



THE RICE INSTITUTE

CONCERNING A GENERALIZATION OF CONVEXITY  
FOR FUNCTIONS OF ONE VARIABLE

by

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INTRODUCTION. A real-valued function  $y(x)$  defined on an open interval  $(a,b)$  is said to be convex on  $(a,b)$  if and only if for arbitrary  $x_1, x_2$ , with  $a < x_1 < x_2 < b$ ,

$$y(x) \leq A + Bx,$$

$x_1 \leq x \leq x_2$ , where  $A$  and  $B$  satisfy

$$y_1 = A + Bx_1$$

$$y_2 = A + Bx_2,$$

with  $y_j = y(x_j)$ ,  $j = 1, 2$ . Our purpose here is to generalize this definition of convexity and to examine some of the implications of the more general definition. In what follows, all values of  $x$  which appear will be understood to be in the open interval  $(a,b)$ . Accordingly, any statements made regarding the continuity and differentiability of the functions considered are made relative to  $(a,b)$ .

PRELIMINARIES AND DEFINITIONS. Let  $\phi_1(x), \phi_2(x) \in C$  and such that the linear system

$$(1) \quad \begin{aligned} y_1 &= A\phi_1(x_1) + B\phi_2(x_1) \\ y_2 &= A\phi_1(x_2) + B\phi_2(x_2), \end{aligned}$$

$x_1 < x_2$ , is solvable uniquely for  $A$  and  $B$ .

Definition 1.

$$F(x_1, x_2) = \begin{vmatrix} \phi_1(x_1) & \phi_2(x_1) \\ \phi_1(x_2) & \phi_2(x_2) \end{vmatrix}.$$

Definition 2.  $\phi_1$  and  $\phi_2$  are said to satisfy condition

(A) if and only if  $F(x_1, x_2) > 0$  for arbitrary  $x_1 < x_2$ .

Theorem 1. A necessary and sufficient condition that (1) be solvable uniquely for  $A$  and  $B$  is that (A) holds.

Proof of Necessity. Let  $x_1^* < x_2^*$  be two particular points in  $(a, b)$ . Since (1) is solvable uniquely for A and B, either (a)  $F(x_1^*, x_2^*) > 0$  or (b)  $F(x_1^*, x_2^*) < 0$ . If (b) holds, rename  $\phi_1$  and  $\phi_2$  so that  $F(x_1^*, x_2^*) > 0$ . Hence we can assume that (a) holds. Since  $F(x_1, x_2)$  is continuous and non-zero on the connected set  $a < x_1 < x_2 < b$ , it cannot change sign there, for if it did, it would have to take on the value zero. Hence  $F(x_1, x_2) > 0$  for  $a < x_1 < x_2 < b$ .

The proof of sufficiency is immediate.

Theorem 2. If (A) holds, then  $\phi_1$  and  $\phi_2$  have no common zero in  $(a, b)$  and  $\phi_j$ ,  $j = 1, 2$ , has at most one zero in  $(a, b)$ .

The proof of this theorem is immediate from Definition 2.

Theorem 3. If (A) holds and  $\phi_1, \phi_2 \in C^1$ , then  $W(x) \geq 0$ ,

where

$$W(x) = \begin{vmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(x) & \phi_2'(x) \end{vmatrix}.$$

Proof. Take  $x_2 > x$ .

$$\begin{aligned} \begin{vmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1(x_2) & \phi_2(x_2) \end{vmatrix} &= \begin{vmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1(x_2) - \phi_1(x) & \phi_2(x_2) - \phi_2(x) \end{vmatrix} \\ &= (x_2 - x) \begin{vmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(\xi_1) & \phi_2'(\xi_2) \end{vmatrix}, \end{aligned}$$

where  $x < \xi_1 < x_2$  and  $x < \xi_2 < x_2$ . Since (A) holds,

$$\begin{vmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(\xi_1) & \phi_2'(\xi_2) \end{vmatrix} > 0.$$

Taking the limit as  $x_2 \rightarrow x$ ,  $W(x) \geq 0$ .

