THE GENERATION OF WAVES OF DISCONTINUITY

BY CONTINUOUS MOTION OF A PISTON

by

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The object of this thesis is to give the solution of a problem in the motion of a perfect gas, to study the singularities of this solution and see what form it must assume in the neighborhood of such a singularity.

The physical model which represents the problem is an infinitely long tube of uniform cross section, having walls of a non-heat-conducting material and fitted with a movable piston at one end. The cylinder is filled with a perfect gas which is assumed to produce no friction with the walls of the cylinder or within itself and to conduct no heat internally.

If the piston is now moved analytically into the gas, it is required in the first part of the problem to give the velocity, pressure, and specific volume of the gas in motion near the origin. These will be given, with a discussion of the extent of the region in which they represent the solution of the problem. It will be shown that for a certain kind of analytic motion of the piston, a singularity arises in this solution of a type that suggests a new solution in which the boundary between the gas in motion and the gas at rest is a shock wave. This is a wave across which there is a discontinuity in the velocity, pressure, and specific volume. In the second part of the problem, the new solution will be found in
the form of a power series. The method of computing the successive derivatives will be indicated with a specific example illustrating the results.

This problem, in particular the first part, has been discussed by several writers. Among them, J. Hadamard, in his *Leçons sur la Propagation des Ondes*, has treated the problem of replacing the solution in the neighborhood of a singularity of the type studied in this paper. His treatment was, however, in a different system of coordinates and the results given in the reference mentioned above were incorrect.

A notion of fundamental importance in the method used here is that of characteristic curve. A characteristic curve of a system of partial differential equations of the first order may be defined as a curve in the space of the independent variables on which specification of the values of the dependent variables does not uniquely determine their derivatives. This means that two integrals of the system may take on the same values on a characteristic curve but otherwise be different. That is, solutions of the equations may be joined together along characteristic curves. This property will be necessary in finding the solution in the first part of the problem and in fitting this solution to the one for
the last part.

Let the velocity u, pressure p, and the specific volume v be functions of the position and time, x and t. Then the differential equations which define the motion of a frictionless fluid are

\[ (1) \quad u \frac{\partial u}{\partial x} + \frac{\partial p}{\partial t} + v \frac{\partial v}{\partial x} = 0 \]

\[ (2) \quad v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x} - v \frac{\partial v}{\partial t} = 0 \]

Equation (1) expresses the conservation of momentum,

\[ \frac{D(u \Delta x)}{D t} = - \frac{\partial p}{\partial x} \Delta x \]

and (2) the conservation of mass,

\[ \frac{D (\Delta x)}{D t} = 0 \]

where \( \Delta x \) is the volume element of the fluid when the cross section of the cylinder is taken to have area equal to 1 and \( D/Dt \) means the total time derivative or differentiation following the motion of the gas. Equation (2) is called the equation of continuity.

Since there is no heat conduction in the gas or through the cylinder walls, we may assume the adiabatic gas law to hold initially. Then

\[ p = p(v) = k v^{-\gamma} \]

and we will choose the units so that the constant k is
equal to 1. This may be done by assuming $p$ equal to 1 and $v$ equal to 1 in the gas at rest. Under this assumption $p$ is a function of $v$ alone and (1) may be written

$$(1) \quad u \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + v \rho' \frac{\partial v}{\partial x} = 0.$$  

Consider now the system of four equations composed of (1) and (2) and the equations for the total differentials of $u$ and $v$: 

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial t} dt,$$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial t} dt.$$  

Let the differentiation with respect to $x$ and $t$ be along an arbitrary curve in the $x,t$ plane. Then this curve is a characteristic of the equations (1) and (2) only if the system of four equations does not uniquely determine the derivatives of $u$ and $v$. The determinant of the coefficients must then be zero and we find the result to be the differential equations defining two real families of curves in the $x,t$ plane:

$$\begin{align*}
(3) \quad \frac{dx}{dt} &= u + v (-\rho')^{1/2} \\
(4) \quad \frac{dx}{dt} &= u - v (-\rho')^{1/2}.
\end{align*}$$

Thus along these curves, for a given set of values of $u$ and $v$, we can find more than one set of values for their
derivatives provided the equations are consistent.

The consistency relation is obtained by setting any one of the other four determinants of the four by five matrix of the system equal to zero. Thus

\[ \frac{dx}{dt} = u + v \frac{du}{dv} \]

must hold for both of the families (3) and (4).

From the relation between \( p \) and \( v \),
\[ (-p')^2 = \gamma \frac{v^2}{v^2} v - \frac{x^2}{2} \]

From (5) and (3) we find

\[ \frac{du}{dv} = (-p')^2 = \gamma \frac{v^2}{v^2} v - \frac{x^2}{2} \]

and from (5) and (4)
\[ \frac{dv}{d\omega} = -(-p')^2 = -\gamma \frac{v^2}{v^2} v - \frac{x^2}{2} \]

In consequence it follows that if \( u \) and \( v \) are given on a curve in the \( x,t \) plane, the condition that their first derivatives \( \text{not be} \) determined is that either (6) or (7) hold. More generally it can be shown that if the values of \( u \) and \( v \) are given together with their first \( n-1 \) derivatives on such a curve, the condition that their \( n \)-th derivatives not be determined is the same.

