan^2 \theta_s}}\right) d\theta$$

$$G_C(\theta, r_s, Z) = \int L_{nn} \phi_a(C) \left(\frac{f_0 - z}{\sqrt{f_0^2 + (f_0 - z)^2 \tan^2 \theta_s}}\right) d\theta$$

Now the total Doppler profile characteristic of the apparatus is found by summing the contributions from the three regions of the lens. Since $L_{nn}$ is a function of $z$, it is clear that $G_D$ must be evaluated numerically. We have made several such numerical calculations; representative results for $G_D$ for
various apparatus dimensions are shown in Figure 10. Clearly, a resolution of $1:10^6$ is attainable with apparatus of moderate dimensions. This resolution, however, does not include the effects of the velocity spread in the ion beam, discussed in Section III. In that discussion it was noted that $\beta$ was defined to one part in $10^3$ and thus $\lambda$ was defined to one part in $10^4$ ($\beta \approx 0.05$). Thus the $\beta$ definition, rather than the optical system, sets the resolution limit to about one part in $10^4$ to a few parts in $10^5$ (depending on factors such as foil thickness and ion mass). Following the procedure outlined, we define $G_\beta (\lambda - \lambda_\rho)$ as the wavelength profile due to spread in $\beta$. After normalizing to unity we may combine it with (27), (30), and (31) to give the Doppler profile resulting from all effects, $G_D (\lambda - \lambda_\rho)$

$$G_D (\lambda - \lambda_\rho) = \frac{N_\beta}{E_{nm}} \int d\lambda [ G_A (\lambda - \lambda_\rho, r_5) + G_B (\lambda - \lambda_\rho, r_5) + G_C (\lambda - \lambda_\rho, r_5) ] G_\beta (\lambda - \lambda_\rho)$$

(32)

where $N_\beta$ normalizes $G_\beta$ to unity, and $1/E_{nm}$ normalizes $[G_A + G_B + G_C]$ to unity. From Figure 10 and the associated resolution we see that the collimator may easily be adjusted to give a rectangular profile such as equation (16a) earlier by designing a system with a rectangular profile $G_D > G_\beta$. 
LENGTH = 4.4 m  
DIAMETER = 4 cm  
BEAM DEFLECTED

LENGTH = 1 m  
DIAMETER = 4 cm  
F.C.DIAMETER = 2 cm

FIGURE 10
Then

\[ G_D(\lambda' - \lambda_p, r_s) = \frac{N_\beta}{E'_{nn}} \int_{\lambda_{mnw}}^{\lambda_{mnw}} G_\beta(\lambda' - \lambda) \, d\lambda \]  

(33)

and the same optimization arguments hold, namely the collimator dimensions should be adjusted to give a rectangular profile with a width equal to the full width of the \( G_\beta \) profile.

The calculation of the final line shapes proceeds as in Section I, using (7) or (7a). The optimum case is that for which \( G_L \) and the collimator profile are rectangular with \( \frac{b}{D} = \Delta \lambda_D = \Delta \lambda_\beta \). The exact value for less idealized profiles requires a complete calculation, but is approximately equal to the full width of \( G_\beta \). The illuminance profile follows by multiplying the result of (7) by \( E'_{nn} \), as the normalization equation, (26), indicates. The total photoelectric flux follows by solving (21) numerically for the given \( E'_{nn}, (\xi' - \xi(\lambda_p)_0 r_s) \) profile, weighting the \( r_s \) contributions appropriately. This is best left for the consideration of specific systems, and will not by discussed here. It is more useful here to compare the peak illuminance and total photoelectric flux available with this collimating system to that available in the \( \theta = \frac{\pi}{3} \) configurations. To do this we will make the wide slit approximation and assume the \( \Delta \lambda \) of the collimator to be much greater that \( G_\beta \) and thus give an
approximately rectangular $G_D$. Then we may apply equation (17b) to obtain (for $\Delta \lambda/D = M_b$):

