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OPTIMAL COMPENSATION AND THE CIRCLE CONDITION

by

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ABSTRACT

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A desirable property of an optimal system with full state feedback was shown by Kalman to be satisfaction of a circle condition in the $A(j\omega)$ plane where $A(s)$ is the loop gain function (return ratio) of the optimal system. This thesis demonstrates that this same property is satisfied by systems which are designed to be optimal using output feedback. The implications of this result are applied to the problem of system stability in the presence of a time-varying, nonlinear gain at the plant input. This thesis also investigates the optimality of dual mode control for systems using output feedback.
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CHAPTER I
INTRODUCTION

1.1 Problem Formulation:

Much work has been done to develop an analytical design of optimal controllers for linear, time-invariant systems with incomplete state variable measurability. (See for example [1], [2]. However, it is clear that a physical realization for a functional representation of a dynamic controller is not, in general, unique. Indeed, one such realization will often possess properties different from those of another, much in the same way that two dynamically different systems may have the same transfer function. It shall be the purpose of this thesis to examine this problem with the aim of constructing a realization of the compensator design function presented in [1] which will have physical significance beyond the nominal design for optimality. Of course, the realization chosen will depend upon certain desirable properties which motivate that construction.

Motivation for the realization presented in this thesis stems from a standard desirable property, that the closed-loop configuration remain stable in the presence of certain time-varying nonlinearities in the loop, specifically and significantly, a time-varying nonlinear gain at the plant input. This problem was considered by Moore in [3], and will presently be investigated for compensators of the type introduced in [1].
A second problem, to be considered in Chapter III, concerns system stability in the presence of saturating nonlinearities at the plant input. Optimality of such systems will be discussed in light of the compensator realization introduced in Chapter II, as will the stability considerations implied therefrom.
2.1 Background and the Circle Condition

Some of the earliest work on the stability of certain nonlinear, time-varying closed-loop systems was done by Narendra and Goldwyn [4] and Zames [5]. They considered a negative feedback system with a linear time-invariant part with transfer function $H(s) = \sigma/y$ in the forward path and a time-varying nonlinear gain $\eta(\sigma,t)$ in the feedback path.

This work is particularly applicable to the problem at hand, where $\xi(u,t)$ is the time-varying nonlinear gain and the cascaded plant and compensator combination (designed for optimality when $\xi = 1$) represents the linear, time-invariant part. See Figure 1.

Zames [5] formulated, for such time-varying nonlinearities, the Circle Theorem (a generalized Nyquist stability criterion) which essentially states for the above problem that if $\xi(u,t)u/u$ is inside a sector bounded by $a$ and $b$, (which will be denoted, for purposes of notation, $\xi \in [a,b]$), and if the frequency response of the transfer function of the time-invariant part (the loop gain function of cascaded plant and compensator) avoids a "critical region" in the complex plane, then the closed loop system is stable; if $a > 0$, then the critical region is a circle whose center is halfway between $-1/a$ and $-1/b$, and whose diameter is less than or equal to the distance between these points.
Fig. 1: $A_{opt}(s)$ is the loop gain of cascaded plant and compensator.
Of course, a primary objective for the compensator realization is to design an optimal feedback system whose control law minimizes a quadratic, constant-coefficient loss function containing no cross products of the state variables with the control variable (a usual assumption to restrict the class of performance indices). Hence it is necessary to impose the optimality constraint given by Kalman [6], that the frequency response of the loop gain function $A(s)$ avoid the unit circle centered at $-1$.

If $a$ and $b$ are chosen such that the critical stability circle of Zames is contained within or is coincident to the optimality circle of Kalman, then if the frequency response of the loop gain function $A(s)$ avoids the unit circle centered at $-1$, the optimal closed-loop system can allow any $\varepsilon \in [\frac{1}{2}, \infty)$ and remain stable. Satisfaction of this circle condition at the plant input will thus be the deciding criterion for the chosen realization, to be presented in the following sections.