We may note from (3) and (4) that such discontinuities in the derivatives are propagated with sound velocity as has been shown by Hadamard.

On integrating (6) and (7) we have the two following
relations between \( u \) and \( v \) which apply to the families (3) and (4) respectively:

\[
\begin{align*}
(3) \quad \nu &= \frac{2}{1-\delta} \gamma^{\frac{1}{2}} v^{\frac{l-\gamma}{2}} + \frac{2c}{1-\delta} \\
(4) \quad \nu &= -\frac{2}{1-\delta} \gamma^{\frac{1}{2}} v^{\frac{l-\gamma}{2}} + \frac{2c'}{1-\delta}.
\end{align*}
\]

In particular, the initial boundary between the gas in motion and the gas at rest must be a characteristic since the solution \( u=0, p=1, \) and \( v=1 \) in the gas at rest must be joined to the solution for the region of motion of the gas, which takes on the values \( u=0, p=1, v=1 \) on the boundary but must obviously be otherwise different. And since the information that the piston has begun to move must be transmitted to the gas at rest, this boundary must have positive slope at the origin and therefore belongs to the family (3) on which \( dx/dt \) is positive when \( u=0. \)

In the neighborhood of the origin the members of the family (4) cross the boundary of the gas at rest and of course, completely fill out a region near the origin. Then the constant \( c' \) in equation (9), which was derived from (4), must be the same for every member of the family and may be evaluated at the boundary of the gas at rest. On letting \( \rho = 1/v, \) we see that
\[ \left[ \frac{dP}{d\rho} \right]_{\rho=1}^{\gamma} = \gamma \left( \frac{P'}{\rho'} \right)_{\rho=1}^{\gamma} = \gamma^{\frac{1}{2}} = c' = c_0, \]

\( c_0 \) being the adiabatic speed of sound in the gas at rest.

Thus (9), which becomes

\[ v(\omega) = \left( 1 + \frac{\gamma - 1}{2\gamma} u \right)^{\frac{1}{\gamma - 1}}, \]

shows that \( u \) is a function of \( v \) alone near the origin.

Using this in (8) we have

\[ u = \frac{c + c_0}{1 - \gamma} \]

a constant on each member of (3). Substituting for \( v(-p')^{\frac{1}{2}} \)

in (3) from (9), (3) is seen to define a family of straight lines with parameter \( u \):

\[ \frac{dx}{dt} = \frac{\gamma + 1}{2} u + c_0. \]

Integrating,

\[ x = \left( \frac{\gamma + 1}{2} u + c_0 \right) t - \varphi(u), \]

where \( \varphi(u) \) is to be determined by the motion of the piston path.

The boundary of motion of the gas is obtained by putting \( u \) equal to zero in (10). This line, \( x = c_0 t \), places the initial disturbance at the head of a sound wave with velocity \( c_0 \). By analogy the other characteristics of the family (3) are sound waves with velocity relative to the cylinder wall equal to the slope defined by (3).
The family of lines (10) also fills out the region of motion near the origin and with \( f(\omega) \) properly determined, gives \( u \) implicitly as a function of \( x \) and \( t \). Equation (9) then gives \( v \) and consequently \( p \) as functions of \( x \) and \( t \).

Let the piston path be given by \( x = F(t) \). In the problem studied here \( F(t) \) is to be analytic with \( F(0) = 0 \), \( F'(0) = 0 \), and \( F''(0) > 0 \). There is to be a third condition on the derivatives of \( F(t) \) involving \( F'''(0) \) but this will be given later.

If \( F'''(0) \) is greater than zero, \( u = F'(t) \) on the piston path will have a positive-valued analytic inverse function, \( t = \psi(u) \). Then on the piston path \( x \) and \( t \) may be written as follows:

\[
x = \int F'(u) \, dt = \int u \, \psi'(u) \, du = u \, \psi(u) - \int \psi(u) \, du
\]

\[
t = \psi(u).
\]

Substituting these values in (10) determines \( \psi(u) \) so that the lower bound of the gas in motion is the piston path. The result is:

\[
\varphi(u) = \int \psi(u) \, du + \left( \frac{d-1}{2} u + C \right) \psi(u)
\]

\[
(11) \quad \varphi(0) = 0.
\]

The equations for the family (4) can now be given parametrically in \( u \). From (4) and (9) we have for the family (4):
\[
\frac{dx}{dt} = \frac{3-x}{2}u - C_0.
\]

With \( x_\Gamma \) and \( t_\Gamma \) for \( x \) and \( t \) on (4) as functions of \( u \),
\[
\frac{dx_\Gamma}{du} = \left( \frac{3-x}{2}u - C_0 \right) \frac{dt_\Gamma}{du}.
\]

The derivatives here are total derivatives with respect to \( u \) and we may eliminate \( dx_\Gamma/du \) with (10). From (10),
\[
\frac{dx_\Gamma}{du} = \frac{\delta+1}{2}t_\Gamma - \varphi'(u) + \left( \frac{\delta+1}{2}u + C_0 \right) \frac{dt_\Gamma}{du},
\]

which substituted into the expression above for \( dx_\Gamma/du \) gives
\[
\frac{dt_\Gamma}{du} + \left( \frac{\delta+1}{2} \right) \frac{t_\Gamma}{(\gamma-1)u + 2C_0} = \frac{\varphi'(u)}{(\gamma-1)u + 2C_0}.
\]

