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OPTIMAL CONTROL WITH INCOMPLETE STATE MEASUREMENTS

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ABSTRACT

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This thesis considers the problem of designing a controller for a linear plant, which, using output measurements only, will make the plant state variables follow optimally any input signal belonging to a known class of signals.

The dynamic order of the controller is determined by the observability properties of the plant and by the class of input signals. The structure of the controller is determined from the requirement that the closed loop system consisting of plant and controller be an optimum system.

Two approaches to the optimization procedure are indicated:

1) Optimizing the dynamics of an augmented system consisting of plant, controller and a dynamic system generating the input signals, to minimize appropriate errors.

2) Regarding the general problem as one of optimum regulation of errors desired to be minimized.

The results allow the linear servomechanism problem to be treated, for the first time, in a realistic manner.

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INTRODUCTION

1. PROBLEM FORMULATION

Consider the plant described by

$$\underline{x}\,(i+1) = \underline{g}\,(\underline{x}\,(i),\ \underline{u}\,(i),\ i) \qquad (1\text{-}2)$$

$$\underline{y}\,(i) = \underline{f}\,(\underline{x}\,(i),\ \underline{u}\,(i),\ i) \qquad (1\text{-}2)$$

and the dynamic system (henceforth called the input system),

$$\underline{r}\,(i+1) = \underline{p}\,(\underline{r}\,(i),\ i) \qquad (1\text{-}3)$$

$$\underline{s}\,(i) = \underline{q}\,(\underline{r}\,(i),\ i) \qquad (1\text{-}4)$$

$\underline{x}\,(\cdot)$ and $\underline{r}\,(\cdot)$ are the state vectors of the plant and the input system, $\underline{y}\,(\cdot)$ and $\underline{s}\,(\cdot)$ are the respective measurable outputs, $\underline{u}\,(\cdot)$ is a control variable, and $\underline{g}$, $\underline{f}$, $\underline{p}$, $\underline{q}$ are vector functions of appropriate dimension.

A general problem in optimal control theory is the determination of a sequence of controls $\{\underline{u}^*(j)\}$ $j = 0, 1, \ldots N$ so that the cost functional

$$I = \sum_{i=o}^{N} F\,(\underline{x}\,(i),\ \underline{r}\,(i),\ \underline{u}\,(i),\ i) \qquad (1\text{-}5)$$

is minimized. If $\underline{r}\,(i) = o\ \forall\ i$ the problem is known as the optimum regulator problem; otherwise it is a servomechanism problem.

The closed-loop solution to this problem, which for engineering reasons is the desirable solution, would yield

$$\underline{u}^*(i) = \underline{G}\,(\underline{x}(i),\ \underline{r}(i),\ i)$$

$\underline{x}$ denotes that x is a vector.

The optimal control $\underline{u}^{*}(\cdot)$ could be implemented if it were possible to determine all the components of $\underline{x}(i)$ and $\underline{r}(i)$ from the measurements $\underline{y}(j)$ and $\underline{s}(j)$, $j \leq i$. In general, this will not be possible. The problem of achieving optimal control under these conditions is the concern of this thesis.

The major part of the thesis will be devoted to the case of a linear plant with a quadratic performance index and the measurements will be assumed to be linear $\underline{x}$, $\underline{u}$, and $\underline{r}$. For this case, a procedure will be prescribed which will result in a closed-loop optimal system of order $n + p$. It will then be shown how the optimal control law corresponding to the optimal closed-loop system can be generated by a p^{th} order dynamic controller operating on the measurements $\underline{y}(i)$.

2. BACKGROUND

A dynamic controller designed as above is similar to the compensator of classical control theory. In both cases the input to the dynamic element is $\underline{y}(\cdot)$ the plant output, and the output of the dynamic element is $\underline{u}(\cdot)$ the plant input. However while the classical method of designing compensators depends on transfer function manipulation, and the satisfaction of frequency-domain specifications such as band width, phase-margin, M-peak etc. the dynamic controller proposed here is determined solely by the requirement that the closed-loop system by optimal. Also, here the compensator parameters can be determined analytically, as opposed to the trial and error procedures involved in reshaping the root-loci or the Nyquist plot. The most important advantage of the modern method is that it can be easily applied to design controllers for multivariable, non-stationary, and non-linear systems.

Some of the earliest work done in the design of optimum overall systems subject to the constraint of incomplete state measurements is due to Joseph and Tou [8], who showed that a linear plant with an additive white, Gaussian noise disturbance can be optimally controlled using only the output measurements. Here optimality was defined as the minimization of the expected value of a quadratic cost functional involving the state and control vectors. Joseph and Tou showed that the optimal control system consisted of 1) an optimal estimator driven by the available outputs and generating an optimal estimate of the state vector, and 2) a deterministic optimal controller which uses the optimal estimate of the state vector and generates the optimal control signal. The optimal estimator is the familiar Kalman filter and can be designed according to methods given in [9].

For the deterministic problem, Luenberger [6],[7] proposed that a control law involving a linear functional of the state vector could be implemented, by estimating this linear functional using an observer driven by the available plant outputs. The lowest dynamic order of such an observer is p, where $p+1$ is the observability index [7] of the plant.

The idea of reconstructing the state vector with a low-order dynamic system is attractive, but it is clear that if the observer's estimate of the state vector were used in a control system which was optimized on the assumption of availability of the exact state vector, the resulting closed-loop system would not be optimal and would almost always yield a positive increase in cost over the minimum [18]. In fact, it was shown in [18] that an arbitrarily small increase in cost due to the observer can be obtained for all initial states only if the observer is of order n. However it is clear that even if this higher order observer were used, the resulting overall system could not be optimal without the observer possessing the undesirable characteristics of a differentiator.

The theory of compensator design developed in [1], and in this thesis is concerned with the design of systems required to be optimum in the overall sense, while satisfying the constraint of incomplete state measurements. It turns out that the dynamic order of the compensator is the same as the order of an observer. The important difference between observers and compensators, however, is that the p^{th} order dynamics of the compensator are determined from the requirement that the closed-loop system consisting of plant and compensator be optimal, while the observer dynamics are arbitrary. In the next chapter, it will be shown that this requirement of optimality determines the structure of the compensator.

PROBLEM STATEMENT AND SOLUTION

1. PROBLEM STATEMENT

Let the linear, stationary, discrete-time single input, controllable and observable plant be described by

$$\underline{x}(i+1) = A\,\underline{x}(i) + \underline{b}\,u_1(i) \qquad (2.1a)$$

$$\underline{y}(i) = C^T \underline{x}(i) \qquad (2.1b)$$

where,

$\underline{x}(\cdot)$ is an n-vector, the plant state,

$u_1(\cdot)$ is the scalar control signal

$\underline{y}(\cdot)$ is an m-vector of measurable outputs

A is an n x n constant matrix, the transition matrix

b is a constant n x 1 vector

C is an n x m, $m \angle n$ of rank m

C^T is the transpose of C.

The output $\underline{y}(i)$ of the plant is required to "follow" an arbitrary signal $\underline{s}(i)$ optimally. This is the statement of the servomechanism problem in its most general form.

