RICE UNIVERSITY

The Deduction of Phase Equivalent Potentials:  
A New Method and its Application to n- α 
Elastic Scattering

by

William Albert Pearce

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Abstract

Various methods for the deduction of nuclear potentials from the phase shifts are discussed. The Swan method is developed in detail and several tests are examined. Finally the method is applied to n-α scattering. The procedure for smoothing relatively scattered data is discussed and potentials for the interaction are deduced.
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INTRODUCTION

The general problem of investigating the nuclear force is one which has been pursued with vigor for many years. One approach is to study the potential which gives rise to that force and hence will explain the scattering. The body of literature devoted to the deduction of such a potential is quite large and cannot adequately be reviewed here. It will suffice us merely to indicate the more important methods and techniques which have been proposed in the past and to outline a new method due to P. SWAN. This Swan method is concerned not with finding the Nuclear Potential, but with the somewhat more modest task of finding a set of central potentials (one for each partial wave) which will give the scattering found in nature as reflected in the elastic scattering phase shifts.

The Swan method will be formulated and tested in section II. We wish to determine how well it will work and to demonstrate its application.

In section III, we apply this method to n-α elastic scattering. We explain the techniques necessary to treat real data in which there is scatter. Some of the limitations and difficulties of the present formulation are brought to light. Phase equivalent potentials for S, P_{1/2}, and P_{3/2} partial waves are found and briefly discussed.

The details of the derivations of the equations used, the descriptions of the computer programs and calculational techniques, and other comments have been relegated to a set of appendices for the sake of readability and completeness.
I. OTHER METHODS

Perhaps the first real progress in the attempt to establish the relationship between the scattering phase shifts and the potential was made by N. LEVINSON in 1949. He showed that two potentials which decrease rapidly enough, give rise to no bound states, and which give the same phase shifts for all angular momenta are identical. Later that same year, BARGMANN made a study of phase equivalent potentials and introduced a technique for constructing them. His method consisted of giving the Jost function (see for example WU and OHMURA or GOLDBERGER and WATSON) a simple polynomial form and examining the families of potentials associated with it. However, the analysis rapidly becomes difficult so that one is really restricted to particularly simple algebraic forms for the Jost function.

Later, the GEL'FAND LEVITAN method was developed as is described below. More recently, OHMURA proposed a variational method for the derivation of potentials from phase shift data. If one can, by some means, obtain an initial approximation to the potential, one can improve it by application of a technique due to HYLLERASS.
GEL'FAND LEVITAN METHOD

In 1951, GEL'FAND and LEVITAN presented an integral equation which relates the phase shifts as functions of energy to another function from which the scattering potential can be obtained. The phase shifts are to be described by what is called a spectral function and the potential is derived from the kernel which solves their integral equation. The method in more physical form has been formulated by JOST and Kohn, Blazek, Kay and Moses and others. A fairly clear presentation of it may be found in Wu and Ohmura. R.G. Newton has applied it in simple cases and has formulated it for non-central potentials.

For simplicity, let us consider S-wave scattering by a central potential for relatively low energies so that the scattering is well described by the Schrödinger equation:

1-1) \[ \left( \frac{d^2}{dr^2} + k^2 \right) \phi = U(r) \phi \]

In order that equations which we want to use be guaranteed to have unique solutions, we must assume that the potential \( U(r) \) is such that

\[ \int_0^\infty r^n |U(r)| \, dr < \infty \quad n = 1, 2 \]

We can choose two independent solutions to 1-1) that have the asymptotic behavior
\[ f(\pm k, r) \xrightarrow{r \to \infty} e^{\pm ikr}. \]

We may take the general solution as a linear combination of these; we can choose this general solution so that

\[ \phi(k, 0) = 0, \quad \frac{\partial \phi}{\partial r} \bigg|_{r=0} = 1. \]

Such a combination is

\[ \phi = \frac{1}{2i} \left[ f(k) f(-k, r) - f(-k) f(k, r) \right] \]

where

\[ f(k) = f(k, 0). \]

Asymptotically we have

\[ \phi \to -\frac{1}{2i} f(-k) \left[ e^{-ikr} - \frac{f(k)}{f(-k)} f(k, r) \right]. \]

We have from the asymptotic behavior of \( f(k, r) \) that

\[ f(-k, r) = f^*(k, r) \quad \text{and} \quad f(-k) = f^*(k). \]

\( f(k, r) \) is, then, the Jost function. The phase shift is related to it by

\[ e^{2i\sigma} = \frac{f(k)}{f^*(k)}, \]

so that

\[ \sigma = \mu\left[ \text{Im} f(k) \right] = \text{arg} f(k). \]

Now, suppose for simplicity that there are no bound states. The asymptotic behavior of \( \phi \) is

\[ \phi(k, r) \to \frac{f(k)}{k} \sin(kr + \sigma). \]

Now, \( \lim_{|k| \to \infty} f(k) = 1 \) for \( \mu(k) < 0 \), so by the Cauchy theorem we have
\[ \ln f(k) = \frac{j}{2\pi} \int_{-\infty}^{\infty} \frac{R_{\kappa} f(k)}{k - \rho} \, d\rho \]

and

\[ \psi = \frac{j}{2\pi} \int_{-\infty}^{\infty} \frac{\ln f(k)}{k - \rho} \, d\rho \]

For real \( p \) we can push the poles into the upper half plane to get

\[ \ln f(p) = \frac{j}{\pi} \int_{0}^{\infty} \frac{\delta(k) \, dk}{\rho - k - i\epsilon} \]

Thus we have \( f(k) \) in terms of the phase shift. Notice, however, that we need to know \( \delta(k) \) for all energies from zero to infinity. This last is one of the major difficulties encountered when one attempts to use this method.

The Gel'fand Levitan theorem is the following: Define \( D(r, r') \), the spectral weight function by

\[ D(r, r') = \frac{\pi}{\rho} \int_{0}^{\infty} \sin kr \sin kr' \left[ 1 - f(k) \right] \, dk \]

If we assume that the Gel'fand Levitan equation

\[ K(r, r') = -D(r, r') - \int_{0}^{r} k(r, r'') D(r'', r') \, dr'' \]

has a unique solution for \( K(r, r') \), then the scattering potential which yields the phase shifts is given by

\[ U(r) = 2 \frac{d}{dr} k(r, r) \]
Further, the wave function is given by

\[ \phi(k, r) = \frac{\sin k r}{k} \int_0^r K(r, r') \frac{\sin k r'}{k} \, dr' . \]

If there are bound states, the above needs some slight modifications, but we need not treat them here. However, the major result is that upon the introduction of bound states we find that we can only deduce families of potentials, all members of which yield the same phase shifts. Of such a family, we can single out one potential if we wish to make an auxiliary demand, namely that the potential be such that

\[ \lim_{r \to \infty} e^{\mu r} U(r) = 0 \]

for any finite value of \( \mu \).

There are several difficulties one might encounter in attempting to apply such a method to a real experimental set of phase shifts. First of all, one never knows \( S(k) \) over its whole range. This means that we would have to extrapolate the experimental phases to infinity (i.e. the real part of \( S(k) \) since we have not made any attempt to treat inelastic scattering). Second, having suitably defined \( f(k) \) via this extrapolation, we would have to solve two integral equations. This can be done formally by iteration, a process which in this case can be shown to converge. However, in practice such a solution is very difficult to carry through with any precision.

The major importance of this approach is that from it
one can obtain the conditions under which a potential can be found. This approach then, constitutes the formal mathematical solution to the inverse scattering problem, the problem of finding the potential (see also AGRANOVIČ and MARCHENKO).

THE OHMURA METHOD

In 1956, OHMURA proposed a variational method for finding potentials which is similar in form to the one to be used here. One begins with the Schrödinger equation (for simplicity, treat $S$-waves only):

$$\left[ \frac{d^2}{dr^2} + k^2 + V(r) \right] \psi(k, r) = 0$$

with the boundary conditions

$$\psi(k, 0) = 0, \quad \psi(k, r) \rightarrow C \sin(kr + \delta(k)), \quad \text{as } r \rightarrow \infty$$

then, on multiplication by $\psi$ and integration over all $r$, one gets a stationary expression for $V(r)$:

$$-\int_0^\infty \psi(k, r) \left[ \frac{d^2}{dr^2} + k^2 \right] \psi(k, r) \, dr = \int_0^\infty \psi^2(k, r) \, V(r) \, dr.$$

An example of how this might be used to determine $V(r)$ is quoted by OHMURA and has been more thoroughly investigated by PEARCE. This scheme is as follows: Choose the form of the wave function to be
\[
\phi(kr) = (1 - e^{-kr}) \cos kr + \cot \delta \sin kr
\]

and parameterize the potential by

\[
V(r) = \sum_{j=1}^{\infty} a_j e^{-\lambda_j r}
\]

It is supposed that such an expansion should converge rapidly enough that a fairly good approximation can be obtained by taking only a few terms, say N. Now, if one evaluates equation 1-8) for several well separated values of k, say \(k_j\), one can obtain a set of N simultaneous equations which may be solved for the \(a_j\).

As it stands, the Ohmura method seems to be unsuited for practical use. There are some modifications which might improve the situation, though they have not been investigated. First of all, the set of linear equations tends to be rather poorly conditioned. One would expect to be able to improve the conditioning by expanding the integrands in powers of \(k^2\), doing the integrations explicitly, and then equating like powers of \(k^2\). Another weakness of the method is the fact that no means is provided for determining the range parameters in the potential, \(\{\lambda_j\}\). If one gave the \(\lambda_j\) a systematic form e.g., \(\lambda_j = 1/(jb)\), one could solve the N simultaneous equations for N-1 terms in the potential and a basic range parameter \(b\).

Further, the error in the potential produced by this method should behave as the square of the error in the wave
function. If one could improve the wave function, one might expect that one could obtain a better potential. Possibly a wave function of the form used in the Swan method would improve things; i.e.

\[ \phi(k, r) = \cos \delta_2 \int \mathcal{E}(r) - q_s(k, r) N_s(kr) \]

with

\[ q_s(k, r) = (1 - e^{-\frac{r}{l}})^{2\lambda + 1} \left[ 1 + e^{-\frac{r}{l}} \sum_{m=1}^{\infty} \mathcal{E}_m(kr)^m \right]. \]

Whether or not this would indeed effect an improvement has not been investigated. However, as the method is a variational one without bounds, the potential which it determines might prove to be unstable with the use of this somewhat more sophisticated wave function. It would seem that the inherent instabilities and ill-conditioning are worsened by the fact that equation 1-8 is quadratic in the wave function. This means that the potential will be badly determined near the origin, where the wave function is small.
A quite different approach may be made than the mathematical one. If the potential is not too strong, one can get a reasonable approximation to the wave function by distorting a plane wave in some fashion within the range of the potential. By properly using this approximate solution to the wave equation, one can get an approximation to the potential.

First, it is necessary to determine the way in which one must distort the plane wave. Consider the radial Schrödinger equation describing elastic two particle scattering in the center of mass coordinate frame:

\[ 2-1) \left[ \frac{d^2}{dr^2} + k^2 - \frac{\ell(\ell+1)}{r^2} - U(r) \right] \psi_\ell(r) = 0, \]

where \( \psi_\ell \) is the radial wave function for the \( \ell \)th partial wave and \( U(r) \) is the reduced potential

\[ U(r) = \frac{2m}{\hbar^2} V(r), \]

with \( V(r) \) the usual potential function. Consider also the corresponding equation for zero potential:

\[ 2-2) \left[ \frac{d^2}{dr^2} + k^2 - \frac{\ell(\ell+1)}{r^2} \right] \phi(r) = 0, \]

and let \( \psi_\ell \) and \( \phi_\ell \) be the wave functions for zero energy. These satisfy
Cross multiply these and subtract to get rid of explicit mention of the potential:

\[- \left[ \phi'' \psi_0 - \phi'' \psi - \psi'' \psi + \psi'' \psi \right] = \left( k^2 - \frac{\ell (\ell+1)}{r^2} \right) \phi \psi_0 - \left( - \frac{l (l+1)}{r^2} \right) \phi \psi_0 \]

\[- \left[ k^2 - \frac{\ell (\ell+1)}{r^2} - U \right] \psi \psi_0 + \left( - \frac{l (l+1)}{r^2} - U \right) \phi \psi_0 \]

\[= k^2 \left( \phi \psi_0 - \psi \psi_0 \right) . \]

Now integrate both sides:

\[- \left[ \phi' \phi_0 - \phi' \phi - \psi' \psi_0 + \psi' \psi \right]_0^\infty = k^2 \int_0^\infty (\phi \psi_0 - \psi \psi_0) \, dr . \]

We can simplify this if we agree to normalize \( \psi \) at \( \infty \) to match \( \phi \). Note that the upper limit on the left side gives zero contribution. We know too that \( \psi \) must be zero at the origin, leading to

\[2-5) \quad \phi \phi' - \phi' \phi = k^2 \int_0^\infty (\phi \psi_0 - \psi \psi_0) \, dr . \]

This will be the basic equation from which we can determine
an approximation to $\psi$.