Theorem 4. If (A) holds,  $\phi_1, \phi_2 \in C^2$  and  $W(x_0) = 0$ , then

$W'(x_0) = 0$  and  $W[\phi_1'(x_0), \phi_2'(x_0)] = 0$ , where  $W[\phi_1'(x_0), \phi_2'(x_0)]$  is the Wronskian of  $\phi_1'$  and  $\phi_2'$  evaluated at  $x_0$ .

Proof. Since  $\phi_1, \phi_2 \in C^2$  then  $W(x) \in C^1$ . From Theorem 3,  $W(x) \geq 0$ . Hence if  $W(x_0) = 0$ , then  $W'(x_0) = 0$ .

$$\begin{vmatrix} \phi_1(x_0) & \phi_2(x_0) & \phi_1(x_0) \\ \phi_1'(x_0) & \phi_2'(x_0) & \phi_1'(x_0) \\ \phi_1''(x_0) & \phi_2''(x_0) & \phi_1''(x_0) \end{vmatrix} = 0,$$

so that

$$\begin{aligned} \phi_1(x_0)W[\phi_1'(x_0), \phi_2'(x_0)] &= \phi_1'(x_0)W'(x_0) - \phi_1''(x_0)W(x_0) \\ &= 0, \end{aligned}$$

which implies that  $W[\phi_1'(x_0), \phi_2'(x_0)] = 0$  if  $\phi_1(x_0) \neq 0$ . Suppose  $\phi_1(x_0) = 0$ , then from Theorem 2,  $\phi_2(x_0) \neq 0$ . But

$$\begin{aligned} \phi_2(x_0)W[\phi_1'(x_0), \phi_2'(x_0)] &= \phi_2'(x_0)W'(x_0) - \phi_2''(x_0)W(x_0) \\ &= 0, \end{aligned}$$

which again implies that  $W[\phi_1'(x_0), \phi_2'(x_0)] = 0$ .

Definition 3. Let  $\phi_1, \phi_2 \in C$  and satisfy (A) on  $(a, b)$ .

If  $y(x)$  is a real-valued function defined on  $(a, b)$ , then  $y(x)$  is said to be convex on  $(a, b)$  if and only if for arbitrary  $x_1, x_2$ , with  $a < x_1 < x_2 < b$ ,

$$y(x) \leq A\phi_1(x) + B\phi_2(x),$$

$x_1 \leq x \leq x_2$ , where  $A$  and  $B$  satisfy

$$(2) \quad \begin{aligned} y_1 &= A\phi_1(x_1) + B\phi_2(x_1) \\ y_2 &= A\phi_1(x_2) + B\phi_2(x_2) \end{aligned}$$

with  $y_j = y(x_j)$ ,  $j = 1, 2$ .

Noting Theorem 1, it is readily seen that this definition of convexity is a special case of a more general definition of

convexity by E. F. Beckenbach [1]\*.

Solving (2) for A and B and performing a few simple algebraic manipulations we see that  $y(x)$  is convex if and only if

$$y(x) \leq \frac{F(x_1, x)}{F(x_1, x_2)} y(x_2) + \frac{F(x, x_2)}{F(x_1, x_2)} y(x_1),$$

$x_1 \leq x \leq x_2$ . Since  $F(x_1, x_2) > 0$ , we see that  $y(x)$  is convex if and only if

$$\begin{vmatrix} \phi_1(x_1) & \phi_2(x_1) & y(x_1) \\ \phi_1(x) & \phi_2(x) & y(x) \\ \phi_1(x_2) & \phi_2(x_2) & y(x_2) \end{vmatrix} \geq 0,$$

$x_1 \leq x \leq x_2$ .

Definition 4. Let  $\phi_1, \phi_2 \in C$  and satisfy (A) on  $(a, b)$ .

If  $y(x)$  is a real-valued function defined on  $(a, b)$ , then  $y(x)$  is said to be convex on  $(a, b)$  if and only if for arbitrary  $x_1, x_2, x_3$ , with  $a < x_1 \leq x_2 \leq x_3 < b$ ,

$$\begin{vmatrix} \phi_1(x_1) & \phi_2(x_1) & y(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & y(x_2) \\ \phi_1(x_3) & \phi_2(x_3) & y(x_3) \end{vmatrix} \geq 0.$$

From the preceding discussion, it is evident that Definitions 3 and 4 are equivalent.