2.2 Problem Statement

Consider the controllable observable plant with input $u$, output $y$, and state $x$:

$$\dot{x} = Ax + bu$$

$$y = Cx$$  \hspace{1cm} (2.2.1)

where $u$ is a real scalar, $x$ and $y$ are $n$- and $m$- vectors respectively, and $A$, $b$, and $C$ are matrices of appropriate dimensions. Let $\varepsilon$ denote any arbitrary nonlinear time-varying
gain within the sector \([\frac{1}{2}, \infty)\). The purpose of this chapter, then, shall be to answer the question raised by Moore in [3] — Is a closed-loop system of optimum controller in cascade with (2.2.1), with \(u\) replaced by \(\xi u\), asymptotically stable?

Consider the compensator introduced by Pearson [1], [7], whose transfer function matrix is given by

\[ G_c(s) = \frac{\sum_{i=0}^{p} \beta_i s^i}{s^p + \sum_{i=0}^{p-1} \alpha_i s^i} \quad \text{where} \quad \beta_i = \begin{bmatrix} \beta_1^i & \ldots & \beta_m^i \end{bmatrix} \quad \text{(2.2.2)} \]

If a compensator of the form (2.2.2) is to satisfy the above closed-loop stability property, then an equivalent problem statement is to find a realization of (2.2.2) such that the closed-loop system of plant in cascade with the compensator satisfies the circle condition (see [6]):

\[ |1 + A(j\omega)|^2 \geq 1 \quad \text{for all} \ \omega \quad \text{(2.2.3)} \]

where \(A(j\omega)\) is found by breaking the loop at \(u\) (see Figure 2) and calculating the loop gain at that point.

The problem of determining the coefficients of (2.2.2) was solved by Pearson (see [1], Theorem 1). The \(\beta_i's\) are specified by the solution of

\[ [C' A'C' \ldots (A')^p C'] \begin{bmatrix} \beta_0 \\ \vdots \\ \beta_p \end{bmatrix} = \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix} \quad \text{(2.2.4a)} \]
For these $\beta_i$, the $\alpha_i$ are given by

\[
\alpha_i = k_{n+i+1} - \sum_{j=i+1}^{p} \beta_j CA^{j-i-1}b \quad i=0,1, \ldots, p-1
\]

(2.2.4b)

where the $k_i$'s represent optimal feedback gains for the compensated system.

2.3 Solution

Consider the optimal system (see [1]) characterized by full state variable feedback:

\[
\begin{align*}
\dot{x} &= Ax + bu_1 \\
\dot{u}_1 &= u_2 \\
\vdots \\
\dot{w} &= \dot{u}_p = - \sum_{i=1}^{n} k_i x_i - \sum_{i=1}^{p} k_{i+n} u_i
\end{align*}
\]

(2.3.1)

Let $A_{opt}(s)$ denote the return ratio of (2.3.1) measured by breaking the loop at $w$. This loop gain function satisfies (2.2.3) as long as no cross terms between $x$ and $w$ or $u_i$ and $w$ appear in the performance index. The following proposition shows that for a compensator of the form (2.2.2), there exists a realization of the form

\[
\begin{align*}
\dot{z} &= \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \ldots & 0 & 1 \\
\end{bmatrix} z - \begin{bmatrix} \alpha_{p-1} \\
\vdots \\
\alpha_0 \\
\end{bmatrix} u - \begin{bmatrix} \beta_{p-1} \\
\vdots \\
\beta_0 \\
\end{bmatrix} y \\
u &= z_1 - \beta_p y
\end{align*}
\]

(2.3.2)
such that (2.3.2) in cascade with (2.2.1) satisfies the circle condition (2.2.3).

**PROPOSITION:** Let the \( \alpha_i \) and \( \beta_i \) be computed according to equations (2.2.4a) and (2.2.4b). Then using the compensator realization (2.3.2), shown in Figure 2, it follows that \( A(s) = A_{\text{opt}}(s) \), where \( A(s) \) is the loop gain of the compensated system.