We can choose the normalization of $\phi$ so that

$$\phi(r) = \cot \delta_0 \mathcal{J}_0(kr) - \mathcal{N}_0(kr)$$

where

$$\mathcal{J}_0(kr) = kr J_0(kr)$$
$$\mathcal{N}_0(kr) = kr N_0(kr)$$

and $J_0(kr), N_0(kr)$ are the usual spherical Bessel and Neumann functions (see SCHIFF, p. 77): 

$$J_0(kr) = \sqrt{\frac{\pi}{2kr}} J_{\alpha + \frac{1}{2}}(kr)$$
$$N_0(kr) = \sqrt{\frac{\pi}{2kr}} (-1)^{\delta + 1} J_{\beta - \frac{1}{2}}(kr).$$

The unscattered wave must contain only components regular at the origin. The effect of the potential will be not only to distort the wave but also to introduce some of the irregular solution; therefore, let us write

$$\psi(r) = \cot \delta_0 \mathcal{J}_0(kr) - \mathcal{N}_0(kr) \mathcal{N}_0(kr)$$

as our approximation to the wave function. What conditions must be met by our distorting function $\mathcal{N}_0(kr)$? That $\mathcal{N}_0$ must asymptotically approach unity at infinity was determined by our normalization at infinity. $\psi(r)$ must also be regular at the origin and, in fact, behave like $r^{\delta}$ near zero. However, $\mathcal{N}_0$ behaves like $r^{-\delta}$ at the origin so that $\mathcal{N}_0(kr)$ must go to zero at least as fast as $r^{2\delta}$.
We want, in addition, to make $g_0(t, r)$ as flexible as possible so that by determining the constants contained in it we may get as accurate a wave function as possible. One such choice, complicated enough to be reasonably flexible and simple enough to be tractable, is

$$2-7) \quad g_0(t, r) = (1 - e^{-\gamma r})^{2t+1} \left[ 1 + e^{\gamma r} \sum_{n=0}^{\infty} \xi_n (k r)^{2n} \right].$$

The set $\{\gamma, \xi_n\}$ must then be determined by making use of equation 2-5). We will not be able to deduce the whole set of $\xi_n$'s but will be limited by the extent of our knowledge of the phase shifts. We can expand in powers of $k^2$ and do the integrals explicitly. We then equate like powers of $k^2$ so that equation 2-5) becomes a set of simultaneous equations to be solved for $\{\gamma, \xi_n\}$. The equations take the following form (where $n$ takes the values 0, 1, 2, ..., N-1, and the $\{C_n\}$ are determined from the phase shifts by

$$2-8) \quad \frac{k^{2t+1}}{[(2t+1)!]^2} \cos \delta = \sum_{m=0}^{N-1} C_m (-1)^{m+1} \frac{k^{2m}}{2^m}$$

For $n = 0$ we have

$$2-9) \quad C_1 + \frac{2C_0}{2t+1} \sum_{\nu=1}^{2t+1} (-1)^{\nu+1} \frac{1}{(\nu)!} \frac{1}{(\gamma\nu)!} - \delta_{30} \frac{3}{2\gamma}$$

$$+ \left( 1 - \delta_{30} \right) \frac{1}{(2t+1)^2} \left[ \frac{\gamma^{2t+1}}{(2t+1)!} \sum_{\nu=2}^{2t+1} (-1)^{\nu} \frac{(2\nu+2)(\nu, \nu\nu)}{\nu \mu} \right] = 0.$$
and for \( n \, 1, 2, \ldots, N-1 \)

\[
C_{n+1} \, (-1)^n + \frac{C_0 \, (2n+1)!!}{\prod_{\mu=0}^{2n+2}} \sum_{\nu=0}^{2n+1} \frac{(-1)^\nu \, (2\nu+1)}{(2\nu+1)!! \nu!!} \frac{1}{(\sigma_0)^{2n+2}} \\
+ \frac{(-1)^n \, (2n-1)!!}{\prod_{\mu=0}^{2n-2}} \sum_{m=0}^{2n-1} \frac{C_m \, (2n-2m+1)!!}{(2\nu+1+2n-2m)!!} \frac{(-1)^\nu \, (2\nu+1)}{(2\nu+1)!! \nu!!} \frac{1}{(\sigma_0)^{2n-2m+2}} \\
+ \left( \psi \, \nu \leq 1 \right) \frac{(2n-1)!!}{(2\nu+1) (2n)!!} \frac{\sum_{\nu=0}^{2n-2} \frac{(-1)^\nu \, (2\nu+2)}{(2\nu+1)!! \nu!!} \frac{1}{(\sigma_0)^{2n-2\nu+1}}}{\prod_{\mu=0}^{2n-2}} \\
2-10
\]

\[
= \left\{ \begin{array}{l}
C_0 \, (2n+1)!! \sum_{\nu=0}^{2n+1} \frac{(-1)^\nu \, (2\nu+1)}{(2\nu+1)!! \nu!!} \frac{1}{(\sigma_0)^{2n+2}} \\
- \left( \psi \, \nu \leq 1 \right) \frac{1}{2\nu+1} \frac{\gamma^{2n-2\nu+1}}{(2\nu+1)!! \nu!!} \sum_{\nu=0}^{2n+1} \frac{(-1)^\nu \, (2\nu+2)}{(2\nu+1)!! \nu!!} \frac{1}{(\sigma_0)^{2n-2\nu+1}} \\
- \left( \psi \, \nu \geq \ell \right) \frac{1}{2\nu+1} \sum_{\nu=0}^{2n+1} \frac{(-1)^\nu \, (2\nu+2)}{(2\nu+1)!! \nu!!} \frac{(2n-2\nu)!!}{(\sigma_0)^{2n-2\nu+1}} \\
\end{array} \right\} \\
\chi \left[ \sum_{m=1}^{n} \frac{E_m}{(2n-2m)!! \prod_{\mu=0}^{2n-2-2m}} \right].
\]

The details of the derivation of these rather messy equations will, for convenience, be given in appendix A.

We note that the equation for \( n = 0 \) contains only \( \gamma \) and that it is transcendental in \( \sigma \) if \( \ell > 0 \). This means that we are compelled to accept a numerical rather than an analytic solution for \( \gamma \). For \( \ell = 0 \) the equation is a
quadratic and as such possesses two solutions. Is it possible a priori to specify which root will be the physically meaningful one? No, the testing of the procedure revealed that one root or the other would be good, but only its use throughout the whole process could determine whether or not it was a good root. Experience quickly showed that only one of the roots was capable of giving physical results; i.e. of yielding a really phase-equivalent potential.

Having obtained a value for $\gamma$, we may use it to solve the remaining $N-2$ equations which are linear in the $\ell_m$'s. Thus, we have our approximation to the wave function.

We need some equation involving both the potential and the wave function. To get this, start as usual with the Schrodinger equation:

$$2-1) \quad \psi'' + \left[ k^2 - \frac{\ell(\ell+1)}{r^2} - U(r) \right] \psi = 0$$

Corresponding to this there is the equation without the potential:

$$2-2) \quad \phi'' + \left[ k^2 - \frac{\ell(\ell+1)}{r^2} \right] \phi = 0$$

Now eliminate the unwanted parts from the equations by cross multiplying and subtracting, then integrating to get

$$\int_0^R \left[ \psi'' \phi - \phi'' \psi \right] dr = \int_0^R U(r) \phi \psi dr$$
Integrating the left side by parts and requiring

\[ \psi(0) = 0 = \phi(0) \]

we get

\[ \psi' \phi - \phi' \psi \bigg|_0^R = \int_0^R \psi \phi \ u(r) \]

Now we can take advantage of our knowledge of the asymptotic forms of \( \phi \) and \( \psi \):

\[ \psi \rightarrow \cos \delta \ J_0(kr) - \sin \delta \ N_0(kr) \rightarrow \cos \left[ kr - \left( k+1 \right) \frac{\pi}{2} + \delta \right] \]

\[ \phi \rightarrow \ J_0(kr) \rightarrow \cos \left[ kr - \left( k+1 \right) \frac{\pi}{2} \right] \]

and so

\[ \psi' \rightarrow -k \sin \left[ kr - \left( k+1 \right) \frac{\pi}{2} + \delta \right] \]

\[ \phi' \rightarrow -k \sin \left[ kr - \left( k+1 \right) \frac{\pi}{2} \right] \]

Consequently we can take the limit as \( R \rightarrow \infty \) and put the asymptotic forms into the equation to get

\[ -k \sin \left[ kr - \left( k+1 \right) \frac{\pi}{2} + \delta \right] \cos \left[ kr - \left( k+1 \right) \frac{\pi}{2} \right] + k \sin \left[ kr - \left( k+1 \right) \frac{\pi}{2} \right] \]

\[ \times \cos \left[ kr - \left( k+1 \right) \frac{\pi}{2} + \delta \right] \]

or

\[ -k \sin \delta = \int_0^\infty \psi \phi \ u(r) \ dr \]

where we have taken the limit \( R \rightarrow \infty \) on the right side.

We can now change the normalization of \( \psi \) by dividing through by \( \sin \delta \):

\[ -1 = \frac{k}{k} \int_0^\infty \psi \phi \ u(r) \ dr \]

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where now, we represent \( \psi \) by our distorted wave approximation:

\[
\psi = \cot \delta_j \, J_j(kr) - g_j(kr) \, N_j(kr)
\]

Since \( \phi = J_j(kr) \), the equation can be rearranged:

\[
\cot \delta_k = \frac{\int_0^\infty g_k N_j J_j U(r) \, dr - 1}{\int_0^\infty J_j^2 U(r) \, dr}
\]

Realizing that this equation is valid only if \( \delta \neq 0 \) for \( k > 0 \) (if this were not the case, we could use the corresponding formula for \( \tan \delta \) or a combination of the two), we can proceed to solve for \( U(r) \). The easiest way to do this, perhaps, is to parameterize \( U(r) \) and try then to determine the best choice for the parameters. One good choice that may prove flexible enough and which still leaves us with integrals that can be evaluated is

\[
U(r) = \sum_\rho \, U_\rho \, \exp \left( -r / f(\rho) b \right)
\]

where the \( \{ U_\rho \} \) and \( b \) are to be determined. We do not yet need to specify what \( f(\rho) \) is, we can leave it subject to change in order to make the parameterization as flexible as possible. In practice, we found that the choice \( f(\rho) = \rho \) was indeed about as good as any; particularly so since it never proved feasible to take more than a few terms in the series. Now, we can insert our form for \( U(r) \) and, knowing
\[ g_z(k,r) \] we can expand the integrands in series in powers of \( k^2 \), do the integrals explicitly, and equate like powers of \( k^2 \). Thus we can arrive at a set of \( N \) equations: for \( n = 0, 1, \ldots, N-1 \),

\[
\sum_{\rho=1}^{N-1} U_{\rho} \left\{ \left( (2\rho+1)!! \right)^2 (-1)^{\rho} \sum_{m=0}^{n-1} \frac{(2m+2\rho)!}{m!} \frac{(2\rho+2m+2)!}{(2\rho+m+1)!} \frac{\left[ f(\rho) b \right]^{2N+3}}{m!(2\rho+2m+1)!} \right\}
\]

\[ 2-14 \]

\[
- (2\rho+1)!! \sum_{m=0}^{n-1} \frac{(2m-2\rho)!}{(n-m)!} \frac{(2\rho+1)!}{(2\rho+2m+1)!} \prod_{m=0}^{n-m-1} (2\rho+1-2m) \]

\[
\chi \sum_{\nu=0}^{2N+1} \left( \frac{(-1)^{\nu} (2\nu+1)}{2^\nu} \right) \left[ C_m + (1 - C_m) \varepsilon_m \right] \left[ \frac{1}{f(\nu) b + (\nu + 1 - C_m) \chi} \right]^{2N+2} = \delta^\nu_0 \]

where \( \gamma, \varepsilon_m \) are parameters in the distortion function

\[
g_z(k,r) = (1 - e^{-\gamma r})^{2\rho+1} \left[ 1 + e^{-\gamma r} \sum_{m=1}^{\infty} \varepsilon_m (k \nu)^{2m} \right]
\]

and where the \( \{ C_m \} \) represent the phase shifts:

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\[
\frac{k^{2\rho+1} \cos \delta_k}{(2\rho+1)!!} = \sum_{m=0}^{\infty} C_m k^{2m} (-1)^{m+1}
\]

It is because we are only able to determine \( C_0 \ldots C_N \) that we have only \( N \) equations. Knowing only \( N+1 \) of the \( C_m \) 's also restricts the number of \( \varepsilon_m \) 's in the distorting function to \( N-1 \).

