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RECONSTRUCTIONS OF SIGNALS OF A KNOWN CLASS FROM A GIVEN SET OF LINEAR MEASUREMENTS

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ABSTRACT

RECONSTRUCTIONS OF SIGNALS OF A KNOWN CLASS FROM A GIVEN SET OF LINEAR MEASUREMENTS

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This paper investigates the error in reconstructions of a signal based on a given, finite set of linear measurements, and presents two schemes that, if there is available a priori knowledge of the class of signals of which the measured signal is a member, can achieve a reduction of this error beyond the best that could be done without such knowledge. The error measure used is the supremum over the class of the $\mathcal{L}_2$-distance between a signal and its reconstruction. The essence of the proposed reconstruction techniques is a coordinate transformation from the sampling subspace to a new reconstruction subspace known to be efficient for representation of signals of the given class. This study makes application of the theory of extremal subspaces and n-widths of signal classes originated by Kolmogorov.

Results are applied to the much-studied class of time-concentrated, bandlimited signals. The measurement process is here assumed to be the convenient one of Nyquist rate time-sampling. For this problem, plots of the error bounds and of several test functions and their reconstructions are presented, both for the proposed reconstructions, and for conventional cardinal sampling theorem reconstructions.

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I. INTRODUCTION

Suppose that a signal f is subject to a finite set of linear measurements that results in a collection of n real numbers. These numbers in some sense partially characterize the signal, that is they could be used as a basis for various approximate reconstructions of f. This paper will investigate the error committed by such reconstructions and present two algorithms that, assuming there is available a priori knowledge of the "class" of signals of which the measured signal is a member, can achieve a reduction of this error beyond the best that could be done without such knowledge.

An example problem to which this theory is applied is diagrammed in Fig. 1. Given, finite-energy, time-limited signals are first bandlimited by a prefilter, usually taken of sufficient bandwidth not to cause intolerable distortion of the signals, and then sampled by the conventional Nyquist rate (or faster) time sampling for a finite number of samples. It is desired to perform a linear reconstruction of the sampled signal with the greatest possible accuracy. It may be desired merely to determine the greatest accuracy with which the reconstruction could be performed, and thereby measure the accuracy with which the sample vector itself may be said to characterize the signal. Of course, on the basis of bandlimitation alone, a cardinal reconstruction using sinc functions weighted by the sample values could be performed. However, if we also take account of the fact that the sampled signal is known to be time-concentrated - being a filtered version of a strictly time-limited signal - an improved reconstruction would seem to be possible. The theory we develop provides an approximate numerical answer to such questions. In deriving that theory, the point of view will be that of finding a linear operation on the samples to generate accurate coordinates for a good set of reconstruction functions. However the same problem may also be viewed as finding good interpolating functions to use with the original samples as coefficients.

The Measurement Process: Certainly the most common measurement process is that of Nyquist rate or faster time sampling of bandlimited signals. Other time-sampling schemes such as non-uniform spacing of samples, or sampling of the function and its derivative have been investigated. More generally, the signal to be measured may be passed through a bank of linear filters and the outputs sampled to yield the inner-product of the signal with the time reversed impulse responses of the filters. A simple example of this would be a bank of bandpass filters which would measure the sinusoidal Fourier coefficients of periodic signals. Generalized Fourier coefficients in other orthonormal bases can be implemented in similar fashion by Kautz's technique[1] for specifying impulse responses which constitute an orthonormal set and are realizable by lumped linear networks. This technique has been applied by Young and Huggins for representation of electrocardiogram by orthogonal exponentials.[2]

Generalizing, then, the measurement process, which will be called "sampling", will be modeled by considering a measurement b_j to be an inner product of the function f with an appropriate function $\varphi_j \in \mathcal{L}^2$, i.e. $b_j \triangleq (f,\varphi_j)$. (Recall that the Reisz representation theorem guarantees that if b_j is any bounded linear functional it may be so represented.) The sampling process is thus specified by a set of functions $\{\varphi_j\}$ of $\mathcal{L}^2$. Note that in the case of Nyquist rate time sampling of bandlimited functions, the φ_j are the sinc functions. The $\{\varphi_j\}$ will be called the sampling functions and the n-dimensional linear manifold they span, the sampling subspace, denoted Φ_n. The sampling process is thus the operation P_{Φ_n} of orthogonal projection of f onto Φ_n. If the $\{\varphi_j\}$ are orthonormal, then $P_{\Phi_n}(f) = \sum_{j=1}^{n} (f_j,\varphi_j)\varphi_j$. The sampling subspace Φ_n is thus the object which defines the measurement process, and the sampling functions may be assumed orthonormal with no loss of generality.

The Reconstruction Process: In discussions of measurement and reconstruction of signals, it is often an underlying assumption not explicitly mentioned that the

forms of the measurement and reconstruction processes are ridigly tied together, in that linear measurement amounts to projection onto a subspace and linear reconstruction is thus necessarily done in that same subspace. Of itself, projection onto a subspace Φ_n destroys all information about the signal orthogonal to Φ_n and the best reconstruction of the signal is indeed its projection. If the subspace used for such sampling and reconstruction is dictated, as is commonly the case, by the need for simplicity of implementation of sampling,*then it is quite unlikely that it is in any sense optimum for reconstruction if the signals are members of a known class. Use of the a priori knowledge of the class of signals being sampled might be used to restore some of the information lost by the sampling process, and permit an improved reconstruction in another subspace "closer" to the class.

This idea has been applied often before in a heuristic fashion in the special case of time sampling of functions of a known type. There the problem is to find efficient interpolating functions to reconstruct the signal between its samples. In other words, with a specified sampling subspace, a better reconstruction subspace is sought. Richards' note[3] is a good example. He suggests a positive interpolating function to replace the traditional sinc function when the signal being reconstructed is known by physical reasoning to be strictly positive.

In examples such as the above, no precise measure of performance is indicated. Our criterion will be that suggested by Huggins[4] of minimizing the error taken over the entire class of signals. The measure of the distance between individual signals will be any $\mathcal{L}_2$ norm of their difference.

Signal Classes: We will be assuming that the collections of possible signals,

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*An interesting piece of work where this was not strictly true is Young & Huggins' work with representation of EKG's by exponentials. The choice of exponentials, they explain, was made partly for ease of implementation, and partly because of the similarity in form of exponentials and EKGs, which suggested that an exponential basis would provide efficient reconstruction.

or signal classes, that we are discussing are well-described mathematically; that is, that certain of their mathematical properties, discussed in the sequel, can be determined. As is almost inevitably the situation in a modeling problem, there is a necessity to compromise between the mathematical elegance of the model and its relevance to real-world signal classes.

A good example of the type of class description well suited to the analysis we propose was presented by Root[5] in a somewhat different context. He considered the problem of modeling the class of all output signals from a known linear communications channel. The model he proposed might be applied equally well to the output space of, say, any stable linear filter subject to limited energy inputs. Root suggested that this class of signals may be viewed as the image of the unit ball in $\mathcal{L}_2$ under a Hilbert-Schmidt operator. The resulting class is a compact ellipsoid. The problem we have taken as an example is of this type. In Figure 1 the outputs of the bandlimiting filter belong to a class of bandlimited and time-concentrated signals. A thorough study of this particular class was presented by Slepian, Landau, and Pollak[6,7,8] who rigorously explained the idea that the class of signals,

$$P = \{f:\ f = Bx \quad ,\ x(t) = 0\ \ |t| > \frac{T}{2}\ ,\ \int_{-T/2}^{T/2} x^2(t)dt \leq 1\} \qquad * \qquad (1)$$

with most of their energy in the time interval $|t| \leq T/2$ and bandlimited to [-B,B] cycles per second is essentially 2BT dimensional. Their results indicate that this intuitive notion is correct; that 2BT numbers do serve to specify members of this class accurately <u>if</u> those numbers are the Fourier coefficients of the signal in the prolate spheroidal wave function (PSWF) basis. They further show that an equivalent result in terms of time samples is <u>not</u> true.

<u>Error Measures and Widths of Classes</u>: Our objective is the efficient linear reconstruction of signals of a known class from their projection on a specified subspace. This is to be accomplished using <u>a priori</u> knowledge about the signals,

*Where the operator B denotes strict bandlimitation.

in the form of the class description, to choose a good subspace for reconstruction. The fidelity of the reconstruction scheme will be measured as follows: Call the original n-vector of samples* $\underset{\sim}{b}_1 = (b_1 \ \ldots \ b_n)^T$ where $b_j = (f, \varphi_j), j = 1, \ldots, n$ and the $\{\varphi_j\}_{=1}^{n}$ are the sampling functions. Further let the reconstruction $\hat{f} \triangleq \sum_{i=1}^{m} \hat{a}_i \psi_i$, where $\{\psi_i\}_{i=1}^{m}$ are the orthonormal reconstruction functions, and the m-vector of coordinates $\underset{\sim}{\hat{a}}_1 \triangleq (a_1 \ \ldots \ a_m)^T = \Gamma(\underset{\sim}{b}_1)$ where Γ is any mapping from ℓ_n^2 to ℓ_m^2. Then our measure of performance of this scheme will be the supremum of the reconstruction error over the entire class of signals

$$\epsilon \triangleq \sup_{f \in C} \|f - \hat{f}\| \tag{2}$$

The principal result of this paper is the introduction of two specific reconstruction schemes and the bounding of the associated ϵ. Two quantities defined in 1936 by A.N.Kolmogorov[9] are fundamental properties of the class, and provide the terms in which the bounds are stated.