If nothing is known about $\underline{s}(i)$, the problem has no meaningful solution. It will be assumed, in what follows, that the class of signals $\underline{s}(i)$ can be accurately represented as the outputs of a fictitious dynamic system, the input system, represented by

$$\underline{r}(i+1) = M\,\underline{r}(i) \qquad (2.2a)$$

$$\underline{s}(i) = D^T \underline{r}(i) \qquad (2.2b)$$

where

$\underline{r}(\cdot)$ is a q -vector, the state of the input system

M is a $q \times q$ constant matrix

D^T is an $m \times q$ constant matrix

The problem, then, is of designing a controller with inputs $\underline{y}(i)$, $\underline{s}(i)$, and output $u_1(i)$, so that a quadratic performance index representing tracking errors is minimized, subject to the constraints (2.1 - 2.2).

The optimization of servomechanisms to classes of inputs was first considered in [2] by Kalman and Koepcke. The optimal control for a tracking system was shown to depend on the plant state as well as the state of the input system. Since the input system is an entirely fictitious system the only "measurable" signal of this system is the output $\underline{s}(i)$, the physical signal which the plant is required to track. Such a control law can therefore not be realized without using a compensator. In [14] Kalman treated the optimum linear servomechanism problem, but the optimal control was again unrealizable because, the signal to be tracked was required to be known for all time.

The next section demonstrates how the definition of an appropriate performance index yields an optimum control law that can be realized by a controller operating only on output signals.

2. SOLUTION

A. Augmented State Vector Approach.

a) Finite-time problem.

It will become obvious in what follows, that the optimum controller will be required to be a dynamic system. Assuming the controller is of dynamic order p, its state variables can be defined by

$$u_1(i+1) = u_2(i) \qquad (2.3)$$

$$\vdots \qquad \vdots$$

$$u_p(i+1) = u_{p+1}(i)$$

where $\underline{u} = \begin{bmatrix} u_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ u_p \end{bmatrix}$ is the state vector of the controller.

The following additional definitions are also made:-

$$\underline{w} = \begin{bmatrix} \underline{x} \\ \underline{u} \end{bmatrix} \qquad (2.4)$$

$$\underline{w}(i+1) = \bar{A}\,.\underline{w}(i) + \underline{\bar{d}}\, u_{p+1} \qquad (2.5)$$

$$\underline{z} = \begin{bmatrix} \underline{w} \\ \underline{r} \end{bmatrix} \qquad (2.6)$$

$$\underline{z}(i+1) = F\,\underline{z}(i) + \underline{h}\, u_{p+1}(i) \qquad (2.7)$$

$$F = \begin{bmatrix} \bar{A} & 0 \\ 0 & M \end{bmatrix} \qquad \underline{h} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow n+p^{th} \text{ entry} \qquad (2.8)$$

$$J = \sum_{i=1}^{N} \underline{z}^T(i)\, Q\, \underline{z}(i) + \gamma\, u_{p+1}^2(i-1) \qquad (2.9)$$

Minimization of J subject to (2.4 - 2.8) will be assumed to be satisfactory performance for the tracking system. The matrix Q is determined by the tracking errors that are to be minimized.

The optimum control law resulting from the optimization problem mentioned above is of the form

$$u^{*}_{p+1}(i) = -\underline{k}^{T}(i, N)\,\underline{z}(i) = -\underline{k}_1^{T}(i)\,\underline{x}(i) - \underline{k}_2^{T}(i)\,\underline{u}(i) - \underline{k}_3^{T}(i)\,\underline{r}(i) \tag{2.10}$$

where the matrices $\underline{k}_1$, $\underline{k}_2$, $\underline{k}_3$ can be determined from the dynamic programming equations involved in the optimization.

The equations of the controller, with inputs $-\underline{y}(i)$, $-\underline{s}(i)$ and output $u_1(i)$, may be written as

$$\begin{aligned} u_1(i+1) &= u_2(i) \\ &\;\vdots \\ u_p(i+1) &= u_1(i+p) = -\sum_{k=1}^{m}\sum_{j=0}^{p_2}\delta_j^k(i)\,s_k(i+j) \\ &\quad - \sum_{k=1}^{j}\sum_{j=0}^{p_1}\beta_j^k(i)\,y_k(i+j) \\ &\quad - \sum_{j=1}^{p}\alpha_j(i)\,u_j(i) \end{aligned} \tag{2.11}$$

For optimality of the compensated system, the following relations obtained, by expanding $y_k(i+j)$, $s_k(i+j)$ in terms of $y_k(i)$, $s_k(i)$, $k = 1, \ldots m$, must hold,

$$[C \; (A^T)C \; \ldots \; (A^T)^{p_1} C] \begin{bmatrix} \beta_0^1(i) \\ \vdots \\ \beta_0^m(i) \\ \vdots \\ \beta_{p_1}^1(i) \\ \vdots \\ \beta_{p_1}^m(i) \end{bmatrix} = \underline{k}_1(i) \qquad (2.12)$$

$$\sum_{j=1}^{p_1} \sum_{k=1}^{m} \beta_j^k(i) \, c_k^T \sum_{s=1}^{j} A^{j-s} \underline{b} \, u_s(i) + \sum_{j=1}^{p} \alpha_j(i) \, u_j(i)$$

$$= K_2^T(i) \, \underline{u}(i) \qquad (2.13)$$

and $$[D \;\; M^T D \; \ldots \; (M^T)^{p_2} D] \begin{bmatrix} \delta_0^1(i) \\ \vdots \\ \delta_0^m(i) \\ \vdots \\ \delta_{p_2}^1(i) \\ \vdots \\ \delta_{p_2}^m(i) \end{bmatrix} = \begin{bmatrix} k_3^1(i) \\ \vdots \\ k_3^q(i) \end{bmatrix} \qquad (2.14)$$

The parameters β_j^k and δ_j^k can be determined (in general, nonuniquely) from equations (2.12) and (2.14) if the coefficient matrices have respective ranks n and q. If β_j^k is known, α_j is determined from equation (2.13). Since the plant and the input system are observable, there exists numbers p_1^* and p_2^* such that for all p_1, p_2 satisfying

$$p_1 \geq p_1^*$$
$$p_2 \geq p_2^*$$

the rank condition will be satisfied. In addition, for physical realizability, the following inequalities must hold

$$p \geq p_1 \qquad p \geq p_2.$$

Therefore the minimum order controller that can be designed is

$$p^* = \max \{ p_1^*, \; p_2^* \}$$

b) Infinite-time problem.

With the augmented state vector method, the performance index for an infinite-time tracking problem is

$$J = \sum_{i=1}^{\infty} \underline{z}^T(i) \, Q \, \underline{z}(i) + \gamma \, u_{p+1}^2(i-1) \qquad (2.16)$$

which is to be minimized subject to (2.4 - 2.8).

From (2.8) it is obvious that the augmented system (2.7) is not a controllable system, and therefore an optimal control minimizing J in (2.16) does not necessarily exist [4]. The following theorem (proved in the Appendix) states the condition under which an optimum control will exist.

Theorem 1.

A unique optimum control minimizing J in (2.16) exists if
1) the eigenvalues of M lie within the unit circle, and
2) the plant (2.1) is completely controllable. The optimum control law is $u^*_{p+1}(i) = -\underline{k}^{*T}\underline{z}(i)$ where $\underline{k}^* = \lim_{N \to \infty} \underline{\bar{k}}(i, N)$ and $\bar{k}(i, N)$ is the control law for an N-stage tracking problem.

Once the optimal control law is found it can be implemented by a compensator. The parameters of the compensator can be determined by equations analogous to (2.10-14).

The next section discusses an alternative method of designing a servomechanism, namely, through error variables.