**Proof:** Consider first the realization shown in Figure 2. Open the loop at point \( P \). Then by inspection, the loop gain is

\[
A(s) = \frac{1}{s^P} \left[ \sum_{i=0}^{p-1} \alpha_i s^i + \sum_{i=0}^{p} \beta_i s^i G(s) \right]
\]

Now let \( G_k(s) = k'(sI-A)^{-1}b \), where \( k \) is given by (2.2.4a), be the transfer function from \( u_1 \) to \( k'x \) for the system (2.3.1). Then for this system, the loop gain at \( w \) is

\[
A_{\text{ops}}(s) = \frac{k_{n+1}}{s^p} + \frac{k_{n+2}}{s^{p-1}} + \ldots + \frac{k_n+p}{s} + \frac{1}{s^p} G_k(s)
\]

Substituting (2.2.4a),

\[
A_{\text{opt}}(s) = \frac{1}{s^p} \left\{ \sum_{i=0}^{p-1} k_{n+i+1}s^i + \left[ \beta_0 \ldots \beta_p \right] \left[ \begin{array}{c} c \\ \vdots \\ cA^{p} \end{array} \right] (sI-A)^{-1}b \right\}
\]

\[
= \frac{1}{s^p} \left[ \sum_{i=0}^{p-1} k_{n+i+1}s^i + \sum_{i=0}^{p} \beta_i cA^i x(s) \right]
\]
And now, utilizing (2.2.4b) and substituting for $CA^ix(s)$ (where $Cx(s) = G(s)$),

$$A_{\text{opt}}(s) = \frac{1}{sp} \left[ \sum_{i=0}^{p-1} \alpha_i s^i + \sum_{j=1}^{p} \beta_j CA^{j-i-1}b s^i \right]$$

$$+ \beta_0 G(s) + \sum_{j=1}^{p} \beta_j (s^j G(s) - \sum_{i=0}^{j-1} CA^{j-i-1}b s^i)$$

and since

$$\sum_{j=1}^{p} \sum_{i=0}^{j-1} \beta_j CA^{j-i-1}b s^i = \sum_{j=1}^{p} \beta_j CA^{j-1}b + s \sum_{j=2}^{p} \beta_j CA^{j-2}b +$$

$$+ \ldots + s^i \sum_{j=i+1}^{p} \beta_j CA^{j-i-1}b + \ldots + s^{p-1} \beta_p CA^{p-1}b$$

$$= \sum_{i=0}^{p-1} s^i \sum_{j=i+1}^{p} \beta_j CA^{j-i-1}b$$

it follows that

$$A_{\text{opt}}(s) = \frac{1}{sp} \left[ \sum_{i=0}^{p-1} \alpha_i s^i + \sum_{i=0}^{p} \beta_i s^i G(s) \right] = A(s)$$

Q.E.D.

2.4 Example

Realize an optimal compensator for the system described by

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
whose control law minimizes the integral

\[ I = \int_{0}^{\infty} (x_a' Q x_a + u^2) \, dt \]

where \( x_a = [x, u] \) and \( Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 21 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \).

The solution to the augmented matrix Ricatti equation is

\[
P = \begin{bmatrix} 4 & 6 & 4 & 1 \\ 6 & 20 & 15 & 4 \\ 4 & 15 & 20 & 6 \\ 1 & 4 & 6 & 4 \end{bmatrix}
\]

which yields an optimal control law \( u^* = -k' x \), where

\[ k' = [1 \ 4 \ 6 \ 4] \]

(This control law will place the poles of the closed-loop system all at -1).