The first of these quantities is the deviation of the class C from a subspace Ψ_n, denoted $\delta(C, \Psi_n)$. Suppose that in our general reconstruction scheme described above Γ was optimal, that is $\hat{a}_i = (f, \psi_i)$, which are well known to be the optimal expansion coefficients for f if the $\{\psi_i\}$ are orthonormal. The resulting value of ϵ provides the definition

$$\epsilon = \delta(C, \Psi_m) \triangleq \sup_{f \in C} \inf_{\hat{a}_i} \|f - \sum_{i=1}^{m} \hat{a}_i \psi_i\| \tag{3}$$

Thus $\delta(C, \Psi_m)$ provides a lower bound on the error of any scheme for approximating members of C with signals in Ψ_m.

The second quantity is the m-width of the class C, $d_m(C)$, which is the least error any m-dimensional linear reconstruction can accomplish in any subspace:

$$d_m(C) = \inf_{\{\psi_i\}} \sup_{f \in C} \inf_{\hat{a}_i} \|f - \sum_{i=1}^{m} \hat{a}_i \psi_i\| = \inf_{\Psi_m} \delta(C, \Psi_m) \tag{4}$$

The subspace which realizes $d_m(C)$ is called the extremal subspace for C.

*Subscripts on sample and co-ordinate vectors, such as $\underset{\sim}{b}_1$, are introduced here since they will be needed later.

We note that for our example of time concentrated, bandlimited signals in Fig. 1, the class P (see (1)) was shown by Landau and Pollak to have an extremal subspace spanned by the prolate spheroidal wave functions satisfying the integral equation

$$\lambda_i X_i(t) = \int_{-T/2}^{T/2} X_i(s) \frac{\sin 2\pi B(t-s)}{\pi(t-s)} ds \tag{5}$$

furthermore

$$d_m(P) = \lambda_{m+1}^{\frac{1}{2}} \tag{6}$$

We note also, for future reference, that if $f \in P$ is expanded in the extremal basis $\{X_i\}_{i=1}^{\infty}$, the "principal axes" of this ellipsoidal class, as

$$f = \sum_{i=1}^{\infty} (f, X_i) X_i \tag{7}$$

then it can be shown that, in terms of the coordinates (f, X_i), class membership may be expressed as

$$f \in P \Leftrightarrow \sum_{i=1}^{\infty} (f, X_i)^2/\lambda_i \leq 1 \quad f \in \mathcal{B} \tag{8}$$

where $\mathcal{B}$ is the space of bandlimited functions.

We now seek to show the feasibility of schemes which use convenient sampling processes and yet succeed in reconstructing the functions sampled with almost the accuracy possible only with extremal basis sampling.

II. RECONSTRUCTION SCHEMES AND ERROR BOUNDS

The idea of the m-width of a class provides the theoretical limit with which m real numbers serve to determine a signal of that class, in the sense of linear reconstructions. It also indicates which numbers achieve this limit, those being the coordinates of the signal in the m-dimensional extremal subspace. Unfortunately these coordinates-the inner products of the signal with the m orthonormal extremal basis functions - are generally quite difficult to measure by any direct processing of the signal. They may be impossible to obtain by

analog processing, and may require implementing the inner-product integral digitally. It would thus be highly desirable to discover an exact mapping from n more or less arbitrary samples into these preferred coordinates. This cannot generally be done; there will usually be some error. The presence of this error destroys the optimality of extremal basis reconstruction; the optimal linear reconstruction lies in some unknown subspace which is neither the sampling subspace nor the extremal subspace.

We are thus faced with the dilemma of choosing between using exact coordinates in a subspace not especially well suited for representation of the class of signals in question; or using the extremal subspace with only approximate coordinates. It seems intuitively reasonable that if, for a particular class and sampling subspace, it is possible to make the coordinate estimation error "sufficiently small", we may use the approximate representation in the extremal basis with the assurance that its accuracy is very nearly optimum. (This intuitive notion is made precise by Lemma 1).

The above discussion indicates why we intend to choose the extremal subspace for reconstruction. It seems likely to be near optimal even with errors in the coordinates, since it is by definition optimal if the coordinates are exact. However, there may be reasons for specifying reconstruction in a subspace other than the extremal one. Any subspace with a deviation from the class significantly smaller than the deviation of the sampling subspace could justifiably be considered. As it is nowhere necessary in the following development that the extremal basis be used for reconstruction, we shall preserve this generality by not making this assumption. Results will thus refer to the deviation $\delta(C, \Psi_m)$ which will become $d_m(C)$ if the reconstruction subspace Ψ_m is extremal.

We now state Lemma 1, which motivates the remainder of our work of estimating coordinates in the reconstruction subspace with small error. The proof is given in Appendix 1.

Lemma 1: Let the signal class $C \subset \mathcal{L}_2$, and $\{\psi_i\}_{i=1}^{\infty}$ be an orthonormal basis spanning C. Let $\{\varphi_j\}_{j=1}^{n}$ be an orthonormal set such that $\Phi_n \subset \Psi_n$. Denote $b_i = (f,\varphi_i)$ and $a_i = (f,\psi_i)$ further $\underset{\sim}{a}_1 = (a_1 \ldots a_m)^T$ and $\underset{\sim}{b}_1 = (b_1 \ldots b_n)^T$. Let the m-vector $\hat{\underset{\sim}{a}}_1 = F(\underset{\sim}{b}_1)$ for some F, and form $\hat{f} = \sum_{i=1}^{m} \hat{a}_i \psi_i$. Then

$$\delta^2(C,\Psi_m) \leq \sup_{f\in C} \|f - \hat{f}\|^2 \leq \delta^2(C,\Psi_m) + \sup_{f\in C} \sum_{i=1}^{m} (a_i - \hat{a}_i)^2 \tag{9}$$

Lemma 1 leads us to consider the problem of estimation of the coordinates of a function in a given basis from an incomplete set of coordinates (samples) in another basis. To facilitate discussion of the coordinate estimation schemes we propose, we introduce the following notation: C is a class of functions, $\{\psi_i\}_{i=1}^{\infty}$ and $\{\varphi_j\}_{j=1}^{\infty}$ are orthonormal sets of functions which span $\Psi_\infty = \Phi_\infty \supset C$. That is for all $f\in C$, we may expand

$$f = \sum_{j=1}^{\infty} (f,\varphi_j)\varphi_j = \sum_{i=1}^{\infty} (f,\psi_i)\psi_i \tag{10}$$

Denote $\underset{\sim}{M}$ the matrix whose elements are (ψ_i,φ_j) i, j=1, ... ∞

Partition $\underset{\sim}{M}$ and $\underset{\sim}{M}^{-1} = \underset{\sim}{M}^T$ as follows,

$$\underset{\sim}{M} = \left[\begin{array}{c|c} \underset{\sim}{P} & \underset{\sim}{Q} \\ \hline \underset{\sim}{R} & \underset{\sim}{S} \end{array}\right] \tag{11}$$

$$M^{-1} = \left[\begin{array}{c|c} \underset{\sim}{P}^T & \underset{\sim}{R}^T \\ \hline \underset{\sim}{Q}^T & \underset{\sim}{S}^T \end{array}\right] \tag{12}$$

where $\underset{\sim}{P}$ is nxn, $\underset{\sim}{Q}$ and $\underset{\sim}{R}$ are semi-infinite and $\underset{\sim}{S}$ is infinite. $\underset{\sim}{M}$ transforms the coordinates of a vector f in the φ-basis into the coordinates in the ψ-basis. The above partition of $\underset{\sim}{M}$ induces a partition of the coordinate vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, where $a_i = (f,\psi_i)$ and $b_j = (f,\varphi_j)$. Thus $\underset{\sim}{a} = \underset{\sim}{M}\,\underset{\sim}{b}$ is partitioned

$$\left[\begin{array}{c} \underset{\sim}{a}_1 \\ \hline \underset{\sim}{a}_2 \end{array}\right] = \left[\begin{array}{c|c} \underset{\sim}{P} & \underset{\sim}{Q} \\ \hline \underset{\sim}{R} & \underset{\sim}{S} \end{array}\right] \left[\begin{array}{c} \underset{\sim}{b}_1 \\ \hline \underset{\sim}{b}_2 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c} \underset{\sim}{b}_1 \\ \hline \underset{\sim}{b}_2 \end{array}\right] = \left[\begin{array}{c|c} \underset{\sim}{P}^T & \underset{\sim}{R}^T \\ \hline \underset{\sim}{Q}^T & \underset{\sim}{S}^T \end{array}\right] \left[\begin{array}{c} \underset{\sim}{a}_1 \\ \hline \underset{\sim}{a}_2 \end{array}\right] \tag{13, 14}$$

That is $\underset{\sim}{a}_1$ and $\underset{\sim}{b}_1$ are n-vectors, $\underset{\sim}{b}_2$ and $\underset{\sim}{a}_2$ are infinite. The following interpretation will be given these vectors:

$\underset{\sim}{b}_1$ is a given vector of samples, that is Φ_n is the sampling subspace

$\underset{\sim}{a}_1$ is a vector of coordinates in the reconstruction subspace Ψ_n. (We shall also consider $\Psi_m \subset \Psi_n$ for $m < n$ as reconstruction subspaces, by using only the first m components of $\underset{\sim}{a}_1$, $(a_1 \ \ldots \ a_m)$).