$$E_{c_{\|}} = \frac{t}{2n^2} \left[ C \int_0^{f_0 - f_{0_{\min}}} \exp \left( - \sum_m \frac{A_{nm}}{v} z' \right) dz' \right] \frac{\pi}{4} (2 \partial_a \sin \theta_a)$$

where $\pi/4$ corrects the solid angle subtended by the grating for the masking of the corners discussed earlier. Also the use of some form of beam deflection (Figure 7b) has been assumed to eliminate the central Faraday Cup. If this is not done, an additional factor must be included to correct for the area of the lens blocked by the Faraday Cup. The ratio of $E_{c_{\|}}$ to $E_{c_{\perp}}$, (equation (19) with $\Delta \lambda/D = M_b$) is then

$$\frac{E_{c_{\|}}}{E_{c_{\perp}}} = \frac{\pi}{4} \frac{\int_0^{f_0 - f_{0_{\min}}} \exp \left( - \sum_m \frac{A_{nm}}{v} z \right) dz}{\int_0^{f_0 - f_{0_{\min}}} \exp \left( - \sum_m \frac{A_{nm}}{v} z_{m_{\min}} \right) dz}$$

where $z_{min}$ is a reference point, most conveniently chosen such that $z_{min} > h$ to ensure a measurable image size. The exact value for the gain depends on $\sum_m \frac{A_{nm}}{v}$, $d$, and $z_{min}$. It can be less than unity in some cases of large $\sum_m \frac{A_{nm}}{v}$ and $d$, or of small $f_0 - f_{0_{\min}}$. Generally, however, it will be on the order
of 10. (E.g., for the 4686 Å line of He II, at 1 MeV, with a $f_0 - f_{\text{omin}}$ distance of 5 cm, and $h = d = 0.3$ cm, equation (35) yields a gain of 12.4.) The lines formed in the end-on configuration are larger than those formed in the side-on case so that the gain in signal to noise, $(\bar{N}_s - \bar{N}_b)\sqrt{A}$ (where $N_s$ is the mean signal illuminance, $N_b$ the mean background illuminance, and $A$ the image area$^{28}$) is even greater, typically another factor of 5 when good resolution is achieved.

The total photoelectric flux follows from (34) by integrating over the slit. In this approximation the integral is evaluated by multiplying the image weight $M'M'h$ by the slit width $b$, and the $k$ for this approximation, $k = 3/4$.

$$F_{\parallel} = \frac{3}{4} \left( \frac{\Delta \lambda}{DM} \right) t \left( \frac{\alpha}{2} \sin \frac{\theta_a}{2} \right) \frac{\pi}{4} C d M''$$

$$\int_{f_0 - f_{\text{omin}}}^{f_0} \exp \left( -\sum \frac{A_{mn}}{\sqrt{\alpha}} z' \right) d z' \quad (36)$$

Equation 1 may be used to express this in terms of $\Delta \lambda$ :

$$F_{\parallel} = \frac{3}{4} \left( \frac{\Delta \lambda}{DM} \right) t \left( \frac{\alpha}{4} \right) \tan \left[ \cos^{-1} \left( 1 - \frac{\Delta \lambda}{\lambda_0 \delta_3} \right) \right]$$

$$C d \int_{0}^{M'} \exp \left( -\sum \frac{A_{mn}}{\sqrt{\alpha}} z' \right) d z' \quad (37)$$

---

where small angle approximations have been used and \( f_0 - f_{\text{omin}} \) has been assumed approximately equal to the length of the beam available in the \( \tilde{\theta} = \pi/2 \) case. From equation (25a) we then have as the relative gain

\[
\frac{F_{\|}}{F_{\perp}} = \frac{\pi}{4} \frac{\tan[C_0(1 - \frac{\Delta \lambda}{\lambda \delta \beta})]}{\left(\frac{\Delta \lambda}{\lambda \delta \beta}\right)} \tag{38}
\]

This function is plotted in Figure 11, and gives particularly large gains in the region of \( \Delta \lambda / \lambda \) less than \( 10^{-3} \). The approximations used in the derivations of (38) naturally begin to break down in the region of \( \Delta \lambda / \lambda \sim 1:10^4 \), as one approaches the width of \( \delta \beta \). As noted before in Section I, these effects are not serious and will probably not cause a loss of more than 50% in the values given in Figure 11.