The set of equations is linear in the \( N-1 \) values of \( U_\rho \) and is of high order in \( b \), the intrinsic range parameter.
Hence, we can arrive at a numerical solution for $b$ by trying several values for it, solving the first $N-1$ equations for $\{U_{\rho}\}$ and inserting these values along with the trial value for $b$ in the last equation. If the equation is satisfied, we have a solution.

Now, what will be our procedure in practical detail? We need to know the phase shifts for a given partial wave as a function of energy over as wide an energy range as possible. It was noticed years ago (BLATT and JACKSON) that $k^{2\ell+1}\cot \delta_{\ell}$ is a much simpler function of energy than is $\delta_{\ell}$. Therefore, we convert our phase shifts to this form, or rather, to

$$2-8) \quad \frac{k^{2\ell+1}c_{0}+\delta_{\ell}}{[(2\ell+1)!]^2} = \sum_{m=0}^{\infty} C_{m} k^{m}(-1)^{m+1}$$

and plot it as a function of $k^2$. Now, we need to know the coefficients in the series. Clearly the best that we can do is to obtain only a finite number of the $C_{m}$. Therefore, we do a least squares fit to determine them. In practice, it pays to weight the low energy data points heavily in order to determine the first few $C_{m}$'s accurately. This weighting is necessary because the least squares procedure produces a fit to the over-all curve, but we want to require that $C_{0}$ and $C_{1}$ be particularly accurate (they alone determine $\gamma$ in the distortion function and are therefore very influential in the determination of the basic range parameter $b$); i.e. we want an especially good fit in the neighbor-
hood of the origin.

Having thus obtained a set of shape parameters \( \{ C_m \} \), we can obtain \( \{ \varepsilon_m, \gamma \} \) and thus \( \{ U_{\rho}, b \} \), the potential. There may be multiple solutions for \( \gamma \), and for each one, there may be multiple solutions for \( \{ U_{\rho}, b \} \). How does one choose between them? The first criterion is to look at the determinant of the matrix of coefficients (in the solution of the set of linear equations). If the value is relatively small, the equations will be slightly ill-conditioned and the answers may be slightly ridiculous when examined more closely. Often this criterion is not sufficient, then the only recourse is to carry through the process for each set of answers, solve the Schrodinger equation numerically to derive the set of phase shifts corresponding to the deduced potential, and then compare these with the initial data. Thus far, it has been observed that the process will produce only one really good answer.
ITERATION TECHNIQUE

We notice that if we take a potential and from it generate a set of phase shifts and thus a set of shape parameters \( \{ C_n \} \), substitute them into the potential equations 2-12) along with our test potential, we can solve for the distortion parameters in the wave function \( \{ \xi_\omega, \gamma \} \). However, if we use this set \( \{ C_n \} \) in equations 2-10) and thereby determine \( \{ \xi_\omega, \gamma \} \), the two sets of values do not quite agree. The disagreement is not bad but it might be surprising. The answer to this conflict seems to be the essential inflexibility of the wave function parameterization. The most straightforward way to overcome this difficulty is merely to guess at a more complicated form for the wave function and try again. This does not work very well unless there is some really systematic way of increasing the wave function flexibility. It is now that being somewhat sly will pay great dividends. We notice that when we start at the beginning, knowing only the phase shifts, or, equivalently, the set \( \{ C_n \} \), carry out the potential deduction scheme, and use this deduced potential to find its corresponding set of phase shifts and, thereby, a set of \( \{ C_n' \} \) that the two sets \( \{ C_n \} \) and \( \{ C_n' \} \) do not quite match. Our task is to modify the wave function parameters so that they will match or at least come closer to matching. We can do this formally if we take the derivatives of the equations for the potential 2-12) and replace calculus
increments by finite increments:

\[
\sum_{\mu=1}^{N-1} \Delta U_\mu \left\{ (2\mu+1)!! \left\{ (-1)^{\mu} \sum_{m=0}^{\infty} \frac{(2m+2\mu)!!(2m+2\mu+2)!}{m!(2m+1)!(2m+2m+1)!!} \left[ \frac{f(\mu)}{b} \right]^{2m+2\mu+3} \binom{\mu}{m} \right\} \right. \\
- \left. (2\mu+1)!! \sum_{m=0}^{\infty} \frac{(2m-2\mu)!(2m+1)!!}{(\mu-m)!!(2m+2\mu+1)!(2m+2m+1)!!} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}}{\binom{\mu}{\nu} b^\nu} \frac{[\delta_{\mu0} + (1-\delta_{\mu0}) \epsilon_m]}{1 + (\nu + 1 - \delta_{\mu0}) \gamma} \right]^{2m+2\mu+3} \right. \\
+ \Delta b \sum_{\mu=1}^{N-1} U_\mu \left\{ (2\mu+1)!! \left\{ (-1)^{\mu} \sum_{m=0}^{\infty} \frac{(2m+2\mu)!!(2m+2m+2)!(2m+2\mu+2)!}{m!(2m+1)!(2m+2m+1)!!} \left[ \frac{f(\mu)}{b} \right]^{2m+2\mu+3} \binom{\mu}{m} \right\} \right. \\
- \left. (2\mu+1)!! \sum_{m=0}^{\infty} \frac{(2m-2\mu)!(2m+1)!!}{(\mu-m)!!(2m+2\mu+1)!(2m+2m+1)!!} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}}{\binom{\mu}{\nu} b^\nu} \frac{[\delta_{\mu0} + (1-\delta_{\mu0}) \epsilon_m]}{1 + (\nu + 1 - \delta_{\mu0}) \gamma} \right]^{2m+2\mu+3} \right. \\
= \sum_{\mu=1}^{N-1} U_\mu \left\{ (2\mu+1)!! \left\{ (-1)^{\mu} \sum_{m=0}^{\infty} \frac{(2m+2\mu)!!(2m+2\mu+2)!(2m+2\mu+2)!!}{m!(2m+1)!(2m+2m+1)!!} \left[ \frac{f(\mu)}{b} \right]^{2m+2\mu+3} \binom{\mu}{m} \right\} \right. \\
- \left. (2\mu+1)!! \sum_{m=0}^{\infty} \frac{(2m-2\mu)!(2m+1)!!}{(\mu-m)!!(2m+2\mu+1)!(2m+2m+1)!!} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}}{\binom{\mu}{\nu} b^\nu} \frac{[\delta_{\mu0} + (1-\delta_{\mu0}) \epsilon_m]}{1 + (\nu + 1 - \delta_{\mu0}) \gamma} \right]^{2m+2\mu+3} \right. \\
= \sum_{\mu=1}^{N-1} U_\mu \left\{ (2\mu+1)!! \left\{ (-1)^{\mu} \sum_{m=0}^{\infty} \frac{(2m+2\mu)!!(2m+2\mu+2)!(2m+2\mu+2)!!}{m!(2m+1)!(2m+2m+1)!!} \left[ \frac{f(\mu)}{b} \right]^{2m+2\mu+3} \binom{\mu}{m} \right\} \right. \\
- \left. (2\mu+1)!! \sum_{m=0}^{\infty} \frac{(2m-2\mu)!(2m+1)!!}{(\mu-m)!!(2m+2\mu+1)!(2m+2m+1)!!} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}}{\binom{\mu}{\nu} b^\nu} \frac{[\delta_{\mu0} + (1-\delta_{\mu0}) \epsilon_m]}{1 + (\nu + 1 - \delta_{\mu0}) \gamma} \right]^{2m+2\mu+3} \right. \\
= \sum_{\mu=1}^{N-1} U_\mu \left\{ (2\mu+1)!! \left\{ (-1)^{\mu} \sum_{m=0}^{\infty} \frac{(2m+2\mu)!!(2m+2\mu+2)!(2m+2\mu+2)!!}{m!(2m+1)!(2m+2m+1)!!} \left[ \frac{f(\mu)}{b} \right]^{2m+2\mu+3} \binom{\mu}{m} \right\} \right. \\
- \left. (2\mu+1)!! \sum_{m=0}^{\infty} \frac{(2m-2\mu)!(2m+1)!!}{(\mu-m)!!(2m+2\mu+1)!(2m+2m+1)!!} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}}{\binom{\mu}{\nu} b^\nu} \frac{[\delta_{\mu0} + (1-\delta_{\mu0}) \epsilon_m]}{1 + (\nu + 1 - \delta_{\mu0}) \gamma} \right]^{2m+2\mu+3} \right. \\

where we make the identification \( \Delta C_n = C_n - C_n' \). These \( \{ \Delta C_n \} \)'s may be interpreted to be essentially implicit corrections to the wave function. Now we can solve equations 2-15) for \( \{ \Delta U_\mu \} \) and \( \Delta b \), the corrections to the potential parameters. The equations are linear in \( \{ \Delta U_\mu \} \) and \( \Delta b \) so that solution is simple. The second approximation to the
potential (or, as we will call it, the iterated solution) is then

\[ U_z(r) = \sum_{\rho} \left( U_\rho + \Delta U_\rho \right) e^{-\frac{r}{f(\rho)(b+\Delta b)}} \]

With the data which is now available for n-α elastic scattering, it does not appear feasible to take this iteration technique further. However, some thought has been given to this and the few preliminary calculations which have been done seem to indicate that the following scheme does work: Take the iterated potential, derive phase shifts and a set of \( \{ C_n^{''} \} \) from it; put these in the potential equations (2-12) and solve for a corrected set of wave function distortion parameters, \( \{ \epsilon', \gamma' \} \). Then, use these along with the correct set \( \{ C_n \} \) in the potential equations to solve for a new set of potential parameters. This may now be repeated as many times as desired.

The limitations on this process have not been fully investigated, but they appear to be the restriction that the \( \{ \Delta C_n \} \) must be small and the fact that there is a certain inherent error introduced when a set \( \{ C_n \} \) is obtained by taking a least squares fit to a curve.
TESTS

The method is quite susceptible to a closed, consistent testing procedure. One may write down a likely looking potential, derive a set of phase shifts from it and proceed as if these phases were initial data. At the end, one has two checks on correctness: Does the deduced potential match the known exact potential function? Does the set of phases obtained from the deduced potential match the initial data? Many test cases were tried but only representative ones need be examined here.

TEST S-A

Take as an initial potential*

\[ \sum_{n=1}^{\infty} u_n e^{-r/b_n} \]

with \( \{ u_n \} = 0.5, -0.65 \)

\( b = 0.65 \) .

This potential has a small cusp and its inner shape is highly dependent upon the cancellation of two comparable terms. Its outer region is essentially exponential in behavior. We will cheat a little bit and only seek two exponential solutions... but we cheat for a purpose; namely, that we wish as clear a view as possible as to what errors will be introduced when we must determine two nearly equal but can-

* We will use this form for all the potentials that follow.
celling quantities; i.e. will ill-conditioning spoil the answer?

Having derived phase shifts for the energy range $0 < k^2 < 2.0$, we fit the $k \cot \delta$ expansion with

$$\{ C_n \} = -1.151301, 3.55791, -0.076553, -0.0058378 .$$

We obtained for the wave function parameters:

$$\gamma = 0.570639$$

$$\{ \epsilon_n \} = -0.29995, -0.12134$$

and obtained as the solution

$$b = 0.63931$$

$$\{ U_n \} = 0.45326, -0.65175$$

with a determinant of coefficients of $-1.8$ showing that the equations did seem to be well-conditioned. The numerical values of the potential coefficients do not seem to be really impressively close, but to judge clearly, one must look at comparative plots of the potentials. One thing which, perhaps, is significant is that the longer range term is determined more precisely than the shorter range one. This is quite characteristic of the general behavior of the method. It is to be expected anyway because we know that the outer parts of the potential will have a greater effect on the phase shifts than the inner parts and therefore that knowledge of the phases over a limited energy range will (or should) allow us to determine the outer
Figure 1.
Test S-A Potentials

Exact potential: \( U(r) = 0.5 \ e^{-r/0.65} - 0.65 \ e^{-r/1.3} \)

Initial solution: \( U(r) = 0.453 \ e^{-r/0.64} - 0.652 \ e^{-r/1.28} \)

Iterated solution: \( U(r) = 0.454 \ e^{-r/0.653} - 0.633 \ e^{-r/1.306} \)
TEST S-A POTENTIALS

RADIUS (x10^13 CM)

DEPTH (K^2)

EXACT POTENTIAL

ITERATION

INITIAL SOLUTION
Figure 2.
Test S-A Phase Shifts
Phases for exact potential are compared with those deduced from the initial solution. Phases due to iterated solution cannot be distinguished from exact phases on this scale.
TEST S-A
PHASE SHIFTS

PHASES FROM
EXACT POTENTIAL

PHASES FROM
DEDUCED POTENTIAL

PHASE SHIFT (RADIANS)

ENERGY (K^2)

0.5 1.0 1.5 2.0
parts of the potential more precisely than the inner parts.