From equation (2.2.4a),

\[
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \beta_0^1 \\ \beta_0^2 \\ \beta_1^1 \\ \beta_1^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}
\]

\[ \beta_0^1 = 1 \quad \beta_1^1 = 4 \\
\beta_0^2 = 6 \quad \beta_1^2 = 0 \]
From equation (2.2.4b),

\[ \alpha_0 = k_4 = \sum_{j=1}^{1} \beta_j c_b \]

\[ = k_4 - [4 \ 0] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \]

\[ = 4 \]

Hence the realization of Figure 3 follows:
Fig. 3: Realization (2.3.2) of compensator (2.2.2) for the given example
2.5 Discussion

One note of particular importance and added benefit to this design is that the number of amplifiers required for this realization can be significantly decreased. Since breaking the loop at the point where the circle condition is satisfied breaks all of the feedback loops, placing one amplifier of amplification $K$ at point $P$ in Figure 2 will allow reduction in the magnitude of all $\alpha$'s and $\beta$'s in the compensator design by a factor of $K$.

Hence setting $K = \max (\alpha_1, \beta_1)$ will allow an optimal compensator realization in which all compensator parameters are less than or equal to one. This is tantamount to multiplying and dividing $A(s)$ by $K$, thus leaving the return ratio unchanged. In many cases this could mean the use of $2p$ fewer amplifiers than utilized in the previous construction. Also, from the above theory, it is guaranteed that if the amplifier gain $K$ should drift, the closed-loop system will remain stable as long as $K$ remains in the sector $[\frac{1}{2}, \infty)$. To illustrate this, the realization of the above example could be modified as shown in Figure 4.

In connection with this property, it is of interest to contrast this realization with that of the observer design introduced by Luenberger in [2]. In general it is not possible to reduce all observer parameters to values less than or equal to one by placing an amplifier $K$ at point $P$,
Fig. 4: Modification emphasizing frugal use of amplifiers
where the stability properties hold, since point P is not common to all observer feedback loops.
CHAPTER III
DUAL MODE CONTROL

3.1 Background and the Saturation Constraint

The preceding chapter has considered the problem of closed-loop stability of optimal systems in the presence of sector nonlinearities at the plant input. It was concluded that if $\delta(u,t)$ remains in the sector $\left[\frac{1}{k}, \infty\right)$, a compensator designed for optimality could be realized such that the closed-loop system is stable when $u$ is replaced by $\delta(u,t)u$. In this chapter we will consider the case in which $\delta(u,t)$ is some saturating-type gain. This in effect limits the magnitude of the input to the plant. In general, this could cause closed-loop instability since for sufficiently large initial conditions the gain $\delta(u,t)$ will go outside the sector $\left[\frac{1}{k}, \infty\right)$. More precisely, recalling our notation, the function $\delta(u,t)u$ plotted vs. $u$ goes outside the sector bounded by a line of slope $\frac{1}{k}$ and one of slope $\infty$. See Figure 5.

For such systems, then, the plant control will be dual mode; i.e., of the form

$$ u_p = \begin{cases} ku(x,t) & |u| \leq 1/k \\ \text{sgn} u(x,t) & |u| \geq 1/k \end{cases} $$

The problem of minimizing a quadratic performance index subject to the constraint (2.2.1) and the saturation constraint on the control has been discussed in [8] with respect to determining when a dual mode control law is optimal. In [8], Rekasius and Hsia developed a necessary and
Fig. 5: Saturating gain $\xi$, where

$$\xi u = \begin{cases} 
1 & u > \frac{1}{k} \\
ku & |u| \leq \frac{1}{k} \\
-1 & u < -\frac{1}{k}
\end{cases}$$
sufficient condition for the optimality of a dual mode control law. In this chapter an equivalent frequency domain representation of this condition will be developed in terms of the loop gain function. It will be shown that this condition representation in the frequency domain will provide greater insight into the nature of systems which are optimal using dual mode control. Next, we will discuss the problem of determining when a compensated system with realization (2.3.2) is optimal to dual mode control. Finally, the implications of this condition with regard to closed-loop sensitivity will be discussed and an example presented.