An immediate consequence of Definition 4 is that if  $y_1(x)$  and  $y_2(x)$  are convex, then  $y_1(x) + y_2(x)$  is convex.

The above definition of convexity is a generalization of the chordal definition of ordinary convexity. We will now relate the generalized definition to the support and operator aspects of ordinary convexity.

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\*Numbers in brackets refer to the bibliography at the end of this paper.

PRINCIPAL THEOREMS.

Theorem I. Let  $\phi_1, \phi_2, y \in C^1$  and let (A) hold. A necessary and sufficient condition that  $y(x)$  be convex is that for arbitrary  $x_1, x_2$ ,

$$\begin{vmatrix} \phi_1(x_1) & \phi_2(x_1) & y(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & y(x_2) \\ \phi_1'(x_2) & \phi_2'(x_2) & y'(x_2) \end{vmatrix} \geq 0.$$

Proof of Necessity.

Case 1.  $x_1 < x_2$ . Choose  $x_3 > x_2$ , then

$$\begin{vmatrix} \phi_1(x_1) & \phi_2(x_1) & y(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & y(x_2) \\ \phi_1(x_3) & \phi_2(x_3) & y(x_3) \end{vmatrix} = \begin{vmatrix} \phi_1(x_1) & \phi_2(x_1) & y(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & y(x_2) \\ \phi_1(x_3) - \phi_1(x_2) & \phi_2(x_3) - \phi_2(x_2) & y(x_3) - y(x_2) \end{vmatrix} = (x_3 - x_2) \begin{vmatrix} \phi_1(x_1) & \phi_2(x_1) & y(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & y(x_2) \\ \phi_1'(\xi_1) & \phi_2'(\xi_2) & y'(\xi_3) \end{vmatrix},$$

where  $x_2 < \xi_j < x_3$ ,  $j = 1, 2, 3$ . Since  $y(x)$  is convex,

$$\begin{vmatrix} \phi_1(x_1) & \phi_2(x_1) & y(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & y(x_2) \\ \phi_1'(\xi_1) & \phi_2'(\xi_2) & y'(\xi_3) \end{vmatrix} \geq 0.$$

Taking the limit as  $x_3 \rightarrow x_2$ , we obtain,

$$\begin{vmatrix} \phi_1(x_1) & \phi_2(x_1) & y(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & y(x_2) \\ \phi_1'(x_2) & \phi_2'(x_2) & y'(x_2) \end{vmatrix} \geq 0.$$

Case 2.  $x_1 > x_2$ . Choose  $x_3 < x_2$ , then

$$\begin{vmatrix} \phi_1(x_3) & \phi_2(x_3) & y(x_3) \\ \phi_1(x_2) & \phi_2(x_2) & y(x_2) \\ \phi_1(x_1) & \phi_2(x_1) & y(x_1) \end{vmatrix} = (x_3 - x_2) \begin{vmatrix} \phi_1'(\xi_1) & \phi_2'(\xi_2) & y'(\xi_3) \\ \phi_1(x_2) & \phi_2(x_2) & y(x_2) \\ \phi_1(x_1) & \phi_2(x_1) & y(x_1) \end{vmatrix}$$

where  $x_3 < \xi_j < x_2$ ,  $j = 1, 2, 3$ . Since  $y(x)$  is convex and  $x_3 < x_2$ ,

$$\begin{vmatrix} \phi_1'(\xi_1) & \phi_2'(\xi_2) & y'(\xi_3) \\ \phi_1(x_2) & \phi_2(x_2) & y(x_2) \\ \phi_1(x_1) & \phi_2(x_1) & y(x_1) \end{vmatrix} \leq 0,$$

so that

$$\begin{vmatrix} \phi_1(x_1) & \phi_2(x_1) & y(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & y(x_2) \\ \phi_1'(\xi_1) & \phi_2'(\xi_2) & y'(\xi_3) \end{vmatrix} \geq 0.$$

Taking the limit as  $x_3 \rightarrow x_2$ , we obtain,

$$\begin{vmatrix} \phi_1(x_1) & \phi_2(x_1) & y(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & y(x_2) \\ \phi_1'(x_2) & \phi_2'(x_2) & y'(x_2) \end{vmatrix} \geq 0,$$

which completes the proof of necessity.