In an attempt to show what our iteration technique could do, we fit the \( k \cot \delta \) curve for the deduced potential and used

\[
\{ A, C_n \} = 0.0095, 0.0481, -0.06109, -0.013245
\]

to obtain an iterated potential

\[
b = 0.6532
\]

\[
\{ U_n \} = 0.45426, -0.63252
\]

The phase shifts yielded by this potential were a very precise fit to the exact phases. To get a better idea of the magnitude of the error, we converted the phases to the \( k \cot \delta \) form and extracted the following shape parameters:

\[
\{ C_n \} = -0.15396, 3.57053, -0.035017, 0.003409
\]

These are, as expected, an improvement.

**TEST S-B**

What happens when there is a repulsive core, a core less hard, to be sure, than the pure hard core for which we know that the method will break down, but nonetheless, repulsive? What happens when we have an essentially long-ranged potential? What happens when we try to fit the potential with fewer terms than the exact potential has?

To answer these questions, we used as our initial (exact) potential
Figure 3.

Test S-B Potentials

Three exponential potential approximated with two term solution.

Exact potential: \( U(r) = e^{-r/1.2} - 0.75 e^{-r/2.4} + 0.1 e^{-r/3.6} \)

Solution: \( U(r) = 0.895 e^{-r/1.08} - 0.577 e^{-r/2.16} \)
TEST S-B
POTENTIALS

RADIUS (x 10^5 CM)

EXACT POTENTIAL
SOLUTION

DEPTH (M)

0.2
0.1
0.0

-0.1
\[ b = 1.2 \]
\[ \{ U_n \} = 1.0, -0.75, 0.1 \]

Having derived a set of phase shifts for the energy range \( 0 < k^2 < 2.0 \), we fitted the \( k \cot \delta \) curve with

\[ \{ C_n \} = 0.034673, 3.7967, -2.4631, -0.49361 \]

Thence we obtained

\[ \gamma = 0.34161 \]
\[ \{ \xi_n \} = -0.26935, -0.11929 \]

and found a solution

\[ b = 1.08536 \]
\[ \{ U_n \} = 0.89489, -0.57728 \]

Examining plots of this and the exact potential and also of the phases deduced from this in comparison with those due to the exact potential we see that the errors are about 2%. This is quite acceptable and is about what one would expect from the method.

**TEST P-A**

The equations which we must solve for the P-wave potentials are really much more complicated than the corresponding ones for S-waves. We should be able then to check their validity with a simple example. Too, we must try to evaluate the effect of a new complication: the centrifugal barrier which rapidly damps the wave function near the origin.
Figure 4.

Test P-A Potentials

Essentially a single exponential potential approximated by a single term solution.

Exact potential: \( U(r) = -3.88 \ e^{-r/1.03} + .00036 \ e^{-r/2.06} \)

Solution: \( U(r) = -3.155 \ e^{-r/1.1087} \)
Because of this, we should expect the inner details of the potential to be effectively washed out and that our approximation should reproduce the outer portions of the potential with greater accuracy than the inner portions.

As a first, rather simple example, consider what is essentially a single exponential potential:

\[ b = 1.03 \]
\[ \{ U_n \} = -3.88, .00036 \]

Taking phase shifts over the energy range \( 0 < k^2 < 2.0 \) we fit the \((k^3 \cot \delta)/9\) expansion with

\[ \{ C_n \} = -.005049, .006476, -.06339 \]

Thence we get

\[ \gamma = .48068 \]
\[ \varepsilon = -.19297 \]

and a solution

\[ b = 1.1087 \]
\[ U_1 = -3.15505 \]

This is roughly correct but it is not very good near the origin. Most of the error here, we believe, is due to the inaccuracy of the least squares fit to the \((k^3 \cot \delta)/9\) curve. Indeed an examination of the fitting curve and the data being fit revealed that the fit was most in error near the origin. But it is at the origin that we most need
accuracy. This illustrates the necessity of weighting the low energy points in the \((k^{3} \cot \delta)/9\) curve heavily before taking the least squares fit.

**TEST P-B**

We should like now to test a potential with a repulsive core and an outside attractive well. The well should be deep enough to yield a net attractive potential at reasonably low energies. This, of necessity, must be a strong potential and as such should be much more subject to accurate approximation by this method. This time, we will take care that our least squares fits are accurate near the origin. We take as our potential

\[ b = 0.3607 \]

\[ \{U_{n}\} = 35.2343, -8.77616 \]

Taking phase shifts for the energy range \(0 < k^{2} < 0.5\) we convert them to \((k^{3} \cot \delta)/9\) and weight the low energy points \((k^{2} < 0.08)\) very heavily before doing the least squares fit. Taking the fit, we obtain

\[ \{C_{n}\} = -0.021318, 0.0287005, -1.3274, 0.020562 \]

Then we obtain

\[ \gamma = 0.69657 \]

\[ \{\varepsilon_{n}\} = 1.78393, -1.28190 \]

and a solution
Figure 5.
Test P-B Potentials.
Hard core potential with outside attractive well: P- wave test (iterated solution not shown)
Exact potential:
\[ U(r) = 35.2343 \ e^{-r/0.3607} - 8.77616 \ e^{-r/0.7212} \]
Initial solution:
\[ U(r) = 40.2434 \ e^{-r/0.3498} - 9.8481 \ e^{-r/0.6996} \]
Figure 6.
Test P-B Phase Shifts
Phase shifts derived from potentials in test P-B: exact potential, initial solution, iterated solution.
TEST P-B PHASE SHIFTS

PHASE SHIFT (RADIANS)

ENERGY (K^2)

INITIAL SOLUTION

EXACT ITERATION
b = .34984
\{ U_n \} = 40.2434, -9.84811.

When evaluated numerically and plotted in comparison with the exact potential, this is seen to be a very good result, giving phase shifts having a maximum error of about 5%.

As a rather strict measure of agreement, the fit of \((k^3 \cot \delta)/9\) for the deduced potential phase shifts was taken and the shape parameters thus determined to be:

\{ C_n \} = -.021835, .02484, -.12594, .017121.

Thus we can say with some assurance that the method appears to work and that the codes must be reasonably error free.
III. THE $n-\alpha$ CASE

Because of the fact that the method treats a scattering event as if only two particles were involved, one would expect it to work best for interactions between two (possibly composite) particles, both of which are tightly bound. For simplicity, the interaction should be one whose phase shifts obey $\delta < \pi$ for $k^2 > 0$ (although $\delta = \pi$ for $k^2 = 0$ is permissible). In addition, the interaction should be one about which sufficient data of adequate accuracy is available.

Consider the $n-\alpha$ interaction. The neutron is uncharged. This means that the treatment is free of the complications which accompany Coulomb interaction. In addition, the alpha particle is very tightly bound (its binding energy is 28 MeV., with the highest binding energy per pair of any nucleus: 4.7 MeV. (DE BENEDETTI)). Electron scattering experiments (HOFSTADTER) indicate that it is very well localized, with an rms radius of about $1.61 \times 10^{-13}$ cm. Thus one might be justified in treating $\text{He}^4$ as a single particle.

Conveniently too, the $n-\alpha$ interaction has no bound state. There is a resonance (Adair, DE BENEDETTI) in the total cross section at .95 MeV in the center of mass system with a half width of about .5 MeV. Why this resonance is interpreted to be the virtual state $\text{He}^5$ can be seen by doing a rough calculation: The transit time of the neutron across the alpha particle is about $\hbar/\Delta E = t \sim 10^{-21}$ sec.
but since, for the neutron, $1 \text{ MeV} = 2 \times 10^7 \text{ cm/sec.}$, the neutron could traverse a region of free space the size of the alpha particle in about $10^{-22} \text{ sec.}$ Hence the idea of the momentary formation of He ($^5$) (argument given by DE BENEDETTI).

Since this resonance seems to be mainly accounted for by the $P_{3/2}$ partial wave, we expect that partial wave potential to be attractive, and because of spin-orbit coupling, we expect it to be somewhat deeper and more attractive than the $P_{1/2}$ potential (these characteristics are reflected explicitly in the phase shifts).

Because of the relatively long time the neutron spends essentially inside the alpha particle, one might expect any possible spin-orbit effects to be strongly in evidence, and indeed upon examination of the reported n-α phase shifts, one notes the considerable splitting between the $P_{1/2}$ and the $P_{3/2}$ partial waves. The origin of such a large splitting is somewhat of a mystery, for the Thomas term due to relativistic effects and the many-body spin-orbit force together are too small to account for it (FUJITA and MIYAZAWA). Calculations have been made by TERASAWA using meson theoretic and Serber potentials with second order perturbation theory. He considered the deformation of the closed shell nuclear core induced by the tensor interactions between the nucleons in the core and the incident neutron, taking account of the fact that the Pauli principle restricts the deformation. With all this care, only about
half of the splitting was accounted for. The present method is ideal for dealing with this interaction since, by deriving a separate potential for each partial wave, we have included the spin-orbit effects implicitly in the deduced potential. We have thus avoided the necessity of postulating some definite form for the spin-orbit part of the interaction. One might then obtain the spin-orbit interaction merely by taking the difference of the $P_{3/2}$ and $P_{1/2}$ potentials.

There is only one major drawback to the application of the present method to $n$-$\alpha$ scattering. This is the fact that the available phase shifts are not very good. There are two major difficulties: there is much scatter in the data, and the phase shifts are only known over a relatively short energy range. One might, by applying the correct smoothing techniques, overcome the former difficulty, but the latter remains a problem. The only solution, aside from new experimental work, would be to extend the phase shift curves in a reasonable, but nonetheless arbitrary fashion. To do this really does not seem to be justifiable. The very limited range of energy over which the phase shifts are known does indeed lead to some difficulty (particularly in the $S$-wave potential) as will be seen later.
DATA

Before we can ever begin to derive potentials, we need a consistent, smoothly varying, accurate set of phase shifts as initial data. It is here that most of the trouble is encountered. Several groups have published more or less complete sets of n-\(\alpha\) phases over the years covering, on the whole, a very limited energy range. Most sets of phases are for energies equal to or less than 12 MeV. The chief reason for this appears to be the fact that for this energy range the scattering can be adequately described using S and P waves only, while for higher energies, D waves should be included. However these D waves apparently cannot be determined accurately from existing scattering data. It is because of these ambiguities at higher energies that we have neither tried to include D waves in the analysis nor have we tried to use the higher energy data points (at 17.8 and 20.9 MeV).

The sets of phases which we have examined are those due to DODDER and GAMMEL, SEAGRAVE, HUBER and BALDINGER, CLEMENTEL and VILLI, DEMANINS et al., AUSTIN et al., and BERZTISS. Austin has given a rather complete review of all of these except those of Berztiss. Berztiss, however, used Austin's phases as initial guesses in his analysis so that these two sets of phases agree fairly well in their common region.

The phases of HUBER and BALDINGER, DODDER and GAMMEL
and of SEAGRAVE are based on fairly old and therefore necessarily rather inaccurate scattering data, but they do give a rough, qualitative standard of agreement for the others. The phases derived by DEMANINS et al. are purported to be precision values. Indeed their $S$ and $P_{3/2}$ wave points are in agreement with others; however, their (low energy) values for the $P_{1/2}$ phases are much too small and in disagreement with others. Even if these values could be taken to be correct, we could not use them for there are not enough points given to incorporate them into a smoothly varying function of energy. The phases of Berztiss form a rather complete set over the energy range we shall consider here. However, his low energy points for $S$ and $P_{1/2}$ partial waves have too much scatter to be of use. The phases of Clementel and Villi can be used to fill in the regions where Berztiss' phases seem unacceptable. It is to be remembered that Clementel and Villi's phases are relatively old and were deduced with the requirement that the phase shifts be a continuous function of energy. Although this in itself is quite valid, it is believed that some systematic error was thereby introduced (particularly in the $P_{1/2}$ phases below 1 MeV). It is unfortunate that because they quoted no probable errors, we have no way of checking whether or not this apparent error is within the bounds of what might be expected.