VI. Experimental Verification of Conclusion

It is desirable to verify the conclusion of the preceding discussion and equation (37). Accordingly the experiment diagrammed in Figure 12 was set up on the beam tube of the Rice University 5.5 MeV Van de Graaff accelerator. The accelerator provided approximately 0.3 \( \mu \)A of He\(^+\) at 1 MeV into the target chamber. The beam diameter was approximately 4 mm at the point of observation. The He II transition at 4686 Å was selected for study. Excitation was provided by a commercial carbon foil\(^{29}\) with an area density of 10 \( \mu \)gm/cm\(^2\).
The $\bar{\theta} = 0$ configuration was set up using the Rice University Jarrell-Ash 1 meter Czerney-Turner spectrograph. The collimating lens and Faraday Cup were set up to give a magnification of $M'' = 5$ and a Faraday Cup correction factor of 0.85. The line shape is shown in the inset to Figure 12, and is approximately rectangular with $\Delta \lambda = 1 \text{ Å}$. The blaze efficiency was listed by the manufacturer as 81% at 10,000 Å first order or 5,000 Å second order. This is quite close to the line used (4686 Å He II Doppler shifted to 4580 Å) and may be used. The spectrograph slits were set at 0.4 mm. An E.M.I. type 6256 photomultiplier was an S-11 photocathode was used as the detector. Because of the small diameter (1 cm) of the photosensitive surface, only 0.39 of the total exit slit flux could be detected. If the photomultiplier was not exactly aligned in the housing (probably true) this factor could be decreased to 0.2 or less.

The $\bar{\theta} = \frac{\pi}{2}$ configuration was simulated by using the same photomultiplier with a Baird Atomic interference filter with the bandpass centered on the 4686 Å He II line and a peak transmittance of 50%. An aperture was designed to correspond to a collimating system with $M' = 1$ and $\Delta \lambda = 1 \text{ Å}$, with a rectangular profile. As no slit was employed, the observed photoelectric flux of the line was 4 mm/0.4 mm times that

29 Yissum Research and Development Co., Hebrew University, Jerusalem, Israel.
actually available to the spectrograph. Further, the length of beam visible was about twice that visible in the $\vec{\theta}=0$ case.

Measurements of the total flux were made using the same EMI type 6256 photomultiplier for both cases. The raw photomultiplier currents were:

- $3 \times 10^{-8}$ Amp., $\vec{\theta} = 0$ (real spectrograph)
- $5 \times 10^{-8}$ Amp., $\vec{\theta} = \pi/2$ (uncorrected simple aperture)

Applying the correction factors,

\[
\frac{F_{p||}}{F_{p\perp}} = \frac{\text{(raw current)}}{(\text{spectrograph eff.})(\text{slit fraction})}
\]

\[
F_{p||} = \frac{3 \times 10^{-8}}{(0.81)(0.39)} = 9.5 \times 10^{-8} \text{ Amp.}
\]

\[
F_{p\perp} = \frac{(\text{raw current})(\text{slit factor})(\text{beam length factor})}{(\text{filter eff.})}
\]

\[
F_{p\perp} = \frac{(5 \times 10^{-8})(0.4/4.0)(1/2)}{(0.5)} = 5 \times 10^{-9} \text{ Amp.}
\]

Then the experimental ratio is

\[
\frac{F_{p||}}{F_{p\perp}} = \frac{9.5 \times 10^{-8}}{5 \times 10^{-9}} = 19
\]

The theoretical value is, from equation (38) and including the factor for the Faraday Cup,

\[
\frac{F_{p||}}{F_{p\perp}} = \frac{\pi}{4} \tan \cos^{-1} \frac{0.99}{0.01} = 14
\]

The discrepancy, although large, is well within the experimental errors of the rather crude procedures used. The gain, of 19, is impressive and confirms the qualitative and (to a lesser extent) quantitative predictions of equation (38).
We also made a photographic exposure to test the predicted illuminance gains. No quantitative results were obtained. It was our feeling, however, that the gains were qualitatively the same as equation (35) when compared with earlier plates taken with another spectrograph (A Meinel spectrograph manufactured by Astro mechanics, with a f/29 collimator and f/2 effective speed). The most striking advantage was that the size of the image was increased from a microscopic 0.5 mm to a full 2 cm, easily detectable at a glance even though the line was less dense.