B. Optimization Using Error Variables.

An optimum servomechanism can be regarded as a dynamic system in which the control drives certain errors to zero in an optimal fashion. These errors are the differences between the plant outputs and the desired outputs. Under suitable assumptions it is possible to write down the equations of a dynamic system in which the state variables are error variables and the control variable is the same as the plant input. Optimal tracking can now be defined to be synonymous with optimal regulation of this error system. The optimal control will be a function of the error variables. If the relationship between the error variables and the state variables of the plant and compensator, and the input state is known, the optimal control is a function of these state variables and can be realized by choosing the compensator order and parameters exactly as in section A. The advantage of this method of design lies in the fact that the theory of the regulator problem which is well-developed can be used to guarantee optimality and stability of the closed-loop compensated system, for a wide class of inputs. The error system which is to be optimized in this design procedure is of dimension $n + p$, whereas the augmented system method calls for an optimization problem of dimension $n + p + q$. This may mean that computer solution of the dynamic programming equations involved with optimizing the error system involves considerably less storage requirements than the method using the augmented system approach.

Also, using an error system, the servomechanism can be made to follow almost any class of inputs; in particular, for an infinite-time control process the restriction that the input class be necessarily generated from a stable system is removed. However, it will turn out that errors in a multiple-output plant which is required to track optimally a given class of signals, cannot be defined completely arbitrarily.

A single-input plant will be considered. The plant and compensator equations are

$$\underline{x}(i+1) = A\,\underline{x} + b\,u_1(i) \quad (2.17)$$

$$\underline{y}(i) = C^T\,\underline{x}(i) \quad (2.18)$$

$$\underline{u}(i+1) = B\,\underline{u}(i) + \underline{d}\,u_{p+1}(i) \quad (2.19)$$

B is a p x p matrix and $\underline{u}(\cdot)$ is a p vector of compensator states.

The composite system can be written as

$$\underline{\bar{w}}(i+1) = \bar{A}\,\underline{w}(i) + \underline{\bar{d}}\,u_{p+1}(i), \quad \underline{y} = \bar{C}^T\,w(i) \quad (2.20)$$

where

$$\underline{w} = \begin{bmatrix} \underline{x} \\ \underline{u} \end{bmatrix} \quad (2.21) \qquad \bar{A} = \begin{bmatrix} A & bo \\ o & B \end{bmatrix} \qquad \underline{\bar{d}} = \begin{bmatrix} o \\ \vdots \\ o \\ \underline{d} \end{bmatrix} \quad (2.22)$$

and

$$\bar{C}^T = \begin{bmatrix} C^T & 0 \end{bmatrix} \quad (2.23)$$

is an m x (n + p) matrix.

A dynamic error system driven by the control $u_{p+1}(i)$ can now be defined by

$$\underline{\tilde{e}}(i+1) = \tilde{A}\,\underline{\tilde{e}}(i) + \underline{\tilde{d}}\,u_{p+1}(i) \quad (2.24)$$

$$\underline{e}(i) = C^T\,\tilde{e}(i) = \underline{y}(i) - \underline{s}(i) \quad (2.25)$$

where $\underline{\tilde{e}}$ is an (n + p) x 1 vector,

$\tilde{A}$ is an (n + p) x (n + p) constant matrix,

$\underline{\tilde{d}}$ is an (n + p) x 1 vector,

$\tilde{C}^T$ is an (m) x (n + p) constant matrix,

$\underline{e}$ is the physical error vector,

$\underline{y}(i)$ is the plant output, and

$\underline{s}(i)$ is the desired plant output.

The error variables $\tilde{e}$ are related to the states $\underline{w}$ and $\underline{r}$ through

$$\tilde{e}(i) = T_1\,\underline{w}(i) + T_2\,\underline{r}(i) \quad (2.26)$$

where T_1 is an $(n+p) \times (n+p)$ matrix and T_2 is an $(n+p) \times q$ matrix. Substituting (2-26) into (2.24) and using (2.20), (2.18) and (2.2) yields the set of equations

$$T_1 \bar{A} = \tilde{A} T_1 \qquad (2.27)$$

$$T_1 \bar{d} = \tilde{d} \qquad (2.28)$$

$$T_2 M = \tilde{A} T_2 \qquad (2.29)$$

$$\tilde{C}^T T_1 = \bar{C}^T \qquad (2.30)$$

$$\tilde{C}^T T_2 = -D^T \qquad (2.31)$$

T_1 will be required to be invertible. This will ensure that any stability statements about $\tilde{\underline{e}}$ will also be applicable to $\underline{w}$. It will also guarantee that controllability of the $\underline{w}$ system will ensure controllability of the error system; controllability of the error system will in turn guarantee the existence and uniqueness of an optimum control u^*_{p+1} for an infinite-time control interval. It is also clear from these remarks that the error system has to be of order $n+p$. If it were any lower, stability predictions could not be made; if it were higher it would be uncontrollable or unobservable or both [15] and the existence of an optimal control would not be assured [16].

Assuming T_1 is invertible, the equations (2.28-31) can be written as

$$\tilde{A} = T_1 \bar{A} T_1^{-1} \qquad (2.32) \qquad\qquad T_1^{-1} T_2 M = \bar{A} T_1^{-1} T_2 \qquad (2.35)$$

$$\tilde{d} = T_1 \bar{d} \qquad (2.33)$$

$$\tilde{C}^T = \bar{C}^T T_1^{-1} \qquad (2.34) \qquad\qquad \bar{C}^T T_1^{-1} T_2 = -D^T \qquad (2.36)$$

It is clear that a non zero matrix solution for $T_1^{-1}\ T_2$ in (2-35) does not necessarily exist. Moreover, even if a non zero solution to (2-35) existed, this solution need not satisfy (2-36), for a given matrix D. It is obvious therefore that the matrix D cannot be arbitrarily chosen. This in turn means that the errors which the servomechanism can drive to zero cannot be arbitrarily defined; these definitions must be such that equations (2-35) and (2-36) are satisfied with a non zero matrix $T_1^{-1}\ T_2$. Equation (2-35) has a non zero solution for $T_1^{-1}\ T_2$ if and only if M and $\bar{A}$ have at least one eigenvalue in common [17]. Also the solution will depend on k arbitrary parameters and will have rank k if k of the eigenvalues are common. Obviously the maximum rank $T_1^{-1}\ T_2$ can have is q , i.e. when all the eigenvalues of M are included in $\bar{A}$. In that case the q arbitrary parameters along with equation (2-36) can be used to determine the freedom in choosing the matrix D. The maximum arbitrariness in the specification of D is therefore limited by these considerations. Once D is chosen subject to these restrictions, the parameters specifying $T_1^{-1}\ T_2$ are fixed and so it is determined. The choice of any non-singular T_1 determines T_2. $\tilde{A}$, $\tilde{d}$ and $\tilde{e}$ are then determined from (2.32-34). The set of eigenvalues of $\bar{A}$ can be made to include those of M by including all eigenvalues of M not contained in A, in the matrix B. Since the eigenvalues of $\bar{A}$ are made up of eigenvalues of A and of B, all the eigenvalues of M will have been included in $\bar{A}$ by this procedure.

The ability to solve the servomechanism problem using an error system depends crucially on the fact that the matrix $\bar{A}$ can be made to possess all the eigenvalues of M. It is always possible to achieve this condition since the open-loop poles of the compensator can be arbitrarily chosen. This freedom is utilized to choose the eigenvalues of B, to be precisely those which enable the definition of certain meaningful (and nontrivial) errors which the optimum compensated system can drive to zero.