Integrating,
\[
(12) \quad t_\Gamma = \left[ (\gamma-1)u + 2C_0 \right]^{\frac{\delta+1}{2(\gamma-1)}} \left\{ \int_0^{\gamma-1} \varphi'(u) \left( \frac{\delta+1}{2}u + C_0 \right)^{\frac{3-\gamma}{2(\gamma-1)}} \right\} du + K,
\]

where \( K \) is to be determined by the intersection of the characteristic with the line \( x = C_0 t \). \( x_\Gamma \) is found by putting \( t_\Gamma \) into (10).

The path of a particle or cross section of the gas is found similarly. Letting \( x_p \) and \( t_p \) be \( x \) and \( t \) on the particle path, we have by definition
\[
\frac{dx_p}{du} = u \frac{dt_p}{du},
\]
and eliminating \( dx_p/du \) with (10) and integrating, \( t_p \) as a function of \( u \) is:
\[
(13) \quad t_p = \left( \frac{\delta-1}{2}u + C_0 \right)^{\frac{\delta+1}{1-\delta}} \left\{ \int_0^{\gamma-1} \varphi'(u) \left( \frac{\delta-1}{2}u + C_0 \right)^{\frac{2-\gamma}{1-\delta}} \right\} du + K.
\]
where $K$ is determined by the initial position of the particle. $x_p$ is given by substitution of (13) into (10).

Figure 1 shows the general appearance of the solution in the $x,t$ plane near the origin. It may be noticed that any one of the particle paths may be regarded as a piston path if desired.

Figure 1 is suggestive of the type of singularity that (10) may have. Since the slope of the members of (3) increases when the slope of the piston path increases, these characteristics will cross somewhere above the envelope when the piston path has a point of inflection and intersects the piston path only when there is an infinite acceleration.

"On the other hand, if $F''(t)$ is always negative, the envelope is below the piston path and the solution (10) holds. We might also note that the envelope moves off to infinity when the piston path has a point of inflection and intersects the piston path only when there is an infinite acceleration."
piston path, that is, somewhere in the region of space-time occupied by the gas. At points where two of the characteristics (3) cross one set of values for x and t corresponds to two values of u and (10) no longer serves to give u as a single valued function of x and t. The boundary of the region where the characteristics cross is an envelope of the family (3). This envelope is determined by the two equations

\[ x - \left( \frac{\gamma + 1}{2} u + c_0 \right) t + \varphi(u) = 0 \]
\[ \frac{\partial}{\partial u} \left[ x - \left( \frac{\gamma + 1}{2} u + c_0 \right) t + \varphi(u) \right] = 0. \]

Letting \( x_E \) and \( t_E \) be the coordinates of points on the envelope, we have:

\[ x_E = \frac{2}{\gamma + 1} \left[ \left( \frac{\gamma + 1}{2} u + c_0 \right) \varphi'(u) - \varphi(u) \right] \]
\[ t_E = \frac{2}{\gamma + 1} \varphi'(u). \]  

(14)

If \( \varphi(u) = 0 \), then \( \varphi'(u) \) may be written

\[ \varphi'(u) = \frac{t_0}{t} \left( \frac{\gamma + 1}{2} u + c_0 \right) - x_0, \]

with \( x_0 = c_0 t_0 \) since \( \varphi(\omega) = 0 \), and (10) reduces to

\[ x - x_0 = (t - t_0) \left( \frac{\gamma + 1}{2} u + c_0 \right), \]

a family of straight lines through the point \( x_0, t_0 \). The envelope here is the point \( x_0, t_0 \). This case will not be discussed.

If \( \varphi''(u) \neq 0 \), there will be an envelope which is not just a single point. From (11),

\[ \varphi''(u) = \gamma \varphi'(u) + \left( \frac{\gamma - 1}{2} u + c_0 \right) \varphi''(u). \]
Now \( \psi(t) \) is the inverse function of \( u = F'(t) \) and

\[
\psi'(u) = \frac{1}{F''(u)} , \quad \psi''(u) = -\frac{F'''(u)}{[F''(u)]^3},
\]

so that

\[
\psi''(F'(u)) = \frac{1}{[F''(u)]^3} \left\{ \gamma[F''(u)]^2 - \left( \frac{x-1}{2} F'(u) + C_0 \right) F'''(u) \right\}
\]

If the piston path remains analytic, \( F''(t) \) remains finite and the condition that there be an envelope which is not a point is

\[
\gamma[F''(u)]^2 - \left( \frac{x-1}{2} F'(u) + C_0 \right) F'''(u) \neq 0.
\]

In particular when \( u = 0 \), \( F'(t) = 0 \) and

\[
(15) \quad \psi''(\sigma) = \frac{1}{[F''(\sigma)]^3} \left\{ \gamma[F''(\sigma)]^2 - C_0 F'''(\sigma) \right\} \neq 0
\]

means that there will be an envelope which meets the line \( x = C_0 t \) at the point \( Q \) found by putting \( u = 0 \) in (14). The coordinates of \( Q \) are:

\[
\chi = \frac{2C_0}{\delta+1} \psi'(\sigma) = \frac{2C_0}{\delta+1} \frac{1}{F''(\sigma)} > 0
\]

\[
t = \frac{2}{\delta+1} \psi'(\sigma) = \frac{2C_0}{\delta+1} \frac{1}{F''(\sigma)} > 0.
\]

Note here that if \( F''(0) = \pm \infty \), the solution (10) breaks down at the origin.