The problem with which we shall be concerned is to estimate $\underset{\sim}{a}_1$ given $\underset{\sim}{b}_1$, and to bound the error which results when the approximate coordinates $\hat{\underset{\sim}{a}}_1$ are used to reconstruct f.

Perhaps the most obvious thing to do is use the truncated expansion $\hat{\underset{\sim}{a}}_1 = \underset{\sim}{P}\,\underset{\sim}{b}_1$, with an error $\underset{\sim}{Q}\,\underset{\sim}{b}_2$. We shall refer to this as Method 1. This seems reasonable, since this estimate is optimal in the sense that the least upper bound on the normalized error

$$\frac{1}{\|f\|}\sum_{i=1}^{m}(a_i - \hat{a}_i)^2 \triangleq \frac{1}{\|f\|}\sum_{i=1}^{m}\Big|(f,\psi_i) - \sum_{j=i}^{n} c_{ij}(f,\varphi_j)\Big|^2 \qquad (15)$$

is realized by $c_{ij} = (\psi_i, \varphi_j)$.[10]

Geometrically, Method 1 simply takes the projection of the signal on the sampling subspace, and further projects this projection onto the reconstruction subspace. Recall however that the motivation for using a new reconstruction subspace is that it is known to be close to the region in $\mathcal{L}_2$ from which the original signal is known to come. Thus a more reasonable thing to do might be to seek that element of the reconstruction subspace whose samples coincide with those given, something like approximate inversion of the projection operation. This technique will be called Method 2.

With reference to the matrix formulation, Method 1 amounted to assuming the remaining samples $\underset{\sim}{b}_2$ were zero, producing an error $\underset{\sim}{Q}\underset{\sim}{b}_2$. Method 2 assumes instead

that the original signal lay exactly in the reconstruction subspace, that is that $\underset{\sim}{a}_2$ is zero. Solving the resulting equation gives the estimate $\hat{\underset{\sim}{a}}_1 = (\underset{\sim}{P}^T)^{-1}\ \underset{\sim}{b}_1$, and an error, $(\underset{\sim}{P}^T)^{-1}\ \underset{\sim}{R}^T\underset{\sim}{a}_2$ (assuming $\underset{\sim}{P}$ is nonsingular). Since the $\underset{\sim}{a}_2$ coordinates are small for all $f\epsilon C$, and the same does not hold for $\underset{\sim}{b}_2$, it would seem that Method 2 is always superior to Method 1. This is not true because of a greater sensitivity of Method 2 to the relative alignment of the sampling and reconstruction subspaces. More will be said on this subject shortly.

We now present two theorems which bound the maximum error over the class in reconstructions done by Methods 1 and 2. The statements of both theorems assume the notation just introduced. Proofs are given in Appendix 1.

Theorem 1: Method 1 Error: Let $\{\psi_i\}_{i=1}^{\infty}$ and $\{\varphi_j\}_{j=1}^{\infty}$ be such that $\Psi_\infty = \Phi_\infty \supset C$. If from the n-vector of samples $\underset{\sim}{b}_1$ we form $\hat{\underset{\sim}{a}}_1 = \underset{\sim\sim}{P}\underset{\sim}{b}_1$ and expand $\hat{f} = \sum_{i=1}^{m} \hat{a}_i\ \psi_i$, then the error

$$\sup_{f\epsilon C} \|f - \hat{f}\|^2 \leq \delta^2(C,\Psi_m)+\delta^2(C,\Phi_n).\min\{1, \sum_{i=1}^{m} [1- \sum_{j=1}^{n} (\varphi_j,\psi_i)^2]\} \qquad (16)$$

Theorem 2: Method 2 Error: If we take $\hat{\underset{\sim}{a}}_1 = (\underset{\sim}{P}^T)^{-1}\underset{\sim}{b}_1$ and form $\hat{f} = \sum_{i=1}^{m} \hat{a}_i\psi_i$, then for $m \leq n$

$$\sup_{f\epsilon C}\|f-\hat{f}\|^2 \leq \delta^2(C,\Psi_m) + \lambda\ (m,n\ )\delta^2(C,\Psi_n).\min\{1, \sum_{j=1}^{n} [1- \sum_{i=1}^{n} (\varphi_j,\ \Psi_i)^2]\} \qquad (17)$$

where $\lambda\ (m,n)$ is the largest eigenvalue of $\underset{\sim}{M} = \underset{\sim}{T}^T\underset{\sim}{T}$ and $\underset{\sim}{T}$ is the m x n matrix comprising the first m rows of $(\underset{\sim}{P}^T)^{-1}$.

These two bounds are stated in as nearly parallel form as possible, but they are sufficiently different that any general comparison is impossible. The factor λ (m,n) complicates the second expression, and nothing can be said in general regarding its behavior with m and n. The first term in both bounds is just the deviation of the class from the reconstruction subspace. In the second terms, however, Method 1 is related to the deviation from the sampling

subspace, $\delta(C,\Phi_n)$, whereas Method 2 is related to the smaller term $\delta(C,\Psi_n)$. The factor with the double summation measures the degree of alignment of the sampling and reconstruction subspaces for specific dimensions m and n.

The usefulness of these bounds lies not in any generalizations about the behavior of Method 1 and 2 that can be made for all applications, rather in that for any particular application, the bounds can actually be evaluated on a digital computer for various m and n, and an evaluation of the proposed scheme can be made.

Considerable insight into the behavior of these two schemes and the behavior of their error bounds may be gained by considering a simple example in a two dimensional space. Figure 2 shows a two dimensional, ellipsoidal class C,* the one-dimensional sampling subspace Φ_1 and extremal reconstruction subspace Ψ_1. The 1-width $d_1(C) = \delta(C,\Psi_1)$ and the deviation $\delta(C,\Phi_1)$ of the class from the sampling subspace are also indicated. For this problem the change-of-basis matrix which we have called $\underset{\sim}{M}$ is given by

$$\underset{\sim}{M} = \begin{bmatrix} (\tilde{\psi}_1,\varphi_1) & (\tilde{\psi}_1,\varphi_2) \\ (\tilde{\psi}_2,\varphi_1) & (\tilde{\psi}_2,\varphi_2) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{18}$$

where θ is the angle between Φ_1 and Ψ_1 (Note that the basis vectors φ_1,φ_2 and $\tilde{\psi}_1,\tilde{\psi}_2$ are not shown, φ_1 is a unit vector in Φ_1,φ_2 is a unit vector orthogonal to it, similarly $\tilde{\psi}_1$ & $\tilde{\psi}_2$).

We assume that the only data regarding $f \in C$ is the sample $(f,\varphi_1) = b_1$ given by the distance D(0,b). We wish to estimate $a_1 = (f,\tilde{\psi}_1) = D(0,c)$. The matrix $\underset{\sim}{P}$ is simply cos θ. Method 1 uses $\hat{a}_1 = P\, b_1 = b_1 \cos\theta$; the Method 1 approximation to point c is point d. Method 2 uses $\hat{a}_1 = (P^T)^{-1} b_1$ or $b_1/\cos\theta$, which gives point e. That is, Method 1 merely projects onto Ψ_1 the projection of f on Φ_1. Method 2 finds the member of Ψ_1 whose projection on Φ_1 is the same as $b_1 = (f,\varphi_1)$, on the assumption that the actual f is known to be close to Ψ_1,

*The class P in (1) is an infinite-dimensional, compact ellipsoid.

by virtue of its class membership. If we call the distance $D(c,d) \triangleq \epsilon_1$, $D(c,e) \triangleq \epsilon_2$, $D(a,b) \triangleq h$, and $D(a,c) \triangleq s$, we find by direct application of a little trigonometry, the coordinate errors

$$\epsilon_1 = h \sin\theta \leq \delta(C,\Phi_1) \sin\theta \tag{19}$$

$$\epsilon_2 = s \tan\theta \leq d_1(C) \tan\theta \tag{20}$$

These bounds are in fact the same as the second terms of Theorems 1 & 2, which give

$$\epsilon_1^2 \leq \delta^2(C,\Phi_1). \quad \min\{1, [1 - (\varphi_1, \psi_1)^2]\} = \delta^2(C,\Phi_1) \cdot [1 - \cos^2\theta]$$
$$= \delta^2(C,\Phi_1) \sin^2\theta \tag{21}$$

$$\epsilon_2^2 \leq \lambda_1(m,n)\, d_1^2(C) \min\{1, [1 - (\varphi, \psi_1)^2]\}$$

$$= \frac{1}{\cos^2\theta}\, d_1^2(C) \cdot [\sin^2\theta] = d_1^2(C) \cdot \tan^2\theta \tag{22}$$

(Recall that the eigenvalue of a 1x1 matrix [a] is just a.)