In general, the matrix $T_1^{-1} T_2$ depends on arbitrary parameters. However, when the plant has an output from which it is completely observable, the error corresponding to this output, say, the i^{th} output can be specified by

$$c_i^T T_1^{-1} T_2 = - d_i^T \tag{2.37}$$

Then the existence of a unique matrix $T_1^{-1} T_2$ satisfying (2.37) and (2.35) is guaranteed by the following theorem (proved in the Appendix).

<u>Theorem 2.</u>

a) If there exists a solution to (2.35) and (2.37) then

$$O_1 T_1^{-1} T_2 = - O_2 \tag{2.38}$$

where

$$O_1 = \begin{bmatrix} c_i^T \\ c_i^T (\bar{A}) \\ \vdots \\ c_i^T (\bar{A})^{n+p-1} \end{bmatrix} \quad (2.39) \qquad O_2 = \begin{bmatrix} d_i^T \\ d_i^T M \\ \vdots \\ d_i^T M^{n+p-1} \end{bmatrix} \tag{2.40}$$

b) If (i) $(\bar{A}, c_i)$ is an observable pair (2.41)

and (ii) $\{\lambda_i (\bar{A})\} \supset \{\lambda_j (M)\}$ $i = 1, .. n+p$ $j = 1, ...q$

then the unique solution to (2.35) and (2.37) is $T_1^{-1} T_2 = -O_1^{-1} O_2$.

Once a unique $T_1^{-1} T_2$ is found, choice of any nonsingular T_1 determines a unique T_2. As before, $\tilde{A}$, $\tilde{d}$, $\tilde{C}$ are determined from (2.32-34).

The unspecified part of the matrix D is now uniquely determined from (2.36) and dictates what the definition of the remaining (m-1) errors have to be. The optimum tracking problem has now been reduced to a problem of optimum regulation of the error system. An appropriate performance index is

$$J = \sum_{k=1}^{\infty} \tilde{\underline{e}}^T(i) \quad Q \ \tilde{\underline{e}}(i) + \gamma u_{p+1}^2 (i-1) \tag{2.42}$$

A unique optimal control u_{p+1}^* which minimizes J exists if the error system (2.24) is controllable. But (2.24) is a similarity transformation of (2.20), which is always controllable if the plant (2.17) is controllable. The invertibility of T_1 and controllability of the plant therefore guarantees existence of the optimal control law. This is of the form

$$u_{p+1}^*(i) = -\underline{k}_1^T \tilde{\underline{e}}(i) = -\underline{k}_1^T T_1 \underline{w}(i) - \underline{k}_1^T T_2 \underline{r}(i) \tag{2.43}$$

$$= -k_{11}^T \underline{x}(i) - \underline{k}_{12}^T \underline{u}(i) - \underline{k}_2^T \underline{r}(i) \tag{2.44}$$

The optimal control (2.44) is of the same form as in equation (2.8) and the compensator parameters can be determined from equations analogous to (2.10), (2.11) and (2.12). The order p of the compensator has to satisfy the same conditions as in section section A.

Stability (asymptotic) of the error system is ensured by choosing Q in (2.42) so that

$$Q = H^T H, \quad H = n \times (n+p) \quad n = \text{rank } Q \tag{2.45}$$

and $(\tilde{A}, H)$ is an observable pair. Since T_1 is invertible asymptotic stability of $\underline{w}$ is guaranteed by this same condition.

Remark: Obviously both the methods of servomechanism design reduce to the design of a regulator if $\underline{r}(i) = o \quad \forall i$. The order of the controller is determined, in this case, only by the observability properties of the plant.

Three numerical examples are presented below to illustrate the theory.

EXAMPLE 1 REGULATOR DESIGN

Consider the plant

$$x_1(i+1) = x_2(i)$$
$$x_2(i+1) = x_3(i)$$
$$x_3(i+1) = x_1(i) + x_2(i) + x_3(i) + u_1(i)$$
$$y(i) = x_1(i)$$

The plant is observable from x_1 and the observability index is $= 3$. Therefore a second order compensator can be designed. The compensator state variables are defined by

$$u_1(i+1) = u_2(i)$$
$$u_2(i+1) = u_3(i)$$

The performance index is chosen to be

$$I = \sum_{i=1}^{\infty} 5\, x_1^{\,2}(i) + x_2^{\,2}(i) + x_3^{\,2}(i) + u_3^{\,2}(i-1)$$

The optimum control law is

$$u_3(i) = -\underline{k_1}^T \underline{x} - \underline{k_2}^T \underline{u}$$

where

$$k_1^T = [2.1010 \quad 3.1930 \quad 4.1010]$$
$$k_2^T = [2.1010 \quad 1.0920]$$

The compensated system can be written as

$$u_3(i) = -\beta_0\, y(i) - \beta_1\, y(i+1) - \beta_2\, y(i+2) - \alpha_0\, u_1(i) - \alpha_1\, u_1(i+1)$$

$$= -\beta_0\, x_1(i) - \beta_1\, x_2(i) - \beta_2\, x_3 - \alpha_0\, u_1 - \alpha_1\, u_2$$

Equating coefficients of $\underset{\sim}{x}$, and $\underline{u}$ in the optimal and compensated systems

$$k_{11} = \beta_0 = 2 \cdot 1010 \qquad k_{21} = \alpha_0 = 2 \cdot 1010$$

$$k_{12} = \beta_1 = 3 \cdot 1930 \qquad k_{22} = \alpha_1 = 1 \cdot 0920$$

$$k_{13} = \beta_2 = 4 \cdot 1010$$

The compensator transfer function is

$$G_c(z) = \frac{-u_1(z)}{y(z)} = \frac{\beta_0 + \beta_1 z + \beta_2 z^2}{(\alpha_0 + \alpha_1 z + z^2)}.$$

Servomechanism Design

Both examples in this section are concerned with the control of an inverted pendulum hinged to a movable cart. Control is exerted on the system by the current-input to a servo-motor which drives the cart.

The equations of motion (linearized) of the continuous system are

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = 16\, x_1 + 5\, u_1$$

$$\dot{x}_3 = x_4$$

$$x_4 = x_1 - 2\, u_1$$

where

x_1 is the angular position of the pendulum (in radians),

x_2 is the angular velocity (rads./sec.) of the pendulum,

x_3 is the cart position,

x_4 is the cart velocity, and u_1 is the current fed to the motor driving the cart.

The measurable outputs of the plant are $y_1 = x_1$ and $y_2 = x_3$.

The sampled-data version of the plant equations is

$$\underline{x}\ (\underline{i+1}\ T) = \phi_A\ (T)\ \ \underline{x}\ (i\ T) + h\ (T)\ u_1\ (i\ T)$$

where, with a sampling period $T = 0.1$ seconds

$$\phi_A\ (T) = \begin{bmatrix} 0.4108 & 0.1027 & 0 & 0 \\ 1.6430 & 0.4108 & 0 & 0 \\ 0.0368 & 0.0002 & 1 & 0.1 \\ 0.1027 & 0.0368 & 0 & 1 \end{bmatrix}$$

and $$h\ (T) = \begin{bmatrix} -0.0101 \\ -0.0405 \\ 0.5000 \\ 0.5100 \end{bmatrix}$$

EXAMPLE 2

The pendulum is required to follow any signal of the form

$$r\ (t) = r_1(o)\ e^{-\alpha t} \cos w\ t + r_2\ (o)\ e^{-\alpha t} \sin w\ t$$

for a given α and w and $\forall$ $r_1(o)$ and $r_2\ (o)$.