Equation (15) above is the condition on \( F''''(0) \) mentioned on page 8.

The behavior of the envelope in the neighborhood of
\( Q \) can be determined by looking at its curvature there.

From (14),

\[
\frac{\partial^2 x_E}{\partial t^2} = \left( \frac{\psi'(u)}{2} \right)^2 \phi''(u).
\]

The case where \( \psi'(u) = 0 \) is not to be considered here.

Suppose \( \psi''(u) < 0 \). Then the envelope bends downward into the gas in motion as shown in Figure II.

Consider the quantities

\[
\Delta x = x_E - F(\psi(u))
\]

\[
\Delta t = t_E - \psi(u)
\]

which are the differences between \( x \) and \( t \), respectively, on the envelope and the piston path along a characteristic (3). From (10), (11), and (14),

\[
\Delta x = \left( u + \frac{2C_0}{\delta + 1} \right) \left( u + \frac{2C_0}{\delta - 1} \right) \frac{\psi'(u)}{2}
\]

\[
\Delta t = \left( u + \frac{2C_0}{\delta - 1} \right) \frac{\psi'(u)}{2}
\]
Since $\psi'(u)=1/F''(t)$ is positive for all $t$ sufficiently near the origin, $A_x$ and $A_t$ will not be zero simultaneously, that is, the envelope does not meet the piston path. Therefore $x_E$ and $t_E$ have a relative minimum somewhere above the piston path. From (14) this is seen to be where $\psi''(u)=0$. But from the expression for the curvature of the envelope, this point is also a cusp. This has been indicated in Figure II. This type of singularity presents a more difficult problem than the one to be treated in this paper.

The problem to which the remainder of this discussion applies is that of extending the solution beyond the point $Q$ when $\psi'(\phi)$ is greater than zero. Figure III shows the nature of the envelope in this case. The solution (10) becomes double-valued in the region between $x=C_0 t$ and the envelope. But this does not mean that the solution (10) will hold everywhere below this initial characteristic, as will be seen.

Consider the first order derivatives of $u$, $p$, and $v$. Since $v$ and consequently $p$ are functions of $u$ alone, both the partial derivatives for each quantity involve the factor $\partial^2 u$. But $\partial^2 u$ is zero on the envelope. This suggests that the functions $u$, $p$, and $v$ themselves become discontinuous at points to the right of $Q$. This hypothesis may be given some physical justification if we
think of the piston as generating sound waves continuously with continuous increase in velocity due to increase in velocity of the medium. These waves overtake one another in the vicinity of the point Q and produce a discontinuous change in the pressure across the wave front. If we find \( \frac{\partial p}{\partial x} \) from (9) and (10) and evaluate it on the wave \( x = c_0 t \), the result is:

\[
\frac{\partial p}{\partial x} \bigg|_{t = \frac{x}{c_0}} = \frac{2Y}{\gamma + 1} \left[ x - \frac{2x}{\gamma + 1} \frac{1}{F''(c)} \right]^{-1}.
\]

The denominator here is negative for points to the left of Q and approaches zero through negative values as x approaches the x coordinate of the point Q. Thus the pressure gradient becomes negatively infinite at Q and the continuation of the solution must be expected to involve a pressure discontinuity beyond that point. Such a discontinuity is referred to as a shock wave.

This wave must meet the line \( x = c_0 t \) at Q and have an initial velocity \( c_0 \), the speed of sound in the gas at rest. For such a wave, the Rankine- Hugoniot equations express the conservation of mass and momentum. They are, respectively,

\[
(18) \quad m = (U - u_2) \frac{1}{\sqrt{2}} = (U - u_1) \frac{1}{\sqrt{2}},
\]

\[
(19) \quad m(u_i - u_2) = \rho_i - \rho_2,
\]

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where \( m \) is the mass passing through the wave front in unit time, \( U \) is the velocity of the wave and the subscripts refer to conditions on opposite sides of the wave front.

Near the origin the conservation of energy was stated when we assumed \( \rho_U = 1 \). This is the integral of the first law of thermodynamics

\[
\frac{D \mathcal{E}}{Dt} + \rho \frac{D \mathcal{v}}{Dt} = \frac{D Q}{Dt} ,
\]

under the assumption that \( DQ/Dt = 0 \) and with \( \mathcal{E} \), the internal energy, given by the perfect gas law, \( \mathcal{E} = \rho U^2 (\gamma - 1) \).