VII. Summary

We have shown that the effectiveness of the high-energy atomic-beam technique in spectroscopy can be enhanced by using an optical collecting system colinear with the beam. We have found that for typical atomic transitions, there can be a gain in illuminance at a photographic plate on the order of 10 over the perpendicular optic axis case, and maintaining the same resolution. There is also a simultaneous increase in the image size. Furthermore, photoelectric gains of 10-50 are possible for resolutions of 1:10^4. The limit on resolution in high energy atomic beam experiments is not imposed by the optical collecting systems, but rather by the energy spread in the excitation foil. This resolution limit is on the order of 1:10^4, using current foil technology.

30 The Meinel was loaned to Rice by the U.S. Air Force Aerospace Research Laboratory, Wright-Patterson A.F.B., Ohio.
Appendix A:
The "Axicon"

Another rather ingenious device to collect light from a fast moving beam has been proposed by Bickel.\textsuperscript{31} The device, which he calls an "axicon" is shown in Figure 13a. The rays of light emitted perpendicular to the axis of the conical mirror are reflected parallel to the axis. The lens then focuses these rays at the primary focal point. As may easily be shown by the usual constructions for image size\textsuperscript{32}, a ray deviating from parallel to the lens axis by an angle \( \delta \) will fall at a radial distance \( \rho \) from the focal point given by

\[
\rho = f_1 \tan \delta
\]  

(39)

when \( f_1 \) is the focal length of the lens. If the elements are properly aligned on the same axis the radial distance will correspond to the angle of emission \( \delta \) of a photon, where \( \delta \) is the angle between the perpendicular from the optic axis to the point on the conical mirror which reflected the photon, and the path taken by the photon itself. This is diagramed in Figure 13a. Let it be emphasized that no attempt is being made to form an image in the usual sense, so that the light pattern at the focus is a function of \( \delta \) and \( f_1 \) only. Which ray goes where is made irrelevant by the cylindrical symmetry of the system.

\textsuperscript{32} E.g. Figure 44 of F.A. Jenkins and H.E. White, \textit{op. cit.}, p. 56.
Naturally, by equation (1), the angle of emission corresponds to a Doppler shift of wavelength of the photon as given by (1), with $\pi/2 + \delta = \theta$, provided that the photon is emitted in the plane defined by the optic axis and the perpendicular to this point of reflection. Consideration of Figure 13b will show that points in the beam outside this plane may also have a deviation from the central ray of $\delta$, and hence fall a distance $\rho$ from the focus, with a Doppler shift corresponding to some $\theta \in \frac{\pi}{2} - \delta < \theta < \frac{\pi}{2} + \delta$, as given by equation (1).

Thus the light flux falling a distance $\rho$ from the focus has a Doppler profile, rather than a well defined shift, as a result of the finite beam diameter even if the beam axis falls exactly on the optic axis. This profile may be found by integrating $L^v_{nn}$ over the segment of a conical shell formed by the intersection of the beam with the conical shell of angle $\delta$ about the perpendicular to the beam axis at the point of reflection. This unnormalized profile may be calculated for any representative cone, again by the cylindrical symmetry of the system and the form of $L^v_{nn}$. As this calculation is not needed here, it will be omitted.