The linear system that will general signals of this form is

$$\dot{\underline{r}} = M\ \underline{r}$$

$$s = r_1$$

where

$$M = \begin{bmatrix} o & 1 \\ -(1+w^2) & -2\alpha \end{bmatrix}$$

The sampled-data version of these equations are, with $\alpha = 1$, and $W = 10$, and $T = 0.1$

$$\underline{r}\ (\underline{i+1}\ T) = \phi_M\ (T)\ \ \underline{r}\ (i)$$

$$s\ (i) = r_1\ (i)$$

$$\phi_M\ (T) = \begin{bmatrix} 0.5650 & 0.0761 \\ -7.6900 & 0.4124 \end{bmatrix}$$

The augmented state-vector method will be used in the design, since the input system is stable.

The observability index of both the plant and the input system is = 2 and therefore the dynamic order of the compensator required = 1. The compensator state is defined by

$$u_1 \ (i+1) = u_2 \ (i)$$

The performance index is

$$J = \sum_{i=1}^{\infty} \underline{z}^T(i) \ Q \ \underline{z}(i) + \gamma \ u_2^2(i-1)$$

with

$$\underline{z}(i) = \begin{bmatrix} \underline{x}(i) \\ u_1(i) \\ \underline{r}(i) \end{bmatrix}$$

$$Q = \begin{bmatrix} 9 & 0 & 0 & 0 & 0 & -9 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -9 & 0 & 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \gamma = 1.$$

The optimum control law is

$$u_2(i) = -k_1 x_1 \ -k_2 x_2 \ -k_3 x_3 \ -k_4 x_4 \ -k_5 u_1 \ -k_6 r_1 \ -k_7 r_2$$

with $k_1 = -0.04525$

$k_2 = -0.05235$

$k_3 = 0.8020$

$k_4 = 0.8578$

$k_5 = 4.3680$

$k_6 = -0.007098$

$k_7 = -0.000169$

The compensator equation is

$$u_1(i+1) = -\sum_{k=1}^{2}\sum_{j=0}^{1} \beta_j^k \, y_k(i+j) - \alpha_o u_1 - \sum_{j=o}^{i} \delta_j \, \beta(i+j)$$

and the coefficients α_o, β_j^k, δ_j are found to be

$\beta_o^1 = 0.4867$

$\beta_o^2 = -7.7760$

$\beta_1^1 = -0.5264$

$\beta_1^2 = 8.5780$

$\alpha_o = 0.07368$

$\delta_o = -0.005843$

$\delta_1 = -0.002221$

EXAMPLE 3

The inputs considered in this example will consist of steps, ramps and accelerations and the design will be carried out through the use of an error system.

The input system is represented by

$$\underline{r}(i+1) = M\ \underline{r}(i) \qquad s_1 = r_1 \qquad s_2 = r_3$$

with

$$M = \begin{bmatrix} 1 & 0.1 & 0.005 \\ 0 & 1 & 0.1 \\ 0 & 0 & 1 \end{bmatrix}$$

The eigenvalues of M are 1,1,1. The plant has two poles at 1, and therefore the compensator state can be defined as

$$u_1(i+1) = u_1(i) + 0.1\, u_2(i)$$

The observability index for the plant as well as the input system is 2, and therefore a 1st order compensator will be designed.

The error system can be written as

$$\underline{\tilde{e}}(i+1) = \bar{A}\ \underline{\tilde{e}} + \underline{d}\ u_2(i)$$

with

$$\bar{A} = \left[\begin{array}{c|c} \phi_A(T) & h(T) \\ \hline 0 \qquad 0 & 1.0 \end{array}\right]$$

$$\underline{d} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.1 \end{bmatrix}$$

where T_1 in (2.32) has been chosen to be the identity matrix.

The solution to the equation $T_2 M = \bar{A}\ T_2$ now yields

$$T_2 = \left[\begin{array}{c|c|c} 0 & 0 & a_1\, t_3^{\,1} \\ 0 & 0 & a_2\, t_3^{\,1} \\ t_3^{\,1} & t_3^{\,2} & t_3^{\,3} \\ 0 & t_3^{\,1} & b_1\, t_3^{\,1} + b_2\, t_3^{\,2} \\ 0 & 0 & b_3\, t_3^{\,1} \end{array}\right]$$

where

$$a_1 = -0{\cdot}01107$$

$$a_2 = -0{\cdot}04428$$

$$b_1 = -0{\cdot}9295$$

$$b_2 = 1{\cdot}0$$

$$b_3 = 0{\cdot}1951$$

and $t_3^{\,1}$, $t_3^{\,2}$, $t_3^{\,3}$ are arbitrary.

Therefore

$$\tilde{C}^T\, T_2 = -D^T = \begin{bmatrix} 0 & 0 & a_1\ t_3^{\,1} \\ t_3^{\,1} & t_3^{\,2} & t_3^{\,3} \end{bmatrix}$$

and D can be defined by choosing $t_3^{\,1}$, $t_3^{\,2}$, $t_3^{\,3}$.

If the pendulum is required to follow a step input the term $t_3^{\,1}$ has to be non-zero. This automatically means that the cart position has to follow are acceleration input.

In this example, the choice

$$t_3^{\,1} = t_3^{\,2} = t_3^{\,3} = 1 \quad \text{is made,}$$

and

$$\underline{e} = \tilde{C}^T . \underline{\tilde{e}} = \tilde{C}^T w - D^T r = \underline{y} - \underline{s}$$

i.e. $$\underline{e} = \begin{bmatrix} x_1 + a_1 \cdot r_3 \\ x_3 + (r_1 + r_2 + r_3) \end{bmatrix}$$

The performance index is

$$J = \sum_{i=1}^{\infty} 9e_1^{\,2}(i) + e_2^{\,2}(i) + u_2^{\,2}(i-1)$$

The optimum control law is

$$u_2(i) = -k_1 \tilde{e}_1 - k_2 \tilde{e}_2 - k_3 \tilde{e}_3 - k_4 \tilde{e}_4 - k_5 \tilde{e}_5$$

$k_1 = -0.0515$

$k_2 = -0.03753$

$k_3 = 0.8188$

$k_4 = 0.5997$

$k_5 = 3.971$

Rewriting the above equation

$$u_2 = -k_1 x_1 - k_2 x_2 - k_3 x_3 - k_4 x_4 - k_5 u_1 - k_6^* r_1 - k_7^* r_2 - K_8^* r_3$$

where

$k_6^* = 0.8188$

$k_7^* = 1.4185$

$k_8^* = 1.6380$

The compensator equation is

$$u_2(i) = -\sum_{k=1}^{2} \sum_{j=0}^{1} \beta_j^k y_k(i+j) - \alpha_0 u_1 - \sum_{k=1}^{2} \sum_{j=0}^{1} \delta_j^k s_k(i+j)$$