Hadamard has shown in his *Lecons sur la Propagation des Ondes* that the above thermodynamic equation still applies for the gas in continuous motion. The energy equation for the shock wave however, must take into account the kinetic energy differences across the wave. Here we must say that the work done in passing through the wave front is equal to the change in kinetic energy plus the change in internal energy. The energy equation with the terms written in this order is now

\[
(20) \quad \rho \mathcal{u}_1 - \rho \mathcal{u}_2 = \frac{m(u_1^2 - u_2^2)}{2} + \frac{m}{\gamma - 1} (\rho v_1 - \rho v_2) v_2^2 .
\]

Eliminating \( u_1 \) and \( u_2 \) with (18) and (19) we have the energy equation for the shock wave as given by Hadamard:

\[
(21) \quad \rho v_1 - \rho v_2 = \frac{\gamma - 1}{2} (\rho + \rho_2) (v_2^2 - v_1^2) .
\]
It is clear now that if we conserve energy across the shock wave that entropy is not conserved. For if we assume equation (21) to hold, the product $pv^\gamma$ is no longer a constant across the shock wave but varies with $p$ and $v$. There is a discontinuous increase in entropy which varies in amount as the shock wave moves down the tube.

At points other than those on the shock wave the first law of thermodynamics will hold. That is,

$$\frac{DE}{Dt} + p \frac{Dv^\gamma}{Dt} = 0$$

or

$$p v^\gamma = k,$$

where $k$ is 1 in the gas at rest but is a different constant for each cross section of the gas whose initial position is to the right of $Q$. The value of the constant depends on the strength of the shock wave when it reaches the initial position of the particle.

Since the singularities in the derivatives must be expected to be present in the new solution, the continued use of $x$ and $t$ as the independent variables is inconvenient. However the solution (9) and (10) indicates that the solution is analytic in $u$ and $t$ near $Q$ and we accordingly now introduce these as the independent variables. To illustrate the transformation of equations
(1) and (2), consider \( v \). \( v = v(x, t) \) will become \( v = v(u, T) \) with \( t = T \) and \( x = x(u, T) \). Then

\[
\frac{\partial v}{\partial u} = \frac{\partial v}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial v}{\partial x} \frac{\partial x}{\partial t} \frac{\partial t}{\partial u},
\]

\[
\frac{\partial v}{\partial T} = \frac{\partial v}{\partial x} \frac{\partial x}{\partial T} + \frac{\partial v}{\partial x} \frac{\partial x}{\partial t} \frac{\partial t}{\partial T}.
\]

The Jacobian of the transformation \( J \) is

\[
J = \begin{vmatrix}
\frac{\partial x}{\partial u} & \frac{\partial x}{\partial t} \\
\frac{\partial x}{\partial T} & \frac{\partial x}{\partial t}
\end{vmatrix} = \frac{\partial x}{\partial u}.
\]

since \( \partial t/\partial T = 1 \) and \( \partial t/\partial u = \partial T/\partial u = 0 \). Solving for \( \partial v/\partial x \) and \( \partial v/\partial t \) we have:

\[
\frac{\partial v}{\partial x} = \frac{\partial v}{\partial u} / \frac{\partial x}{\partial u},
\]

\[
\frac{\partial v}{\partial t} = \left( \frac{\partial x}{\partial u} \frac{\partial x}{\partial t} - \frac{\partial x}{\partial t} \frac{\partial x}{\partial u} \right) / \frac{\partial x}{\partial u}.
\]

From \( x = x(u, T) \) we find with \( x \) fixed that

\[
\frac{\partial x}{\partial u} \frac{\partial u}{\partial t} + \frac{\partial x}{\partial T} \frac{\partial T}{\partial t} = 0
\]

or

\[
\frac{\partial u}{\partial t} = - \frac{\partial x}{\partial t} / \frac{\partial x}{\partial u},
\]

and with \( T \) fixed

\[
1 - \frac{\partial x}{\partial u} \frac{\partial u}{\partial x} = 0
\]

or

\[
\frac{\partial x}{\partial u} = 1 / \frac{\partial u}{\partial x}.
\]

The derivatives of \( p \) may be found in the same way in which those of \( v \) were found.

Equations (1) and (2) when written for \( u \) and \( t \) as
the independent variables are now respectively:

\[(23) \quad u - \frac{\partial x}{\partial t} + \nu \frac{\partial \rho}{\partial u} = 0\]

\[(24) \quad (u - \frac{\partial x}{\partial t}) \frac{\partial \rho}{\partial u} + \frac{\partial x}{\partial u} \frac{\partial \rho}{\partial t} - \nu = 0.\]

The energy equation is now

\[D(\rho v^2) / Dt = 0\]

where \(D/Dt\) is the total time derivative, the entropy being constant on the particle path. Carrying out the differentiation and transforming to the variables \(u\) and \(t\) we have:

\[(25) \quad (u - \frac{\partial x}{\partial t})(\gamma \rho \frac{\partial v}{\partial u} + \nu \frac{\partial \rho}{\partial u}) + \frac{\partial x}{\partial u}(\gamma \rho \frac{\partial v}{\partial t} + \nu \frac{\partial \rho}{\partial t}) = 0.\]

Figure IV shows how the various solutions for \(x\), \(p\), and \(v\) may be fitted together near \(Q\).
There will be two solutions for $x$, $p$, and $v$, one for each of the regions I and II indicated in Figure IV, the solution I fitting with the solution (10) across $\Gamma$. The theoretical grounds for this picture are as follows:

Region II is between the shock wave $S$ and the particle path $P$ through $Q$. Thus $P$ is the last particle for which $p\sqrt{\gamma} = 1$ and in II equations (25), (24), and (25) are to be solved subject to the boundary conditions (18), (19), and (21) on $S$. On the other hand, below $P$, $p\sqrt{\gamma} = 1$, and in consequence we have to expect a different solution there. Finally we cannot expect the solution in I to agree with (10) since the latter is based on an analysis of the equations (10) and (9) which no longer applies.