We take then, as initial data, the phases of Berztiss, supported by those of Clementel and Villi.
TABLE I.
Phase Shift Data Used

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<td>119.2 B</td>
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<tr>
<td>5.54</td>
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* C denotes Clementel and Villi, B denotes Berztiss as source.
TABLE I. (cont'd)

<table>
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<th>P(_{3/2})</th>
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<td>-74.5 B</td>
<td>52.49B</td>
<td>103.0 B</td>
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Initially, plots of these phases seem to indicate relative smoothness and accuracy (see figs. 10, 12, 13). However it is not in this form that we can use them. We only have theoretical grounds for using them in the form \( k^{2l+1} \cot \delta \) or in the corresponding form involving \( \tan \delta \). It is when our chosen data is converted to this form and plotted against \( k^2 \) (energy, that is) that the wildness of the scatter really becomes evident (see figs. 7, 8, 9). Now, the most straightforward thing to do is to do a least squares fit, making a compromise between smoothness of the fitting line and precision of the fit. Actually, one is forced to such a compromise by the irregularity and the widness of the spacing of the data. It is, however, important that the first two shape parameters obtained by this fit be especially accurate. It is in this low energy region however that the general scatter in the data is the greatest and therefore that the correct fit is most ambiguously defined. We are aided by what is essentially effective range theory which tells us that \( k^{2l+1} \cot \delta \) should have an especially simple behavior near the origin (i.e. at low energies). For S-waves this behavior is prescribed to be a straight line. For P-waves, the linear term in the equation for the curve is generally very small compared with the quadratic term so that the behavior in the low energy is not precisely linear. Observation of a number of \( k^3 \cot \delta \) curves (at least for exponentially decaying potentials) indicates that the simple shape is that of a parabola open-
ing upward. The parabola may or may not be centered about zero energy, it may have its center above or below the axis.

So, the correct thing to do (if indeed there is such a thing when the scatter in the data is so great) is to fit the low energy points of the data with the correct smooth curve, straight line or parabola, move the data onto the curve and then adjoin to this the higher energy data. Perhaps this procedure seems drastic, but we believe it to be justified in view of the data. Now the question arises: how low is low energy? How many points of the data may be smoothed without throwing away the essential shape of the curve? In answer to this, one can only try; being careful that the fit does not seem to deviate from the general trend of the data too much.

Now, to carry out our scheme. The S-wave phases converted to \( k \cot \delta \) may be seen in fig. 7, along with the curve to which they were moved to smooth them. To check whether the smoothed phases were still actually representative, the smoothed points were converted back to the \( \delta \) form again. The comparison was very satisfactory. The smoothing curve was

\[
k \cot \delta = -0.4004 + (0.69308) k^2
\]

and is shown in the figure.

As has been indicated, there appears to be some systematic error in the low energy \( P_{1/2} \) phases. This is very clearly seen in the \( k^3 \cot \delta /9 \) plot. Accordingly, we
moved those points below 1 MeV up, onto a rough, hand drawn parabola before attempting to fit them with the smoothing curve. This amounted to assuming a maximum error in the phase shift of about $1.4^\circ$. This in fact does not seem unreasonable. This done, the phases were smoothed by the curve

$$(k^3 \cot \delta)/9 = 0.005108 + (0.006174) k^2 + (0.06435) k^4$$

which is shown in the figure.

Best results in the $P_{3/2}$ case were obtained by not including the three lowest energy points among those to be smoothed. The resonance point, naturally, was heavily weighted when doing fits since it is a point that is known with comparatively high accuracy. The smoothing curve was

$$(k^3 \cot \delta)/9 = 0.0011976 - (0.03364) k^2 + (0.03521) k^4$$

and, again, is shown in the figure.
Figure 7.

S-Wave $k \cot \delta$

$k \cot \delta$ as a function of energy for $n-\alpha$ scattering. Dashed curve gives the line to which the scattered points were moved when smoothing the data.

Energy units are such that $k^2 = 1$ corresponds to 25.65 MeV. in the center of mass or to 32.0625 MeV. in the lab.
Figure 8.

$P_{1/2} = \text{wave}(k^3 \cot \delta_{1/2})/9$ as a function of energy. Parabolic smoothing curve is shown.
\[ \frac{1}{2} \text{ WAVE} \]

\[ k^3 \cot \frac{s}{g} \]

\[ k^3 \cot \frac{s}{g} = g \times 10^2 \text{ cm}^{-3} \]

\[ \text{ENERGY} (k^2) \]

\[ \text{SMOOTHING CURVE} \]
Figure 9.

$P_{3/2} = \text{wave}(k^3 \cot \delta_{3/2})/9$ as a function of energy. Parabolic smoothing curve is shown.
$P^{3/2}$ WAVE

$E_{\text{N}} \sim \frac{1}{K^3 \cot \frac{8}{9}}$
RESULTS

S-WAVE POTENTIAL

The n-α S-wave phase shifts have been said to be well described by hard sphere scattering. Indeed, according to the data used here, this is nearly the case. HODGSON puts the hard sphere radius at about $2.4 \times 10^{-13}$ cm.; however we find that our low energy data is fit better by a hard core of radius $2.5 \times 10^{-13}$ cm. (the value indicated by the scattering length) for energies as high as 5 MeV (in the lab). The higher energy phases, however, deviate from what would be expected from a hard core potential.

The phases were smoothed linearly in the manner described in the previous chapter, but when we tried to fit them with the $k \cot \delta$ expansion it was found that $C_0 C_1$ could be either just greater than or just less than $9/32$ depending on how the data was weighted (i.e. we might not or might be able to find solutions for the wave function parameters $\epsilon_\alpha, \gamma$ ...see Appendix C: $\gamma$-Finder). This indicates that the Swan method as formulated here might not work very well because a hard core annihilates the wave function inside its radius and this cannot be considered to be a distortion of the wave. Indeed it is a case which is at the very limit of its domain of applicability. With this in mind, we sought solutions for the potential, taking care to weight the first few points heavily so that $C_0 C_1$ would be less than $9/32$. We fitted the $k \cot \delta$ curve with four parameters initially and searched for solutions. We obtained no acceptable sol-
olutions. The solutions which came closest to fitting the data had a small basic range and depended on the approximate cancellation of two large terms for their basic shape. Because of this, it was thought that the situation might be improved if we sought a three exponential solution. We hoped that the additional flexibility of three terms might enable us to find acceptable solutions.

Accordingly, we fitted the data with the following shape parameters:

\[
\{ c_n \} = \{ 0.39988, 0.64754, -1.1401, -6.6831, -8.9787 \}
\]

From which two solutions for \( \{ \xi_n, \gamma \} \) could be obtained; the first of these was

\[
\gamma = 0.83201
\]
\[
\{ \xi_n \} = \{ 0.12227, -4.684, 12.347 \}
\]

which yielded a solution:

\[
b = 0.10731
\]
\[
\{ U_n \} = \{ -19168.064, 4500.237, -618.303 \}
\]

with a determinant of coefficients \( -4.3 \times 10^{-9} \). This was then an ill-conditioned solution yielding a bad set of phase shifts. While pursuing further solutions, it was noticed that there appeared to be almost a solution at

\[
b = 0.3
\]
\[
\{ U_n \} = \{ -114.237, 12.142, 2.6 \}
\]
No further solutions were obtained from this choice for $\gamma$, so we examined the other choice:

$$\gamma = 1.4844$$
$$\{\epsilon_n\} = -60.6419, -35.883, -15.114$$

These are huge and seem to indicate that the method is hard put to distort the wave function sufficiently in the proper fashion. Solutions for the potential were found to be

$$b = .3816$$
$$\{U_n\} = -31.3692, 3.8943, 1.0555$$

$$b = .47355$$
$$\{U_n\} = -20.0576, 4.5164, .08117$$

and

$$b = 1.8068$$
$$\{U_n\} = .65701, -.004878, .0000934$$

The first of these yielded phase shifts of the correct shape and qualitative behavior, but of too small a magnitude. The second again yielded phases of the correct shape but having much too large a magnitude. The third yielded a phase curve that was negative but having the wrong qualitative shape and too small a magnitude. Thus, none of the solutions obtained was really acceptable.

Realizing that the method was beginning to break down in this case, we decided to try to find a potential yielding the correct phases without using the method explicitly.
We took as an initial guess the "almost solution" which was mentioned earlier, i.e.

\[ b = 0.3 \]

\[ \{ U_n \} = -11.237, 12.142, 2.6 \]

The phases produced by this potential were too small but of the correct shape. Their value at the maximum energy for which we have experimental data was -0.97 rad. To increase this magnitude, we made the trial solution more repulsive:

\[ b = 0.3 \]

\[ \{ U_n \} = -11.237, 14.14, 0.26 \]

The phases retained the correct shape, but still had a small magnitude: \[ |\delta|_{\text{max}} = 1.055 \text{ rad.} \] Proceeding in this manner, slowly increasing the height of the potential barrier and comparing the resultant phase shifts with the data, we finally arrived at

\[ b = 0.3 \]

\[ \{ U_n \} = -11.237, 28.54, 2.6 \]

which yielded phases of the correct shape and magnitude.

As is glaringly evident, there is a great deal of arbitrariness in this potential, for the phases have little shape, differing only slightly from pure hard core phase shifts. It could be disturbing to note that our final estimate of the potential has a deep interior well. This is, of course, not real, for the potential which our incident particles, having a maximum energy of only 9.6 MeV (in c.m.)
see is only a rather tall, thick potential hill. They cannot be effected by the inner parts of the potential. This well is, then, a numerical effect caused by our efforts to build the potential from this rather inflexible series of exponentials:

\[ U(r) = \sum_{\ell} U_\ell e^{-r/\rho b} \]

The well appears because the effective potential must be steep and repulsive at a point about 2 fm. from the origin and this is obtained by the cancellation of two large, short range terms, the outer repulsive, and the inner term necessarily attractive. To show the validity of this belief that the inner portions of the potential are indeed effectively unseen, we extracted a set of phase shifts for our potential modified by replacing the well by a repulsive square barrier of height 2.5 BeV and having a 1.2 fm. radius (both the barrier and the inner well are shown as dotted curves in the figure). As expected, this made negligible difference to the low energy phases and deepened the high energy phases by 45° (about 75% difference). This small perturbation may be taken as proof that for the energy range dealt with here, the incident particle (and hence the phase shifts) are not effected by the inner part of the potential. Conversely, the inner parts of the potential cannot be determined knowing the phase shifts only over the rather limited energy range considered here.
Figure 10.

n-α S-wave phase shift data compared to the phase shifts corresponding to the solution potential.
Figure 11.

n-\(\pi\) S-wave solution potential showing maximum energy of the phase shift data. Inner well (dashed) can be replaced by square barrier shown here as a dashed line:

Barrier height 2.5 BeV., well depth at origin 2.1 BeV.

Solution potential:

\[ U(r) = -114.237 \ e^{-r/0.3} + 28.54 \ e^{-r/0.6} + 2.6 \ e^{-r/0.9} \]
P-WAVE POTENTIALS

We naturally expect the potential representing the $P_{3/2}$ interaction to be attractive and relatively deep, for this is the partial wave which is primarily responsible for the resonance in the cross section at .95 MeV.

The $P_{3/2}$ data was smoothed in the manner described in the previous chapter and the phase shifts were fitted with the $(k^3 \cot \delta)/9$ expansion. The shape parameters were thus found to be

$$\{ C_n \} = -0.010262, -0.027250, 0.021875, 0.11296$$

These gave wave function parameters:

$$\chi = 0.55575$$
$$\{ \varepsilon_n \} = 4.7518, -2.9797$$

The large size of these $\{ \varepsilon_n \}$ are further indication of the strength of the potential since they are a rough measure of the distortion of the wave function.

Two solutions were found:

$$b = 0.766301$$
$$\{ U_n \} = -10.913, -0.8622$$

(determinant of coefficients here was .23)

and

$$b = 1.187545$$
$$\{ U_n \} = -4.27605, -0.011682$$

(determinant of coefficients was 19.2). When the phase shifts were derived for each of these potentials it was
observed that (as might have been expected from the conditioning of the equations as shown by the value of the determinant) only the second solution is good. See fig 14 for a plot of the potential.

When the derived phase shifts were converted to the "shape independent" form, i.e. to \( (k^3 \cot \delta) / 9 \), the shape parameters were determined by least squares fitting to be

\[
\{ C_n \} = \{-0.011227, -0.026328, -0.043841, -0.003394\}
\]

\( C_0 \) and \( C_1 \) are fairly close to what they should be so it was decided to try to iterate. In doing so, we used

\[
\{ \Delta C_n \} = \{0.0009656, -0.00067, 0.02, 0.0\}
\]

since \( C_2 \) and \( C_3 \) were found to be so badly off. Thus, we obtained

\[
b = 1.1669 \\
\{ U_n \} = -4.5889, -0.01735
\]

This solution as well as its corresponding phase shifts are shown in figures 12, 14. As can be seen, the over-all fit to the phases is fairly good although the deduced phases do not pass through the resonance point at \( 0.95 \text{ MeV} \) (error here of about \( 4.5^\circ \) or \( 0.119 \text{ MeV} \) in the center of mass). When the shape parameters were determined, fitting \( (k^3 \cot \delta) / 9 \), we found

\[
\{ C''_n \} = \{-0.0088999, -0.02945, -0.03654, 0.003899\}
\]

Looking at this two term potential an noting the relative smallness of the second term tempts one to ask how essential this second term really is. To check this, we calcul-
ated the phase shifts due to the first term alone. It was found that the overall fit was not as good as for the phases due to both terms, but that the single-term phase curve did pass somewhat closer to the resonance point.