and the coefficients are β_j^K, δ_j^K, and α_0 are

β_0^1 = 0.03241

β_0^2 = −0.51782

β_1^1 = −0.03771

β_1^2 = 0.59970

α_0 = −0.9024

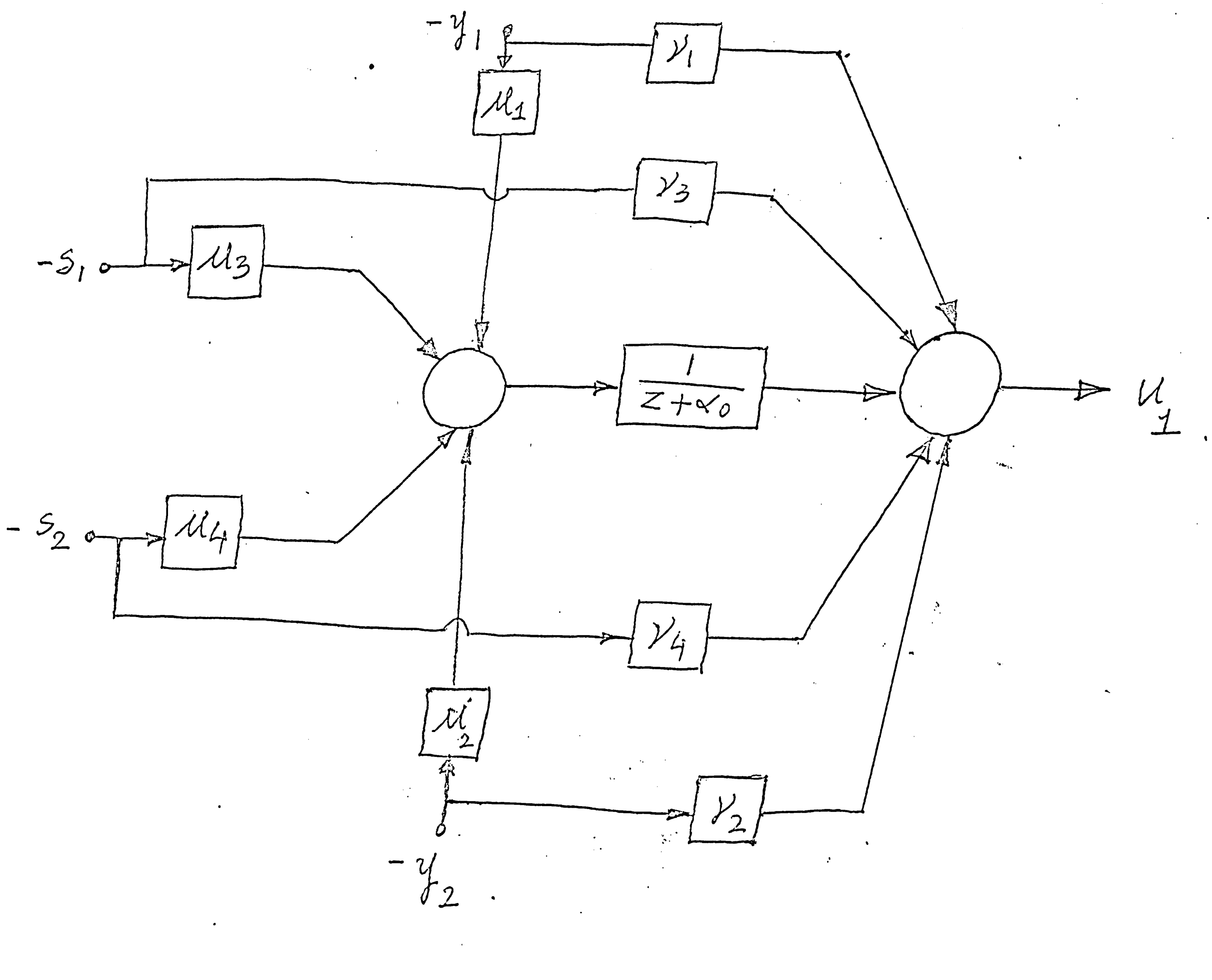
δ_1^2 0.5997

δ_0^2 = − 0.51782

δ_1^1 = 0.0190

δ_0^1 = arbitrary = 0.

The compensator can be schematically represented as follows:



$$\mu_1 = \beta_o^1 - \beta_1^1 \alpha_0 = -0.001620$$

$$\mu_2 = \beta_o^2 - \beta_1^2 \alpha_0 = 0.023349$$

$$\mu_3 = \delta_o^1 - \delta_1^1 \alpha_0 = 0.07122$$

$$\mu_4 = \delta_o^2 - \delta_1^2 \alpha_0 = \mu_2$$

$$\nu_1 = \beta_1^1, \quad \nu_2 = \beta_1^2, \quad \nu_3 = \delta_1^1 \quad \nu_4 = \delta_1^2 = \nu_2$$

Note, that the error corresponding to x_3 could be arbitrarily defined since the second row of D^T could be arbitrarily specified by choosing t_3^1, t_3^2 and t_3^3. The plant is completely observable from x_3 and therefore this is in conformity with Theorem 2.

CHAPTER 111: SUMMARY AND CONCLUSIONS

It has been shown in this thesis that optimal control (in the sense of a regulator or servomechanism) of a linear plant with respect to a quadratic performance index can be achieved using only output measurements. The controller is a dynamic system in cascade with the plant. Its order is determined by the observability properties of the plant; its structure from the requirement of optimality.

The treatment of the servomechanism problem leads to several basic conclusions. First, for a finite-time control interval it is possible to design controllers to make any linear system track optimally any signal generated by another (known) linear system. The class of input signals can be arbitrarily large.

For the infinite-time tracking problem, the class of inputs that can be optimally tracked is reduced if the definition of the errors that the tracking operation optimizes is to be unrestricted. This class is the class of inputs generated by a stable system. This is the conclusion of Th. 1. Conversely, when the class of inputs is unrestricted, the definition of errors that the control signal can optimize is restricted. However, it is important to note that the error corresponding to an observable output can be defined arbitrarily. Theorem 2 is relevant in this context. The mathematical confirmation of the intuitive fact that an observable output can be made to track any signal generated from a linear system, the modes of which are included in the composite plant, is appealing.

POSSIBLE EXTENSIONS AND FURTHER WORK

Since the realization of a controller depends only on the observability properties of the system, this technique of compensator design can be extended to observable nonlinear systems with nonlinear control laws. The order of the compensator will depend on the number of output measurements required to reconstruct the nonlinear functional of the state. In general this will depend on the plant as well as the functional. The nonlinear control law may result from using specific optimal control methods or may be experimentally determined from simulations. Since the procedure for synthesizing a controller is now known a detailed examination of the characteristics of nonlinear plants and feedback control laws may yield a useful design procedure in these cases.

A straight forward extension of the method of design can be made to a multi-input (r inputs) linear plant, and will result in a compensator of order rp . It would be useful to determine if a lower order compensator can be used to ensure optimality, in this case.

In the design of the compensator for the servomechanism problem the matrix $T_1^{-1} T_2$ contains arbitrary parameters. This makes it difficult to compute this matrix with a computer. A computer method to determine all the arbitrary parameters of this matrix (perhaps by using the procedure in [17]) would save a lot of drudgery and yield a design procedure that would be completely computerized.

The fact that in the solution of the servo problem, an observable output of the plant could be made to follow any signal by including the appropriate poles in the compensator and optimizing the error system suggests that this may be a possible approach to the design of multivariable control systems which satisfy certain "noninteracting" constraints.

APPENDIX 1

PROOF OF THEOREM 1

The performance index (2.16) can be written as

$$J \sum_{i=1}^{N} \underline{w}^T(i) \; Q_{11} \; \underline{w}(i) + 2\underline{w}^T(i) \; Q_{12} \; \underline{r}(i)$$
$$\underline{r}^T(i) \; Q_{22} \; \underline{r}(i) + \gamma u^2_{p+1}(i-1) \qquad (1)$$

Where Q_{11}, Q_{12}, Q_{22} correspond to an appropriate partition of Q, and use has been made of the fact $Q = Q^T$.