The solution in I should, however, fit continuously with (10) across some curve through $Q$ and by virtue of our initial discussion of the properties of characteristics, it follows that this curve must be the characteristic of (10) of negative slope through $Q$. This is the curve which we have denoted by $\Gamma$. As the figure indicates we now take $Q$ as the origin of coordinates.

We turn first to a detailed discussion of the boundary conditions. To begin we rewrite the equations (18), (19), and (21) taking $u_2 = 0$, $p_2 = 1$, $v_2 = 1$, corresponding to the gas at rest, and $u_1 = u$, $p_1 = p$, and $v_1 = v$, corresponding to the gas in motion.
\( m = \mathcal{U} = (\mathcal{U} - \omega) \frac{1}{\mathcal{U}} \)

\( \mu = \mu_1 \)

\( \mu_1 = \frac{\nu - 1}{2}(p + 1)(1 - \nu) \).

\[ S \text{ may be assumed to be given parametrically in } u \text{ in power series form:} \]
\[ t_s = \sum_{n=1}^{\infty} M_n u^n \]
\[ x_s = \sum_{n=1}^{\infty} N_n u^n \]
\[ \frac{dx_s}{dt_s} = \mathcal{U} \]
\[ \frac{dx_s}{dt_s}/u = \frac{N_1}{M_1} = C_0. \]

The particle path \( P \) may also be given parametrically in power series in \( u \). Here
\[ \frac{dx_p}{dt_p} = u \]
and \[ \frac{dx_p}{du} = u \frac{dt_p}{du}. \]

Since the solutions for \( x \) for regions I and II are to be joined across \( P \), we have:
\[ \frac{dx(u, t_p)}{du} = \frac{\partial x}{\partial u} + \frac{\partial x}{\partial t} \frac{dt_p}{du} = u \frac{dt_p}{du}. \]

It will be seen that \( \partial x/\partial t \) is not zero at the origin \( Q \), and because of the singularity at \( Q \), \( \partial x/\partial u \) is zero and \( dt_p/du \) is therefore zero there. This means that the power series for \( t_p \) begins like \( u^2 \) and that the one for \( x_p \) like \( u^3 \). We have:
\[ t_p = \sum_{n=2}^{\infty} \beta_n u^n \]
\[ x_p = \sum_{n=3}^{\infty} \kappa_n u^n. \]
The functions $p$ and $v$ of $u$ and $t$ for the regions $I$ and $II$ become identities in $u$ when $t$ is replaced by $t_p$.

In region $I$, as indicated above, we have equations (23) and (24) again but instead of (25) we have $pv = 1$. The solution in $I$ is to be joined to the solution (10) across $\Gamma$. $q(u)$ may be expanded in a power series in $u$ and (10) becomes, with the origin now the point $Q$:

$$
\chi(u, t) = C_0 t + \frac{r+1}{2} u t - \sum_{n=2}^{\infty} \frac{q^{(n,0)}(u)}{n!} u^n.
$$

From (9)

$$
\nu(u) = \left(1 + \frac{r-1}{2} u\right)^{\frac{2}{1-r}}
$$

behind $\Gamma$. Letting $t_\Gamma$ be $t$ on the characteristic $\Gamma$, we may expand in a power series in $u$ from (12). Across $\Gamma$ $\chi(u, t_\Gamma)$ in $I$ must be identical with (10) when $t$ is replaced by $t_\Gamma$. Similarly $\nu(u, t_\Gamma)$ must be an identity in $u$ with (9).

We assume power series of the following form, there being one of each kind for the region $I$ and $II$, except that in $I$, \( p = v^{-x} \):

$$
\chi = \sum_{j,k=0}^{\infty} a_{j,k} u^j t^k, \quad \nu = \sum_{j,k=0}^{\infty} b_{j,k} u^j t^k, \quad p = \sum_{j,k=0}^{\infty} c_{j,k} u^j t^k.
$$
We note here that \( v(0,0) = p(0,0) = 1 \) and \( x(0,0) = \partial x(0,0)/\partial u = 0 \).