Proof of Sufficiency. We wish to show that for arbitrary

$x_1, x_2, x_3$ , with  $x_1 < x_2 < x_3$ ,

$$\begin{vmatrix} \phi_1(x_1) & \phi_2(x_1) & y(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & y(x_2) \\ \phi_1(x_3) & \phi_2(x_3) & y(x_3) \end{vmatrix} \geq 0.$$

Now for any  $t \in (a, b)$  we are given that,

$$\begin{vmatrix} \phi_1(x_1) & \phi_2(x_1) & y(x_1) \\ \phi_1(t) & \phi_2(t) & y(t) \\ \phi_1'(t) & \phi_2'(t) & y'(t) \end{vmatrix} \geq 0,$$

which can be written,

$$y'(t)F(x_1, t) - y(t)F'(x_1, t) + y(x_1)W(t) \geq 0,$$

where the prime denotes differentiation with respect to  $t$ .

If  $t \neq x_1$ , then

$$\frac{y'(t)F(x_1, t) - y(t)F'(x_1, t)}{F^2(x_1, t)} \geq -y(x_1) \frac{W(t)}{F^2(x_1, t)},$$

that is,

$$\frac{d}{dt} \left[ \frac{y(t)}{F(x_1, t)} \right] \geq -y(x_1) \frac{W(t)}{F^2(x_1, t)},$$

so that

$$\frac{y(x_3)}{F(x_1, x_3)} - \frac{y(x_2)}{F(x_1, x_2)} \geq -y(x_1) \int_{x_2}^{x_3} \frac{W(t)}{F^2(x_1, t)} dt.$$

But if  $x > x_1$ ,

$$\int_{x_2}^x \frac{W(t)}{F^2(x_1, t)} dt = \frac{F(x_2, x)}{F(x_1, x)F(x_1, x_2)},$$

which is easily proved by differentiation. Hence

$$\int_{x_2}^{x_3} \frac{W(t)}{F^2(x_1, t)} dt = \frac{F(x_2, x_3)}{F(x_1, x_3)F(x_1, x_2)},$$

so that

$$\frac{y(x_3)}{F(x_1, x_3)} - \frac{y(x_2)}{F(x_1, x_2)} \geq -y(x_1) \frac{F(x_2, x_3)}{F(x_1, x_3)F(x_1, x_2)}.$$

Since  $F(x_1, x_2) > 0$  and  $F(x_1, x_3) > 0$  by (A), we have

$$\begin{vmatrix} \phi_1(x_1) & \phi_2(x_1) & y(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & y(x_2) \\ \phi_1(x_3) & \phi_2(x_3) & y(x_3) \end{vmatrix} \geq 0,$$

which completes the proof of sufficiency.



We note that if  $W(x_0) > 0$ , the condition

$$\begin{vmatrix} \phi_1(x) & \phi_2(x) & y(x) \\ \phi_1(x_0) & \phi_2(x_0) & y(x_0) \\ \phi_1'(x_0) & \phi_2'(x_0) & y'(x_0) \end{vmatrix} \geq 0,$$

for arbitrary  $x$ , is equivalent to the statement that

$$y(x) \geq A\phi_1(x) + B\phi_2(x)$$

where  $A$  and  $B$  satisfy

$$y_0 = A\phi_1(x_0) + B\phi_2(x_0)$$

$$y_0' = A\phi_1'(x_0) + B\phi_2'(x_0),$$

with  $y_0 = y(x_0)$ ,  $y_0' = y'(x_0)$ ; i.e. the curve  $y(x)$  lies always above or on the curve  $A\phi_1(x) + B\phi_2(x)$  tangent to  $y(x)$  at  $x_0$ .