The parameter of immediate interest is the illuminance of the image plane at $f$ as a function of $\rho$, and therefore as a function of the full Doppler width. To compute this we will calculate the total flux emitted within a deviation angle $\delta$ as a function $\delta'$. Consider Figure 13b again. All the
flux falling on the generatrix of the axicon through the point of reflection indicated (A) with $\sigma < \sigma'$ originates in the shaded area of the beam. By cylindrical symmetry the shape of this volume does not alter as the point A is rotated around the axicon. Then there are two ways to calculate the flux collected with $\sigma \leq \sigma'$. The first is to note that each point in the region $r_s \leq q r_b$ may emit into a solid angle

$$\Delta \Omega = 2 \pi \int_{\pi/2 - \sigma}^{\pi/2 + \sigma} \sin \theta \, d\theta = 4 \pi \sin \sigma'$$  \hspace{1cm} (40)

Then note that each point in the region $r_s > q r_b$ is restricted to some solid angle $2 \phi_A \sin \sigma'$, $\phi_A$ determined by the total angle through which the point A may be rotated while the point remains in the shaded region of Figure 14. The second, and simpler, approach is to note that the shape of the shaded area does not change as A is rotated through $2\pi$. Therefore the fraction of the beam contributing to the flux is a constant for all positions of A. We may therefore assume an effective volume $V_{ef}$ of beam emitting as in (40):

$$V_{ef} = 4 \int_0^R \int_0^{R \tan \sigma} \int_0^{(r_b^2 - x^2)^{1/2}} (r_b^2 - x^2)^{1/2} \, dy \, dx \, dz$$  \hspace{1cm} (41)

(when $\sigma$ and $r_b$ are small enough that the sides of the shaded region are effectively parallel.) Multiplying (41) by $\mathcal{F}_{nn, (z)}$ and the fraction collected, $\Delta \Omega / 4\pi = \sin \sigma'$, and letting $R \tan \sigma' = q r_b$, $0 \leq q \leq 1$ to allow the results to be expressed
in a generalized form, we have \( F'_{nn} \):

\[
F'_{nn} = 2 \frac{rb^3}{R} \left\{ q \left[ q (1-q^2)^{1/2} + \sin^{-1} q \right] \right\}

\[
\left\{ C \int_{z_{min}}^{z_{max}} e^{xp \left[ -\sum_{m} \frac{Amn}{\nu} z \right]} \, dz \right\}
\]

(42)

where (14) has been used for \( F'_{nn} \).

In any real case we will be accepting a fixed \( q \) with the spectrograph. This requires a constant \( R \) for a given \( \delta' \), which is not true as soon as \( z \) is varied, as from Figure 13a,

\[ R = R_{min} + (z - z_0) \]

when \( R_{min} \) is the radius of the axicon at \( z_0 \). A 20% variation in \( R \) corresponds to 20% of the area of a narrow slit having a Doppler profile wider than \( 2 \nu_0 \gamma / \beta \delta' \). Choosing this as an arbitrary cutoff point, we have a condition on the limits in the integral:

\[ z_{max} - z_{min} = 0.2R \]

Where \( R \) is the average value of the radius of the axicon. If we take the obvious choice of \( z_{min} = 0 \)

\[ z_{max} = 0.2R \]

(43)

The illuminance of the image is just

\[ E'_{nn} = \frac{dF'_{nn}}{dA} = \frac{dF'_{nn}}{dq} \frac{dq}{dA} \]

(44)

Now from (39), the area of the image, \( A \), is
Noting that $f_1$ and $R$ are related by the focal ratio of the spectrograph, $f/ = f_1/2R$

$$A = 4\pi q^2 r_b^2 (f/)^2$$

$$\frac{dA}{dq} = 8\pi qr_b^2 (f/)^2 \quad (45)$$

Differentiating (42),

$$\frac{dF_{nn'}}{dQ} = 2 \frac{r_b^3}{R} \left[ 2 q (1-q^2)^{1/2} + \frac{q-2}{(1-q^2)^{1/2}} + \sin^{1/2} q \right]$$

$$\left[ \sum_{m=1}^{\pm\infty} \exp \left( \frac{\pi N m}{\sqrt{q}} \right) \frac{Z_{max}}{Z_{min}} \right] (46)$$

Then by (44), (45), and (46)

$$E_{nn'} = \frac{1}{4\pi} \frac{r_b}{(f/)^2 R} \left[ 3 (1-q^2)^{1/2} + \frac{\sin^{1/2} q}{q} \right]$$

$$\left[ \sum_{m=1}^{\pm\infty} \exp \left( -\frac{\pi N m}{\sqrt{q}} \right) \frac{Z_{max}}{Z_{min}} \right] \quad (47)$$

The $q$ term (first term in brackets) is plotted in Figure 14a. The profile actually continues out beyond the limits of $K = 1$, but this region is of no interest.