The determination of the coefficients has been carried out to terms of degree high enough to show the difference between the solutions for I and II. Letting \( A = \phi'(0), \ B = \phi''(0) \), etc., the results are as follows:

In region I:

\[
X(u,t) = C_0 t - \frac{A}{2!} u^2 + \frac{8}{2} u t - \frac{B}{3!} u^3 - \frac{C}{4!} u^4
\]

\[
+ \frac{(1+t)^4}{\delta(4)^6 A^2} u^3 t + \frac{(1+t)^4 (3t - 5)(5t + 1)}{2 \gamma(4)^7 A^3} t^4
\]

\[
- \frac{D}{5!} u^5 + \frac{27(1+t)^4 (3t - 5)}{C_0^3 (4)^7 A} u^3 t^2
\]

\[
+ \frac{g(1+t)^4 (5-3t)}{\gamma^2 (4)^7 A^3} \left\{ \frac{C_0 (3 + 5t) A + 2t B}{C_0 (3 + 5t) A + 2t B} \right\} u^2 t^3
\]

\[
+ q_{14} u t^4 + q_{05} t^5 - \frac{E}{6!} u^6 + \cdots
\]

\[
V(u,t) = 1 - \frac{1}{C_0} u + \frac{1 + \delta}{4} u^2 - \frac{1 + \delta}{12 C_0} u^3 + \frac{g(1+t)^4 (5-3t)}{2 C_0^3 (4)^5 A^3} t^3
\]

25
\[
\frac{(1+\gamma)(3\gamma-1)}{96\gamma} u^4 - \frac{27(1+\gamma)^4(5-3\gamma)}{2\gamma^2(4)^6 A^2} u^2 t^2 + \frac{9(1+\gamma)^5(5-3\gamma)}{2\gamma^2(4)^6 A^3} u t^3 \\
+ b_4 t^4 + \frac{(1+\gamma)(1-2\gamma)(3\gamma-1)}{480\gamma} u^5 + \frac{27(1+\gamma)^4(5-3\gamma)}{2\gamma^5(4)^7 A} u^4 t \\
+ \frac{9(1+\gamma)^4(3\gamma-5)}{\gamma^3(4)^7 A^3} \left\{ c_0 (3-\gamma) A + 2\gamma B \right\} u^3 t^2 + b_{23} u^2 t^3 + b_{14} u t^4 \\
+ b_5 t^5 + \frac{1+\gamma}{90\gamma^3(4)} \left\{ y(4)^5(3\gamma-1)(3-5\gamma)(1-2\gamma)^2 + (5)(27)(1+\gamma)^3(5-3\gamma) \right\} u^6 + \ldots
\]

\[ p = \nu - \gamma \]

Here and in the results to follow the \( a^s, b^s, \) and \( c^s \) are used to designate undetermined coefficients.

The particle path through \( \theta \):  

\[ t_p = \frac{A}{2c_0} u^2 + \frac{1}{12\gamma} \left\{ 2c_0 B + (1-3\gamma) A \right\} u^3 + \]

\[ + \frac{1}{48\gamma} \left\{ 2\gamma C + 2c_0 (1-2\gamma) B + (1-2\gamma)(1-3\gamma) A \right\} u^4 + \]

\[ + \frac{1}{48\gamma^2} \left\{ 48c_0 D + 2\gamma(3-5\gamma) C + 2c_0 (1-2\gamma)(3-5\gamma) B + (1-2\gamma)(1-3\gamma)(3-5\gamma) A \right\} u^5 + \ldots. \]

\[ \chi_p = \sum_{n=3}^{\infty} d_n u^n \quad \text{with} \quad n \cdot d_n = (n-1) \beta_{n-1}. \]
In region II:

\[ x(u,t) = C_0 t - \frac{A}{2!} u^2 + \frac{B}{3!} u^3 - \frac{C}{4!} u^4 \]

\[ \left( \frac{9(1+r)^4(3y-5)}{\gamma(4)^6 A^2} \right) u^3 t^3 + \left( \frac{27(1+r)^5(5y-7)}{2 \gamma(4)^7 A^3} \right) t^4 - \frac{D}{5!} u^5 \]

\[ + \left( \frac{27(1+r)^4(3y-5)}{C_0^3(4)^7 A} \right) u^3 t^2 \]

\[ \left( \frac{9(1+r)^4(3y-5)}{\gamma^2(4)^7 A^3} \right) \left\{ 3C_0(5y+3)A + 2rB \right\} u^2 t^3 \]

\[ + q_4 u^4 t^4 + q_5 u^5 t^5 - \frac{E}{6!} u^6 + \frac{27(1+r)^4(5y-3)}{\gamma^2(4)^8} u^5 t + \ldots \]

\[ u(u,t) = 1 - \frac{1}{C_0} u + \frac{1+r}{4\gamma} u^2 - \frac{1+r}{12C_0} u^3 + \frac{9(1+r)^4(5y-3)}{2C_0^3(4)^5 A^3} \]

\[ \left( \frac{(1+r)(3y-1)}{2\gamma} \right) u^4 + \frac{27(1+r)^4(11-13y)}{2(4)^6 \gamma^2 A^2} u^2 t^2 + \frac{9(1+r)^4(45y^2-14y-27)}{2(4)^6 \gamma^2 A^3} u^3 t^3 \]

\[ - \frac{(3)^5(1+r)^5}{\gamma(4)^8 A^5} \left\{ \frac{A - 75y^2 - 58y - 51}{12C_0} + \frac{(1+r)B}{C_0} \right\} t^4 \]