3.2 Problem Statement

Consider the optimization problem of minimizing the performance index

$$I = \int_0^\infty (x'Qx + u^2) \, dt$$

subject to (2.2.1) with the additional constraint that

$$|u| \leq 1$$

Solving the optimization problem without the control magnitude constraint yields the optimal control $u^* = -k'x$.

Rekasius and Hsia in [8] show that the control law

$$u_c = \begin{cases} u^* & |u^*| \leq 1 \\ \text{sgn } u^* & |u^*| \geq 1 \end{cases} \quad (3.2.1)$$
is the optimal solution to the constrained optimization problem if and only if there exists a real number \( \alpha < 0 \) such that the following condition is satisfied:

\[
\alpha k' = k'(A+bk')
\]  

(3.2.2.)

It is desired to exhibit the implications of this condition in the frequency domain with particular regard to compensated systems of the form (2.3.2) in cascade with (2.2.1).

3.3 Solution

PROPOSITION 2: A necessary and sufficient condition for equation (3.2.2) to be satisfied is that the loop gain function satisfy the condition

\[
G_k(s) = \frac{k'b}{s-\alpha+k'b}
\]

(3.3.1)

for some real \( \alpha < 0 \).

Proof: a) Necessity:

First, multiply both sides of (3.2.2) by \((sI-A)^{-1}b\) to get

\[
\alpha k'(sI-A)^{-1}b = k'A(sI-A)^{-1}b + k'bk'(sI-A)^{-1}b
\]

but since \(k'(sI-A)^{-1}b = G_k(s),\)

\[
k'A(sI-A)^{-1}b + k'b = sG_k(s)
\]

so

\[
\alpha k'(sI-A)^{-1}b = sG_k(s) - k'b + k'bG_k(s)
\]

which implies that

\[
k'b = (s-\alpha+k'b)G_k(s)
\]

or

\[
G_k(s) = \frac{k'b}{s-\alpha+k'b}
\]
b) Sufficiency;

If \((s-\alpha+k'b)G_k(s) = k'b\)
then \(\alpha G_k(s) = sG_k(s) - k'b + k'bG_k(s)\)
But \(sG_k(s) - k'b = k'A(sI-A)^{-1}b\).
So \(\alpha G_k(s) = k'A(sI-A)^{-1}b + k'bG_k(s)\)
or \(\alpha k'(sI-A)^{-1}b = k'A(sI-A)^{-1}b + k'b(\alpha k'(sI-A)^{-1}b\)
Now if \((A,b)\) is controllable,

\[ (sI-A)^{-1}b = \frac{1}{q(s)} \begin{bmatrix} 1 \\ s \\ \cdot \\ \cdot \\
\end{bmatrix} \]

where \(q(s) = \text{det} (sI-A)\), and \(P\) is an \(nxn\) invertible matrix.

Hence

\[ [\alpha k' - k'A - k'b(\alpha k')] P \begin{bmatrix} 1 \\ s \\ \cdot \\ \cdot \\
\end{bmatrix} = 0 \quad \text{for all} \ s. \]

thus

\[ \alpha k' - k'A - k'b(\alpha k') = 0 \]

since \(P\) is nonsingular.

Q.E.D.

Note that when a plant and compensator are used in cascade to minimize the performance index

\[ I = \int_0^\infty (x_a'Q_ax_a + \omega^2) \, dt \]

where \(x_a' = [x' \, u_1 \, \ldots \, u_p]\), subject to (2.2.1) and \(|\omega| \leq 1\), the condition becomes, by natural extension to the
augmented optimal system,

\[ A(s) = \frac{k'b}{s-\alpha+k'b} \]  

(3.3.2)

where \( k \) and \( b \) are now \((n+p)\)-vectors corresponding to the augmented system. (From now on, unless explicit mention is made to the contrary, our notation will refer to the augmented system).