We can now see how Method 1 and 2 behave for different degrees of coincidence between sampling and reconstruction subspaces, as measured by the angle θ. The width $d_1(C)$ is always less than $\delta(C, \Phi_1)$, but for $0 \leq \theta \leq 90°$ $\tan\theta > \sin\theta$. For small θ however they are nearly equal; so there will exist some range of θ about zero where Method 2 is better.

III. APPLICATION

To illustrate the application of this theory to a specific problem with some engineering significance, we have used the class P introduced in (1). This class has already been thoroughly analyzed, and provides a ready application, without digression to compute the widths and extremal subspace of a new class. The new results we present regarding the class P are the error bounds of figures 3,4 & 5 which show that even when ordinary time sampling is used, Method 2 permits a reconstruction from these samples the error of which behaves very much as though the entire process of sampling and reconstruction were done in the extremal subspace. This suggests that the class P may be considered 2BT dimensional even when the 2BT numbers are time samples, and not only when they are inner

products with prolate spheroidal wave functions.

Fortran programs were written for a Burroughs B5500 computer to generate the appropriately normalized (as in [6]) X_i from their expansion coefficients.[11,12] Since the sampling process is taken to be the usual one for bandlimited functions, Nyquist-rate time sampling, the sampling functions φ_i are thus the normalized sinc functions

$$(2B)^{\frac{1}{2}}\text{sinc}_n(t) \triangleq (2B)^{\frac{1}{2}} \frac{\sin(2\pi Bt - n\pi)}{2\pi Bt - n\pi} \quad -\infty < n < \infty \tag{23}$$

The sampling theorem implies that the inner product of a bandlimited function f with $\text{sinc}_n(t)$ is just $f(\frac{n}{2B})(2B)^{\frac{1}{2}}$. Thus elements of the matrix $\underset{\sim}{M}$ are readily formed from a knowledge of the values of the reconstruction functions at the sampling instants. The indexing of the sampling functions was modified to run from zero to $+\infty$, with lowest indices being nearest the origin. Thus

$$\varphi_i(t) = (2B)^{\frac{1}{2}}\text{sinc}_n(t) \quad \text{where } i = \begin{cases} 0 & n=0 \\ 2n-1 & n > 0 \\ -2n & n < 0 \end{cases} \tag{24}$$

This reindexing simply produced a more logical ordering of the sampling points for a function known to be time concentrated about the origin; the most important samples are first, so that truncation after any given number of samples is least harmful.

The bounds of Theorems 1 and 2 were evaluated for various numbers n of time samples and m of coordinates estimated. The results are presented in Figures 3 and 4, which plot the value of the error bound versus m, with n as a parameter. Of course for both sets of curves, for a given m, the bound is strictly decreasing with increasing n. This decline is asymptotically limited by the m width of the class, however, since as the coordinate estimation becomes perfect, the only error is due to the limited number of coordinates estimated. The curves indicate that when such saturation occurs, estimation of a larger number of coordinates is warranted by the number of time samples available.

The Method 1 bounds (Fig. 3) show that for a given number of samples greater than the essential dimension ($2BT \approx 4$ for this case) the number of coordinates estimated should not be extended arbitrarily however, for the error eventually increases when attempting to estimate those coordinates whose range of magnitude is smaller than the possible error. A distinct optimal value of m occurs for each n. The number m giving this minimum does slowly increase with increasing n, as the coordinate estimation becomes more accurate.

Method 2 is by its nature limited to estimating $m \leq n$ coordinates*. The superior performance of Method 2 in this case is reflected in the location (see Fig. 4) of the minimum error at the m=n point, which implies that as many coordinates should be estimated as the method allows. This comment applies actually only after n has exceeded the essential dimension of the class. In fact for $n < 2BT$ the curves <u>increase</u> with m, and a minimum number of coordinates should be used.

Using the data of Figures 3 and 4, both methods were optimized with respect to m for each n by picking the value yielding the minimum point on the curves. With the understanding then that the methods estimate the optimal number of coordinates for the number n of samples given, error bounds were plotted versus n for both methods (see Fig. 5). It is clear that for the problem being considered, the situation is as suggested by the geometric interpretation shown in Fig. 2. That is, the sampling and reconstruction subspaces are sufficiently close that Method 2 significantly outperforms Method 1. In fact the Method 2 bound behaves much like the m-width of the class, while the Method 1 bound looks like the deviation from the sampling subspace just shifted by a scale factor less than 1. Examination of Theorem 1 suggests that this will be the case if

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*The obvious generalization of Method 2 to arbitrary m and n by replacing $(\underset{\sim}{P}^T)^{-1}$ by a pseudo inverse is currently being investigated.

the coefficient of $\delta^2(P,\Phi_n)$ is relatively constant, and if $\delta^2(P,\Phi_n)$is significantly larger than $\delta^2(P,\Psi_m) = d_m(P)$, which is clearly true in this case. In Theorem 2, on the other hand, one would expect the term $\delta^2(P,\Psi_m) = d_m(P)$ to dominate, provided λ (m,n) does not become large, as is the case here.

The final phase of this application consisted of actually generating some typical members of P, and reconstructing them from their time samples by 1) the cardinal sampling theorem, 2) Method 1, 3) Method 2. Generation of the functions was done by merely specifying arbitrary fractions of the maximum allowable amount of each of the first nine PSWF coordinates. Provided the resulting vector of coordinates $\underset{\sim}{a}$ satisfied the class constraint $\sum a_i^2/\lambda_i \leq E = 1$, the function was assembled as that linear combination of the PSWF's, $f = \sum_{i=1}^{9} a_i X_i$. The nine functions used here very well approximate the entire infinite dimensional class, since the essential dimension of the class is only four, and $d_9(P)$ is extremely small.

Figures 6 and 7 show two such functions and their reconstructions derived from five time samples. Reconstruction error was computed for Methods 1 and 2 from a knowledge of the PSWF coordinates, exact and estimated. The error for the sampling theorem reconstruction could not be calculated other than by numerical integration over $(-\infty,\infty)$, and this was not done. The errors for reconstructions from various numbers of samples are summarized below. From inspection of the corresponding plots, it was considered obvious that the sampling theorem reconstruction of these functions never performed any better than Method 1. (Just as this is clearly the case in Figs. 6 and 7.)

	# of Samples	Method 1 Error	Method 2 Error
Function A	3	0.191×10^{-0}	0.127×10^{-0}
	4	0.104×10^{-1}	0.457×10^{-1}
	5	0.952×10^{-2}	0.150×10^{-2}
	6	0.575×10^{-2}	0.175×10^{-3}
	7	0.460×10^{-2}	0.283×10^{-4}

Function B	5	0.453×10^{-1}	0.236×10^{-3}

The example presented here is for a class with an essential dimension of about four. The factor limiting the dimension of the class used was that tabulated expansion coefficients for PSWF's were available only for values of the parameter $c = \pi BT$ in the range corresponding to such low dimensions. Examination of the tabulated values of $\lambda_n = d_n^2(C)$ [1] for large values of c shows that once the essential dimension of the class is exceeded, $d_n(C)$ exhibits the same sharp decrease, regardless of how great that essential dimension is. This leads us to believe that the error curves we have obtained would exhibit similar behavior even for quite high dimensional classes of this family.

IV. CONCLUSIONS

This paper has developed two coordinate transformation techniques for classes of signals, making feasible the independent choice of subspaces for sampling and reconstruction of signals of the class. Thus for example the sampling subspace may be selected for ease of implementation and the reconstruction subspace for efficiency and accuracy of reconstruction. In particular, our suggestion for a potentially efficient reconstruction subspace is the extremal subspace for the class, as defined by Kolmogorov. Error bounds presented for the two methods allow numerical evaluation of specific problems where the class is mathematically well described. Further, the form of the bounds suggests two intuitive criteria for success of the methods, which may be applied even if the bounds cannot be evaluated: The class of signals should be essentially finite-dimensional, that is its m-width should decrease rapidly for large m; and the proposed sampling and reconstruction subspaces should be somewhat well aligned.