The illuminance $E$ may be increased by using a smaller axicon (i.e. by increasing $r_b/R$). The Doppler spread in increased however, requiring the use of very small $q$, and hence
very small image size, to attain the illuminance and resolution gains. By equations (1) and (39), and the definitions preceding (42)

\[ \Delta \lambda = 2 \lambda_0 \gamma \beta \frac{q_r b}{K} \]

\[ \rho' = \frac{q_r b}{K} = 2 (f/\rho) q_r b \]

combining,

\[ R = \frac{\rho' \lambda_0 \gamma \beta}{(f/\rho) \Delta \lambda} \]  \hspace{1cm} (48)

This establishes the minimum "axicon" radius for a given length of slit \( \rho' \) and total Doppler spread \( \Delta \lambda \). Then assuming the peak value of 4 for the \( q \) expression

\[ E_{nn'} = \frac{r_b \Delta \lambda}{\pi (f/\rho) \rho' \lambda_0 \gamma \beta} \left[ C \int_{z_{min}}^{z_{max}} \exp \left( -\sum \frac{A_{nm}}{v} z \right) dz \right] \]

Assuming wide slits and the optimum resolution condition \( b = \Delta \lambda / MD \), we have by (17b)

\[ E_{nn'} = \frac{r_b \Delta \lambda}{\pi (f/\rho) \rho' \lambda_0 \gamma \beta M^2} \left[ C \int_{z_{min}}^{z_{max}} \exp \left( -\sum \frac{A_{nm}}{v} z \right) dz \right] \]

\hspace{1cm} (49)

This may be compared to the case with the lens at \( \theta = 0 \) by taking the ratio of (49) and (34), assuming the limits given by (43) and (48):
\[
\frac{E''_{nn'i}}{E''_{nn'i, H}} = \frac{8}{\pi^2} \frac{r_b}{\rho'} \frac{\Delta \lambda}{\lambda_0 \gamma \beta} \frac{1}{D_Z} \left[ \int_0^{\rho'} \exp \left( -\sum_{m} \frac{A_{nm}}{v} z \right) dz \right] \left[ \int_0^{f_0 - f_{mn}} \exp \left( -\sum_{m} \frac{A_{nm}}{v} z \right) dz \right]
\]

(50)

Assuming an f/10 spectrograph, \( r_b = 3 \) mm, \( \Delta \lambda / \lambda_0 \gamma \beta = 10^{-3} \), \( \rho' = 0.3 \) mm and the integrals to be equal,

\[
\frac{E''_{nn'i}}{E''_{nn'i}} = (0.8)(10)(2 \times 10^{-3})(10) = 0.16
\]

So that the ratio is less than 1 for reasonable \( \Delta \lambda / \lambda \). Also, the area of the image formed by the lens at \( \theta = 0 \) is larger, so that the signal to noise formula given in the discussion following (34) favors the lens. Also, the limits on the integrals are not really comparable as \( f_0 - f_{mn} \) can be made much larger than the \( z_{\max} \) used in (50). This can favor the lens by a considerable factor which depends on the lifetime to velocity ratio, \( \sum_{m} \frac{A_{nm}}{v} \), of the line. The total photoelectric flux available is found by integrating (47) over the entrance slit. From the formulas preceding (48) the usable length of slit is

\[
\frac{L}{z} = 2 \frac{f}{r} \frac{\Delta \lambda}{\lambda_0 \gamma \beta}
\]