$\underline{w}(i)$ evolves according to

$$\underline{w}(i+1) = \bar{A}\,\underline{w}(i) + \bar{\underline{d}}\; u_{p+1}(i) \qquad (2)$$

where

$$\bar{A} = \begin{bmatrix} A & bo \\ o & B \end{bmatrix} \qquad \bar{d} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

As shown in Chapter II, the assumption of controllability of (A,b) (assumption (ii) in the theorem) implies controllability of $(\bar{A},\bar{d})$ provided B is defined as in (2-22). Controllability of $(\bar{A},\bar{d})$, implies [3] , that a "dead-beat" controller can be designed for the $\underline{w}$ system.

Let the control law which achieves "dead-beat" control of (2) be denoted by $\left\{ u^o_{p+1}(i) \right\}$ $i = 0,1, \ldots N$ where $N \leq n+p$

Next, it will be shown that the third term in (1) is bounded due to condition (i) of the theorem.

Let $\underline{r}(i) \quad Q_{22} \quad \underline{r}(i) = - \left[V(i+1) - V(i) \right]$ (3)

where $V(i) \triangleq \underline{r}^T(i) \quad P\,\underline{r}(i)$ (4)

P is a symmetric matrix.

Then $V(i+1) = \underline{r}^T(i+1) \; P \, \underline{r}(i+1) = r^T(i) \; M \; P \, M \, \underline{r}(i)$

and $r^T(i) \; Q_{22} \, r(i) = \underline{r}(i) \left[M \; P \, M - P \right] \underline{r}(i)$ (5)

Q 22 is symmetric and therefore

$$- Q_{22} = M^T P \, M - P \qquad (6)$$

Q_{22} is positive semidefinite. This fact, along with assumption (i) of the theorem implies [13] that P satisfying (6) is positive semidefinite.

Now $\sum_{i=o}^{\infty} \underline{r}^T(i) \; Q_{22} \, r(i) = V(o) - V(\infty)$ (7)

$= V(o)$

since $V(\infty) = o$ by assumption (i).

Therefore

$$\sum_{i=o}^{\infty} \underline{r}^T(i) \; Q_{22} \, r(i) = r^T(o) \; P \; r(o) \qquad (8)$$

Using the control $\left\{ u^o_{p+1}(i) \right\}$ yields a cost

$$J\left\{u^o_{p+1}(i)\right\} = \sum_{i=1}^{N} \underline{w}^T_o(i) \; Q_{11} \, \underline{w}_o(i) + 2 \, \underline{w}^T_o(i) \; Q_{12} \, \underline{r}(i) + \underline{r}^T(o) \; P \, \underline{r}(o) \qquad (9)$$

Thus $J\left\{u^o_{p+1}\right\} < \infty$. (10)

The dead-beat control law is a feed-back control law and therefore (10) can be written as

$$J\left\{u^o_{p+1}\right\} = \left[w^T_o(o) \;\; r^T(o) \right] \; R^o \begin{bmatrix} w_o(o) \\ r(o) \end{bmatrix} \qquad (11)$$

Now consider the cost function

$$J_N \left(z(o), u(\cdot)\right) = \sum_{i=1}^{N} z^T(i) \; \bar{Q} \; z(i) + \gamma u_{p+1}^2 (i-1) \qquad (12)$$

Define

$$V_{N-j}(z(j) = \min_{u_{p+1}(j), \ldots u_{p+1}(N-1)} \sum_{i=j+1}^{N} \underline{z}^T(i) \; \bar{Q} \; \underline{z}(i) + \gamma u_{p+1}^2 (i-1) \qquad (13)$$

By the principle of optimality,

$$V_{N-j}\left(z(j)\right) = \min_{u_{p+1}(j)} \left\{ \underline{z}^T(j+1) \; \bar{Q} \, z(j+1) + \gamma u_{p+1}^2 (j) \; V_{N-j-1} \; z(j+1) \right\} \qquad (14)$$

Using (14) it is easily established by induction [19], that

$$V_{N-j}\left(z(j)\right) = \underline{z}^T(j) \quad P(N-j) \quad \underline{z}(j) \qquad (15)$$

where P (N-j) is a positive semidefinite matrix.

The optimal control minimizing J_N in (12) can easily be shown to be

$$u^o(j) = - \underline{k}^T(j, N) \; \underline{z}(j)$$
$$k^T(j, N) = - \frac{\bar{d}^T P(N-j-1) \bar{A}}{\gamma + \bar{d}^T P(N-j-1) \bar{d}} \qquad (16)$$

where

$$P(N-j) = \bar{Q} + \bar{A}^T \; P(N-j-1) \; \bar{A} - \frac{\bar{A}^T \; P(N-j-1) \; \bar{d} \; \bar{d}^T P(N-j-1) \bar{A}}{\gamma + \bar{d}^T P(N-j-1) \bar{d}} \qquad (17)$$

subject to P (o) = o

Obviously

$$V_N\left(z(o)\right) = \underline{z}^T(o) \quad P(N) \quad \underline{z}(o) \leq J\left\{u_{p+1}^o\right\} = z^T(o) \, R^o \; z(o) \quad \forall \; z(o) \qquad (18)$$

Also

$$V_{N+1}\left(z(0)\right) = \min_{u_{p+1}(0)\cdots u_{p+1}(N)} \left\{ \sum_{i=1}^{N+1} \underline{z}^T(i)\,\bar{Q}\,\underline{z}(i) + \gamma u_{p+1}^2(i-1) \right\}.$$

$$= \min_{u_{p+1}(0),\cdots u_{p+1}(N-1)} \left\{ \sum_{i=1}^{N} z^T(i)\,\bar{Q}\,z(i) + \gamma u_{p+1}^2(i-1) \right. \tag{19}$$

$$\left. + \min_{u_{p+1}(N)} \left(\gamma u_{p+1}^2(N) + z^T(N+1)\,Q\,z(N+1) \right) \right\}$$

$$\geq V_N\left(z(0)\right) \tag{20}$$

Therefore $z^T(o)\ P_N\ z(o)$ is a bounded (18) and nondecreasing (20) sequence and therefore converges $\forall\ z(o)$

$$\lim_{N\to\infty} z^T(o)\ P_N\ z(o) = z^T(o)\ \hat{P}\ z(o) \quad \forall\ z(o)$$

and therefore $\lim_{N\to\infty} P_N = \hat{P}$ (21)

Taking limits in both sides of equation (17)

$$\hat{P} = Q + \bar{A}^T\ \hat{P}\ \bar{A} - \frac{\bar{A}^T\ \hat{P}\ \bar{d}\ \bar{d}^T\ \hat{P}\ \bar{A}}{\gamma + \bar{d}^T\ P\ \bar{d}} \tag{22}$$

Let $u^*(j) = -\dfrac{\bar{d}^T\ \hat{P}\ \bar{A}}{\gamma + \bar{d}\ \hat{P}\ \bar{d}}\ \underline{z}(j)$ (23)

First, it will be shown that

$$J\left(z(o),\ u^*(\cdot)\right) = \lim_{N\to\infty} J_N\left(z(o),\ u^*(\cdot)\right) = \underline{z}^T(o)\ \hat{P}\ \underline{z}(o) \tag{24}$$

To show this let

$$J^N\left(z(o),\ u(\cdot)\right) = \underline{z}^T(N)\ \hat{P}\ \underline{z}(N) + J_N\left(z(o)\ u(\cdot)\right) \tag{25}$$

The optimal control law for J^N in (25) is obviously $u^*(j)$.