As is ultimately the case with any theory, the primary limitation on this work arises in the modeling problem. It is this area that further work should

attack most urgently. Good mathematical models for realistic signal classes are essential if any theory such as the foregoing is to succeed in practical application.

Extensions of the ideas presented here might include determination of the optimal reconstruction subspace relative to a specified sampling subspace and class. Another study of interest would be the quantization of coordinates to estimate the ϵ-entropy of classes, and measure the reduction, if any, in entropy produced by the coordinate transformations suggested.

Appendix I - Proofs of Theorems

Lemma 1: Let the signal class $C \subset \mathcal{L}_2$ and $\{\psi_i\}_{i=1}^{\infty}$ be an orthonormal basis spanning C. Let $\{\varphi_j\}_{j=1}^{n}$ be an orthonormal set such that $\Phi_n \subset \Psi_\infty$. Denote $b_j = (f,\varphi_j)$ and $a_i = (f, \psi_i)$ further $\underset{\sim}{a}_1 = (a_1 \ \ldots \ a_m)^T$ and $\underset{\sim}{b}_1 = (b_1 \ \ldots \ b_n)^T$. Let the m vector $\underset{\sim}{\hat{a}}_1 = F(\underset{\sim}{b}_1)$ for some F, and form $\hat{f} = \sum_{i=1}^{m} \hat{a}_i \psi_i$. Then

$$\delta^2(C,\Psi_m) \leq \sup_{f\in C} \|f-\hat{f}\|^2 \leq \delta^2(C,\Psi_m) + \sup_{f\in C} \sum_{i=1}^{m} (a_i - \hat{a}_i)^2 \tag{25}$$

Proof: The first inequality is true by definition. The second may be seen by decomposing the vector error $f-\hat{f}$ into two orthogonal components, one in Ψ_m and one orthogonal to it.

$$\|f-\hat{f}\|^2 = \|f-P_{\Psi_m} f\|^2 + \| P_{\Psi_m} f-\hat{f}\|^2 \tag{26}$$

Noting that $\|P_{\Psi_m} f-\hat{f}\|^2 = \sum_{i=1}^{m} (a_i-\hat{a}_i)^2$ and taking the supremum over the class on both sides of (26) noting $\sup \Sigma \leq \Sigma \sup$ gives the desired result.

The next lemma provides a bound on the coordinate error for Method 1, $\sup_{f\in C} \sum_{i=1}^{m} (a_i - \hat{a}_i)^2$.

Lemma 2: If $a_i = (f,\psi_i)$ and $\hat{a}_i = \sum_{j=1}^{n} (f,\varphi_j)(\varphi_j,\psi_i)$ for $f\in C$ where $\{\varphi_j\}_{j=1}^{n}$ and $\{\psi_i\}_{i=1}^{m}$ are orthonormal sets of functions, then

$$\sup_{f\in C} \sum_{i=1}^{m} (a_i - \hat{a}_i)^2 \leq \delta^2(C,\Phi_n) \cdot \min\{1, \sum_{i=1}^{m} [1 - \sum_{j=1}^{n} (\psi_i,\varphi_j)^2]\} \tag{27}$$

Proof: $$\delta^2(C,\Phi_n) \geq \|f - \sum_{j=1}^{n} (f,\varphi_j)\varphi_j\|^2 \geq \| \sum_{i=1}^{m} ([f - \sum_{j=1}^{n} (f,\varphi_j)\varphi_j],\psi_i)\psi_i\| \tag{28}$$

The first inequality follows by definition; the second noting that the norm of a function is greater than or equal to the norm of any of its perpendicular projections. Thus

$$\delta^2(C,\Phi_n) \geq \| \sum_{i=1}^{m} [(f,\psi_i) - \sum_{j=1}^{n} (f,\varphi_j)(\varphi_j,\psi_i)]\psi_i\|^2 \tag{29}$$

That is

$$\delta^2(C,\Phi_n) \geq \sum_{i=1}^{m} (a_i - \hat{a}_i)^2 \qquad f \in C \tag{30}$$

A second inequality completes the proof: note that

$$(a_i - \hat{a}_i) = \sum_{j=n+1}^{\infty} (\varphi_j,\psi_i)(f,\varphi_j) \leq \Big[\sum_{j=n+1}^{\infty} (\psi_i,\varphi_j)^2\Big]^{\frac{1}{2}}\Big[\sum_{j=n+1}^{\infty} (f,\varphi_j)^2\Big]^{\frac{1}{2}} \tag{31}$$

by the Schwartz inequality. Thus

$$(a_i - \hat{a}_i) \leq \Big[\|\psi_i\|^2 - \sum_{j=1}^{n} (\psi_i,\varphi_j)^2\Big]^{\frac{1}{2}}\Big[\|f\|^2 - \sum_{j=1}^{n} (f,\varphi_j)^2\Big]^{\frac{1}{2}} \tag{32}$$

Noting that $\Big[\|f\|^2 - \sum_{j=1}^{n} (f,\varphi_j)^2\Big]^{\frac{1}{2}} \leq \delta(C,\Phi_n)$ and $\|\psi_i\| = 1$ we have

$$(a_i - \hat{a}_i) \leq \delta(C,\Phi_n)\Big[1 - \sum_{j=1}^{n} (\psi_i, \varphi_j)^2\Big]^{\frac{1}{2}} \tag{33}$$

Squaring and summing i = 1, ..., m then, we obtain

$$\sum_{i=1}^{m} (a_i - \hat{a}_i)^2 \leq \delta^2(C,\Phi_n)\sum_{i=1}^{m}\Big[1 - \sum_{j=1}^{n} (\psi_i,\varphi_j)^2\Big] \qquad f \in C \tag{34}$$

Equation (30) taken with (34) furnish the desired result.

We can now bound the net error of Method 1, combining Lemma 1 and 2:

Theorem 1: Let $\{\psi_i\}_{i=1}^{\infty}$ and $\{\varphi_j\}_{j=1}^{\infty}$ be such that $\Psi_\infty = \Phi_\infty \supset C$. If from the samples (f,φ_j) j = 1, ..., n we form coefficients

$$\hat{a}_i = \sum_{j=1}^{n} (f,\varphi_j)(\varphi_j,\psi_i) \text{ and expand } \hat{f} = \sum_{i=1}^{m} \hat{a}_i\psi_i \text{ then the error}$$

$$\sup_{f\in C} \|f - \hat{f}\|^2 \leq \delta^2(C,\Psi_m) + \delta^2(C,\Phi_n)\min\Big\{1, \sum_{i=1}^{m}\Big[1 - \sum_{j=1}^{n} (\varphi_j,\psi_i)^2\Big]\Big\} \tag{35}$$

The next lemma provides a bound on the coordinate error for Method 2.

Lemma 3: From the n samples $\underset{\sim}{b}_1$ form $\hat{\underset{\sim}{a}}_1 = (\underset{\sim}{P}^T)^{-1}\underset{\sim}{b}_1$.

Then the error, for any $m \leq n$

$$\sup_{f\in C} \sum_{i=1}^{m} (a_i - \hat{a}_i)^2 \leq \lambda\ (m,n)\delta^2(C,\Psi_n).\min\{1, \sum_{j=1}^{n}[1 - \sum_{i=1}^{n}(\psi_i,\varphi_j)^2]\} \qquad (36)$$

where $\lambda\ (m,n)$ is the largest eigenvalue of $\underset{\sim}{M} = \underset{\sim}{T}^T\underset{\sim}{T}$ and $\underset{\sim}{T}$ is the $m \times n$ matrix comprising the first m rows of $(\underset{\sim}{P}^T)^{-1}$.

Proof: We have $\underset{\sim}{b}_1 = \underset{\sim}{P}^T\underset{\sim}{a}_1 + \underset{\sim}{R}^T\underset{\sim}{a}_2 \Rightarrow \underset{\sim}{a}_1 = (\underset{\sim}{P}^T)^{-1}\ \underset{\sim}{b}_1 - (\underset{\sim}{P}^T)^{-1}\ \underset{\sim}{R}^T\underset{\sim}{a}_2 \qquad (37)$

thus the m-vector of errors in the first $m \leq n$ coordinates a_i is given by $\underset{\sim}{T}\ \underset{\sim}{R}^T\underset{\sim}{a}_2$. We bound the norm of this error, $\sum_{i=1}^{m}(a_i - \hat{a}_i)^2$ by bounding the norm of the vector $\underset{\sim}{R}^T\underset{\sim}{a}_2$, and using the inequality $\|\underset{\sim}{T}\underset{\sim}{R}^T\underset{\sim}{a}_2\| \leq \|\underset{\sim}{T}\| \cdot \|\underset{\sim}{R}^T\underset{\sim}{a}_2\|$, where $\|\underset{\sim}{T}\| = \sup_{\|\underset{\sim}{x}\|=1} \|Tx\|$. It is well known that $\|\underset{\sim}{T}\|$ is equal to the square root of the largest eigenvalue of $\underset{\sim}{T}^T\underset{\sim}{T}$.[14] To bound $\|\underset{\sim}{R}^T\underset{\sim}{a}_2\|$, notice that the jth component of the n-vector $\underset{\sim}{R}^T\underset{\sim}{a}_2$ is given by

$$\sum_{i=n+1}^{\infty} (\psi_i,\varphi_j)(f,\psi_i) = (b_j - \hat{b}_j) \quad j = 1, \ldots, n \qquad (38)$$

if we call $\hat{b}_j \triangleq \sum_{i=1}^{n}(\psi_i,\varphi_j)(f,\psi_i)$. (39)

A similar quantity has been bounded in Lemma 2. Applying (30) and (33) to $(b_j - \hat{b}_j)$ we obtain,

$$\|\underset{\sim}{R}^T\underset{\sim}{a}_2\|^2 = \sum_{j=1}^{n}(b_j - \hat{b}_j)^2 \leq \delta^2(C,\Psi_n)\min\{1, \sum_{j=1}^{n}[1 - \sum_{i=1}^{n}(\psi_i,\varphi_j)^2]\} \qquad (40)$$

Applying Lemma 3 to Lemma 1 gives the analog of Theorem 1 for Method 2.