(51)
noting that $M^2 q^2 = \xi^2 + \gamma^2$, when $\xi$ and $\gamma$ are measured relative to the point on the image plane corresponding to the focal point on the lens, $(\xi)$ and $\gamma$, $(\xi)$. Then,

$$F_{PA} = \frac{k b}{4\pi^2} \int \frac{d\xi}{M} \int \frac{d\gamma}{b \gamma} \left[ 3(1-q^2)^{1/2} + \frac{\sin \varphi}{l} \right]_{q = \frac{\sqrt{\xi^2 + \gamma^2}}{M}}$$

$$\left[ C \int_{z_{min}}^{z_{max}} \exp \left[ -\sum_m \frac{A_{mn}}{V} \right] dz \right] (52)$$

For purposes of comparison this may be evaluated by assuming the $q$ terms to equal its peak value, 4, so that the integral is evaluated by multiplying (47) by the slit area and the collection factor $K_t$

$$F_{PA} = k t \frac{b \eta}{\pi} \frac{4}{2(2\eta_0 \gamma \beta)} b \left[ C \int_{z_{min}}^{z_{max}} \exp \left[ -\sum_m \frac{A_{mn}}{V} \right] dz \right]$$

assuming $b = \Delta \lambda / DM$,

$$F_{PA} = \frac{k t b}{\pi (\xi)} \left( \frac{2 \Delta \lambda}{\lambda_0 \gamma \beta} \right) \left( \Delta \lambda / DM \right)$$

$$\left[ C \int_{z_{min}}^{z_{max}} \exp \left[ -\sum_m \frac{A_{mn}}{V} \right] dz \right] (53)$$

Now (37) may have the tan $\left[ \cos^{-1} (\Delta \lambda / \lambda_0 \gamma \beta) \right]$ term expanded by series, and to first order it becomes $\left( \frac{2 \Delta \lambda}{\lambda_0 \gamma \beta} \right)^{1/2}$.
Then assuming $K = 3/4$ for the axicon and nothing $f/ = Q_a$

$$\frac{F_{p,\alpha}}{F_{p,\beta}} = \pi \frac{d}{r_b} \left( \frac{2 \Delta \lambda}{\lambda_0 \gamma_\beta} \right)^{-1/2} \left[ \int_0^{f_0} \exp \left( \sum \frac{A_{nm}}{v} z \right) dz \right]$$

with the integrals equal

$$\frac{F_{p,\alpha}}{F_{p,\beta}} = \pi \frac{d}{r_b} \left( \frac{2 \Delta \lambda}{\lambda_0 \gamma_\beta} \right)^{-1/2}$$

In general $d = r_b$ so we may set this factor equal to 1.

Figure 14b plots some curves of $F_{\beta} / F_{p,\alpha}$ as a function of $A_\lambda$ and $\beta$ for $d = r_b$. These curves assume that the limits on the integrals are the same, which requires that a rather large axicon be constructed. For 1 MeV He II with $\sum A_{nm} = 0.2 \text{ cm}^{-1}$ for the 4686 Å line, approximately 10 cm of beam should be imaged to gather most of the available light (10 cm corresponds to two lifetimes, which includes 86.5% of the total emission.) This may easily be done with a lens in the $\theta = 0$ configuration with an apparatus two or three meters long and $\leq 15 \text{ cm}$ diameter. An axicon with equivalent performance must be $10 \text{ cm} / 0.2 = 50 \text{ cm}$ in radius (1 meter diameter), a rather unwieldy and expensive device, which still does not yield either more photoelectric flux or illuminanace than a simple lens in the $\theta = 0$ configuration.

The Axicon possesses the advantage of being able to look at a very short length of beam, necessary for lifetime
measurements by the time-of-flight technique. This advantage, namely the application to lifetime work, is very important. Here resolution is much less important, with 1:10^2 or less frequently acceptable (50 Å in the visible). By the argument in Section III, a moderate sized "axicon" provides a total flux gain of \( \frac{\varphi_0}{2\pi} \), which will usually be greater than 6.5 (f/1 collimator at \( \theta = \pi/2 \)), more reasonable 30 (f/5 collimator at \( \theta = \pi/2 \)). This gain is fully utilized when interference filters are employed to isolate the line. This is particularly valuable in cases such as the 6560 Å (n = 6 to n = 4) line in He II studied by Jordan et al., where the transition is very weak.