Therefore

$$\underline{z}^T(o)\ \hat{P}\ \underline{z}(o) = J^N\left(\underline{z}(o),\ u^*(\cdot)\right) = \underline{z}^T(N)\ \hat{P}\ \underline{z}(N) + J_N\left(\underline{z}_o,\ u^*\right) \qquad (26)$$

$$\therefore\ \underline{z}^T(o)\ P\ \underline{z}(o) \geq J_N\left(\underline{z}(o),\ u^*\right) \geq \underline{z}^T(o)\ P(N)\ \underline{z}(o) \qquad (27)$$

and therefore by (21), (24) follows.

Now $u^*(\cdot)$ is claimed to be the optimal control for the infinite time problem.

To prove this claim, let $u^o(\cdot)$ be some other control and assume that

$$J\left(\underline{z}(o),\ u^*(\cdot)\right) - J\left(\underline{z}(o),\ u^o(\cdot)\right) \geq \delta > o \qquad (28)$$

By (24), $\exists$ some N, such that

$$J\left(\underline{z}(o),\ u^*\right) \leq \underline{z}^T(o)\ P(N)\ \underline{z}(o) + \delta/2 \qquad (29)$$

Also $J\left(\underline{z}(o),\ u^o\right) \geq J_N\left(\underline{z}(o),\ u^o\right) \quad \forall\ u^o$

by the definition of J_N.

$$\therefore\ J\left(\underline{z}(o),\ u^*\right) \geq J_N\left(\underline{z}(o),\ u^o\right) + \delta \qquad (30)$$

and this implies

$$\underline{z}^T(o)\ P(N)\ \underline{z}(o) \gg J_N\left(\underline{z}(o),\ u^o\right) + \delta/2$$

which is a contradiction because

$\underline{z}^T(o)\ P(N)\ \underline{z}(o)$ is the minimum value of J_N.

Therefore the optimal control law is u^*.

Q. E. D.

PROOF OF THEOREM II

Part (a):- Assume there esists a solution $T_1^{-1}\,T_2$ to the equations

$$T_1^{-1}\,T_2\,M = \bar{A}\,T_1^{-1}\,T_2 \qquad (1)$$

and

$$c_i^T\,T_1^{-1}\,T_2 = -\,d_i^T \qquad (2)$$

Multiplying (2) from the right by M and using (1), yields

$$c_i^T(\bar{A})^{1}T_1^{-1}\,T_2 = -\,d_i^T\,(M)^{1} \qquad (3)$$

Repeating this process n+p - 1 times yields the set of equations

$$c_i^T\,(\bar{A})^k\,T_1^{-1}\,T_2 = -\,d_i^T\,M^k\,,\quad k = 1, 2, \dots n+p-1 \qquad (4)$$

Defining

$$O_1 = \begin{bmatrix} c_i^T \\ c_i^T\,\bar{A} \\ \vdots \\ c_i^T(\bar{A})^{n+p-1} \end{bmatrix} \qquad O_2 = \begin{bmatrix} d_i^T \\ d_i^T\,M \\ \vdots \\ d_i^T\,M^{n+p-1} \end{bmatrix}$$

proves part (a).

Part (b):- Assume

(I) $(\bar{A}, c_i)$ is an observable pair

(II) $\{\lambda_i\,(\bar{A})\} \supset \{\lambda_j\,(M)\}$

$i = 1, \dots n+p \qquad j = 1, \dots q$

Let the characteristic equation of M be

$$\lambda^q + \sum_{j=0}^{q-1} \beta_j\,\lambda^j = 0 \qquad (5)$$

Because a matrix satisfies its own characteristic equation [17],

$$M^q + \sum_{j=0}^{q-1} \beta_j\,M^j = 0$$

Multiplying (6) by M^i, yields

$$M^{q+i} + \sum_{j=0}^{q-1} \beta_j M^{j+i} = 0 \tag{7}$$

$$i = 0, 1, 2, \ldots$$

Multiplying (2) from the right by M and using (4) gives

$$c_i^T T_1^{-1} T_2 \quad M = c_i^T \bar{A} T_1^{-1} T_2 \tag{8}$$

This process can be repeated n+p - 2 times to yield

$$c^T \bar{A}^j \; T_1^{-1} T_2 \; M = c_i^T \bar{A}^j \; \bar{A} T_1^{-1} T_2$$

$$j = 0, 1, \ldots n+p-2 \tag{9}$$

Letting $k = j+i$ in (4), multiplying by β_j, and using (7)

gives the relation

$$c_i^T \left[\bar{A}^{q+i} + \sum_{j=0}^{q-1} \beta_j \; \bar{A}^{j+i}\right] T_1^{-1} T_2 = 0 \tag{10}$$

Letting $k = n+p-1$ in (4) and multiplying from the right by M yields

$$\left(c_i^T \; \bar{A}^{n+p-1}\right) T_1^{-1} T_2 \quad M = -d_i^T \quad M^{n+p} \tag{11}$$

Substituting for M^{n+p} in (11) from (7) by putting $i = n+p-q$

in (7), and using (4) again, gives

$$\left(c_i^T \bar{A}^{n+p-1}\right) T_1^{-1} T_2 M = d_i^T \left[\sum_{j=0}^{q-1} \beta_j \; M^{n+p+j-q}\right]$$

$$= \sum_{j=0}^{q-1} \beta_j \, d_i^T \; M^{n+p+j-q} \tag{12}$$

$$= - \sum_{j=0}^{q-1} \beta_j \; c_i^T \; (\bar{A})^{n+p+j-q} \; T_1^{-1} T_2$$

By assumption (ii)

$$\left(\bar{A}^q + \sum_{j=0}^{q-1} \beta_j \bar{A}^j\right)\left(\bar{A}^{n+p-q} + \sum_{j=0}^{n+p-q-1} \alpha_j \; \bar{A}^j\right) = 0 \tag{13}$$

or

$$\left(\bar{A}^{n+p} + \sum \beta_j \bar{A}^{n+p+j-q}\right) + \sum_{j=0}^{n+p-q-1} \alpha_j \left[\bar{A}^{q+j} + \sum_{i=0}^{q-1} \beta_i \bar{A}^{j+i}\right] = 0 . \tag{14}$$

Multiplying by c_i^T from the left and $T_1^{-1} T_2$ from the right gives,

$$c_i^T \bar{A}^{n+p} T_1^{-1} T_2 + \sum_{j=0}^{q-1} \beta_j c_i^T \bar{A}^{n+p\, j-q} T_1^{-1} T_2$$

$$+ \sum_{j=0}^{n\,p\,q-1} c_i^T \alpha_j \left[\bar{A}^{q+j} + \sum_{j=0}^{q-1} \beta_i \bar{A}^{j+i}\right] T_1^{-1} T_2 = 0 \tag{15}$$

By (10), the last set of terms in (15) vanishes, and therefore

$$- \sum_{j=0}^{q-1} \beta_j c_i^T \bar{A}^{n+p+j-q} T_1^{-1} T_2 = c_i^T \bar{A}^{n+p} T_1^{-1} T_2 \tag{16}$$

From (12) and (16),

$$c_i^T \bar{A}^{n+p-1} T_1^{-1} T_2 M = c_i^T \bar{A}^{n+p-1} \bar{A} T_1^{-1} T_2 \tag{17}$$

Using (15) and (9) and assumption (ii) yields

$$T_1^{-1} T_2 M = \bar{A} T_1^{-1} T_2$$

Q. E. D.

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