Theorem 2: Notation as above, if $\hat{\underset{\sim}{a}}_1 = (\underset{\sim}{P}^T)^{-1}\ \underset{\sim}{b}_1$ and $\hat{f} = \sum_{i=1}^{m}\hat{a}_i\psi_i$ for $m \leq n$

$$\sup_{f\in C} \|f - \hat{f}\|^2 \leq \delta^2(C,\Psi_m) + \lambda\ (m,n)\delta^2(C,\Psi_n)\min\{1, \sum_{j=1}^{n}[1 - \sum_{i=1}^{n}(\psi_i,\varphi_j)^2]\} \qquad (41)$$

Appendix II - Computation of Deviations

In our discussion we have assumed that the class of signals is well described, that is that the extremal subspace is known, that the deviation from this subspace (the m-width of the class) is known, and further that the deviation from a given arbitrary sampling subspace is known. The first two of these are known for a variety of classes. The third, however, is not merely an invariant of the class but a property of the class and sampling subspace jointly. Not only does it vary with the sampling subspace, thus requiring a new computation for each sampling scheme to be considered, but even for any one given sampling subspace not identical to the extremal subspace, the computation of this deviation analytically may prove impossible.

In order to be able to use the foregoing theory in a particular problem, we are thus obliged to develop a means of approximating the deviation $\delta(C,\Phi_n)$ from arbitrary Φ_n. We will show how this can be done for a somewhat general category of classes; specifically, classes of the type $C = \{f:\|Lf\| \leq E\}$ where L is a nonsingular real linear operator. Our example class P is of this type.

We first take note of some results which allow an exact computation of $\delta(C,\Phi_n)$ if C is finite dimensional, i.e. $f\in C \Rightarrow f = \sum_{i=1}^{m} a_i x_i$. The operator L has a matrix representation $\underset{\sim}{U}$ in the $\underset{\sim}{x}$ basis, that is, the coordinates of Lf are given by $\underset{\sim}{U}\underset{\sim}{a}$ with $\underset{\sim}{U} = [u_{ij}]$ where $u_{ij} = (Lx_j, x_i)$. The constraint $\|Lf\| \leq E$ defining the class may be expressed in terms of the coordinates $\underset{\sim}{a}$ of f:

$$\|Lf\|^2 = \sum_{i=1}^{m}\sum_{j=1}^{m} a_i a_j (Lx_i, Lx_j) \leq E \tag{42}$$

That is

$$\underset{\sim}{a}^T\underset{\sim}{U}^T\underset{\sim}{U}\underset{\sim}{a} = (\underset{\sim}{U}\underset{\sim}{a},\ \underset{\sim}{U}\underset{\sim}{a}) = (\underset{\sim}{U}^T\underset{\sim}{U}\underset{\sim}{a},\ \underset{\sim}{a}) \le E \tag{43}$$

Note that the quadratic form $(\underset{\sim}{U}^T\underset{\sim}{U}\underset{\sim}{a},\ \underset{\sim}{a})$ is positive definite: It is positive by the positivity of the norm $\|\underset{\sim}{U}\ \underset{\sim}{a}\|^2 = (\underset{\sim}{U}\ \underset{\sim}{a},\ \underset{\sim}{U}\ \underset{\sim}{a})$, and the assumption of non-singularity of L, and hence of $\underset{\sim}{U}$, provides positive definiteness. Furthermore the form $(\underset{\sim}{U}^T\underset{\sim}{U}\underset{\sim}{a},\underset{\sim}{b})$ is symmetric, and linear in either $\underset{\sim}{a}$ or $\underset{\sim}{b}$ by the properties of the inner product. It therefore may be considered an inner product itself, which we shall call

$$(\underset{\sim}{a},\ \underset{\sim}{b})_{\underset{\sim}{U}} \triangleq (\underset{\sim}{U}^T\underset{\sim}{U}\underset{\sim}{a},\underset{\sim}{b}) \tag{44}$$

We will also need the following basic property of quadratic forms: if $\underset{\sim}{A}$ is an n x n matrix, and $\underset{\sim}{A}_s \triangleq \frac{1}{2}[\underset{\sim}{A} + \underset{\sim}{A}^T]$ (the symmetric part of $\underset{\sim}{A}$), then for any n-vector $\underset{\sim}{x}$, we have

$$(\underset{\sim}{x},\ \underset{\sim}{A}\underset{\sim}{x}) = (\underset{\sim}{x},\ \underset{\sim}{A}_s\underset{\sim}{x}) \tag{45}$$

We now can make the following exact calculation of the deviation of a finite dimensional class from an arbitrary sampling subspace:

Lemma 4: Let L be a nonsingular linear operator, $\{\varphi_j\}_{j=1}^n$ be an orthonormal set, and $\{\psi_i\}_{i=1}^m$ $n \le m$ another orthonormal set such that $C \triangleq \{f: \|Lf\| \le E\} \subset \Psi_n$ for a given finite dimensional class C. That is, for $f \in C$ we may write $f = \sum_{i=1}^m a_i\psi_i$.

Denote by $\underset{\sim}{U}$ the m x m matrix representation of L in the ψ basis. Define the matrix $\underset{\sim}{G}_n \triangleq [c_{ik}(n)]$ where $c_{ik}(n) \triangleq \sum_{j=1}^n (\psi_i,\varphi_j)(\psi_k,\varphi_j)$ $i,k, = 1, \ldots, m$. Then if $\underset{\sim}{T}_n \triangleq [(\underset{\sim}{U}^T\underset{\sim}{U})^{-1}\ (\underset{\sim}{I} - \underset{\sim}{G}_n)]$ and $\underset{\sim}{G}_n \triangleq \frac{1}{2}[\underset{\sim}{T}_n + \underset{\sim}{T}_n^T]$, the deviation $\delta^2(C,\Phi_n) = \lambda_1(n)\cdot E$ where $\lambda_1(n)$ is the largest eigenvalue of $\underset{\sim}{G}_n$, for $n \le m$.

Proof:

$$\delta^2(C,\Phi_n) \triangleq \sup_{f\in C}\|f - \sum_{j=1}^n (f,\varphi_j)\varphi_j\|^2 = \sup_{\|Lf\|\le E}\{\|f\|^2 - \sum_{j=1}^n (f,\varphi_j)^2\} \tag{46}$$

$$= \sup_{(\underset{\sim}{U}^T\underset{\sim}{U}\underset{\sim}{a},\underset{\sim}{a}) \le E}\{\sum_{i=1}^n a_i^2 - \sum_{j=1}^n [\sum_{i=1}^m a_i(\psi_i,\varphi_j)][\sum_{k=1}^m a_k(\psi_k,\varphi_j)]\} \tag{47}$$

where we have replaced f by its ψ-representation. Changing the order of summation in the second term gives

$$\delta^2(C,\Phi_n) = \sup_{(\underset{\sim}{U}^T\underset{\sim}{U}\underset{\sim}{a},\underset{\sim}{a}) \le E} \{ \sum_{i=1}^{n} a_i^2 - \sum_{i=1}^{m}\sum_{k=1}^{m} c_{ik}(n) a_i a_k \} \tag{48}$$

$$= \sup_{(\underset{\sim}{U}^T\underset{\sim}{U}\underset{\sim}{a},\underset{\sim}{a}) \le E} \{([\underset{\sim}{I} - \underset{\sim}{C}_n]\,\underset{\sim}{a},\,\underset{\sim}{a})\} \tag{49}$$