Appendix B

The Vacuum Ultraviolet

The systems discussed in the preceding sections have been implicitly confined to the region of $\lambda > 2000 \ \AA$ by the use of lenses (although LiF optics will extend this to about 1250 \AA). Reflection optics are necessary below this, and grazing incidence reflection optics below 400 or 500 \AA.$^{34}$

For the $\theta = 0$ and $\theta = \pi/2$ cases, spherical or parabolic mirrors may be used in any of the usual configurations, such as off axis incidence and reflection in the region of 15° (Herschel), Cassegrain, or Newtonian. This is the most generally applicable approach as the entire spectrum from the far infrared to 500 \AA is accessible by interchanging mirrors of identical focal length with different coatings appropriate to the wavelength regions to be studied.

For the region of 10 \AA $< \lambda < 500$ \AA various grazing incidence collimators may be employed. The most efficient are nested assemblies such as those proposed by Kantor$^{35}$ or the nested elliptical assemblies shown in Figure 15, where they are applied to the $\theta = 0$ case.

The "axicon" may be used as a reflection instrument if the design proposed by Bickel (Figure 13a) is modified by

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combining the function of the conical mirror and the lens. This may be done by using the surface of revolution shown in Figure 16. The diagram on the left half of the parabola shows the well known property of a parabola: the reflection of all rays parallel to the axis onto the focus. Notice that these rays are also perpendicular to the latus rectum (l.r.). Therefore if a surface is formed by revolving the parabola about the latus rectum, it will have the property of reflection all rays of light emitted perpendicular to the l.r. onto the focus. Rays emitted at an angle $\phi$ (as defined in Figure 13 earlier) will fall a distance $\rho$ from the focus, approximately as given in (39). The characteristics are the same as discussed earlier. The length of beam utilized is subject to the same 0.2 $\bar{R}$ restriction. The actual segment of the surface of resolution to be utilized is found by taking the segment from the intercept of the parabola with a line from the focus at an angle $\phi_a$ ($\phi_a$ equal to the acceptance angle of the spectrograph) to the l.r., to a point 0.2 $\bar{R}$ further from the focus (as measured along the l.r.). This system fails at about $\lambda = 500$ $\AA$ as grazing incidence reflections become necessary.

This pure reflection "axicon" will be especially useful in the regions of $\lambda$ below 2400 $\AA$, where narrow band interference filters are not readily available and monochromatizers must be used. The attendant losses due to small slit openings and

other losses make the gains of = 30 available with an "axicon" (Appendix A) particularly attractive.
Appendix C
Glossary of Terms

Flux  As used in this paper, flux will be the number of photons per unit time passing through some area under consideration, such as the entrance slit of a spectrograph or spectrometer. This differs from the usual usage where flux is the energy per unit time passing through the area. The modified definition is used here for the following reasons. First, the energy is related to the wavelength, which varies with the angle of emission when a high velocity beam is used as a source. This variation complicates the formulas derived in this paper and obscures the fundamental relationships. Second, most detectors (especially photomultipliers) respond to quanta rather than total energy. (Albeit with a quantum efficiency which depends on energy, but not usually in a simple way.) This is particularly applicable here as photon counting is a frequently used technique in low-signal experiments such as these. The results may be readily converted by multiplying equation (14) by $\frac{hc}{\lambda \beta (1 - \cos \beta)}$ and carrying out the remainder of the calculations.

Luminance  is the light flux emitted per unit area of source into the $2\pi$ steradians, assuming the source to be isotropic. Isotropic emission is a good approximation for a radiation ion beam (see equation (13)).
Illuminance is the incident light flux per unit area of image.

Line Profile is the flux or illuminance as a function of some parameter such as wavelength or plate coordinate. It may equivalently be regarded as the probability distribution for a photon to have a given wavelength or fall on a given plate coordinate.
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Bibliography