Theorem II. Let  $\phi_1, \phi_2, y \in C^2$  and let (A) hold. Presume also that  $W(x) > 0$  throughout  $(a, b)$ . A necessary and sufficient condition that  $y(x)$  be convex is that

$$\begin{vmatrix} \phi_1(x) & \phi_2(x) & y(x) \\ \phi_1'(x) & \phi_2'(x) & y'(x) \\ \phi_1''(x) & \phi_2''(x) & y''(x) \end{vmatrix} \geq 0.$$

Proof of Necessity. Take  $x_1 > x$ . From Theorem I we have

$$\begin{vmatrix} \phi_1(x) & \phi_2(x) & y(x) \\ \phi_1(x_1) & \phi_2(x_1) & y(x_1) \\ \phi_1'(x_1) & \phi_2'(x_1) & y'(x_1) \end{vmatrix} \geq 0 \text{ and } \begin{vmatrix} \phi_1(x_1) & \phi_2(x_1) & y(x_1) \\ \phi_1(x) & \phi_2(x) & y(x) \\ \phi_1'(x) & \phi_2'(x) & y'(x) \end{vmatrix} \geq 0.$$

Hence,

$$\begin{vmatrix} \phi_1(x) & \phi_2(x) & y(x) \\ \phi_1(x_1) & \phi_2(x_1) & y(x_1) \\ \phi_1'(x) & \phi_2'(x) & y'(x) \end{vmatrix} \leq 0,$$

so that

$$\begin{vmatrix} \phi_1(x) & \phi_2(x) & y(x) \\ \phi_1(x_1) & \phi_2(x_1) & y(x_1) \\ \phi_1'(x_1) & \phi_2'(x_1) & y'(x_1) \end{vmatrix} - \begin{vmatrix} \phi_1(x) & \phi_2(x) & y(x) \\ \phi_1(x_1) & \phi_2(x_1) & y(x_1) \\ \phi_1'(x) & \phi_2'(x) & y'(x) \end{vmatrix} =$$

$$\begin{vmatrix} \phi_1(x) & \phi_2(x) & y(x) \\ \phi_1(x_1) & \phi_2(x_1) & y(x_1) \\ \phi_1'(x_1) - \phi_1'(x) & \phi_2'(x_1) - \phi_2'(x) & y'(x_1) - y'(x) \end{vmatrix} \geq 0.$$

But

$$\begin{vmatrix} \phi_1(x) & \phi_2(x) & y(x) \\ \phi_1(x_1) & \phi_2(x_1) & y(x_1) \\ \phi_1'(x_1) - \phi_1'(x) & \phi_2'(x_1) - \phi_2'(x) & y'(x_1) - y'(x) \end{vmatrix} =$$

$$\begin{vmatrix} \phi_1(x) & \phi_2(x) & y(x) \\ \phi_1(x_1) - \phi_1(x) & \phi_2(x_1) - \phi_2(x) & y(x_1) - y(x) \\ \phi_1'(x_1) - \phi_1'(x) & \phi_2'(x_1) - \phi_2'(x) & y'(x_1) - y'(x) \end{vmatrix} =$$

$$(x_1 - x)^2 \begin{vmatrix} \phi_1(x) & \phi_2(x) & y(x) \\ \phi_1'(\xi_1) & \phi_2'(\xi_2) & y'(\xi_3) \\ \phi_1''(\xi_1') & \phi_2''(\xi_2') & y''(\xi_3') \end{vmatrix},$$

where  $x < \xi_j < x_1$ ,  $x < \xi_j' < x_1$ ,  $j = 1, 2, 3$ . Therefore,

$$\begin{vmatrix} \phi_1(x) & \phi_2(x) & y(x) \\ \phi_1'(\xi_1) & \phi_2'(\xi_2) & y'(\xi_3) \\ \phi_1''(\xi_1') & \phi_2''(\xi_2') & y''(\xi_3') \end{vmatrix} \geq 0.$$

Taking the limit as  $x_1 \rightarrow x$ , we obtain

$$\begin{vmatrix} \phi_1(x) & \phi_2(x) & y(x) \\ \phi_1'(x) & \phi_2'(x) & y'(x) \\ \phi_1''(x) & \phi_2''(x) & y''(x) \end{vmatrix} \geq 0.$$