which can be replaced by the equivalent form

$$\delta^2(C,\Phi_n) = E \sup_{\underset{\sim}{a}} \frac{([\underset{\sim}{I} - \underset{\sim}{C}_n]\,\underset{\sim}{a},\underset{\sim}{a})}{(\underset{\sim}{U}^T\underset{\sim}{U}\,\underset{\sim}{a},\,\underset{\sim}{a})} = E \sup_{a} \frac{((\underset{\sim}{U}^T\underset{\sim}{U})^{-1}(\underset{\sim}{I} - \underset{\sim}{C}_n)\underset{\sim}{a},\underset{\sim}{a})_{\underset{\sim}{U}}}{(\underset{\sim}{a},\,\underset{\sim}{a})_{\underset{\sim}{U}}} \tag{50}$$

in terms of the $\underset{\sim}{U}$-inner product (see (44)). Thus

$$\delta^2(C,\Phi_n) = E \sup_{\underset{\sim}{a}} \frac{(\underset{\sim}{T}_n\underset{\sim}{a},\,\underset{\sim}{a})_{\underset{\sim}{U}}}{(\underset{\sim}{a},\,\underset{\sim}{a})_{\underset{\sim}{U}}} = E \sup_{\underset{\sim}{a}} \frac{(\underset{\sim}{G}_n\underset{\sim}{a},\,\underset{\sim}{a})_{\underset{\sim}{U}}}{(\underset{\sim}{a},\underset{\sim}{a})_{\underset{\sim}{U}}} \tag{51}$$

replacing $\underset{\sim}{T}_n$ by its symmetric part, $\underset{\sim}{G}_n$. But it is well known[14] that this supremum is $\lambda_1(n)$, the largest eigenvalue of $\underset{\sim}{G}_n$.

Application of this result is simplified by picking the basis $\{\psi_i\}_{i=1}^m$ to be the extremal basis for C, $\{\tilde{\psi}_i\}_{i=1}^m$. Golomb[16] shows that the $\tilde{\psi}_i$ are the eigenfunctions of L^*L. Using $\{\tilde{\psi}_i\}$ to represent L and f thus causes the matrix $\underset{\sim}{U}^T\underset{\sim}{U}$ to be diagonal, and the diagonal elements are just $d_i^2(C)$, i = 1, ..., m.

We now use the ability to calculate deviations for finite dimensional classes to approximate deviations for infinite-dimensional classes. This will be done by approximating the given class by its intersection with the m dimensional extremal subspace and calculating the deviation for that m-dimensional class by the previous lemma. The error in this approximation is bounded by the following lemma.

Lemma 5: Let $\{\varphi_j\}_{j=1}^n$ and $\{\tilde{\psi}_i\}_{i=1}^m$ be orthonormal sets of functions, further let $\tilde{\Psi}_m$ by the extremal basis for an arbitrary class of functions, C. Then

$$\delta^2(C \cap \tilde{\Psi}_m, \Phi_n) \le \delta^2(C,\Phi_n) \le \delta^2(C \cap \tilde{\Psi}_m, \Phi_n) + 2d_m^2(C) \tag{52}$$

Proof: The first inequality is true, as $(C \cap \tilde{\Psi}_m) \subset C$

The second inequality is seen noting $\delta(C,\Phi_n) = \sup_{f\in C} \|f - P_{\Phi_n}(f)\|$ and that we may

write $f = P_{\tilde{\Psi}_m}(f) + P^{\perp}_{\tilde{\Psi}_m}(f)$, that is f may be written as the sum of a component in $\tilde{\Psi}_m$ and one orthogonal to it. Thus

$$\|f - P_{\Phi_n}(f)\| = \|f - P_{\tilde{\Psi}_m}(f) - (P_{\Phi_n}(f) - P_{\tilde{\Psi}_m}(f))\|^2 \tag{53}$$

$$= \|f - P_{\tilde{\Psi}_m}(f) - (P_{\Phi_n}[P_{\tilde{\Psi}_m}(f) + P^{\perp}_{\tilde{\Psi}_m}(f)] - P_{\tilde{\Psi}_m}(f))\|^2 \tag{54}$$

$$\leq \|f - P_{\tilde{\Psi}_m}(f)\|^2 + \|P_{\tilde{\Psi}_m}(f) - P_{\Phi_n}(P_{\tilde{\Psi}_m}(f))\|^2 + \|P_{\Phi_n}(P^{\perp}_{\tilde{\Psi}_m}(f))\|^2 \tag{55}$$

$$\leq d_m^2 + \delta^2(C \cap \Psi_m, \Phi_n) + \|P^{\perp}_{\tilde{\Psi}_m}(f)\|^2 \tag{56}$$

$$\leq 2d_m^2 + \delta^2(C \cap \tilde{\Psi}_m, \Phi_n) \tag{57}$$

This error may thus be made small by picking an m such that $2d_m^2(C)$ is small. (There exists such an m.) Thus for class with a known extremal subspace and width, we have a computational technique that allows us to approximate the deviation of the class from arbitrary subspaces with any desired accuracy.

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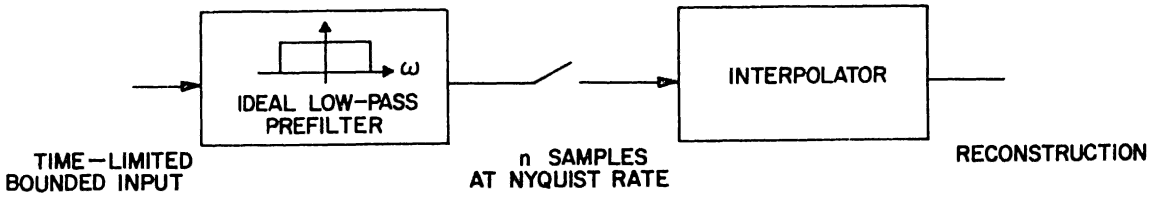

Figure 1: Block diagram for example problem. Given, finite-energy, time-limited input signals are bandlimited, forming the class P. Members of P are then sampled at the Nyquist rate. Using these n samples, the interpolator can be designed to do better than the cardinal reconstruction using sinc functions.

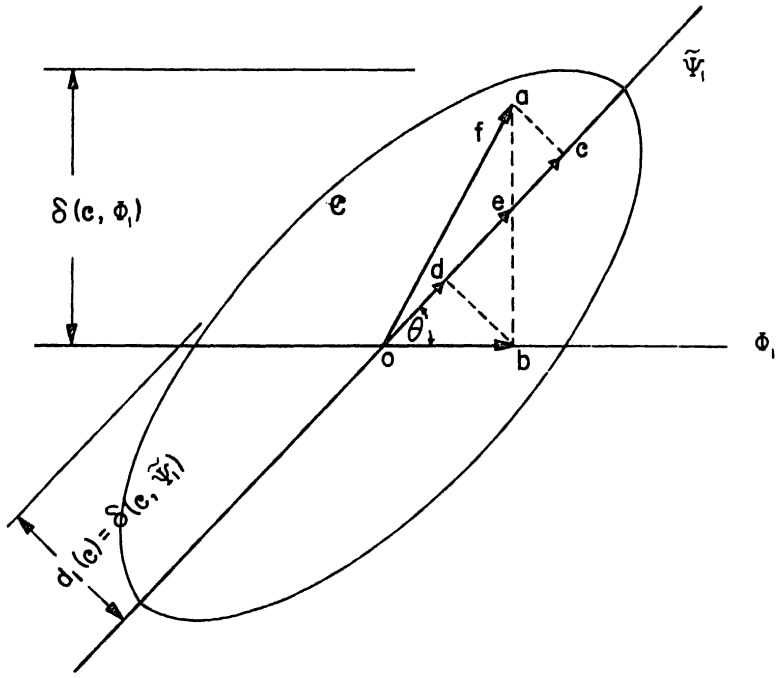

Figure 2: Geometric interpretation of Methods 1 and 2. A signal f of the class C is projected onto Ψ_1 by the sampling operation. The Method 1 reconstruction is pointed; Method 2 gives point e. The best representation of f in Ψ_1 is point c.