The $P_{1/2}$ phases are rather poorly known, particularly in the low energy region where it is important that we know them well. There are at present, too, differing opinions about the character of the higher energy behavior. The phase shifts were smoothed parabolically as described in the previous chapter and the following shape parameters were extracted:

\[
\{ C_n \} = -0.005029, 0.013559, -0.002770, 0.1370
\]

These yielded wave function parameters

\[ \gamma = 0.474 \]

\[ \{ \varepsilon_n \} = 2.5768, -1.7112 \]

which are, as expected, smaller than those for the $P_{3/2}$ wave. Two solutions were found:

\[ b = 0.7755 \]

\[ \{ U_n \} = -9.3663, -0.2872 \]

and

\[ b = 1.0866 \]

\[ \{ U_n \} = -3.3492, -0.009230 \]

of which only the latter was good. The last potential and its corresponding phase shifts may be seen in Figs. 13,14. Shape parameters for the phase shifts were extracted:

\[ \{ C'_n \} = -0.005064, -0.001213, -1.0000, -0.02786 \]
As can be seen, all except \( C_0 \) deviate markedly from those found for the data. This is why iteration in an attempt to improve the potential is useless in this case. The overall fit to the data is, however, fairly good at this stage as may be seen in the figure.
Figure 12.

\( n-\alpha P_{3/2} \) phase shift data compared with phases due to initial solution and to the iterated solution. Note the point at which the data phases pass through 90° denoted as resonance point.
$P_{3/2}$ PHASE SHIFTS

- DATA
- RESONANCE POINT
- INITIAL SOLUTION
- ITERATION

ENERGY ($K^2$) vs. RADIAN
Figure 13.

n-α P₁/₂ phase shift data compared with phases due to the initial solution.
PHASE SHIFTS

---

DEUDED PHASES

---

DATA

---

P/2 PHASE SHIFTS
Figure 14.

n-α P-wave Potentials

$P_{1/2}$ potential solution compared with the initial and iterated $P_{3/2}$ solutions.

$U_{1/2}(r) = -3.3492 \ e^{-r/1.0866} - .00923 \ e^{-r/2.1732}$

Initial $U_{3/2}(r) = -4.276 \ e^{-r/1.19} - .0177 \ e^{-r/2.38}$

Iterated $U_{3/2}(r) = -4.5899 \ e^{-r/1.167} - .01735 \ e^{-r/2.334}$

(energy units such that $k^2 = 1.0$ corresponds to 25.65 MeV.)
APPENDIX A. EPSILON- GAMMA EQUATIONS

We start from the basic equation

A-1) \( \phi \phi' - \phi_0' \phi \bigg|_{r=0} = k^2 \int_0^\infty (\phi \phi_0 - \psi \psi_0) \, dv \)

where we choose \( \phi = \cot \delta \beta (kr) - N_0 (kr) \)

and \( \psi = \cot \delta \beta (kr) - g_0 (kr) N_0 (kr) \)

with \( g_0 (t, r) = (1 - e^{-r})^{2l+1} \left[ 1 + e^{-r} \sum_{m=1}^{\infty} e_m (kr)^{2m} \right] \).

Let us evaluate the left-hand of equation A-1); We have

\[ \phi = \frac{k^2 \cot \delta \beta J_0}{(2l+1)!!} - \frac{k^2 N_0}{(2l+1)!!} \]

and

\[ \phi_0 = - C_0 k^{l+1} + r^{-2} (2l+1) \]

where we have used the expansion

\[ 2-8) \ \frac{k^{2l+1} \cot \delta \beta}{[(2l+1)!!]^2} = \sum_{n=0}^{\infty} C_n (-1)^{n+1} k^{2n} l \]

Now

\[ \phi' = \frac{k^{2l+1} \cot \delta \beta J_{l+1}}{(2l+1)!!} - \frac{k^{l+1} \cot \delta \beta J_{l+1}}{(2l+1)!!} \frac{l}{2l+1} \]

\[ + \frac{k^{l+1} N_0}{(2l+1)!!} \frac{l}{2l+1} \]

and

\[ \phi_0 = - C_0 (l+1) r^l - l \frac{r^{-l-1}}{(2l+1)} \]

so we have

\[ \phi_0 \phi' - \phi \phi_0' \bigg|_{r=0} = \left[ - C_0 k^2 \cot \delta \beta J_{l+1} \frac{l}{(2l+1)!!} \beta e + C_0 \cot \delta \beta \frac{k^{l+1} l}{(2l+1)!!} \right] \]

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where we have used the shorthand $J_\ell(r) = J_\ell$, $\phi_\ell = \delta$

Several terms cancel and we get upon taking the limit as $r \to 0$

\[
\left\{ \begin{array}{c}
- \frac{C_0 k^{\ell+1} r^{\ell+1} N_{\ell+1}}{(2\ell+1)!!} \\
+ \frac{\cot \delta (2\ell+1) k^\ell r^{-\ell-1} J_\ell}{(2\ell+1) (2\ell+1)!!} \\
- \frac{k^\ell r^{-\ell-1} N_{\ell}}{(2\ell+1)!!} \\
\end{array} \right. \quad r \to 0
\]

In taking the limit, we have used the expansions for the Ricatti-Bessel functions (these are essentially the expressions quoted in WATSON, p. 143).
\[ \mathcal{A}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{-n}}{n! \left(2n + 2n + 1\right)!} \]

\[ N_0(z) = -z^{-l} \sum_{n=0}^{\infty} \frac{z^{2n} (2n + 1)!!}{(2n)!! \left(\frac{1}{\nu} + \frac{n}{\nu + 1}\right)} \]

Taking the limit, we have

\[ \phi_0 \phi' - \phi_0 = C_0 + \frac{k^{2l+1} \cos \delta}{(2l+1)!!} + \lim_{r \to 0} \frac{k^{2l+1} \rho^{-\delta} N_0}{(2l+1)!!} \]

The last term is singular at \( r = 0 \) for \( l \geq 0 \), but it will cancel with a similar pole on the other side of the equation and hence need not be cause for concern.

Evaluate now the right-hand side of the equation:

\[
\int_0^\infty \left[ \phi_0 - \psi \right] \, dr = \left[ -C_0 \frac{k^{2l+1} \cos \delta}{(2l+1)!!} \right] + \frac{k^{2l+1} \cos \delta}{(2l+1)!!} + \frac{k^{2l+1} \rho^{-\delta} N_0}{(2l+1)!! (2l+1)!!}
\]

\[ + \frac{C_0 \frac{k^{2l+1} \cos \delta}{(2l+1)!!}}{(2l+1)!!} - \frac{k^{2l+1} \rho^{-\delta} N_0}{(2l+1)!! (2l+1)!!} \]

\[ - \frac{g_0 C_0 \frac{k^{2l+1} \rho^{-\delta} N_0}{(2l+1)!!}}{2l+1)!!} + \frac{g_0 \frac{k^{2l+1} \rho^{-\delta} N_0}{(2l+1)!! (2l+1)!!}}{(2l+1)!! (2l+1)!!} \right] \, dr . \]

Note that the term \( -\frac{k^{2l+1} \rho^{-\delta} N_0}{(2l+1)!! (2l+1)!!} \) cancels the pole term on the left-hand side of the equation unless \( l = 0 \)
since
\[
\frac{d}{dr} \left( \frac{k^{2l+1} r^l N_{l-1}}{(2l+1)(2r+1)!!} \right) \bigg|_0^\infty = \frac{k^{2l+1} r^l N_{l-1}}{(2l+1)!! (2r+1)!!}
\]
and
\[
\frac{d}{dr} \left( \frac{k^{2l+1} r^l N_{l-1}}{(2l+1)(2r+1)!!} \right) \bigg|_0^\infty = 0.
\]

So we can add this zero quantity to the pole term, put it under the integral and we have cancelation. However, if \( l = 0 \), there is no pole on either side of the equation and in fact the term
\[
\frac{k^{2l+1} r^l N_{l-1}}{(2l+1)!! (2r+1)!!}
\]
does not cancel. To signify this, we leave it in our equations multiplied by a Kroneker delta: \( \delta_{l0} \). Remaining, there is for the right-hand side:

\[
\int_0^\infty dr \left\{ \left[ 1 - (1-e^{-\nu r})^{2l+1} \right] \frac{k^{2l+1} \cos \delta \frac{L^2}{(2l+1)!! (2r+1)!!}} {L^2} \right. \\
+ \left. \left[ 1 - g_0 \right] \frac{C_0 k^{2l+1} N_{l-1}}{(2l+1)!!} - \frac{\left[ \delta_{l0} - g_0 (1-e^{-\nu r})^{2l+1} \right] k^{2l+1} N_{l-1}}{(2l+1)!! (2r+1)!!} \right\}.
\]

Now, we must evaluate the integrals:

\[
\int_0^\infty \frac{C_0 k^{2l+1} (1-g_0) r^{2l+1} N_{l-1} dr}{(2l+1)!!}
\]

\[
= \int \left( -C_0 (1-g_0) \right) \sum_{n=0}^{\infty} \frac{k^{2n} \nu^{2n+1}}{(2n)!!} \frac{\nu}{\nu+1} \frac{1}{(2l+1-2\nu)!!}
\]

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and
\[
(1 - g_4) = 1 - (1 - e^{-\gamma r})^{2a+1} - (1 - e^{-\gamma r})^{2a+1} \sum_{\nu=0}^\infty \frac{C_0}{(2\mu)! (2\mu+1)} (-1)^\mu \frac{1}{(\gamma r)^{2\mu+1}}
\]
\[
\sum_{\nu=1}^\infty \frac{(-1)^\nu}{(\nu)^2} (2\nu + 1) e^{-\gamma r} \sum_{\mu=0}^\infty \frac{C_0}{(2\mu)! (2\mu+1)} (-1)^\mu \frac{1}{(\gamma r)^{2\mu+1}}
\]

so we have
\[
\int_0^\infty \frac{\mu^k}{r^k} \sum_{\mu=0}^\infty \frac{C_0}{(2\mu)! (2\mu+1)} (-1)^\mu \frac{1}{(\gamma r)^{2\mu+1}}
\]
\[
\sum_{\nu=1}^\infty \frac{(-1)^\nu}{(\nu)^2} (2\nu + 1) e^{-\gamma r} \sum_{\mu=0}^\infty \frac{C_0}{(2\mu)! (2\mu+1)} (-1)^\mu \frac{1}{(\gamma r)^{2\mu+1}}
\]
\[
= \sum_{\mu=0}^\infty \frac{C_0}{(2\mu)! (2\mu+1)} (-1)^\mu \frac{1}{(\gamma r)^{2\mu+1}}
\]

Also
\[
\int_0^\infty \frac{\mu^k}{r^k} \frac{\cot \delta}{(2\mu+1)! (2\mu+1)} \sum_{\mu=0}^\infty \frac{C_0}{(2\mu)! (2\mu+1)} (-1)^\mu \frac{1}{(\gamma r)^{2\mu+1}}
\]
\[
\sum_{\mu=0}^\infty \frac{C_0}{(2\mu)! (2\mu+1)} (-1)^\mu \frac{1}{(\gamma r)^{2\mu+1}}
\]
\[
= \frac{\mu^k}{r^k} \frac{\cot \delta}{(2\mu+1)! (2\mu+1)} \sum_{\mu=0}^\infty \frac{C_0}{(2\mu)! (2\mu+1)} (-1)^\mu \frac{1}{(\gamma r)^{2\mu+1}}
\]
\[
= \frac{\mu^k}{r^k} \frac{\cot \delta}{(2\mu+1)! (2\mu+1)} \sum_{\mu=0}^\infty \frac{C_0}{(2\mu)! (2\mu+1)} (-1)^\mu \frac{1}{(\gamma r)^{2\mu+1}}
\]

We have left one term to integrate:
\[
\left[ \frac{\sin \delta - g_0 (1 - e^{-\gamma r})^{2a+1}}{(2a+1)!! (2a+1)} \right] k^e \mu^{-e} N_k
\]