Proof of Sufficiency. Take arbitrary  $x_1, x_2, x_3$ , with  $x_1 < x_2 < x_3$ . We can write,

$$\begin{vmatrix} \phi_1(x) & \phi_2(x) & y(x) \\ \phi_1'(x) & \phi_2'(x) & y'(x) \\ \phi_1''(x) & \phi_2''(x) & y''(x) \end{vmatrix} = g(x)$$

where  $g(x) \in C$  and  $g(x) \geq 0$ . Since  $W(x) > 0$ , we see that

$$y''(x) - \frac{W'(x)}{W(x)} y'(x) + \frac{W(\phi_1', \phi_2')}{W(x)} y(x) = \frac{g(x)}{W(x)},$$

which implies that

$$y(x) = \int_{x_0}^x \frac{\phi_1(t) \phi_2(x) - \phi_1(x) \phi_2(t)}{W^2(t)} g(t) dt + C_1 \phi_1(x) + C_2 \phi_2(x),$$

which may be written

$$y(x) = \int_{x_0}^x \frac{F(t, x)}{W^2(t)} g(t) dt + C_1 \phi_1(x) + C_2 \phi_2(x),$$

where  $C_1$  and  $C_2$  are constants of integration.

It is sufficient to show that

$$\begin{vmatrix} \phi_1(x_1) & \phi_2(x_1) & G(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & G(x_2) \\ \phi_1(x_3) & \phi_2(x_3) & G(x_3) \end{vmatrix} \geq 0,$$

where

$$G(x) = \int_{x_0}^x \frac{F(t, x)}{W^2(t)} g(t) dt.$$

We observe that

$$\begin{vmatrix} \phi_1(x_1) & \phi_2(x_1) & G(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & G(x_2) \\ \phi_1(x_3) & \phi_2(x_3) & G(x_3) \end{vmatrix} = \int_{x_0}^{x_1} \frac{F(t, x_1) F(x_2, x_3)}{W^2(t)} g(t) dt - \int_{x_0}^{x_2} \frac{F(t, x_2) F(x_1, x_3)}{W^2(t)} g(t) dt$$

$$\begin{aligned}
& + \int_{x_0}^{x_3} \frac{F(t, x_3)F(x_1, x_2)}{W^2(t)} g(t) dt = \\
& \int_{x_0}^{x_1} \frac{F(t, x_1)F(x_2, x_3) - F(t, x_2)F(x_1, x_3) + F(t, x_3)F(x_1, x_2)}{W^2(t)} g(t) dt \\
& + \int_{x_1}^{x_2} \frac{F(t, x_3)F(x_1, x_2) - F(t, x_2)F(x_1, x_3)}{W^2(t)} g(t) dt \\
& + \int_{x_2}^{x_3} \frac{F(t, x_3)F(x_1, x_2)}{W^2(t)} g(t) dt = \\
& \int_{x_1}^{x_2} \frac{F(x_1, t)F(x_2, x_3)}{W^2(t)} g(t) dt + \int_{x_2}^{x_3} \frac{F(t, x_3)F(x_1, x_2)}{W^2(t)} g(t) dt,
\end{aligned}$$

since from Definition 1, we see that

$$F(t, x_3)F(x_1, x_2) - F(t, x_2)F(x_1, x_3) = F(x_1, t)F(x_2, x_3)$$

so that

$$\begin{aligned}
& F(t, x_1)F(x_2, x_3) - F(t, x_2)F(x_1, x_3) + F(t, x_3)F(x_1, x_2) = \\
& F(t, x_1)F(x_2, x_3) + F(x_1, t)F(x_2, x_3) = 0.
\end{aligned}$$

From (A) we see immediately that

$$\begin{vmatrix}
\phi_1(x_1) & \phi_2(x_1) & G(x_1) \\
\phi_1(x_2) & \phi_2(x_2) & G(x_2) \\
\phi_1(x_3) & \phi_2(x_3) & G(x_3)
\end{vmatrix} \geq 0.$$

The preceding theorem is a special case of a more general theorem proved by M. M. Peixoto [2], but the analysis used here is considerably simpler and more direct.

Theorem III. Let  $\phi_1, \phi_2, y \in C^2$  and let (A) hold. Presume also that  $W(x) > 0$  throughout  $(a, b)$ . A necessary and sufficient

condition that

$$\begin{vmatrix} \phi_1(x_1) & \phi_2(x_1) & y(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & y(x_2) \\ \phi_1'(x_2) & \phi_2'(x_2) & y'(x_2) \end{vmatrix} \geq 0$$

for arbitrary  $x_1, x_2$ , is that

$$\begin{vmatrix} \phi_1(x) & \phi_2(x) & y(x) \\ \phi_1'(x) & \phi_2'(x) & y'(x) \\ \phi_1''(x) & \phi_2''(x) & y''(x) \end{vmatrix} \geq 0.$$

The proof of this theorem is immediate from Theorems I and II.