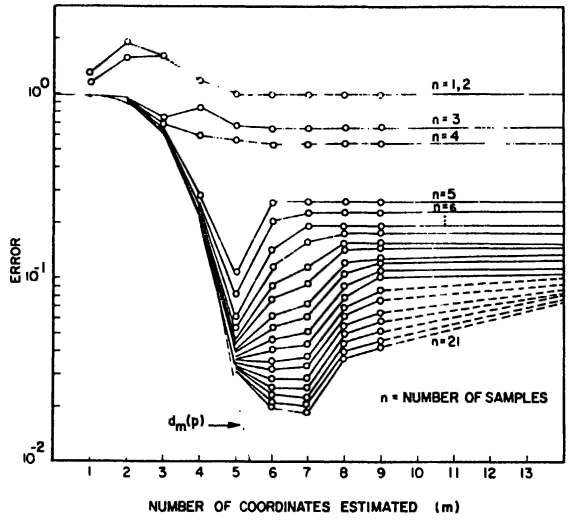

Figure 3

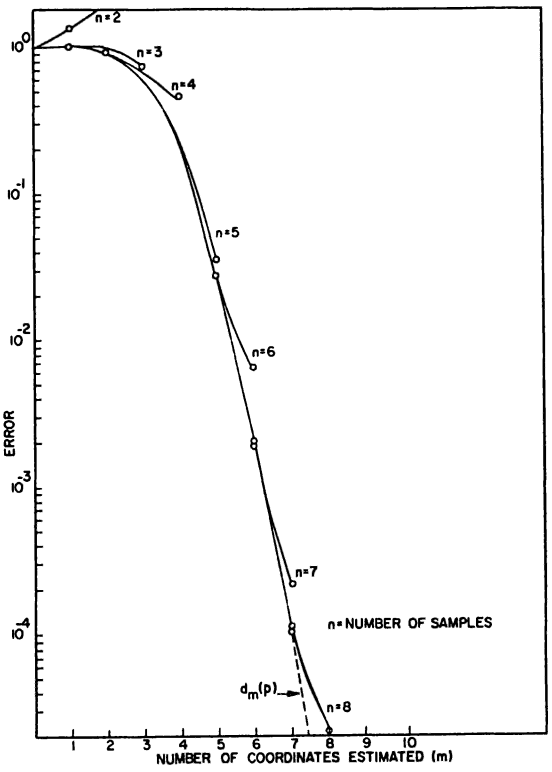

Figure 4

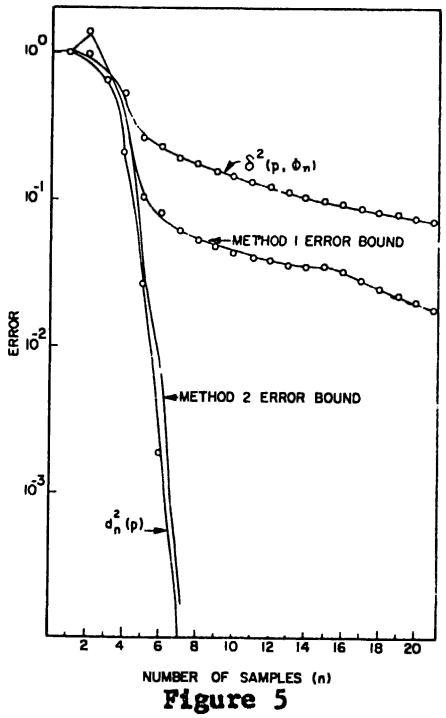

Figure 5

Figure 3: Error bounds for Method 1 for the class P.

Figure 4: Error bounds for Method 2 for the class P.

Figure 5: Error bounds for Methods 1 and 2 assuming the optimal number of coordinates are estimatedfor the given number of samples available. The deviation of the class P from the sampling subspace, and from the extremal (PSWF) subspaces is also shown.

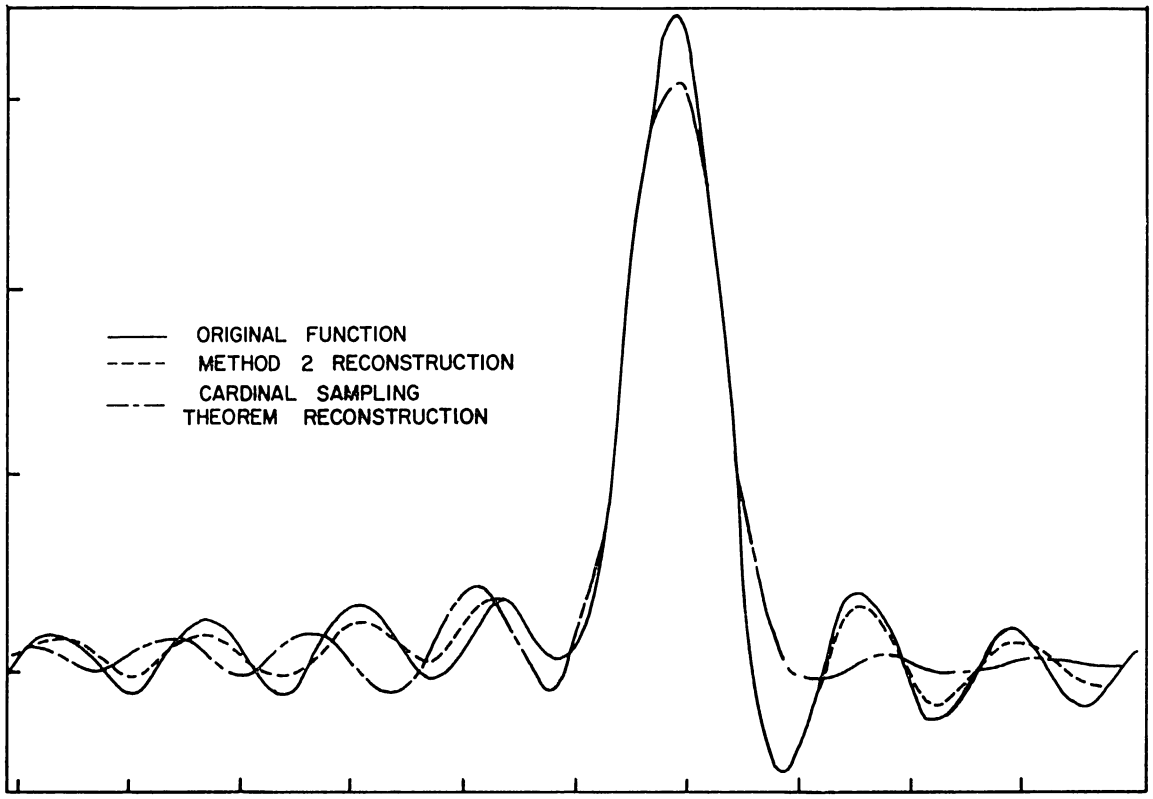

Figure 6: A member of class P (function A) and its reconstruction by Method 2 and by the cardinal sampling theorem. The Method 1 estimate is not shown, for this case it is quite close to Method 2.

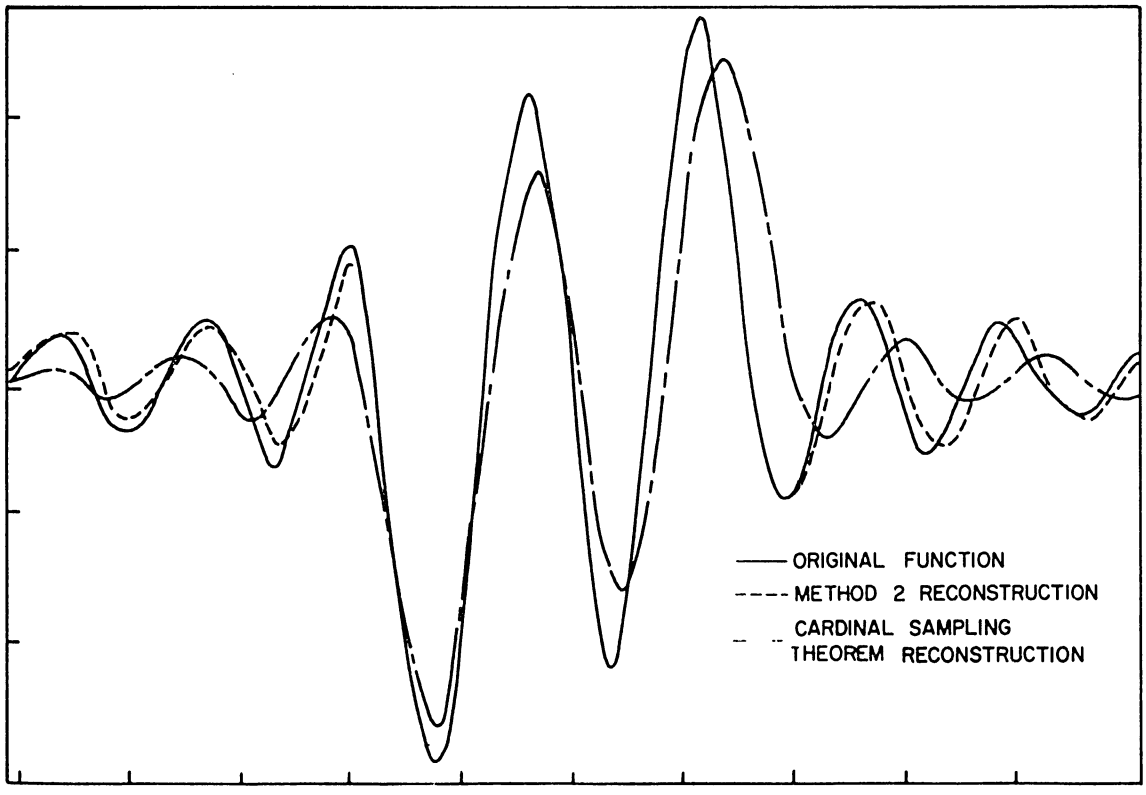

Figure 7: Another member of class P (function B) and reconstructions.