If \( \mu=0 \) this is just
\[ \delta_2 \sum_{n=0}^{\infty} \frac{k_n}{(2n)!!} \frac{1}{\Gamma(1-\nu)} \left[ \sum_{\nu=0}^{4\beta+2} \frac{(-1)^{\nu+1} (4\beta+2) (2\mu)!}{(\nu)!! \Gamma(2\nu+1)} \right] \]

\[ - \sum_{\nu=0}^{4\beta+2} (-1)^{\nu} (4\beta+2) \sum_{\nu=0}^{\infty} \epsilon_n \frac{k_2^{2n}(2n+2\mu)!}{\Gamma(2n+1)!!} \left[ \frac{1}{(2n+1)!!} \right]^{2n+2\mu+1} \]

We have kept the general form in order to be able to see clearly later that this may be easily incorporated into the expression for \( \lambda > 0 \). Now for \( \lambda > 0 \) generally we have to evaluate

\[ \int_0^\infty g(kr) \frac{(1-e^{-kr})^{2\lambda+1} r^{-\lambda} N_2}{(2\lambda+1)!! (2\lambda+1)} \]

which is, for some \( R \) sufficiently large

\[ = \int_0^R g(kr) \frac{(1-e^{-kr})^{2\lambda+1} r^{-\lambda} N_2}{(2\lambda+1)!! (2\lambda+1)} + \int_R^\infty \frac{r^{-\lambda} N_2}{(2\lambda+1)!! (2\lambda+1)} \]

where we have used the asymptotic behavior of \( g \). This is

\[ \int_0^R g(kr) \frac{(1-e^{-kr})^{2\lambda+1} r^{-\lambda} N_2}{(2\lambda+1)!! (2\lambda+1)} = \frac{R^{-\lambda} N_2}{(2\lambda+1)!! (2\lambda+1)} \]

Using the series expansion for \( N_2(kr) \) these may be evaluated:

\[ - \sum_{n=0}^{4-1} \frac{k_n^{2n}}{(2n+1)(2n)!! (2n+1-2\lambda)} \left[ \frac{\gamma^{2n-2\lambda-1}}{(2n-2\lambda)!} \sum_{\nu=0}^{4\beta+2} \frac{(-1)^{\nu} (4\beta+2) \nu! \nu}{\Gamma(2\nu+1)} \right], \]

\[ + \frac{1}{2n-2\lambda+1} R^{2n-2\lambda+1} \]

\[ - \sum_{n=0}^{4\beta+2} \frac{k_n^{2n}}{(2n+1)(2n)!! (2n+1-2\lambda)} \left[ \sum_{\nu=0}^{4\beta+2} \frac{(-1)^{\nu} (4\beta+2) \nu! \nu}{\Gamma(2\nu+1)} \left\{ e^{-x\mu R} \right\} \right] \]

\[ X \sum_{\rho=0}^{2n-2\lambda} \frac{(2n-2\lambda)!}{(2n-2\lambda-\rho)! (x\mu)^{\rho+1}} - \frac{(2n-2\lambda)!}{(x\mu)^{2n-2\lambda+1}} \left\{ R^{2n-2\lambda+1} \right\} \]

\[ 53 \]
\[ + \sum_{n=0}^{\infty} \frac{k^{2n} R^{2n-2\mu+1}}{(2n)!! \prod_{\nu=0}^{n} (2\nu - 1 - \ell) (2\nu + 1)^2} \]

\[ - \sum_{n=1}^{d-1} \frac{k^{2n}}{(2n+1)} \left[ \sum_{\ell=0}^{4n+2} \left( -1 \right)^{\nu} \binom{4n+2}{\nu} \frac{(2n-2\mu)!}{\nu!(\nu+1)!} \right] \]

\[ \times \sum_{m=1}^{n} \frac{\epsilon_{\ell+\mu}}{(2n-2m)!! \prod_{\nu=0}^{m} (2\nu + 1 - \ell)} \]

We notice several things: As \( R \to \infty \) the term in \( e^{-\mu x} R \) goes to zero and all remaining terms in \( R \) cancel (which is as it should be since \( R \) was arbitrary though large). If we never sum over negative indices, this reduces to just the result for \( \ell = 0 \) so that we need make no special provision for \( \ell = 0 \) after all.

Now, we can put all the results together in the equation and equate like powers of \( k^2 \) to get our final equations to be solved for \( \{\epsilon_{\ell}\} \) and \( \nu \):
$$\begin{align*}
C_{n+1} \, (-1)^n &= \frac{C_0 \, (2n+1)!!}{17 \, (2n+2-2\mu)} \sum_{\nu=1}^{2n+1} (-1)^\nu \binom{2\nu+1}{\nu} \frac{1}{(\nu \mu)^{2n+2}} \\
+ \frac{(-1)^{n+1} \, (2n+1)!!}{(2n+1)} \sum_{\nu=0}^{n} \frac{C_{n-\nu} \, (2n-2\nu+1)!!}{(2n+1-2\nu + 2n-2\nu)!!} \sum_{\nu=1}^{2n+1} (-1)^\nu \binom{2\nu+1}{\nu} \frac{1}{(\nu \mu)^{2n-2\nu+2}} \\
+ \left( \psi \, n \leq q-1 \right) \frac{-1}{(2n+1) \times (2n)!!} \prod_{\nu=0}^{n-1} \frac{1}{(2n+1-2\nu)} \left[ \frac{\gamma^{2n-2n-1}}{(2n-2n-1)!} \sum_{\nu=1}^{4n+2} \frac{(-1)^\nu \binom{4n+2}{\nu}}{(\nu \mu)^{2n-2n-1}} \right] \\
+ \left( \psi \, n \geq r \right) \frac{-1}{(2r+1)} \frac{\gamma^{2n-2n-1}}{(2n-2n-1)!} \sum_{\nu=1}^{4n+2} \frac{(-1)^\nu \binom{4n+2}{\nu}}{(\nu \mu)^{2n-2n-1}} \\
+ \left( 1-\delta_{n0} \right) \left\{ C_0 \, (2n+1)!! \sum_{\nu=0}^{n} \frac{(-1)^\nu \binom{2\nu+1}{\nu}}{(\nu \mu)^{2n+2}} \right. \\
+ \left( \psi \, n \leq q-1 \right) \frac{\gamma^{2n-2n-1}}{(2n-2n-1)!} \sum_{\nu=1}^{4n+2} \frac{(-1)^\nu \binom{4n+2}{\nu}}{(\nu \mu)^{2n-2n-1}} \\
+ \left( \psi \, n \geq r \right) \frac{-1}{(2r+1)} \sum_{\nu=0}^{4n+2} \frac{\binom{4n+2}{\nu}}{(\nu \mu)^{2n-2n+1}} \right\} \\
X \left[ \sum_{\mu=1}^{n} \frac{E_m}{(2n-2m)!!} \prod_{\nu=0}^{m-1} \frac{1}{(2\nu+1-2\mu)!} \right].
\end{align*}$$

By taking $n = 0$ we get the equation which must be solved for $\gamma$. 
We begin with the basic equation
\[
\cot \delta = \frac{1 - \frac{1}{M}}{\frac{1}{\kappa} \int_{0}^{\infty} U(r) N_{\lambda} \, dr}
\]
where
\[
\frac{1}{\kappa} \int_{0}^{\infty} U(r) N_{\lambda} \, dr = \frac{1}{\kappa} \int_{0}^{\infty} \mathcal{G}_{\lambda}(k, r) U(r) N_{\lambda} \, dr
\]
and
\[
\mathcal{G}_{\lambda}(k, r) = (1 - e^{-kr})^{2\lambda + 1} \left[ 1 + e^{-kr} \sum_{n=1}^{\infty} \epsilon_{n} (kr)^{2n} \right].
\]

There are some useful expansions:
\[
\frac{n^{2\lambda + 1}}{(2\lambda + 1)!!} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} C_{k} n^{2k}}{k!}
\]
\[
\frac{1}{\kappa} \int_{0}^{\infty} U(r) N_{\lambda} \, dr = - \sum_{m=0}^{\infty} \frac{(kr)^{2m+2}}{(2m+1)!!} \frac{(2\lambda + 2m + 1)!!}{\lambda!} \frac{1}{(2\lambda + 1 - 2\lambda)}
\]
\[
\frac{\delta^{2}}{\kappa^{2}} = \sum_{m=0}^{\infty} \frac{(-1)^{m} (kr)^{2m+2}}{m! (2\lambda + m + 1)!!} \frac{(2m + 2\lambda)!!}{(2\lambda + 2m + 1)!!}
\]

These last two expressions are derived from expansions given by WATSON, p. 147. We parameterize the potential \( U(r) \) by
\[
U(r) = \sum_{\lambda=1}^{\infty} U_{\lambda} e^{-r/(fr^{b})}
\]
and write our equation in the form
\[
\frac{1}{\kappa} \int_{0}^{\infty} U(r) N_{\lambda} \, dr + \frac{1}{\kappa} \int_{0}^{\infty} \mathcal{G}_{\lambda}(k, r) U(r) N_{\lambda} \, dr = 1
\]
Using the expansions, we find that we need two integrals:

\[
\int_0^\infty e^{-\frac{r}{f(\rho)b}} \, d\rho = \sum_{m=0}^{\infty} \frac{(-1)^m k^{2m+2}}{m!} \frac{(2m+2)!}{(2m+2m+1)! (2m+2m+1)!} [f(\rho)b]^{2m+2m+3}
\]

and

\[
\int_0^\infty e^{-\frac{r}{f(\rho)b}} g_n(k,\rho) \, d\rho = -\sum_{m=0}^{\infty} k^{2m+1} (2m+1)! \sum_{n=0}^{\infty} \frac{(2m-2n)!}{(m-n)!} \frac{(2m+1)!}{(2m+2m-2n+1)!} \times
\]

\[
\sum_{\nu=0}^{\infty} \frac{(-1)^\nu (2\nu+1) \left[ \delta_{n0} + (1-\delta_{n0}) \rho \right] (2m+1)!}{\left( \frac{1}{f(\rho)b} + \gamma (\nu+1-\delta_{n0}) \right)^{2m+2}}
\]

The equation is then

\[
\sum_{\rho=1}^{\infty} U_\rho \left\{ \frac{1}{k} \left[ k^{-2(2\rho+1)} \frac{(2\rho+1)!}{(2\rho+1)!} \sum_{m=0}^{\infty} (-1)^m C_m k^{2m+2m+2} \frac{2m+2m+2}{m! (2m+2m+1)! (2m+2m+1)!} \right. \right.
\]

\[
- \frac{1}{k} \sum_{m=0}^{\infty} k^{2m+1} (2m+1)! \sum_{n=0}^{\infty} \frac{(2m-2n)!}{(m-n)!} \frac{(2m+1)!}{(2m+2m-2n+1)!} \times
\]

\[
\sum_{\nu=0}^{\infty} \frac{(-1)^\nu (2\nu+1) \left[ \delta_{n0} + (1-\delta_{n0}) \rho \right] (2m+1)!}{\left( \frac{1}{f(\rho)b} + \gamma (\nu+1-\delta_{n0}) \right)^{2m+2}} \}
\]

\[
= 1
\]

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Equating like powers of $k^2$, we obtain the system of equations to be solved for $\{ U_\rho, b \}$: (for $n = 0,1, \ldots, N$)

$$
\sum_{\rho=1}^{N} U_\rho \left\{ \left( \frac{2Q+1}{2} \right)! \right\}^2 (-1)^n \sum_{m=0}^{n} \frac{(2m+2Q+1)!! \, (2Q+2m+2)! \, \left\{ f(n) \, b \right\}^{2Q+2m+2} \, C_{n-m}}{m! \, (2Q+m+1)! \, (2Q+2m+1)!!}
$$

$$
- (2Q+1)!! \sum_{m=0}^{n} \frac{(2n-2m)! \, (2n+1)! \, \left[ \prod_{\mu=0}^{n-m} (2Q+2n-2\mu+1)!! \right]}{[(n-m)!]^2 \, (2Q+2n-2m+1)!! \, \prod_{\mu=0}^{n-m} (2Q+1-2\mu)}
$$

$$
\sum_{\nu=0}^{2Q+1} (-1)^\nu \left( \frac{2Q+1}{2} \right) \left[ S_{\nu} + (-S_{\nu}) \, E_{\nu} \right] \left[ \frac{1}{f(\nu) \, b} + (\nu+1 - S_{\nu}) \, \gamma \right]^{2Q+2}
$$

$$
= \delta_{m,0}
$$
APPENDIX C. PROGRAMS AND SPECIAL METHODS

TRANSCENDENTAL EQUATION SOLVER

One of the simplest and perhaps the oldest of the methods for solving an equation numerically is the Regula Falsi technique. We must cast the equation in the form \( f(x) = 0 \).