3.4 Implications of Dual Mode Optimality in the Frequency Domain

Note that (3.3.2) is more than a restatement of (3.2.2). In light of the realization presented in the previous chapter, it has one highly significant consequence. Since satisfaction of (3.3.2) guarantees the optimality of dual mode control at \( w \) in the optimal system, and since \( A_{opt}(s) = A(s) \) by Proposition 1, then utilization of realization (2.3.2) is such that the properties of the return ratio at \( w \) in the optimal system are identically the properties of the return ratio at \( u \) in the compensated system.

Also note that (3.3.2) implies that the \((n-1)\) zeros of \( k'(sI-A)^{-1} \) must be eigenvalues of \( A \). This can be seen as follows:

Let

\[ k'(sI-A)^{-1}b = \frac{p(s)}{q(s)} = \frac{k'b}{s-\alpha+k'b} \]

where \( q(s) = \det (sI-A) \). This implies that

\[ q(s) = (s-\alpha+k'b) \overline{q}(s). \]
Hence

\[ p(s) = k'b \bar{q}(s). \]

The \( n^{th} \) eigenvalue of \( A \) is, of course, \((\alpha-k'b)\).

Another implication of (3.3.2) is that the system matrix \( A \) must be asymptotically stable. If it were not, calculation of the loop gain would involve pole-zero cancellation at the origin or in the right half plane.

An immediate consequence of this fact is that augmented systems of the form (2.3.1) will not satisfy (3.3.2), since the augmenting string of integrators adds \( p \) open loop poles at zero. Thus it becomes necessary to further generalize (2.3.1) by feeding back the \( u_i \) through nonzero gains to make the open loop compensator eigenvalues nonzero.

Thus in the compensated system, we must make the eigenvalues of (2.3.2) nonzero by feeding back the \( z_i \). Of course, since the compensator (2.3.2) has all its states available to measurement, we can either feed back all the \( z_i \) to \( \dot{z}_p \) or we can simply feed back each \( z_i \) to \( \dot{z}_i \) through some nonzero gain.

The structure of this compensator realization modification will be illustrated in the example of the following section.

One final point needs mentioning. Equation (3.3.2) also implies that to avoid a trivial solution, the plant must have at least one real pole, \( \alpha-k'b \). If \( \alpha-k'b \) is not a plant pole, it is obvious that with a first order compensator, choosing \( k_1 = k_2 = \ldots = k_n = 0, \ k_{n+1} = k'b, \) and
choosing the compensator pole at $\alpha-k'b$, guarantees the satisfaction of (3.3.2) for any general $n^{th}$ order plant. Of course, this is a trivial solution since the plant is essentially bypassed. Such a solution merely asserts the structure of the loop gain for a first order system.

3.5 Example

Consider the linear controllable and observable plant described by

$$
\begin{align*}
A &= \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \\
b &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
c &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\end{align*}
$$

Augmenting with the control variable as in (2.3.1),

$$
\begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w
$$

which has an open loop pole at zero, so we feed back around the compensating integrator,

$$
\begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -\gamma \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w
$$

For the optimal system with full state variable feedback,

$$
A_{opt}(s) = \frac{s^2 + s\left(\frac{k_2}{k_3} + 3\right) + \left(\frac{k_2}{k_3} + \frac{k_1}{k_3} + 2\right)}{(s+1)(s+2)(s+\gamma)}, \quad k_3 \quad (3.5.1)
$$

The problem, then, is to find $k_1$, $k_2$, $k_3$ and $\alpha$ such that

$$
A_{opt}(s) = \frac{k_3}{s-\alpha+k_3} \quad (3.5.2)
$$
Clearly, choosing \( -\alpha + k_3 = \gamma \) leads to the trivial solution. Suppose we choose \( -\alpha + k_3 = 2 \). Equation (3.5.1) yields

\[
\begin{align*}
k_1 &= 0 \\
k_2/k_3 + 2 &= \gamma
\end{align*}
\]

Equation (3.5.2) requires \( k_3 < 2 \).