Theorem IV. If the restriction  $W(x) > 0$  throughout  $(a, b)$  is removed from Theorems II and III, then these theorems no longer hold.

Proof. From Theorem 3,  $W(x) \geq 0$ . Suppose then that there exists an  $x_0 \in (a, b)$  for which  $W(x_0) = 0$ . Define

$$y^*(x) = -|F(x_0, x)|.$$

$y^*(x) \in C^2$ , in fact, by Theorem 4,  $y^{*'}(x_0) = 0$  and  $y^{*''}(x_0) = 0$ . From Definitions 1 and 2,  $y^*(x_0) = 0$  and  $y^*(x) < 0$ ,  $x \neq x_0$ .

The function  $y^*(x)$  satisfies

$$\begin{vmatrix} \phi_1(x) & \phi_2(x) & y^*(x) \\ \phi_1'(x) & \phi_2'(x) & y^{*'}(x) \\ \phi_1''(x) & \phi_2''(x) & y^{*''}(x) \end{vmatrix} \geq 0, \text{ in fact, } \begin{vmatrix} \phi_1(x) & \phi_2(x) & y^*(x) \\ \phi_1'(x) & \phi_2'(x) & y^{*'}(x) \\ \phi_1''(x) & \phi_2''(x) & y^{*''}(x) \end{vmatrix} = 0.$$

Take  $x_1, x_2$ , so that  $x_1 < x_0 < x_2$ . Then

$$\begin{vmatrix} \phi_1(x_1) & \phi_2(x_1) & y^*(x_1) \\ \phi_1(x_0) & \phi_2(x_0) & y^*(x_0) \\ \phi_1(x_2) & \phi_2(x_2) & y^*(x_2) \end{vmatrix} = \begin{vmatrix} \phi_1(x_1) & \phi_2(x_1) & y^*(x_1) \\ \phi_1(x_0) & \phi_2(x_0) & 0 \\ \phi_1(x_2) & \phi_2(x_2) & y^*(x_2) \end{vmatrix} = \dots$$

$y^*(x_1)F(x_0, x_2) + y^*(x_2)F(x_1, x_0) < 0$ , so that Theorem II does not hold.

Noting that  $W(x)$  cannot be identically zero in any sub-interval of  $(a, b)$  since this would violate (A), take  $x_1, x_2$ , so that  $x_1 < x_0 < x_2$  and  $W(x_2) > 0$ . Then

$$\begin{vmatrix} \phi_1(x_1) & \phi_2(x_1) & y^*(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & y^*(x_2) \\ \phi_1'(x_2) & \phi_2'(x_2) & y^{*'}(x_2) \end{vmatrix} = \begin{vmatrix} \phi_1(x_1) & \phi_2(x_1) & F(x_0, x_1) \\ \phi_1(x_2) & \phi_2(x_2) & -F(x_0, x_2) \\ \phi_1'(x_2) & \phi_2'(x_2) & -F'(x_0, x_2) \end{vmatrix} = \\ \begin{vmatrix} \phi_1(x_1) & \phi_2(x_1) & 2F(x_0, x_1) \\ \phi_1(x_2) & \phi_2(x_2) & 0 \\ \phi_1'(x_2) & \phi_2'(x_2) & 0 \end{vmatrix} =$$

$2F(x_0, x_1)W(x_2) < 0$ , so that Theorem III does not hold.

DISCUSSION. We note that in Theorems I and II, the restriction  $W(x) > 0$  throughout  $(a, b)$  is not needed for the proofs of necessity. In fact, this restriction is used only in the proof of sufficiency for Theorem II. From the proof of necessity for Theorem II, it is easily seen that a proof of necessity for Theorem III would not require this restriction, whereas from Theorem IV, a proof of sufficiency must require it.

## BIBLIOGRAPHY

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