\[ + \frac{(1+r)(1-2y)(3y-1)}{480 C_0 \gamma} u^5 + \frac{27(1+r)^4(29y-27)}{2C_0^5(4)^7 A} u^4 t \]
\[-\frac{27 (1+y)^4}{\gamma^3 (4)^7 A^3} \left\{ C_0 A \left[ (38-5)(23\gamma+13) + 4(5\gamma-3)(2+3\gamma) - 2(11-13\gamma) \right] + 2y(13\gamma-11)B \right\} u^3 t^2
\]

\[+ b_{23} u^2 t^3 + b_{14} u t^4 + b_{05} t^5 \]

\[+ \frac{1+y}{g_0 (4)^8 \gamma^3} \left\{ (4)^5 y(3-5\gamma)(1-2\gamma)(3\gamma-1) + (15)(27)(1+y)^3 (5\gamma-61\gamma) \right\} u^6 + \ldots \]

\[p(u,t) = 1 + C_0 u + \frac{1+y}{4} u^2 + C_0 u^3 + \frac{9(1+y)^4(38-5)}{g_0 (4)^4 C_0 A^3} t^3 \]

\[+ \frac{(1+y)(3-y)}{96 y} u^4 - \frac{27 (1+y)^4(38-5)}{2 \delta (4)^6 A^2} u^2 t^2 + \frac{9(1+y)^4(5\gamma^2-26\gamma-9)}{2 (4)^6 y A^3} u t^3 \]

\[+ \frac{(3)^5 (1+y)^5}{\gamma (4)^6 A^5} \left\{ \frac{1}{12} \left[ y(75\gamma^2-58\gamma-5) - 32(3\gamma-1)(\gamma-1) \right] + \frac{y(\gamma+1)B}{C_0} \right\} t^4 \]

\[+ \frac{(1+y)(8-2)(\gamma-3)}{480 C_0 \gamma} u^5 + \frac{27 (1+y)^4(38-5)}{2 C_0^3 (4)^7 A} u^4 t \]

\[- \frac{9 (1+y)^4}{\gamma^2 (4)^7 A^3} \left\{ C_0 A \left[ (38-5)(15\gamma+13) + 2(11-13\gamma) \right] + 2y(3\gamma-5)B \right\} u^3 t^2 \]

\[+ c_{23} u^2 t^3 + c_{14} u t^4 + c_{05} t^5 \]

\[+ \frac{(1+y)(5-3\gamma)}{g_0 (4)^8 \gamma^2} \left\{ (15)(27)(1+y)^3 + (4)^5 (5-2)(\gamma-2) \right\} u^6 + \ldots \]
The shock wave $S$

$$t_s = \frac{4A}{3(1+\gamma)} u + \left\{ \frac{A}{24C} + \frac{B}{2(1+\gamma)} \right\} u^2$$

$$+ \frac{1}{480 \gamma (1+\gamma)} \left\{ 64 \gamma C + 12 \gamma (3\gamma-1) B + (1+\gamma)(7\gamma-13) A \right\} u^3 + \cdots$$

$$x_s = \frac{4CA}{3(1+\gamma)} u + \left\{ \frac{5}{24} A + \frac{C B}{2(1+\gamma)} \right\} u^2$$

$$+ \frac{1}{480 C (1+\gamma)} \left\{ 64 \gamma C + 4C (19\gamma+7) B + (1+\gamma)(17\gamma-3) A \right\} u^3 + \cdots$$

In the region $II$, 

$$\rho u^{-\gamma} = 1 + \frac{9(\gamma-1)(\gamma+1)^4}{(4\gamma)^4 C A^3} t^3 + \cdots$$

Thus the first term to show the entropy difference is of order $t^3$. This is also the order of the first term in which $v$ in $I$ differs from $v$ in $II$. The first term in which $x$ in $I$ and $x$ in $II$ differ from the solution (10) is that in $ut^3$ and where $x$ in $I$ and $x$ in $II$ are different is that in $t^4$.

To terms of order three, $x_3$ on the envelope and $x_3$ on the shock wave are given as functions of $t$ by:
\[ x_c = C_0 t + \frac{(y+1)^2}{8A} t^2 - \frac{(y+1)^3 B}{48A^3} t^3 + \ldots. \]

\[ x_s = C_0 t + \frac{3(y+1)^2}{32A} t^2 + \frac{g(1+y)}{4C_0} \frac{A^4}{2(4)^4A^3} t^3 + \ldots. \]

Since \( A = \phi''(0) \) is positive, it is seen that the shock wave lies between the envelope and the line \( x = C_0 t \).

To give a graphical illustration of the position of the shock wave, let \( y = 7/5 \) and \( t_g = u \). Then to terms of order three in \( t \) we have:

\[ x_c = 1.1832 t + .4000 t^2 + .0150 t^3. \]

\[ x_s = 1.1832 t + .3000 t^2 + .0507 t^3. \]

Figure V shows \((x_c - x_s) \cdot 10^4\) and \((x_s - C_0 t) \cdot 10^4\) plotted against \( t \) for values of \( t \) small enough that the term in \( t^3 \) may be neglected.

The first two terms of the characteristic (4) through \( Q_0 \), \( x = x_T \), and the particle path through \( Q_0 \), \( x = x_P \), are for the same example, as functions of \( T \):

\[ x_T = -1.183t + .87 t^{3/2} + \ldots. \]

\[ x_P = .54 t^{3/2} + .04 t^2 + \ldots. \]