Choose \( k_3 = 0.5, k_2 = 1, \gamma = 4 \). Thus

\[
W^* = \begin{bmatrix} 0 & 1 & 0.5 \end{bmatrix} x^T
\]

is a control law which minimizes the previously given performance index for some \( Q_a \). Using the expression from [6],

\[
|1 + k'(sI-A)^{-1}b|^2 = 1 + ||H(sI-A)^{-1}b||^2,
\]

we can find an \( H \) such that \( Q_a = H'H \). One such \( H \) is given by

\[
H = \begin{bmatrix} 0 & 0 \\ 3\sqrt{3} & 0 \\ 0 & 3/2 \end{bmatrix} \quad \text{and thus} \quad Q_a = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 9/4 \end{bmatrix}
\]

Satisfaction of (3.3.2) thus guarantees that the optimal system with full state variable feedback is, with this control law, optimal with dual mode control at \( w \).

The problem remains to exhibit the compensated system which is optimal with dual mode control at \( u \), the plant input.

For a first order compensator with nonzero pole \(-\gamma\), equation (2.2.4a) becomes

\[
\begin{bmatrix} C' \ A' C' \end{bmatrix} \begin{bmatrix} \beta_0 + \gamma \beta_1 \\ \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} = \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix}
\]
For a first order compensator, (2.2.4b) remains

\[ \alpha_1 = k_{n+1} - \beta_1 C_b \]

In general, the compensated system will have the structure of Figure 6. For this particular example, \( p = 1, \alpha_0 = .5, \beta_0 = -3, \beta_1 = 1, \) and \( \gamma_0 = 4. \)
Fig. 6: Compensator structure for Dual Mode Optimality
CHAPTER IV
CONCLUSION

4.1 Discussion and Summary

In Chapter II it was shown that there exists a realization of the compensator whose general form is given by (2.2.2) such that the return ratio at the plant input is identically that measured at w for the augmented optimal system. The structure of such a realization is given by (2.3.2) and illustrated in Figure 2. Thus it is possible to construct a compensator realization which, since it is thus able to satisfy the circle condition (2.2.3) at the plant input, can tolerate nonlinearities contained in the sector $[\frac{1}{2}, \infty)$ without inducing closed-loop instability. Another particularly beneficial property of such a realization was shown to be a reduction in the number of amplifiers required for compensator construction. It should be noted that construction of this realization is quite simple, since the required feedback gains are precisely the $\alpha_i$ and $\beta_i$ calculated from equations (2.2.4a) and (2.2.4b). Thus no further calculations are required to determine parameters of the compensated system.

Chapter III considered a second major problem, the case in which $\phi$ is a gain of the saturating type. For such gains and large initial conditions, $\phi$ will go outside the sector $[\frac{1}{2}, \infty)$ somewhere in the saturation region. For a system to be optimal with dual mode control (which is the
result of such a saturating gain), the necessary and sufficient condition

\[ A(s) = \frac{k'b}{s-\alpha+k'b} \quad (4.1.1) \]

where \( A(s) \) is the system loop gain function, must be satisfied. This was shown to be a very stringent condition which serves as a simple test for optimality of systems containing saturating gains at the plant input.

In connection with the results of [4] and [5], it is consequential to note that if condition (4.1.1) is satisfied for \( k'b > 0 \), then the closed-loop system will remain stable for any \( \xi(0,\infty) \). This follows since if \( \xi(0,\infty) \), the "critical circle" becomes the left half plane, which is naturally avoided by the Nyquist diagram of \( A(s) \) with the satisfaction of (4.1.1).

In the interest of further research, it might be consequential to investigate desirable properties of physical realizations of the compensator other than that presented here with regard to similar sensitivity studies at various positions in the loop. For the case of dual mode control, two considerations might be of interest for further work; 1) With respect to system sensitivity, what is the best way to make the compensator poles nonzero, and 2) For what cases could the dual mode solution be considered a feasible suboptimal control?
REFERENCES