Then, if we know two trial values of \( x \), say \( x_1 \) and \( x_2 \) such that \( f(x_1) < 0 \) and \( f(x_2) > 0 \) the Regula Falsi estimate of a better trial value is (see, for example, MERRIMAN):

\[
C-1) \quad x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_2) - f(x_1)}.
\]

To show why this is so, consider similar triangles in the figure.

\[
\frac{x_3 - x_1}{x_2 - x_3} = \frac{|f(x_1)|}{|f(x_2)|}
\]

So we have

\[
C-2) \quad x_3 = \frac{|f(x_1)| x_2 + |f(x_2)| x_1}{|f(x_1)| + |f(x_2)|}
\]

If \( f(x_1) \) is always negative, eq C-2) reduces to eq C-1).

However, for our purposes, we should use eq C-2) for we do not know a priori that \( f(x_1) \) is not positive. There are two distinct advantages to this method: It is guaranteed to converge unless the solution lies at a point of tangency with the axis; it requires only that one evaluate the function, never its derivative.

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\[ Y - \text{FINDER} \]

The equation to be solved for \( Y \) is

\[
0 = C_0 + \frac{2C_0}{2l+1} \sum_{\nu=1}^{2l+1} (-1)^{\nu+1} \frac{(2l+1)!}{(\nu\nu)^2} \left[ (\frac{1}{\nu})^2 - \delta_{20} \left( \frac{1}{2} \frac{1}{\nu^2} \right) \right] + (1-\delta_{20}) \frac{1}{(2l+1)!} \left[ \frac{Y^{2l+1}}{(2l+1)!} \sum_{\nu=2}^{2l+1} (-1)^{\nu+1} \frac{(2l+2)!}{(\nu\nu)!} \nu \lambda_{\nu} \nu \right]
\]

When \( l = 0 \) this is simply a quadratic:

\[
0 = C_0 + 2C_0 \frac{1}{Y^2} - \frac{1}{2} \frac{1}{Y}
\]

and as such has positive solutions for \( Y \) as long as

\[
9/32 < C_0 C_1
\]

This says, in effect, that we can get solutions for \( Y \) for S-wave interactions only if there is no absolutely hard core. We have made this identification of solution criterion with hard core behavior for two reasons: the quantity \( C_0 C_1 = 1/3 \) for a true hard core (see this by expanding \( k \cot (-ka) \) in a power series) which is very near to \( 9/32 \); secondly, an absolutely hard core produces a really major distortion of a plane wave so that our basic approximation begins to break down.

For any case but S-wave, the equation we must solve for \( Y \) is of higher order; we must, on that account, generally use our Regula Falsi technique to obtain solutions. Solving for gamma then consists of evaluating the equation for several trial values of \( Y \) until we find two which are suitable for starting the Regula Falsi iteration.
**EPSILON FINDER**

The equations from which \( \{ \xi_n \} \) are found are quite simple in form, but quite complicated in detail. In fact, we can represent them symbolically as

\[
C-3) \quad f(n,y) = A(n,y) \sum_{m=1}^{M} \xi_m B_m(n,y) \quad \text{for } n = 1, 2, \ldots, N-1,
\]

which is linear in the unknowns \( \xi_m \). The only difficulties encountered are in the calculations of \( f(n,y) \), \( A(n,y) \) and \( B_m(n,y) \); and these difficulties are only due to the algebraic complexity of the expressions. Actual solution of eq. C-3) is accomplished using a standard library routine for the solution of simultaneous linear equations.

**SCHRÖDINGER EQUATION SOLVER**

The equation which is to be solved is

\[
C-4) \quad \psi = [ -k^2 + \frac{q(q+1)}{r^2} + U(r) ] \psi
\]

and this is formally

\[
C-5) \quad \ddot{\psi} = f(r) \psi \quad \text{where the dots refer to differentiation with respect to } r. \text{ Equation } C-5) \text{ is a particularly simple form of differential equation for which there exist especially simple and accurate numerical solution techniques. One of the best of these as given by FOX and GOODWIN is}
\]

\[
C-6) \quad [ 1 - \frac{i}{\hbar} k^2 f(r_0) ] \psi = \left( k^2 + \frac{\xi}{\hbar^2} f(r_0) \right) \psi_0 - \left( i - \frac{i}{\hbar^2} k^2 f(r_0) \right) \psi_1 + \Delta
\]
where \( h \) is the increment between successive points i.e.,
\[ h = |r_1 - r_0|, \]
\( \Delta \) is the residual error
\[ \Delta = \left( -\frac{1}{24} \delta^6 + \frac{1}{720} \delta^8 + \ldots \right) r_0, \]
and \( \delta \) is the central difference operator. In obtaining
this, the differential equation (C-5) was replaced by its
finite difference equivalent ( replace \( \dot{y} \) by its central
difference equivalent, then apply the operator \( (1 - 1/2 \delta^2) \)
throughout to obtain the recurrence relation eq. (C-6) ).

This relation is actually the corrector formula of a pre-
dictor-corrector method, but as Fox and Goodwin have shown,
its extreme accuracy ( error on the order of \( h^6/240 \) )
justifies its use alone, without the predictor formula.

If the solutions to the Schrodinger equation are known
for two successive initial points, further solutions can
be found by applying eq (C-6). In fact, \( \Delta \) is found to be
so small that it can be safely neglected.

A routine for iterating eq. (C-6) was coded. To check
the codes and the validity of assuming the residual error
per iteration, \( \Delta \), to be negligible, the routine was used to
solve \( y'' = -y \) to obtain the sine function. For example,
with \( h = 0.01 \) and \( r = 9.61 \) the computed value is
\[ \sin(9.61) = -0.1841647, \]
whereas the actual value is \(-0.1841678\). Thus, errors appear only in the sixth significant figure
after 960 steps in the integration.

In order to use this, however, two initial values are
needed. These can be obtained by summing a series solution
to the equation. Let us try a solution of the form
\[ \psi = \sum_{n=0}^{\infty} a_n r^{n+p} \]. Substituting this into eq. C-4) gives

\[ \sum_{n=0}^{\infty} \left\{ \frac{(n+p)(n+p+1)}{2} a_n r^{n+p-2} + k^2 a_n r^{n+p} - \ell (\ell+1) a_n r^{n+p+2} \right\} = 0 . \]

Taking \( n = 0 \) and equating like powers of \( r \) gives \( p = \ell + 1 \). Putting this back into eq. C-7) and equating like coefficients of \( r \) gives the recursion relation

\[ a_{n+2} = \frac{U(r) - k^2}{\left( (n+3)(n+2+2) - \ell (\ell+1) \right)} a_n . \]

We can put the solution in a more convenient form

\[ \psi (r) = r^{\ell+1} \sum_{n} a_n (\ell, n) \text{ where } \]

\[ A_{n+2} (\ell, n) = \frac{\left[ U(r) - k^2 \right] r^2}{\left[ (n+1+3)(n+2+2) - \ell (\ell+1) \right]} a_n . \]

The only question, then, in carrying out the summation of the series is that of how many terms need be summed. One is tempted to stop summing when the successive terms seem to be negligibly small. This, however, is a trap, as consideration of the series \( \sum \frac{1}{n} \) clearly reveals. The correct thing to do is to examine a comparison series. The recursion relation C-8) looks very like that of the series for \( e^{-x} \). A routine was written, therefore, to examine the accuracy of evaluating \( e^{-x} \) by summing its series expansion.
It was found that for reasonably large $x$, first that double precision was necessary and second, that, using double precision, fifty terms in the series would give sufficient accuracy. Therefore it was decided to do the same when summing eq C-7).

Finally, tests were made on the accuracy of the whole Schrodinger equation solving system by setting $U(r) = 0$ so that solutions should be $\psi(r) = \sin kr$. Accuracy of better than ten significant figures was obtained.

**PHASE SHIFT FINDER**

Solutions to the Schrodinger equation will have the form

$$\psi_s = C \left[ A_s J_0(kr) - B_s N_0(kr) \right]$$

where $\frac{B_s}{A_s} = \tan \delta$, by definition.

If we solve the equation for two points in space, say $R$ and $R + \Delta$, we can calculate

$$\rho = \frac{\psi_s(R)}{\psi_s(R+\Delta)} = \frac{C_s \left[ J_0(kr) - \tan \delta N_0(kr) \right]}{C_s \left[ J_0(kR+\Delta) - \tan \delta N_0(kR+\Delta) \right]}$$

so that we can solve for $\tan \delta$:

$$C-10) \quad \tan \delta = \frac{J_0(kR) - \rho J_0(kR+\Delta)}{N_0(kR) - \rho N_0(kR+\Delta)}$$

Thus, having coded routines to generate these Ricatti-Bessel functions and a routine to solve the Schrodinger equation,
it is a simple and straightforward task to use them to generate a set of phase shifts.

**POTENTIAL FINDER**

The set of equations which must be solved for \( \{ U_n \} \) and \( b \) as given by eq 2-14) are of the form

\[
\sum_{\rho, \gamma, \ldots, \varepsilon \choose \zeta_n, \ldots, \varepsilon_n, b} U_n = \delta_n
\]

for \( n = 0, 1, \ldots, N-1 \). These are \( N \) simultaneous equations, linear in the \( \{ U_n \} \) but of higher order in \( b \), the basic range parameter. What we seek is a set of values \( \{ U_n, b \} \) which solves all the equations simultaneously. We see that for any given value of \( b \) the first \( N-1 \) of these can be solved for a unique set of \( \{ U_n \} \) but in general these values along with the trial value for \( b \) will not be a solution to the last equation. \( b \) cannot be negative or zero so we may start with some very small initial trial value for it, solve the first \( N-1 \) equations for the corresponding set of potential coefficients \( \{ U_n \} \), and substitute \( \{ U_n, b \} \) in the \( N \)th equation. If \( \{ U_n, b \} \) is actually a solution to the whole set of equations, the right-hand side of this \( N \)th equation will be zero. Thus we can treat the whole procedure as the search for the solution to an equation of the form \( F(b) = 0 \). By systematically choosing trial values for \( b \), we can find two values which straddle a root, and then we can use our Regula Falsi technique to converge to a good approximation to the solution to the solution.
There are two numerical effects which demand some care in the following of this prescription. First, one must examine the conditioning of the first N-1 linear equations. If the determinant of the coefficients of the unknowns is very small, the equations are ill-conditioned. This means that their "solutions" contain few significant figures, and indeed may not have any physical significance whatsoever. Secondly, it may happen that eq 2-14 has a pole (due to ill-conditioning) so that \( F(b_1) > 0 \) and \( F(b_2) < 0 \), \( b_1 < b_2 \) but that \( F(b) \neq 0 \) for \( b_1 < b < b_2 \). That is, a plot of \( F(b) \) may resemble that of \( \tan(x) \) and have two or more branches. Care must be taken, then, to avoid becoming trapped by this circumstance since the Regula Falsi routine will keep looking forever for the root that really is not there.

With these precautions, it is easy to search out all of the solutions to the potential equations. One must keep in mind that numerical effects such as round-off error and the numerical noise introduced by the subtraction of two nearly equal numbers can and often does introduce spurious "solutions." Thus, no solution can really be taken for granted until the phase shifts which it produces have been examined and compared with the original data.
APPENDIX D. NOTATION AND PHYSICAL CONSTANTS

\[ n! \lambda = (n) (n-2) \ldots (n-(n-2)) \]
\[ \binom{n}{\lambda} \text{ is the binomial constant.} \]

Brackets denote a set of quantities; for example,
\[ \{c_n\} \text{ means the set } c_0, c_1, c_2, \ldots \]
\( \delta_\mu \) (one subscript) denotes a phase shift, while \( \delta_{\mu \nu} \) (two subscripts) is the Kronecker delta function:
\[ \delta_{\mu \mu} = 1 \text{ if } \mu = \nu \]
\[ = 0 \text{ if } \mu \neq \nu \]

Mass \( \text{He}^4 \) = \( 37.28337 \times 10^8 \) ev.
\[ = 6.646044 \times 10^{-24} \text{ gm.} \]

Mass \( n \) = \( 1.67482 \times 10^{-24} \text{ gm.} \)

Reduced mass \( \mu = 1/2(2.674254 \times 10^{-24}) \text{ gm.} \)

\[ \frac{2 \mu}{\hbar^2} = 2.4049689 \times 10^{30} \text{ cm}^{-2} \]
\[ \times 10^4 \text{ fm}^{-2} \]

Thus if \( E \) is in MeV, \( \psi E = k^2 \) where \( \psi = 3.89845 \times 10^{-2} \text{ fm}^{-2} \text{ MeV}^{-1} \)

This means that 25.65 MeV in the center of mass (or 32.06 MeV in the lab.) corresponds to \( k^2 = 1 \).

\( \hbar = 1.05450 \times 10^{-27} \text{ erg-sec.} \) and there are \( 1.621 \times 10^{-6} \text{ MeV} \) \( \text{ erg}^{-1} \).
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